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Hyperdeterminantal expressions for Jack functions of rectangular shapes ☆

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Abstract

We derive a Jacobi–Trudi type formula for Jack functions of rectangular shapes. In this formula, we make use of a hyperdeterminant, which is Cayley's simple generalization of the determinant. In addition, after developing the general theory of hyperdeterminants, we give summation formulas for Schur functions involving hyperdeterminants, and evaluate Toeplitz type hyperdeterminants by using Jack function theory.

Keywords: Jacobi–Trudi formula; Hyperdeterminant; Jack function; Schur function; Toeplitz determinant; Pfaffian

1. Introduction

The Schur function associated with a partition λ of length $\leqslant n$ has the determinantal expression

$$s_\lambda = \det(h_{\lambda_i - i + j})_{1 \leqslant i, j \leqslant n},$$

where h_k is the complete symmetric function, i.e., the one-row Schur function $s_{(k)}$. This formula is called the Jacobi–Trudi formula.

Jack functions are symmetric functions indexed by partitions, i.e., Young diagrams. They have one parameter $\alpha > 0$ and include Schur functions as the special case $\alpha = 1$. In this paper, we

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consider the problem of obtaining a Jacobi–Trudi type formula for Jack functions. Lassalle and Schlosser [14,15] have recently derived such a formula for Macdonald functions. Their formula gives an expansion of any Macdonald function with respect to one-row Macdonald functions or elementary symmetric functions. Macdonald functions [20] are generalizations of Jack functions, and so the formula of Lassalle and Schlosser reduces to one for Jack functions as a limit case. However, their formula is very complicated and is not expressed in a determinant-like form. Kerov obtains a determinant expression for Macdonald functions (and so Jack functions) of hook shapes [12]. Also, the other determinantal expression is seen in [13].

We would like to obtain a Jacobi–Trudi formula for Jack functions expressed in a determinant-like form. In particular, the goal of this paper is to obtain the Jacobi–Trudi type formula for Jack functions of *rectangular shapes* when the parameter α is either an integer m or its inverse $1/m$. By Jack functions of a rectangular shape we mean the Jack functions associated with a rectangular Young diagram.

In our formula, we employ a *hyperdeterminant*. Cayley [5] defined some generalizations of the determinant to higher-dimensional arrays. Among them, we here deal with the following simple alternating sum

$$\det{}^{[2m]}(A) := \frac{1}{n!} \sum_{\sigma_1,\dots,\sigma_{2m} \in \mathfrak{S}_n} \operatorname{sgn}(\sigma_1)\cdots\operatorname{sgn}(\sigma_{2m}) \prod_{i=1}^{n} A\big(\sigma_1(i),\dots,\sigma_{2m}(i)\big)$$

for an array $A = (A(i_1,\dots,i_{2m}))_{1 \leqslant i_1,\dots,i_{2m} \leqslant n}$. We call it the hyperdeterminant of A. This polynomial has been studied in [2,7,16,18,19], see also [23]. Specifically, a Cauchy–Binet type summation formula for hyperdeterminants is obtained (see e.g. [2]). Luque and Thibon [18,19] studied the hyperdeterminant analogue of Hankel determinants, which is closely related to Selberg's integral evaluation. Furthermore, Haukkanen [7] and Luque [16] studied a hyperdeterminant analogue of GCD matrices and its extension to semilattices.

Our main result is stated as follows: for the Jack Q-function $Q^{(1/m)}_{(L^n)}$ associated with the partition $(L, L, \dots, L)$ with n components and parameter $\alpha = 1/m$, we obtain the following expression:

$$Q^{(1/m)}_{(L^n)} = \frac{n!(m!)^n}{(mn)!} \det{}^{[2m]}\big(g^{(1/m)}_{L+i_1+\cdots+i_m-i_{m+1}-\cdots-i_{2m}}\big)_{1 \leqslant i_1,\dots,i_{2m} \leqslant n}, \tag{1.1}$$

where $g^{(1/m)}_k$ are one-row Jack Q-functions. If we substitute $m = 1$ in expression (1.1), we obtain the Jacobi–Trudi formula for Schur functions of rectangular shapes $s_{(L^n)} = \det(h_{L+i-j})_{1 \leqslant i,j \leqslant n}$. Thus, our formula (1.1) is the Jacobi–Trudi formula for Jack functions of rectangular shapes.

We give some applications of hyperdeterminants here. In detail, this paper is organized as follows:

In Section 2, we develop the theory of hyperdeterminants and hyperpfaffians, where the hyperpfaffian is a pfaffian analogue of hyperdeterminants. We give a Cauchy–Binet type integral formula involving hyperdeterminants, which was previously essentially obtained in summation form. However, the present integral form yields many applications. We also define a hyperpfaffian and give related integral formulas. These integral formulas will be applied in later sections. Note that our hyperpfaffian differs from that of Barvinok; the explicit relation between these two hyperpfaffians is given in Appendix A.

In Section 3, we obtain summation formulas for Schur functions by employing the formulas obtained in Section 2. We express the summation

$$\sum_{\lambda:\ \ell(\lambda)\leqslant n} s_\lambda\big(\mathbf{x}^{(1)}\big)s_\lambda\big(\mathbf{x}^{(2)}\big)\cdots s_\lambda\big(\mathbf{x}^{(k)}\big), \quad \text{where each } \mathbf{x}^{(i)} \text{ is the set of } n \text{ variables,}$$

as a hyperdeterminant if k is even, or as a hyperpfaffian if k is odd. Our formulas in Section 3 are generalizations of Cauchy's determinant formula, Gessel's formula [6], and Stembridge's formula [24].

In Section 4, we compute Toeplitz hyperdeterminants. A Toeplitz hyperdeterminant is a hyperdeterminant whose entries are given by $d(i_1+i_2+\cdots+i_m-i_{m+1}-\cdots-i_{2m})$, where $d(k)$ are Fourier coefficients for a function defined on the unit circle. We compute values of some Toeplitz hyperdeterminants explicitly by employing the theory of Jack functions and the integral formulas obtained in Section 2. The problem of calculating a Toeplitz hyperdeterminant is then reduced to the problem of evaluating a coefficient in the expansion of a certain symmetric function with respect to Jack functions. Furthermore, we obtain a Toeplitz hyperdeterminant version of the strong Szegö limit theorem. We derive the asymptotic behavior of a Toeplitz hyperdeterminant when the dimension n goes to the infinity.

Finally, we prove our main result (1.1) in Section 5, employing the technique for Toeplitz hyperdeterminants developed in Section 4.

2. Hyperdeterminants and hyperpfaffians

In this section, we first give a Cauchy–Binet type integral formula for hyperdeterminants. We define a new type of hyperpfaffian and obtain some integral formulas making use of this concept. These formulas will be applied in later sections.

We first state the definition of the hyperdeterminant again, see also [5]. Let m and n be positive integers. For an array $A=(A(i_1,\dots,i_{2m}))_{1\leqslant i_1,\dots,i_{2m}\leqslant n}$, the hyperdeterminant of A is defined by the alternating sum

$$\det{}^{[2m]}\big(A(i_1,\dots,i_{2m})\big)_{1\leqslant i_1,\dots,i_{2m}\leqslant n} := \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n^{2m}} \operatorname{sgn}(\sigma)\prod_{i=1}^{n} A\big(\sigma_1(i),\sigma_2(i),\dots,\sigma_{2m}(i)\big),$$

where $\operatorname{sgn}(\sigma)=\operatorname{sgn}(\sigma_1)\cdots\operatorname{sgn}(\sigma_{2m})$ if $\sigma=(\sigma_1,\dots,\sigma_{2m})\in\mathfrak{S}_n^{2m}$. We will sometimes write $(A(i_1,\dots,i_{2m}))_{1\leqslant i_1,\dots,i_{2m}\leqslant n}$ as $(A(i_1,\dots,i_{2m}))_{[n]}$ for short.

The following proposition is a generalization of the Cauchy–Binet formula, and was previously essentially obtained in summation form, see e.g. [2]. The case $m=1$ is the well known Cauchy–Binet formula for determinants.

Proposition 2.1. *Let $(X,\mu(\mathrm{d}x))$ be a measure space and $\{\phi_{i,j}\}_{1\leqslant i\leqslant 2m,\,1\leqslant j\leqslant n}$ functions on X. Assume*

$$M(i_1,\dots,i_{2m}) := \int_X \phi_{1,i_1}(x)\phi_{2,i_2}(x)\cdots\phi_{2m,i_{2m}}(x)\,\mu(\mathrm{d}x)$$

is well defined. Then we have

$$\frac{1}{n!}\int_{X^n}\prod_{i=1}^{2m}\det\bigl(\phi_{i,j}(x_k)\bigr)_{1\leqslant j,k\leqslant n}\cdot\prod_{j=1}^{n}\mu(\mathrm{d}x_j)=\det{}^{[2m]}\bigl(M(i_1,\dots,i_{2m})\bigr)_{[n]}.$$

Proof. From definition of the hyperdeterminant, we see that

$$\begin{aligned}\det{}^{[2m]}\bigl(M(i_1,\dots,i_{2m})\bigr)_{[n]}&=\frac{1}{n!}\sum_{\sigma_1,\dots,\sigma_{2m}\in\mathfrak{S}_n}\operatorname{sgn}(\sigma_1\cdots\sigma_{2m})\prod_{j=1}^{n}\left(\int_X\prod_{i=1}^{2m}\phi_{i,\sigma_i(j)}(x)\,\mu(\mathrm{d}x)\right)\\&=\frac{1}{n!}\int_{X^n}\sum_{\sigma_1,\dots,\sigma_{2m}\in\mathfrak{S}_n}\operatorname{sgn}(\sigma_1\cdots\sigma_{2m})\prod_{j=1}^{n}\prod_{i=1}^{2m}\phi_{i,\sigma_i(j)}(x_j)\cdot\prod_{j=1}^{n}\mu(\mathrm{d}x_j).\end{aligned}$$

