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# On direct sums of Baer modules

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## ABSTRACT

The notion of Baer modules was defined recently. Since a direct sum of Baer modules is not a Baer module in general, an open question is to find necessary and sufficient conditions for such a direct sum to be Baer. In this paper we study rings for which every free module is Baer. It is shown that this is precisely the class of semiprimary hereditary rings. We also prove that every finite rank free  $R$ -module is Baer if and only if  $R$  is right semihereditary, left  $\Pi$ -coherent. Necessary and sufficient conditions for finite direct sums of copies of a Baer module to be Baer are obtained, for the case when  $M$  is retractable. An example of a module  $M$  is exhibited for which  $M^n$  is Baer but  $M^{n+1}$  is not Baer. Other results on direct sums of Baer modules to be Baer under some additional conditions are obtained. Some applications are also included.

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## 1. Introduction

Let  $R$  be a ring with unity. For any class of  $R$ -modules satisfying a certain property, an interesting question is to find necessary and sufficient conditions for a finite or arbitrary direct sum of modules in the class to inherit the property. It is well known that a finite direct sum of injective modules is always injective, while an arbitrary direct sum of injective right modules is injective if and only if the base ring  $R$  is right noetherian. Among some of the interesting generalizations of injectivity, it is known that a finite direct sum of (quasi-) continuous (respectively, quasi-injective) modules is (quasi-) continuous (respectively, quasi-injective) if and only if each direct summand is (quasi-) continuous (respectively, quasi-injective), and relatively injective to the other summands [20]. The case of a direct sum of extending modules to be extending is different. It is well known that the direct sum of extending modules is not always extending. Finding a decent characterization for a direct sum of

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extending modules to be extending remains an open and difficult problem, even though special cases for this result to hold true are present in the literature [11].

Kaplansky [15] introduced the notion of a Baer ring—a concept which has its roots in functional analysis. A ring  $R$  is called Baer (quasi-Baer) if  $l(I) = Re$ , for every  $I \subseteq R$  ( $I$  ideal of  $R$ ), for some  $e^2 = e \in R$ , equivalently  $r(I) = eR$ , for every  $I \subseteq R$  ( $I$  ideal of  $R$ ). A number of research papers have been published on Baer and quasi-Baer rings, see for example [1–6,26]. In a general module-theoretic setting, the concept of Baer modules was introduced recently [21]. A module  $M$  is called Baer [21–23] if for every  $N \leq M$ ,  $l_S(N) = Se$  with  $e^2 = e \in S$ , where  $S$  is the endomorphism ring of  $M$ . It is known that a direct sum of Baer modules is not always Baer; see [21]. The problem of characterizing when is a direct sum of Baer modules a Baer module remains open. In [21] we showed that there is a strong connection between the class of Baer modules and the class of extending modules. A solution to the problem of obtaining a characterization for a direct sum of Baer modules to be Baer could provide an approach for a solution to the similar direct sum problem for the case of extending modules. In this paper one of our aims is to investigate the Baer property of direct sums of copies of a given module. We provide necessary and sufficient conditions for a direct sum of copies of a Baer module to be Baer and characterize rings  $R$  for which every free (projective)  $R$ -module is Baer.