Here, in the second equality, we switch the integral and the product. The integrand on the right-hand side above is equal to

$$\prod_{i=1}^{2m}\left(\sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}(\sigma)\prod_{j=1}^{n}\phi_{i,\sigma(j)}(x_j)\right)=\prod_{i=1}^{2m}\det\bigl(\phi_{i,k}(x_j)\bigr)_{1\leqslant j,k\leqslant n},$$

and so we obtain the claim. $\square$

Remark 2.1. We may obtain a permanent analogue of Proposition 2.1 in a similar way. For functions $\{\phi_{i,j}\}_{1\leqslant i\leqslant m,\,1\leqslant j\leqslant n}$, it holds that

$$\frac{1}{n!}\int_{X^n}\prod_{i=1}^{m}\operatorname{per}^{[2]}\bigl(\phi_{i,j}(x_k)\bigr)_{1\leqslant j,\,k\leqslant n}\cdot\prod_{j=1}^{n}\mu(\mathrm{d}x_j)=\operatorname{per}^{[m]}\left(\int_X\prod_{k=1}^{m}\phi_{k,i_k}(x)\,\mu(\mathrm{d}x)\right)_{1\leqslant i_1,\dots,i_m\leqslant n}.$$

Here $\operatorname{per}^{[m]}$ is a *hyper-permanent*

$$\operatorname{per}^{[m]}\bigl(A(i_1,\dots,i_m)\bigr)_{1\leqslant i_1,\dots,i_m\leqslant n}:=\frac{1}{n!}\sum_{\sigma_1,\dots,\sigma_m\in\mathfrak{S}_n}\prod_{i=1}^{n}A\bigl(\sigma_1(i),\dots,\sigma_m(i)\bigr).$$

We next define a hyperpfaffian. Let $B=(B(i_1,\dots,i_{2m}))_{[2n]}$ be an array satisfying

$$B\bigl(i_{\tau_1(1)},i_{\tau_1(2)},\dots,i_{\tau_m(2m-1)},i_{\tau_m(2m)}\bigr)=\operatorname{sgn}(\tau_1)\cdots\operatorname{sgn}(\tau_m)B(i_1,\dots,i_{2m})\tag{2.1}$$

for any $(\tau_1,\dots,\tau_m)\in(\mathfrak{S}_2)^m$. Here each $\tau_s\in\mathfrak{S}_2$ permutes $2s-1$ with $2s$. We then define the *hyperpfaffian* of B by

$$\operatorname{pf}^{[2m]}(B):=\frac{1}{n!}\sum_{\sigma_1,\dots,\sigma_m\in\mathfrak{E}_{2n}}\operatorname{sgn}(\sigma_1\cdots\sigma_m)\prod_{i=1}^{n}B\bigl(\sigma_1(2i-1),\sigma_1(2i),\dots,\sigma_m(2i-1),\sigma_m(2i)\bigr),\tag{2.2}$$

where $\mathfrak{E}_{2n} := \{\sigma \in \mathfrak{S}_{2n} \mid \sigma(2i-1) < \sigma(2i)\ (1 \leqslant i \leqslant n)\}$. When $m=1$, this expression is that of the ordinary pfaffian $\mathrm{pf}(B)$ of an alternating matrix $B = (B(i,j))_{1 \leqslant i,j \leqslant 2n}$.

Remark 2.2. Our hyperpfaffian is expressed as Barvinok's hyperpfaffian, see Section 6.

Remark 2.3. Let $\{\xi_i\}_{i \geqslant 1}$ be anti-commutative symbols, i.e., $\xi_i \xi_j = -\xi_j \xi_i$, and let Λ be the $\mathbb{C}$-algebra generated by these ξ_i's. For a given array $A = (A(i_1, \ldots, i_{2m}))_{[n]}$, if we put

$$\eta = \sum_{1 \leqslant k_1, \ldots, k_{2m} \leqslant n} A(k_1, \ldots, k_{2m})\, \xi_{k_1} \otimes \cdots \otimes \xi_{k_{2m}} \in \Lambda^{\otimes 2m},$$

then we have $\eta^n = n!\, \det^{[2m]}(A)(\xi_1 \cdots \xi_n)^{\otimes 2m}$. Similarly, if we put

$$\zeta = \sum_{1 \leqslant k_1 < k_2 \leqslant 2n} \cdots \sum_{1 \leqslant k_{2m-1} < k_{2m} \leqslant 2n} B(k_1, \ldots, k_{2m})(\xi_{k_1}\xi_{k_2}) \otimes \cdots \otimes (\xi_{k_{2m-1}}\xi_{k_{2m}}) \in \Lambda^{\otimes m}$$

for a given tensor $B = (B(i_1, \ldots, i_{2m}))_{[2n]}$ satisfying (2.1), then $\zeta^n = n!\, \mathrm{pf}^{[2m]}(B)(\xi_1 \cdots \xi_{2n})^{\otimes m}$.

The following proposition describes the relationship between hyperpfaffians and hyperdeterminants. It will be shown that any hyperdeterminant is a special case of a hyperpfaffian.

Proposition 2.2. *Let $A = (A(i_1, \ldots, i_{2m}))_{[n]}$ be any array. Define the array $B = (B(i_1, \ldots, i_{2m}))_{[2n]}$ as follows. If i_{2s-1} is odd and i_{2s} is even for all $1 \leqslant s \leqslant m$, then $B(i_1, \ldots, i_{2m}) = A(p_1, q_1, \ldots, p_m, q_m)$, where $i_{2s-1} = 2p_s - 1$ and $i_{2s} = 2q_s$ for $1 \leqslant s \leqslant m$. If i_{2s-1} is even and i_{2s} is odd for all $1 \leqslant s \leqslant m$, then $B(i_1, \ldots, i_{2m})$ is defined by* (2.1); *otherwise $B(i_1, \ldots, i_{2m}) = 0$. We then have $\mathrm{pf}^{[2m]}(B) = \det^{[2m]}(A)$.*

Proof. From the alternating property (2.1) and the definition of a hyperpfaffian, we can express a hyperpfaffian as alternating sums on $\mathfrak{S}_n^m$:

$$\mathrm{pf}^{[2m]}(B) = \frac{1}{2^{nm} n!} \sum_{\sigma_1, \ldots, \sigma_m \in \mathfrak{S}_{2n}} \mathrm{sgn}(\sigma_1 \cdots \sigma_m) \prod_{i=1}^{n} B\big(\sigma_1(2i-1), \sigma_1(2i), \ldots, \sigma_m(2i-1), \sigma_m(2i)\big).$$

Because of the definition of $B(i_1, \ldots, i_{2m})$, each term vanishes if there exist $1 \leqslant s \leqslant m$ and $1 \leqslant i \leqslant n$ such that $\sigma_s(2i-1) \equiv \sigma_s(2i) \pmod 2$. We may therefore assume that on the sum of the above equality, only one of $\sigma_s(2i-1)$ and $\sigma_s(2i)$ is odd, say $2r_{2s-1} - 1$, and another is even, say $2r_{2s}$ for any $1 \leqslant s \leqslant m$. If $\sigma_s(2i-1) = 2r_{2s}$ (and therefore $\sigma_s(2i) = 2r_{2s-1} - 1$), we replace $B(\ldots, 2r_{2s}, 2r_{2s-1} - 1, \ldots)$ with $-B(\ldots, 2r_{2s-1} - 1, 2r_{2s}, \ldots)$. Then, for each $1 \leqslant s \leqslant m$, the sequences $(\sigma_s(1), \sigma_s(3), \ldots, \sigma_s(2n-1))$ and $(\sigma_s(2), \sigma_s(4), \ldots, \sigma_s(2n))$ are permutations of $1, 3, \ldots, 2n-1$ and $2, 4, \ldots, 2n$, respectively. Hence we have the expression

$$\mathrm{pf}^{[2m]}(B) = \frac{1}{2^{mn} n!} \sum_{\tau_1, \ldots, \tau_{2m} \in \mathfrak{S}_n} \mathrm{sgn}(\sigma_1 \cdots \sigma_m) \prod_{i=1}^{n} 2^m B\big(2\tau_1(i) - 1, 2\tau_2(i), \ldots, 2\tau_{2m-1}(i) - 1, 2\tau_{2m}(i)\big),$$

where σ_s denotes the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & 2n-1 & 2n \\ 2\tau_{2s-1}(1)-1 & 2\tau_{2s}(1) & \cdots & 2\tau_{2s-1}(n)-1 & 2\tau_{2s}(n) \end{pmatrix} \in \mathfrak{S}_{2n}.$$

Since $\mathrm{sgn}(\sigma_s) = \mathrm{sgn}(\tau_{2s-1})\mathrm{sgn}(\tau_{2s})$, we see that

$$\mathrm{pf}^{[2m]}(B) = \frac{1}{n!} \sum_{\tau_1,\dots,\tau_{2m} \in \mathfrak{S}_n} \mathrm{sgn}(\tau_1 \cdots \tau_{2m}) \prod_{i=1}^{n} A\bigl(\tau_1(i), \dots, \tau_{2m}(i)\bigr) = \det{}^{[2m]}(A). \qquad \square$$

We now give a hyperpfaffian analogue of Proposition 2.1. The case $m = 1$ is well known as the de Bruijn formula [3]. We will make use of this proposition in the proofs of Theorems 3.3 and 3.4 below.

Proposition 2.3. *Let m be an odd positive integer. Let $\{\psi_{i,j}\}_{1 \leqslant i \leqslant m,\, 1 \leqslant j \leqslant 2n}$ be functions on a measure space $(X, \mu(\mathrm{d}x))$ and ϵ be a function on $X \times X$ such that $\epsilon(y, x) = -\epsilon(x, y)$. Suppose that*

$$Q(i_1, \dots, i_{2m}) = \frac{1}{2} \int_{X^2} \epsilon(x, y) \prod_{s=1}^{m} \det\begin{pmatrix} \psi_{s,i_{2s-1}}(x) & \psi_{s,i_{2s-1}}(y) \\ \psi_{s,i_{2s}}(x) & \psi_{s,i_{2s}}(y) \end{pmatrix} \mu(\mathrm{d}x)\, \mu(\mathrm{d}y)$$

is well defined. Then we have

$$\frac{1}{(2n)!} \int_{X^{2n}} \mathrm{pf}\bigl(\epsilon(x_i, x_j)\bigr)_{1 \leqslant i,j \leqslant 2n} \prod_{s=1}^{m} \det\bigl(\psi_{s,j}(x_k)\bigr)_{1 \leqslant j,k \leqslant 2n} \prod_{j=1}^{2n} \mu(\mathrm{d}x_j)$$
$$= \mathrm{pf}^{[2m]}\bigl(Q(i_1, \dots, i_{2m})\bigr)_{[2n]}.$$

Proof. A straightforward calculation gives

$$\mathrm{pf}^{[2m]}\bigl(Q(i_1, \dots, i_{2m})\bigr)_{[2n]}$$
$$= \frac{1}{n!} \sum_{\sigma_1,\dots,\sigma_m \in \mathfrak{E}_{2n}} \mathrm{sgn}(\sigma_1 \cdots \sigma_m) \prod_{j=1}^{n} \Biggl\{ \frac{1}{2} \int_{X^2} \epsilon(x, y)$$
$$\times \prod_{s=1}^{m} \det\begin{pmatrix} \psi_{s,\sigma_s(2j-1)}(x) & \psi_{s,\sigma_s(2j-1)}(y) \\ \psi_{s,\sigma_s(2j)}(x) & \psi_{s,\sigma_s(2j)}(y) \end{pmatrix} \mu(\mathrm{d}x)\, \mu(\mathrm{d}y) \Biggr\}$$
$$= \frac{1}{n!2^n} \int_{X^{2n}} \prod_{j=1}^{n} \epsilon(x_{2j-1}, x_{2j})$$
$$\times \prod_{s=1}^{m} \Biggl\{ \sum_{\sigma \in \mathfrak{E}_{2n}} \mathrm{sgn}(\sigma) \prod_{j=1}^{n} \det\begin{pmatrix} \psi_{s,\sigma(2j-1)}(x_{2j-1}) & \psi_{s,\sigma(2j-1)}(x_{2j}) \\ \psi_{s,\sigma(2j)}(x_{2j-1}) & \psi_{s,\sigma(2j)}(x_{2j}) \end{pmatrix} \Biggr\} \prod_{j=1}^{2n} \mu(\mathrm{d}x_j).$$

Since the well-known expansion (see e.g. Lemma 4 in [9])

$$\det(a_{i,j})_{1\leqslant i,j\leqslant 2n} = \sum_{\sigma\in\mathfrak{E}_{2n}} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} \det\begin{pmatrix} a_{\sigma(2j-1),2j-1} & a_{\sigma(2j-1),2j} \\ a_{\sigma(2j),2j-1} & a_{\sigma(2j),2j} \end{pmatrix},$$

we have

$$\begin{aligned}&\mathrm{pf}^{[2m]}\big(Q(i_1,\dots,i_{2m})\big)_{[2n]}\\ &\quad= \frac{1}{n!2^n}\int_{X^{2n}} \prod_{j=1}^{n} \epsilon(x_{2j-1},x_{2j}) \cdot \prod_{s=1}^{m} \det\big(\psi_{s,j}(x_k)\big)_{1\leqslant j,k\leqslant 2n} \prod_{j=1}^{2n} \mu(\mathrm{d}x_j).\end{aligned} \tag{2.3}$$

On the other hand, by expanding the pfaffian, we see that

$$\begin{aligned}&\frac{1}{(2n)!}\int_{X^{2n}} \mathrm{pf}\big(\epsilon(x_i,x_j)\big)_{1\leqslant i,j\leqslant 2n} \prod_{s=1}^{m} \det\big(\psi_{s,j}(x_k)\big)_{1\leqslant j,k\leqslant 2n} \prod_{j=1}^{2n} \mu(\mathrm{d}x_j)\\ &\quad= \frac{1}{(2n)!n!} \sum_{\sigma\in\mathfrak{E}_{2n}} \int_{X^{2n}} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} \epsilon(x_{\sigma(2j-1)},x_{\sigma(2j)}) \cdot \prod_{s=1}^{m} \det\big(\psi_{s,j}(x_k)\big)_{1\leqslant j,k\leqslant 2n} \prod_{j=1}^{2n} \mu(\mathrm{d}x_j).\end{aligned}$$

Since m is odd, by permuting columns of each $\det(\psi_{s,j}(x_k))_{1\leqslant j,k\leqslant 2n}$, we have

$$\begin{aligned}&= \frac{1}{(2n)!n!} \sum_{\sigma\in\mathfrak{E}_{2n}} \int_{X^{2n}} \prod_{j=1}^{n} \epsilon(x_{\sigma(2j-1)},x_{\sigma(2j)}) \cdot \prod_{s=1}^{m} \det\big(\psi_{s,j}(x_{\sigma(k)})\big)_{1\leqslant j,k\leqslant 2n} \prod_{j=1}^{2n} \mu(\mathrm{d}x_j)\\ &= \frac{1}{n!2^n}\int_{X^{2n}} \prod_{j=1}^{n} \epsilon(x_{2j-1},x_{2j}) \cdot \prod_{s=1}^{m} \det\big(\psi_{s,j}(x_k)\big)_{1\leqslant j,k\leqslant 2n} \prod_{j=1}^{2n} \mu(\mathrm{d}x_j).\end{aligned}$$

Here we use the fact that the cardinality of the set $\mathfrak{E}_{2n}$ is $(2n)!/2^n$. Combining the above expression with Eq. (2.3), we obtain the claim. $\square$

We also obtain another integral formula in a similar way to the previous proposition. The following proposition will be used in the proof of Theorem 4.2.

Proposition 2.4. *Let $\{\psi_{i,j}\}_{1\leqslant i\leqslant 2m,\,1\leqslant j\leqslant 2n}$ be functions on a measure space $(X,\mu(\mathrm{d}x))$. Suppose that*

$$R(i_1,\dots,i_{2m}) = \int_X \prod_{s=1}^{m} \det\begin{pmatrix} \psi_{2s-1,i_{2s-1}}(x) & \psi_{2s-1,i_{2s}}(x) \\ \psi_{2s,i_{2s-1}}(x) & \psi_{2s,i_{2s}}(x) \end{pmatrix} \mu(\mathrm{d}x)$$

is well defined. Then we have

$$\frac{1}{n!}\int_{X^n}\prod_{s=1}^{m}\det\big(\psi_{2s-1,j}(x_k)\mid\psi_{2s,j}(x_k)\big)_{1\leqslant j\leqslant 2n,\,1\leqslant k\leqslant n}\prod_{j=1}^{n}\mu(\mathrm{d}x_j)=\mathrm{pf}^{[2m]}\big(R(i_1,\dots,i_{2m})\big)_{[2n]}.$$

Here $\det(a_{j,k}\mid b_{j,k})_{1\leqslant j\leqslant 2n,\,1\leqslant k\leqslant n}$ *denotes the determinant of the* $2n$ *by* $2n$ *matrix whose* j*th row is given by* $(a_{j,1}\ b_{j,1}\ a_{j,2}\ b_{j,2}\ \dots\ a_{j,n}\ b_{j,n})$.

3. Summation formulas for Schur functions

From the propositions obtained in the previous section, we obtain several summation formulas for Schur functions. We express the summation of the product of Schur functions as a hyperdeterminant or hyperpfaffian.

Let $\mathbf{x}=(\mathbf{x}_1,\dots,\mathbf{x}_n)$ be a sequence of n variables. For a sequence $\alpha=(\alpha_1,\dots,\alpha_n)$ of non-negative integers, we introduce $a_\alpha(\mathbf{x})=\det(\mathbf{x}_i^{\alpha_j})_{1\leqslant i,j\leqslant n}$ as in [20]. In particular, we have $a_\delta(\mathbf{x})=\det(\mathbf{x}_i^{n-j})_{1\leqslant i,j\leqslant n}=V(\mathbf{x})$, where $V(\mathbf{x})=\prod_{1\leqslant i<j\leqslant n}(\mathbf{x}_i-\mathbf{x}_j)$ stands for the Vandermonde product and $\delta=\delta_n=(n-1,n-2,\dots,1,0)$. For a partition $\lambda=(\lambda_1\geqslant\cdots\geqslant\lambda_n)$ of length $\leqslant n$, the Schur function s_λ corresponding to λ is defined by $s_\lambda(\mathbf{x})=a_{\lambda+\delta}(\mathbf{x})/V(\mathbf{x})$, where $\lambda+\delta=(\lambda_1+n-1,\lambda_2+n-2,\dots,\lambda_n)$.