After introducing the basic notions and results in Section 1, our focus in Section 2 is on connections of a Baer module to its endomorphism ring. While the endomorphism ring of a Baer module is always Baer, the converse is not true in general (Theorem 4.1 and Example 4.3 in [21]). We show that every Baer module satisfies an inherent “weak retractability” property. We use this to provide a characterization of a Baer module in terms of its endomorphism ring. Some results on direct sum decompositions of Baer modules are also included. The main results of this paper are in Section 3, where we investigate the validity of the Baer property for a direct sum of copies of a Baer  $R$ -module. We show that every free (projective) right  $R$ -module is Baer if and only if  $R$  is a right semiprimary, right hereditary ring. It is shown that the class of rings for which every finitely generated free (projective)  $R$ -module is Baer is precisely that of rings  $R$  which are right semihereditary, left  $\Pi$ -coherent, equivalently rings  $R$  which are left semihereditary, right  $\Pi$ -coherent, equivalently  $M_n(R)$  is a Baer ring for all  $n \in \mathbb{N}$ . We obtain necessary and sufficient conditions for finite and arbitrary direct sums of copies of a Baer module to be Baer, under some additional conditions. It is shown that if  $R$  is an  $n$ -fir, then  $R^n$  is a Baer module (for a cardinal  $\alpha$ ,  $R$  is called right  $\alpha$ -fir if all  $\alpha$ -generated right ideals of  $R$  are free, of unique rank). We exhibit an example of a module  $M$  such that  $M^n$  is Baer, but  $M^{n+1}$  is not. As an application, we also provide some necessary conditions for a  $\mathcal{K}$ -nonsingular module to be (finitely)  $\Sigma$ -extending.

All our rings have a unity element, and unless otherwise specified, the modules are right  $R$ -modules. For a right  $R$ -module  $M$ ,  $S$  denotes the ring of all  $R$ -endomorphisms of  $M$ , namely  $S = \text{End}_R(M)$ . The notation  $N \leq^\oplus M$  denotes that  $N$  is a direct summand in  $M$ ;  $N \leq^e M$  means that  $N$  is essential in  $M$  (i.e.  $N \cap L \neq 0$  for every  $0 \neq L \leq M$ );  $N \leq M$  means that  $N$  is fully invariant in  $M$  (i.e. for every  $\varphi \in \text{End}(M)$ ,  $\varphi(N) \subseteq N$ );  $E(M)$  denotes the injective hull of  $M$ . We denote by  $M^n$  the direct sum of  $n$  copies of  $M$ . By  $\mathbb{C}$ ,  $\mathbb{R}$ ,  $\mathbb{Q}$  and  $\mathbb{Z}$  we denote the ring of complex, real, rational and integer numbers, respectively;  $\mathbb{Z}_n$  will denote  $\mathbb{Z}/n\mathbb{Z}$ .

We also denote  $r_M(I) = \{m \in M \mid Im = 0\}$ , for  $I \leq S$ ;  $r_R(N) = \{r \in R \mid Nr = 0\}$ ,  $l_S(N) = \{\varphi \in S \mid \varphi N = 0\}$ , for  $N \leq M$ .

**Definition 1.1.** A module  $M$  is called *Baer* if, for every  $N \leq M$ ,  $l_S(N) = Se$  for some  $e^2 = e \in S$ . Equivalently, the module  $M$  is Baer if for every  $I \leq_S S$ ,  $r_M(I) = eM$ , for some  $e^2 = e \in S$  (see [21–24]).

Note that, for  $I \leq_S S$ ,  $r_M(I) = \bigcap_{\varphi \in I} \text{Ker } \varphi$ . In particular, for a Baer module,  $\text{Ker } \varphi \leq^\oplus M$ , for every  $\varphi \in S$  (a module with this property is called a *Rickart* module).

**Definition 1.2.** A module  $M$  is called *extending* (or *CS*) if, for every  $N \leq M$ ,  $\exists N' \leq^\oplus M$  with  $N \leq^e N'$ .

In [21] we defined the relative Baer property specifically for Baer modules. However, we can define a similar property for arbitrary modules, which we more appropriately dub the “relative Rickart” property.

**Definition 1.3.** We call the modules  $M$  and  $N$  *relatively Rickart* if, for every  $\varphi : M \rightarrow N$ ,  $\text{Ker } \varphi \leq^\oplus M$  and for every  $\psi : N \rightarrow M$ ,  $\text{Ker } \psi \leq^\oplus N$ .

The concept of *Rickart modules* will be studied in a subsequent paper.