The Schur functions have the following well-known summation formulas [20, §I-4, Ex. 6], [8]:

$$\sum_{\lambda:\ \ell(\lambda)\leqslant n}s_\lambda(\mathbf{x})s_\lambda(\mathbf{y})=\frac{1}{V(\mathbf{x})V(\mathbf{y})}\cdot\det\left(\frac{1}{1-\mathbf{x}_i\mathbf{y}_j}\right)_{1\leqslant i,j\leqslant n},\qquad \mathbf{y}=(\mathbf{y}_1,\dots,\mathbf{y}_n),\tag{3.1}$$

$$\sum_{\lambda:\ \ell(\lambda)\leqslant n}s_\lambda(\mathbf{x})=\frac{1}{V(\mathbf{x})}\cdot\mathrm{pf}\left(\frac{\mathbf{x}_i-\mathbf{x}_j}{(1-\mathbf{x}_i)(1-\mathbf{x}_j)(1-\mathbf{x}_i\mathbf{x}_j)}\right)_{1\leqslant i,j\leqslant n}.\tag{3.2}$$

Here we assume n is even in expression (3.2). Note that the determinant on the right-hand side of Eq. (3.1) is called Cauchy's determinant. We extend expressions (3.1) and (3.2) to higher degrees.

Theorem 3.1. *Let* $\mathbf{x}^{(i)}=(\mathbf{x}_1^{(i)},\mathbf{x}_2^{(i)},\dots,\mathbf{x}_n^{(i)})$ *be a sequence of variables for each* $1\leqslant i\leqslant 2m$*. For any positive integer* N*, we have*

$$\sum s_\lambda\big(\mathbf{x}^{(1)}\big)s_\lambda\big(\mathbf{x}^{(2)}\big)\cdots s_\lambda\big(\mathbf{x}^{(2m)}\big)=\prod_{i=1}^{2m}\frac{1}{V(\mathbf{x}^{(i)})}\cdot\det{}^{[2m]}\left(\frac{1-(\mathbf{x}_{i_1}^{(1)}\cdots\mathbf{x}_{i_{2m}}^{(2m)})^{n+N}}{1-\mathbf{x}_{i_1}^{(1)}\cdots\mathbf{x}_{i_{2m}}^{(2m)}}\right)_{[n]},$$

where the sum is over all partitions $\lambda=(\lambda_1,\dots,\lambda_n)$ *of length* $\ell(\lambda)\leqslant n$ *and of largest part* $\lambda_1\leqslant N$.

Proof. Let $X=\{0,1,\dots,n+N-1\}$ and $\phi_{i,j}(k)=(\mathbf{x}_j^{(i)})^k$, and apply Proposition 2.1. Here the integrals in Proposition 2.1 are regarded as summations over X. Then, since

$$\sum_{k=0}^{n+N-1}\phi_{1,i_1}(k)\phi_{2,i_2}(k)\cdots\phi_{2m,i_{2m}}(k)=\sum_{k=0}^{n+N-1}\big(\mathbf{x}_{i_1}^{(1)}\cdots\mathbf{x}_{i_{2m}}^{(2m)}\big)^k=\frac{1-(\mathbf{x}_{i_1}^{(1)}\cdots\mathbf{x}_{i_{2m}}^{(2m)})^{n+N}}{1-\mathbf{x}_{i_1}^{(1)}\cdots\mathbf{x}_{i_{2m}}^{(2m)}},$$

we have

$$\det{}^{[2m]}\left(\frac{1-(\mathbf{x}^{(1)}_{i_1}\cdots\mathbf{x}^{(2m)}_{i_{2m}})^{n+N}}{1-\mathbf{x}^{(1)}_{i_1}\cdots\mathbf{x}^{(2m)}_{i_{2m}}}\right)_{[n]}=\sum_{n+N-1\geqslant j_1>\cdots>j_n\geqslant 0}\prod_{i=1}^{2m}\det\bigl(\bigl(\mathbf{x}^{(i)}_p\bigr)^{j_q}\bigr)_{1\leqslant p,q\leqslant n}.$$

Replacing each sequence $(j_1,\dots,j_n)$ with a partition λ by $\lambda_q+n-q=j_q$, we see that

$$\det{}^{[2m]}\left(\frac{1-(\mathbf{x}^{(1)}_{i_1}\cdots\mathbf{x}^{(2m)}_{i_{2m}})^{n+N}}{1-\mathbf{x}^{(1)}_{i_1}\cdots\mathbf{x}^{(2m)}_{i_{2m}}}\right)_{[n]}=\sum_{\substack{\lambda:\ \ell(\lambda)\leqslant n\\ \lambda_1\leqslant N}}a_{\lambda+\delta}\bigl(\mathbf{x}^{(1)}\bigr)a_{\lambda+\delta}\bigl(\mathbf{x}^{(2)}\bigr)\cdots a_{\lambda+\delta}\bigl(\mathbf{x}^{(2m)}\bigr)$$

and the claim follows. $\square$

Corollary 3.2. *Let* $\mathbf{x}^{(i)}$ *be as in Theorem* 3.1. *Then*

$$\sum_{\lambda:\ \ell(\lambda)\leqslant n}s_\lambda\bigl(\mathbf{x}^{(1)}\bigr)s_\lambda\bigl(\mathbf{x}^{(2)}\bigr)\cdots s_\lambda\bigl(\mathbf{x}^{(2m)}\bigr)=\prod_{i=1}^{2m}\frac{1}{V(\mathbf{x}^{(i)})}\cdot\det{}^{[2m]}\left(\frac{1}{1-\mathbf{x}^{(1)}_{i_1}\cdots\mathbf{x}^{(2m)}_{i_{2m}}}\right)_{[n]}. \tag{3.3}$$

Proof. We may assume that each variable $\mathbf{x}^{(i)}_j$ belongs to the open unit disc $\{z\in\mathbb{C}\mid |z|<1\}$. We obtain the claim by taking the limit as $N\to\infty$ in Theorem 3.1. $\square$

One may see expression (3.3) as a simple multi-version of expression (3.1). An odd-product analogue of expression (3.3) is given as follows:

Theorem 3.3. *Let m be an odd positive number and* $\mathbf{x}^{(i)}=(\mathbf{x}^{(i)}_1,\mathbf{x}^{(i)}_2,\dots,\mathbf{x}^{(i)}_{2n})$ *the sequence of variables for each* $1\leqslant i\leqslant m$. *Then we have*

$$\begin{aligned}&\sum_{\lambda:\ \ell(\lambda)\leqslant 2n}s_\lambda\bigl(\mathbf{x}^{(1)}\bigr)s_\lambda\bigl(\mathbf{x}^{(2)}\bigr)\cdots s_\lambda\bigl(\mathbf{x}^{(m)}\bigr)\\&\quad=\prod_{i=1}^{m}\frac{1}{V(\mathbf{x}^{(i)})}\cdot\mathrm{pf}^{[2m]}\left(\frac{\sum_{p=0}^{\infty}\prod_{s=1}^{m}\{(\mathbf{x}^{(s)}_{i_{2s-1}})^{p+1}-(\mathbf{x}^{(s)}_{i_{2s}})^{p+1}\}}{1-\mathbf{x}^{(1)}_{i_1}\mathbf{x}^{(1)}_{i_2}\cdots\mathbf{x}^{(m)}_{i_{2m-1}}\mathbf{x}^{(m)}_{i_{2m}}}\right)_{[2n]}.\end{aligned}$$

Proof. Let $X=\{0,1,2,\dots\}$ and $\psi_{i,j}(k)=(\mathbf{x}^{(i)}_j)^k$. Let ϵ be the alternating function defined by $\epsilon(k,l)=1$ for $k>l$ and apply Proposition 2.3. Then we have

$$\begin{aligned}Q(i_1,\dots,i_{2m})&=\sum_{k>l\geqslant 0}\prod_{s=1}^{m}\bigl\{\bigl(\mathbf{x}^{(s)}_{i_{2s-1}}\bigr)^k\bigl(\mathbf{x}^{(s)}_{i_{2s}}\bigr)^l-\bigl(\mathbf{x}^{(s)}_{i_{2s-1}}\bigr)^l\bigl(\mathbf{x}^{(s)}_{i_{2s}}\bigr)^k\bigr\}\\&=\sum_{p,l\geqslant 0}\prod_{s=1}^{m}\bigl(\mathbf{x}^{(s)}_{i_{2s-1}}\mathbf{x}^{(s)}_{i_{2s}}\bigr)^l\bigl\{\bigl(\mathbf{x}^{(s)}_{i_{2s-1}}\bigr)^{p+1}-\bigl(\mathbf{x}^{(s)}_{i_{2s}}\bigr)^{p+1}\bigr\}\\&=\frac{\sum_{p=0}^{\infty}\prod_{s=1}^{m}\{(\mathbf{x}^{(s)}_{i_{2s-1}})^{p+1}-(\mathbf{x}^{(s)}_{i_{2s}})^{p+1}\}}{1-\mathbf{x}^{(1)}_{i_1}\mathbf{x}^{(1)}_{i_2}\cdots\mathbf{x}^{(m)}_{i_{2m-1}}\mathbf{x}^{(m)}_{i_{2m}}}.\end{aligned}$$

Proposition 2.3 therefore implies that

$$\begin{aligned}\mathrm{pf}^{[2m]}\big(Q(i_1,\dots,i_{2m})\big)_{[2n]} &= \sum_{k_1>k_2>\cdots>k_{2n}\geqslant 0} \mathrm{pf}\big(\epsilon(k_i,k_j)\big)_{1\leqslant i,j\leqslant 2n}\prod_{s=1}^{m}\det\big(\big(\mathbf{x}_p^{(s)}\big)^{k_q}\big)_{1\leqslant p,q\leqslant 2n}\\ &= \sum_{\lambda_1\geqslant\cdots\geqslant\lambda_{2n}\geqslant 0}\prod_{s=1}^{m}\det\big(\big(\mathbf{x}_p^{(s)}\big)^{\lambda_q+2n-q}\big)_{1\leqslant p,q\leqslant 2n}.\end{aligned}$$

Here we have replaced each $(k_1,\dots,k_{2N})$ with a partition $(\lambda_1,\dots,\lambda_{2N})$ by $k_q=\lambda_q+2N-q$. Thus, the theorem follows. □

Recall the Jacobi–Trudi formula for Schur functions

$$s_\lambda=\det(h_{\lambda_i-i+j})_{1\leqslant i,j\leqslant n} \tag{3.4}$$

for any partition λ of length $\leqslant n$. Here h_k is the complete symmetric function

$$h_k(\mathbf{x})=\sum_{\substack{k_1,k_2,\ldots\geqslant 0\\ k_1+k_2+\cdots=k}}\mathbf{x}_1^{k_1}\mathbf{x}_2^{k_2}\cdots$$

in countably many variables $\mathbf{x}=(\mathbf{x}_1,\mathbf{x}_2,\dots)$ if $k\geqslant 0$, or $h_k=0$ otherwise. From Eq. (3.4) we obtain another summation formula for Schur functions.

Theorem 3.4. *Let $\mathbf{x}^{(i)}=(\mathbf{x}_1^{(i)},\mathbf{x}_2^{(i)},\dots)$ for each $i\geqslant 1$. Then, for each positive integer m, we have*

$$\begin{aligned}&\sum_{\lambda:\ \ell(\lambda)\leqslant n} s_\lambda\big(\mathbf{x}^{(1)}\big)\cdots s_\lambda\big(\mathbf{x}^{(2m)}\big)\\ &\quad=\det{}^{[2m]}\Big(\sum_{k\geqslant 0}h_{k-i_1}\big(\mathbf{x}^{(1)}\big)h_{k-i_2}\big(\mathbf{x}^{(2)}\big)\cdots h_{k-i_{2m}}\big(\mathbf{x}^{(2m)}\big)\Big)_{[n]}.\end{aligned} \tag{3.5}$$

Furthermore, for an odd positive integer m,

$$\begin{aligned}&\sum_{\lambda:\ \ell(\lambda)\leqslant 2n} s_\lambda\big(\mathbf{x}^{(1)}\big)\cdots s_\lambda\big(\mathbf{x}^{(m)}\big)\\ &\quad=\mathrm{pf}^{[2m]}\left(\sum_{k,l\geqslant 0}\prod_{s=1}^{m}\det\begin{pmatrix} h_{k+l+1-i_{2s-1}}(\mathbf{x}^{(s)}) & h_{l-i_{2s-1}}(\mathbf{x}^{(s)})\\ h_{k+l+1-i_{2s}}(\mathbf{x}^{(s)}) & h_{l-i_{2s}}(\mathbf{x}^{(s)})\end{pmatrix}\right)_{[2n]}.\end{aligned} \tag{3.6}$$

Proof. Apply Proposition 2.1 to $X=\{0,1,2,\dots\}$ and $\phi_{i,j}(k)=h_{k-j+1}(\mathbf{x}^{(i)})$. Then we have

$$\sum_{k_1>\cdots>k_n\geqslant 0}\prod_{i=1}^{2m}\det\big(h_{k_q-p+1}\big(\mathbf{x}^{(i)}\big)\big)_{1\leqslant p,q\leqslant n}$$

$$= \sum_{\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0} \prod_{i=1}^{2m} \det\bigl(h_{\lambda_q+n-q-(n-p+1)+1}\bigl(\mathbf{x}^{(i)}\bigr)\bigr)_{1 \leqslant p,q \leqslant n}$$

$$= \sum_{\lambda:\ \ell(\lambda) \leqslant n} \prod_{i=1}^{2m} \det\bigl(h_{\lambda_q-q+p}\bigl(\mathbf{x}^{(i)}\bigr)\bigr)_{1 \leqslant p,q \leqslant n}.$$

Here in the first step we have replaced each k_q with $\lambda_q + n - q$ by a partition λ and changed the order of rows of determinants $\det(h_{k_q-p+1}(\mathbf{x}^{(i)}))_{1 \leqslant p,q \leqslant n}$. Hence the claim (3.5) follows from expression (3.4). The second formula (3.6) follows by applying Proposition 2.3 to $X = \{0, 1, \ldots\}$, $\epsilon(k,l) = 1$ for $k > l$, and $\psi_{i,j}(k) = h_{k-j+1}(\mathbf{x}^{(i)})$ in a similar way. □

The cases $m = 1$ in expressions (3.5) and (3.6) have previously been obtained in [6,24], respectively.

4. Toeplitz hyperdeterminants

In this section, we consider a class of hyperdeterminants called Toeplitz hyperdeterminants, and we evaluate them by employing the theory of Jack polynomials. Furthermore, we obtain the strong Szegö limit formula for Toeplitz hyperdeterminants.

4.1. Heine–Szegö formula for Toeplitz hyperdeterminants

Let $f(z)$ be a complex-valued function on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ whose Fourier expansion is given by $f(z) = \sum_{k \in \mathbb{Z}} d(k) z^k$. Then the hyperdeterminant

$$D_n^{[2m]}(f) = \det{}^{[2m]}\bigl(d(i_1 + \cdots + i_m - i_{m+1} - \cdots - i_{2m})\bigr)_{[n]}$$

is called the *Toeplitz hyperdeterminant* of f, see [18].

Theorem 4.1. *For a function $f \in L^1(\mathbb{T})$, we have*

$$D_n^{[2m]}(f) = \frac{1}{n!} \int_{\mathbb{T}^n} f(z_1) f(z_2) \cdots f(z_n) \bigl| V(z_1, \ldots, z_n) \bigr|^{2m} \, \mathrm{d}z_1 \cdots \mathrm{d}z_n,$$

where $\mathrm{d}z_j$ is the Haar measure on $\mathbb{T}$ normalized by $\int_{\mathbb{T}} \mathrm{d}z_j = 1$.

Proof. Apply Proposition 2.1 to functions $\{\phi_{i,j}\}_{1 \leqslant i \leqslant 2m,\, 1 \leqslant j \leqslant n}$ on the measure space $(\mathbb{T}, \mathrm{d}z)$, where

$$\phi_{i,j}(z) = \begin{cases} f(z) z^{n-j} & \text{for } i = 1, \\ z^{n-j} & \text{for } 2 \leqslant i \leqslant m, \\ z^{j-n} & \text{for } m+1 \leqslant i \leqslant 2m. \end{cases}$$

Then, since $\int_{\mathbb{T}} \phi_{1,i_1}(z)\cdots\phi_{2m,i_{2m}}(z)\,\mathrm{d}z = d(i_1+\cdots+i_m-i_{m+1}-\cdots-i_{2m})$, we have

$$D_n^{[2m]}(f) = \frac{1}{n!}\int\limits_{\mathbb{T}^n} f(z_1)\cdots f(z_n)\bigl\{\det\bigl(z_k^{n-j}\bigr)_{1\leqslant j,k\leqslant n}\det\bigl(z_k^{j-n}\bigr)_{1\leqslant j,k\leqslant n}\bigr\}^m\,\mathrm{d}z_1\cdots\mathrm{d}z_n,$$

and upon using $\det(z_k^{n-j})_{1\leqslant j,k\leqslant n}\det(z_k^{j-n})_{1\leqslant j,k\leqslant n} = |V(z_1,\dots,z_n)|^2$, we have the claim. $\square$

The case $m=1$ in this theorem is simply the Heine–Szegö formula for a Toeplitz determinant, see e.g. [4].

We can express $D_n^{[4m]}(f)$ by a hyperpfaffian.

Theorem 4.2. *Toeplitz hyperdeterminants $D_n^{[4m]}(f)$ can be a hyperpfaffian* $\mathrm{pf}^{[2m]}$:

$$D_n^{[4m]}(f) = \mathrm{pf}^{[2m]}\left(\prod_{s=1}^{m}(i_{2s}-i_{2s-1})\cdot d\left((2n+1)m-\sum_{k=1}^{2m} i_k\right)\right)_{[2n]}.$$

Proof. Apply Proposition 2.4 to functions $\{\psi_{i,j}\}_{1\leqslant i\leqslant 2m,\,1\leqslant j\leqslant n}$ on the measure space $(\mathbb{T},\mathrm{d}z)$, where

$$\psi_{i,j}(z) = \begin{cases} f(z)z^{j-n-\frac{1}{2}} & \text{if } i=1,\\ z^{j-n-\frac{1}{2}} & \text{if } i \text{ is odd and } i>1,\\ \bigl(j-n-\frac{1}{2}\bigr)z^{j-n-\frac{1}{2}} & \text{if } i \text{ is even.}\end{cases}$$

Then we have

$$\mathrm{pf}^{[2m]}(R) = \frac{1}{n!}\int\limits_{\mathbb{T}^n} f(z_1)\cdots f(z_n)\bigl|V(z_1,\dots,z_n)\bigr|^{4m}\,\mathrm{d}z_1\cdots\mathrm{d}z_n = D_n^{[4m]}(f)$$

upon using the formula (see e.g. [22, Chapter 11])

$$\det\left(z_k^{j-n-\frac{1}{2}}\;\middle|\;\left(j-n-\frac{1}{2}\right)z_k^{j-n-\frac{1}{2}}\right)_{1\leqslant j\leqslant 2n,\,1\leqslant k\leqslant n} = \prod_{1\leqslant j<k\leqslant n}|z_j-z_k|^4.$$

Here the entry of R is calculated as

$$\begin{aligned} R(i_1,\dots,i_{2m}) &= \int\limits_{\mathbb{T}} f(z)\prod_{s=1}^{m}\left\{\left(i_{2s}-n-\frac{1}{2}\right)-\left(i_{2s-1}-n-\frac{1}{2}\right)\right\}z^{i_{2s-1}+i_{2s}-2n-1}\,\mathrm{d}z\\ &= \prod_{s=1}^{m}(i_{2s}-i_{2s-1})\int\limits_{\mathbb{T}} f(z)z^{i_1+\cdots+i_{2m}-m(2n+1)}\,\mathrm{d}z\\ &= \prod_{s=1}^{m}(i_{2s}-i_{2s-1})\cdot d\bigl(m(2n+1)-i_1-\cdots-i_{2m}\bigr)\end{aligned}$$

and so we obtain the claim. $\square$

In particular, it holds that $D_n^{[4]}(f) = \mathrm{pf}((j-i)d(2n+1-i-j))_{1\leqslant i,j\leqslant 2n}$. For example,

$$D_1^{[4]}(f) = d(0), \qquad D_2^{[4]}(f) = d(2)d(-2) - 4d(1)d(-1) + 3d(0)^2, \ldots.$$

4.2. *Basic properties of Jack functions*

To compute Toeplitz hyperdeterminants, we employ Jack polynomials. We recall the basic properties of Jack functions, see [20, §VI-10] for details. Let $\alpha > 0$ and let $\Lambda(\alpha)$ be the $\mathbb{Q}(\alpha)$-algebra of symmetric functions with variables $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots)$. Let p_k be the power-sum polynomial $p_k(\mathbf{x}) = \mathbf{x}_1^k + \mathbf{x}_2^k + \cdots$ and put $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$ for a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$. Define the scalar product on $\Lambda(\alpha)$ by

$$\langle p_\lambda, p_\mu \rangle_\alpha = \delta_{\lambda,\mu} z_\lambda \alpha^{\ell(\lambda)}$$

for partitions λ and μ. Here $\delta_{\lambda,\mu}$ is Kronecker's delta and $z_\lambda = \prod_{k\geqslant 1} k^{m_k} m_k!$ where $m_k = \#\{i \geqslant 1 \mid \lambda_i = k\}$. We denote a rectangular-shape partition $(k, k, \ldots, k)$ with n components by (k^n).

Let m_λ be the monomial symmetric function. Then Jack P-functions $P_\lambda^{(\alpha)}$ are characterized as homogeneous symmetric functions such that

$$P_\lambda^{(\alpha)} = m_\lambda + \sum_{\mu<\lambda} u_{\lambda\mu} m_\mu \quad \text{with } u_{\lambda\mu} \in \mathbb{Q}(\alpha), \qquad \big\langle P_\lambda^{(\alpha)}, P_\mu^{(\alpha)} \big\rangle_\alpha = 0 \quad \text{if } \lambda \neq \mu,$$

where "$<$" is the dominance ordering. Put

$$c_\lambda(\alpha) = \prod_{(i,j)\in\lambda} \big(\alpha(\lambda_i - j) + \lambda'_j - i + 1\big) \quad \text{and} \quad c'_\lambda(\alpha) = \prod_{(i,j)\in\lambda} \big(\alpha(\lambda_i - j + 1) + \lambda'_j - i\big),$$

where (i, j) run over all squares in the Young diagram associated with λ and $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ is the conjugate partition of λ. The set of functions $\{P_\lambda^{(\alpha)} \mid \lambda \text{ are partitions}\}$ is an orthogonal basis of $\Lambda(\alpha)$. Defining Jack Q-functions by $Q_\lambda^{(\alpha)} = c_\lambda(\alpha) c'_\lambda(\alpha)^{-1} P_\lambda^{(\alpha)}$, the set $\{Q_\lambda^{(\alpha)} \mid \lambda \text{ are partitions}\}$ is its dual basis, i.e., $\langle P_\lambda^{(\alpha)}, Q_\mu^{(\alpha)} \rangle_\alpha = \delta_{\lambda,\mu}$.

We sometimes call the Jack function the *Jack polynomial* if $\mathbf{x}$ is a finite sequence, $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ say. The Jack polynomials also satisfy another orthogonality condition. Define the scalar product by

$$\langle \phi, \psi \rangle'_{n,\alpha} = \frac{1}{n!} \int_{\mathbb{T}^n} \phi(z_1, \ldots, z_n) \psi\big(z_1^{-1}, \ldots, z_n^{-1}\big) \big|V(z_1, \ldots, z_n)\big|^{2/\alpha} \,\mathrm{d}z_1 \cdots \mathrm{d}z_n \tag{4.1}$$

for n-variables symmetric polynomials ϕ, ψ. Then we have the orthogonality

$$\big\langle P_\lambda^{(\alpha)}, Q_\mu^{(\alpha)} \big\rangle'_{n,\alpha} = \delta_{\lambda,\mu} I_n(\alpha) \prod_{(i,j)\in\lambda} \frac{n + (j-1)\alpha - i + 1}{n + j\alpha - i}, \tag{4.2}$$

where

$$I_n(\alpha) := \frac{1}{n!} \int_{\mathbb{T}^n} \left|V(z_1, \dots, z_n)\right|^{2/\alpha} \mathrm{d}z_1 \cdots \mathrm{d}z_n = \frac{\Gamma(n/\alpha + 1)}{n!\Gamma(1/\alpha + 1)^n}. \tag{4.3}$$

The last equality is given in [1, §8] for example.

4.3. Evaluation of Toeplitz hyperdeterminants in terms of Jack functions

Let $\mathbf{1}$ be the function such that $\mathbf{1}(z) = 1$ for any $z \in \mathbb{T}$. From Theorem 4.1 and expression (4.3), we have

$$D_n^{[2m]}(\mathbf{1}) = I_n(1/m) = \frac{(mn)!}{n!(m!)^n}.$$

For a function f on $\mathbb{T}$, denote by $\widehat{D}_n^{[2m]}(f)$ the normalized Toeplitz hyperdeterminant

$$\widehat{D}_n^{[2m]}(f) = \frac{D_n^{[2m]}(f)}{D_n^{[2m]}(\mathbf{1})}.$$

We compute Toeplitz hyperdeterminants employing Jack polynomials. Let f be a function in $L^1(\mathbb{T})$ with the Fourier expansion $f(z) = \sum_{k \in \mathbb{Z}} d(k) z^k$. The value $D_n^{[2m]}(f)$ is independent of $d(k)$ for $|k| > (n-1)m$ because the $d(k)$ do not appear among entries of $D_n^{[2m]}(f)$. Thus, there exists a non-negative integer R such that $D_n^{[2m]}(f) = D_n^{[2m]}(F_R)$, where $F_R(z) = \sum_{k \geqslant -R} d(k) z^k$. Then, from Theorem 4.1, we see that

$$D_n^{[2m]}(F_R) = \frac{1}{n!} \int_{\mathbb{T}^n} \prod_{k=1}^n z_k^R F_R(z_k) \cdot \overline{(z_1 \cdots z_n)^R} \cdot \left|V(z_1, \dots, z_n)\right|^{2m} \mathrm{d}z_1 \cdots \mathrm{d}z_n.$$

We have $P_{(R^n)}^{(\alpha)}(\mathbf{x}_1, \dots, \mathbf{x}_n) = (\mathbf{x}_1 \cdots \mathbf{x}_n)^R$ for any $\alpha > 0$ (see [20, §VI, (4.17)]), while the formal power series

$$S_f(\mathbf{x}_1, \dots, \mathbf{x}_n; R) := \prod_{k=1}^n \mathbf{x}_k^R F_R(\mathbf{x}_k)$$

in $\mathbb{C}[\![\mathbf{x}_1, \dots, \mathbf{x}_n]\!]$ can be expanded with respect to Jack Q-polynomials. Therefore, if we obtain the coefficient of $Q_{(R^n)}^{(1/m)}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ in this expansion, then we obtain the value $D_n^{[2m]}(F_R)$ by the orthogonality (4.2). Indeed, if we denote by $\gamma(f, n, m, R)$ the coefficient of $Q_{(R^n)}^{(1/m)}$ in the expansion of $S_f(\mathbf{x}_1, \dots, \mathbf{x}_n; R)$ with respect to Jack Q-polynomials $Q_\lambda^{(1/m)}$, we have

$$\begin{aligned} D_n^{[2m]}(f) &= \frac{1}{n!} \int_{\mathbb{T}^n} S_f(z_1, \dots, z_n; R) P_{(R^n)}^{(1/m)}\left(z_1^{-1}, \dots, z_n^{-1}\right) \left|V(z_1, \dots, z_n)\right|^{2m} \mathrm{d}z_1 \cdots \mathrm{d}z_n \\ &= \gamma(f, n, m, R) \cdot \left\langle Q_{(R^n)}^{(1/m)}, P_{(R^n)}^{(1/m)} \right\rangle_{n,1/m}'. \end{aligned}$$

The explicit value of $\langle Q^{(1/m)}_{(R^n)}, P^{(1/m)}_{(R^n)}\rangle'_{n,1/m}$ can be obtained from (4.2). Finally, we have the following theorem.

Theorem 4.3. *Let $f \in L^1(\mathbb{T})$ and let R be a non-negative integer such that $D^{[2m]}_n(f) = D^{[2m]}_n(F_R)$. Then we have*

$$\widehat{D}^{[2m]}_n(f) = \gamma(f,n,m,R)\prod_{i=1}^{n}\prod_{j=1}^{R}\frac{im+j-1}{(i-1)m+j},$$

where $\gamma(f,n,m,R)$ is the coefficient of $Q^{(1/m)}_{(R^n)}$ in the expansion of $S_f(\mathbf{x}_1,\dots,\mathbf{x}_n;R)$ with respect to Jack Q-polynomials $Q^{(1/m)}_\lambda$.

As this theorem indicates, the computation of a Toeplitz hyperdeterminant is reduced to the evaluation of the coefficients of a Jack polynomial expansion. However, it is hard to obtain explicit values of $\gamma(f,n,m,R)$ in general. Here we give a few simple examples where $\gamma(f,n,m,1)$ can be explicitly calculated.

Example 4.1. Consider $f(z) = z^a - z^{-1}$ where a is a positive integer. In the notation of Theorem 4.3, we can take $R = 1$. Then $S_f(\mathbf{x}_1,\dots,\mathbf{x}_n;R) = (-1)^n\prod_{k=1}^n(1-\mathbf{x}_k^{a+1})$. The degree of each term in the polynomial $\prod_{k=1}^n(1-\mathbf{x}_k^{a+1})$ is divisible by $a+1$ and therefore $\gamma(f,n,m,R) = 0$ unless $n \equiv 0 \pmod{a+1}$. Henceforth, assume $n \equiv 0 \pmod{a+1}$ and put $n_a = n/(a+1)$. Then the term of degree n in $(-1)^n\prod_{k=1}^n(1-\mathbf{x}_k^{a+1})$ is given by an elementary symmetric polynomial

$$\begin{aligned}(-1)^{n+n_a}e_{n_a}\big(\mathbf{x}_1^{a+1},\dots,\mathbf{x}_n^{a+1}\big) &= (-1)^{n+n_a}(e_{n_a}\circ p_{a+1})(\mathbf{x}_1,\dots,\mathbf{x}_n)\\ &= \sum_{\lambda\vdash n_a}\frac{(-1)^{n+\ell(\lambda)}}{z_\lambda}(p_\lambda\circ p_{a+1})(\mathbf{x}_1,\dots,\mathbf{x}_n).\end{aligned}$$

Here $\circ$ denotes the plethysm product [20, §I-8] and, in the second equality, the formula (2.14′) in [20, §I] is used. Now we use the basic property $p_\lambda \circ p_k = p_{k\lambda}$, where $k\lambda = (k\lambda_1, k\lambda_2, \dots)$, and the expansion formula (obtained from [20, §VI, (10.27)])

$$p_\rho = \alpha^{\ell(\rho)}z_\rho\sum_{\lambda:\ |\lambda|=|\rho|}\frac{\theta^\lambda_\rho(\alpha)}{c_\lambda(\alpha)}Q^{(\alpha)}_\lambda. \tag{4.4}$$

Then we have $\gamma(f,n,m,R) = \sum_{\lambda\vdash n_a}(-1)^{n+\ell(\lambda)}z_\lambda^{-1}m^{-\ell(\lambda)}z_{(a+1)\lambda}\theta^{(1^n)}_{(a+1)\lambda}(\frac{1}{m})/c_{(1^n)}(\frac{1}{m})$. Finally, since $\theta^{(1^n)}_\rho(\alpha) = (-1)^{n-\ell(\rho)}n!z_\rho^{-1}$ (see [20, §VI-10, Ex. 1]) and $n!\sum_{\lambda\vdash n}\alpha^{\ell(\lambda)}z_\lambda^{-1} = \prod_{i=0}^{n-1}(\alpha+i)$, we have

$$\gamma(f,n,m,R) = \frac{\prod_{i=0}^{n_a-1}(im+1)}{(n_a)!m^{n_a}}.$$

Therefore, by Theorem 4.3, the Toeplitz hyperdeterminant of $f(z) = z^a - z^{-1}$ is given by

$$\widehat{D}_n^{[2m]}(f) = \prod_{i=n_a}^{n-1} \frac{im+m}{im+1}$$

if $n \equiv 0 \bmod a+1$, and $\widehat{D}_n^{[2m]}(f) = 0$ otherwise.

Example 4.2. Let $f(z) = \{sz(1-sz)\}^{-1}$ with $0 < |s| < 1$. Then $S_f(\mathbf{x}_1, \dots, \mathbf{x}_n; R) = s^{-n} \prod_{k=1}^n (1 - s\mathbf{x}_k)^{-1}$ with $R = 1$. The term of degree n is the complete symmetric polynomial $h_n(\mathbf{x}_1, \dots, \mathbf{x}_n)$. Hence, by a similar discussion to the last example, we have $\gamma(f, n, m, R) = (-1)^{n-1}(n!m^n)^{-1} \prod_{i=1}^{n-1}(im-1)$. Here the value is independent with s. Therefore the Toeplitz hyperdeterminant is

$$\widehat{D}_n^{[2m]}(f) = (-1)^{n-1} \prod_{i=1}^{n-1} \frac{im-1}{im+1}.$$

In particular, $D_n^{[2]}(f) = 0$ for $n > 1$.

Example 4.3. Let $f(z) = z^{-1}e^z$. Then we have $S_f(\mathbf{x}_1, \dots, \mathbf{x}_n; 1) = \exp(p_1(\mathbf{x}))$ and therefore the term of degree n is $p_{(1^n)}/n!$. It follows by (4.4) that $\gamma(f, n, m, 1) = (m^n n!)^{-1}$, and so the Toeplitz hyperdeterminant is

$$\widehat{D}_n^{[2m]}(f) = \prod_{i=1}^{n-1} (im+1)^{-1}.$$

4.4. *Strong Szegö limit theorem for Toeplitz hyperdeterminants*

We consider the asymptotic limit of a Toeplitz hyperdeterminant of size n in the limit as $n \to \infty$, whence we obtain a hyperdeterminant analogue of the strong Szegö limit theorem.

Theorem 4.4. *Let $f(z) = \exp(\sum_{k \in \mathbb{Z}} c(k) z^k)$ be a function on $\mathbb{T}$ and assume*

$$\sum_{k \in \mathbb{Z}} \bigl|c(k)\bigr| < \infty \quad \textit{and} \quad \sum_{k \in \mathbb{Z}} |k| \bigl|c(k)\bigr|^2 < \infty. \tag{4.5}$$

Then we have

$$\widehat{D}_n^{[2m]}(f) \sim \exp\left(c(0)n + \frac{1}{m} \sum_{k=1}^{\infty} k c(k) c(-k) \right)$$

as $n \to \infty$.

The case $m = 1$ is actually the strong Szegö limit theorem for a Toeplitz determinant, see e.g. [22]. This theorem follows from Theorem 4.1 and the following lemma. The lemma has previously been given in [10,11], while the present author has given a simpler algebraic proof in [21].

Lemma 4.5. *Let f be as in Theorem* 4.4. *Then, for any $\alpha > 0$, we have*

$$\lim_{n\to\infty} \frac{1}{I_n(\alpha)\, e^{nc(0)}} \int_{\mathbb{T}^n} \prod_{j=1}^{n} f(z_j) \left|V(z_1,\dots,z_n)\right|^{2/\alpha} \mathrm{d}z_1 \cdots \mathrm{d}z_n = \exp\left(\alpha \sum_{k=1}^{\infty} kc(k)c(-k)\right).$$

Example 4.4. Let $f(z) = e^{x(z-z^{-1})}$ with $x > 0$. The Fourier coefficients are Bessel functions, see e.g. [1, Eq. (4.9.10)]. Then Theorem 4.4 says $\lim_{n\to\infty} \widehat{D}_n^{[2m]}(e^{x(z-z^{-1})}) = e^{-x^2/m}$.

Example 4.5. Let $f(z) = (1+tz)^{w_1}(1+sz^{-1})^{w_2}$ with complex parameters s, t, w_1, w_2 satisfying $|s|, |t| < 1$. Since $c(k) = w_1(-1)^{k+1}t^k/k$ and $c(-k) = w_2(-1)^{k+1}s^k/k$ for $k > 0$, we have

$$\lim_{n\to\infty} \widehat{D}_n^{[2m]}(f) = \exp\left(\frac{w_1 w_2}{m} \sum_{k=1}^{\infty} \frac{(st)^k}{k}\right) = (1-st)^{-w_1 w_2/m}.$$

Example 4.6. Define $c(k) = |k|^{-1-x}$ if $k \neq 0$ and $c(0) = 0$ with $x > 0$. Then $\sum_{k\in\mathbb{Z}} |c(k)| = 2\zeta(1+x)$ and $\sum_{k\in\mathbb{Z}} |k||c(k)|^2 = 2\zeta(1+2x)$, where $\zeta(s)$ is the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $\operatorname{Re} s > 1$. Therefore $f(z) = \exp(\sum_{k=1}^{\infty}(z^k - z^{-k})/k^{1+x})$ and we have $\lim_{n\to\infty} \widehat{D}_n^{[2m]}(f) = e^{\zeta(1+2x)/m}$.

5. Jacobi–Trudi type formula for Jack functions of rectangular shapes

The main result of the present study is presented in this section. We obtain the Jacobi–Trudi formula for Jack functions of rectangular shapes by employing the Toeplitz hyperdeterminant studied in the previous section.

Put

$$G_{\mathbf{x}}^{(\alpha)}(z) = \prod_{i=1}^{\infty} (1 - \mathbf{x}_i z)^{-1/\alpha}, \qquad E_{\mathbf{x}}(z) = \prod_{i=1}^{\infty} (1 + \mathbf{x}_i z).$$

These functions are the generating functions for one-row Q-functions and one-column P-functions,

$$G_{\mathbf{x}}^{(\alpha)}(z) = \sum_{r=0}^{\infty} Q_{(r)}^{(\alpha)}(\mathbf{x}) z^r, \qquad E_{\mathbf{x}}(z) = \sum_{r=0}^{\infty} P_{(1^r)}^{(\alpha)}(\mathbf{x}) z^r.$$

Put $g_r^{(\alpha)} = Q_{(r)}^{(\alpha)}$. The function $P_{(1^r)}^{(\alpha)}$ is just equal to the elementary symmetric function e_r. We put $g_r^{(\alpha)} = e_r = 0$ unless $r \geqslant 0$.

Consider a shifted Toeplitz hyperdeterminant defined by

$$\widehat{D}_{n;a}^{[2m]}(f) = \widehat{D}_n^{[2m]}\big(z^{-a} f(z)\big), \quad a \in \mathbb{Z}.$$

Then we have the following formula.

Theorem 5.1. *Let m, n, and L be positive integers, and let $\mathbf{x}$ be the sequence of infinitely many variables $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots)$. Then we have*

$$Q^{(1/m)}_{(L^n)}(\mathbf{x}) = \widehat{D}^{[2m]}_{n;L}\big(G^{(1/m)}_{\mathbf{x}}\big), \qquad P^{(m)}_{(n^L)}(\mathbf{x}) = \widehat{D}^{[2m]}_{n;L}(E_{\mathbf{x}}). \tag{5.1}$$

In other words, we have

$$Q^{(1/m)}_{(L^n)} = \frac{n!(m!)^n}{(mn)!} \cdot \det{}^{[2m]}\big(g^{(1/m)}_{L+i_1+\cdots+i_m-i_{m+1}-\cdots-i_{2m}}\big)_{[n]},$$
$$P^{(m)}_{(n^L)} = \frac{n!(m!)^n}{(mn)!} \cdot \det{}^{[2m]}(e_{L+i_1+\cdots+i_m-i_{m+1}-\cdots-i_{2m}})_{[n]}.$$

Proof. We may assume each $\mathbf{x}_j$ is a non-zero complex number and $|\mathbf{x}_j|$ is sufficiently small. We apply Theorem 4.3 to the function $f(z) = z^{-L}E_{\mathbf{x}}(z)$ and $R = L$. Then in the notation of Theorem 4.3 we have $S_f(z_1, \ldots, z_n; L) = \prod_{k=1}^n \prod_{i=1}^\infty (1 + \mathbf{x}_i z_k)$. The dual Cauchy formula for Jack functions [20, §VI-5, 10] says that

$$\sum_{\lambda:\ \ell(\lambda)\leqslant n} Q^{(1/\alpha)}_{\lambda'}(\mathbf{x}) Q^{(\alpha)}_{\lambda}(z_1, \ldots, z_n) = \prod_{i=1}^{\infty}\prod_{k=1}^{n}(1 + \mathbf{x}_i z_k).$$

Therefore we obtain $\gamma(f, n, m, L) = Q^{(m)}_{(n^L)}(\mathbf{x})$ and

$$\widehat{D}^{[2m]}_{n;L}(E_{\mathbf{x}}) = \widehat{D}^{[2m]}_{n}(f) = Q^{(m)}_{(n^L)}(\mathbf{x}) \prod_{i=1}^{n}\prod_{j=1}^{L} \frac{im + j - 1}{(i-1)m + j} = P^{(m)}_{(n^L)}(\mathbf{x}),$$

and so we have obtained the second in Eq. (5.1) to be proved.

Recall the endomorphism ω_α on the $\mathbb{C}$-algebra of symmetric functions (see [20, §VI-10]). It satisfies the duality $\omega_\alpha(P^{(\alpha)}_\lambda) = Q^{(1/\alpha)}_{\lambda'}$, and so the first formula follows from the second formula in Eq. (5.1). $\square$

As a corollary of the previous theorem, we see that

$$Q^{(1/m)}_{(L^n)} \in \mathbb{Q}\big[g^{(1/m)}_{L+i}\colon -m(n-1) \leqslant i \leqslant m(n-1)\big],$$
$$P^{(m)}_{(n^L)} \in \mathbb{Q}\big[e_{L+i}\colon -m(n-1) \leqslant i \leqslant m(n-1)\big].$$

Since the Schur function is the Jack function associated with $\alpha = 1$: $s_\lambda = Q^{(1)}_\lambda = P^{(1)}_\lambda$, the case $m = 1$ in Theorem 5.1 reduces to the well-known Jacobi–Trudi identity and the dual identity for the Schur function of a rectangular shape:

$$s_{(L^n)} = \det(h_{L-i+j})_{1\leqslant i,j\leqslant n}, \qquad s_{(n^L)} = \det(e_{L-i+j})_{1\leqslant i,j\leqslant n}.$$

If α is an even number and its inverse, the Jack function of a rectangular shape can also be expressed by a hyperpfaffian as follows. From Theorems 5.1 and 4.2, we have the following corollary.

Corollary 5.2. *For any positive integers m, L and n, we have*

$$Q^{(1/(2m))}_{(L^n)} = \frac{n!((2m)!)^n}{(2mn)!}\mathrm{pf}^{[2m]}\left(\prod_{s=1}^{m}(i_{2s}-i_{2s-1})\cdot g^{(1/(2m))}_{L+m(2n+1)-i_1-\cdots-i_{2m}}\right)_{[2n]},$$

$$P^{(2m)}_{(n^L)} = \frac{n!((2m)!)^n}{(2mn)!}\mathrm{pf}^{[2m]}\left(\prod_{s=1}^{m}(i_{2s}-i_{2s-1})\cdot e_{L+m(2n+1)-i_1-\cdots-i_{2m}}\right)_{[2n]}.$$

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Appendix A. Barvinok's hyperpfaffian

Our hyperpfaffian $\mathrm{pf}^{[2m]}$ defined by expression (2.2) differs from Barvinok's hyperpfaffian $\mathrm{Pf}^{[2m]}$ in [2]. In this section, we give the explicit relationship between these two hyperpfaffians.

For positive integers m and n, we put

$$\mathfrak{E}_{2mn,2m} = \left\{\sigma \in \mathfrak{S}_{2mn} \;\middle|\; \sigma\big(2m(i-1)+1\big) < \sigma\big(2m(i-1)+2\big) < \cdots < \sigma(2mi)\ (1 \leqslant i \leqslant n)\right\}.$$

Let $M = (M(i_1,\dots,i_{2m}))_{[2mn]}$ be an array satisfying

$$M(i_{\tau(1)},\dots,i_{\tau(2m)}) = \mathrm{sgn}(\tau)M(i_1,\dots,i_{2m})$$

for any $\tau \in \mathfrak{S}_{2m}$. Barvinok [2] defines his hyperpfaffian by

$$\mathrm{Pf}^{[2m]}(M) = \frac{1}{n!}\sum_{\sigma\in\mathfrak{E}_{2mn,2m}}\mathrm{sgn}(\sigma)\prod_{i=1}^{n} M\big(\sigma\big(2m(i-1)+1\big),\sigma\big(2m(i-1)+2\big),\dots,\sigma(2mi)\big),$$

see also [17].

As the following proposition states, our hyperpfaffian $\mathrm{pf}^{[2m]}$ is expressed by $\mathrm{Pf}^{[2m]}$.

Proposition A.1. *Let $B = (B(i_1,\dots,i_{2m}))_{[2n]}$ be an array satisfying condition* (2.1). *Let*

$$M = \big(M(i_1,\dots,i_{2m})\big)_{[2mn]}$$

be the array whose entries $M(i_1,\dots,i_{2m})$ are given as follows: if $i_1 < \cdots < i_{2m}$ and if there exist $1 \leqslant r_1,\dots,r_{2m} \leqslant 2n$ such that $i_{2s-1} = 2n(s-1)+r_{2s-1}$ and $i_{2s} = 2n(s-1)+r_{2s}$ for any $1 \leqslant s \leqslant m$, then $M(i_1,\dots,i_{2m}) = B(r_1,\dots,r_{2m})$. Otherwise define $M(i_1,\dots,i_{2m}) = 0$. Then we have $\mathrm{pf}^{[2m]}(B) = \mathrm{Pf}^{[2m]}(M)$.

Proof. The value $\prod_{i=1}^{n} M(\sigma(2m(i-1)+1),\sigma(2m(i-1)+2),\dots,\sigma(2mi))$ is zero unless the permutation $\sigma \in \mathfrak{E}_{2mn,2m}$ satisfies $2n(s-1)+1 \leqslant \sigma(2m(i-1)+2s-1), \sigma(2m(i-1)+2s) \leqslant 2ns$ for any $1 \leqslant s \leqslant m$ and $1 \leqslant i \leqslant n$. Therefore

$$\begin{aligned}\mathrm{Pf}^{[2m]}(M) = \frac{1}{n!} \sum_{\sigma_1,\dots,\sigma_m \in \mathfrak{E}_{2n}} &\epsilon(\sigma_1,\dots,\sigma_m) \\ &\times \prod_{i=1}^{n} M\bigl(\dots, 2(s-1)n + \sigma_s(2i-1), 2(s-1)n + \sigma_s(2i), \dots\bigr),\end{aligned}$$

where $\epsilon(\sigma_1,\dots,\sigma_m)$ is the signature of the permutation σ defined by

$$\sigma\bigl(2m(i-1) + 2s - 1\bigr) = 2n(s-1) + \sigma_s(2i-1) \quad \text{and}$$
$$\sigma\bigl(2m(i-1) + 2s\bigr) = 2n(s-1) + \sigma_s(2i)$$

for any $1 \leqslant i \leqslant n$ and $1 \leqslant s \leqslant m$. Hence we have $\mathrm{Pf}^{[2m]}(M) = \epsilon(\mathrm{id},\dots,\mathrm{id})\mathrm{pf}^{[2m]}(B)$, and so $\epsilon(\mathrm{id},\dots,\mathrm{id}) = \mathrm{sgn}(\rho)$, where

$$\rho\bigl(2m(i-1) + 2s - 1\bigr) = 2n(s-1) + 2i - 1 \quad \text{and} \quad \rho\bigl(2m(i-1) + 2s\bigr) = 2n(s-1) + 2i.$$

Now it is straightforward to see $\mathrm{sgn}(\rho) = 1$. $\square$

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