**Definition 1.4.** A module  $M$  is said to have the (strong) *summand intersection property*, or (SSIP) *SIP* if, for an (infinite) finite index set  $\mathcal{I}$ , and for every  $M_i \leq^\oplus M$ ,  $i \in \mathcal{I}$ , a class of direct summands of  $M$ ,  $\bigcap_{i \in \mathcal{I}} M_i \leq^\oplus M$ .

**Definition 1.5.** A module  $M$  is said to be  $\mathcal{K}$ -nonsingular if, for every  $\varphi \in S$ ,  $\text{Ker } \varphi \leq^e M \Rightarrow \varphi = 0$ . The module is said to be  $\mathcal{K}$ -cononsingular if, for every  $N \leq M$ ,  $\varphi N \neq 0$  for all  $0 \neq \varphi \in S \Rightarrow N \leq^e M$ .

It is well known that every nonsingular (or even polyform) module is  $\mathcal{K}$ -nonsingular (Proposition 2.10, [21]; Proposition 2.2, [23]), while the converse is not true (Example 2.5, [23]).  $\mathcal{K}$ -nonsingular modules have been studied in detail in [23].

**Remark 1.6.** It is easy to check that every extending module is  $\mathcal{K}$ -cononsingular (Lemma 2.13, [21]).

We present in the following, a number of results from [21] which will be used throughout the paper.

**Theorem 1.7** (Proposition 2.22, [21]). A module  $M$  is Baer if and only if  $\text{Ker } \varphi \leq^\oplus M$ , for every  $\varphi \in S$  and  $M$  has SSIP.

**Proposition 1.8** (Theorem 2.17, [21]). Let  $M$  be a Baer module, and let  $N \leq^\oplus M$ . Then  $N$  is a Baer module.

**Theorem 1.9** (Theorem 2.23, [21]). A module  $M$  is an indecomposable Baer module if and only if  $\text{Ker } \varphi = 0$ , for every  $0 \neq \varphi \in S$ .

Note that the endomorphism ring of an indecomposable Baer module is a domain. The converse is not true, in general ( $\mathbb{Z}_{p^\infty}$  is a counterexample, see Example 4.3 in [21]). See also Proposition 2.11.

**Proposition 1.10** (Proposition 2.25, [21]). Let  $M = \bigoplus_{i \in \mathcal{I}} M_i$  be a Baer module. Then  $M_i$  is a Baer module, and  $M_i$  and  $M_j$  are relatively Rickart, for every  $i, j \in \mathcal{I}$ .

We end this section recalling a result that shows the strong connections between Baer modules and extending modules.

**Theorem 1.11** (Theorem 2.12, [21]). A module is  $\mathcal{K}$ -nonsingular and extending if and only if it is Baer and  $\mathcal{K}$ -cononsingular.

We remark that Theorem 1.11 yields a rich supply of examples of Baer modules. Every nonsingular injective (extending) module is Baer; more generally,  $M/Z_2(M)$  is a Baer module for any extending module  $M$ , where  $Z_2(M)$  is the second singular submodule of  $M$ . The next corollary provides more examples of Baer modules.

**Corollary 1.12.** If  $R_R$  is extending, then every nonsingular cyclic  $R$ -module  $M$  is extending, hence a Baer module.

**Proof.** The fact that  $M$  is extending follows from Proposition 2.7 in [6]. Since  $M$  is extending and nonsingular, hence extending and  $\mathcal{K}$ -nonsingular, we obtain that  $M$  is a Baer module, by Theorem 1.11.  $\square$

## 2. Endomorphism rings and retractability

Since we will focus on free modules in the later part of this paper, we first investigate a useful property satisfied by every free module, namely retractability. It is known from Theorem 4.1 [21], that the endomorphism ring  $S$  of a Baer  $R$ -module  $M$  is always a Baer ring but the converse does not hold true in general (Example 4.3 in [21]). In this section we obtain necessary and sufficient conditions and connections between the Baer property of a module  $M$  and the Baer property of its endomorphism ring  $S$ . We also investigate the property of retractability for a direct sum of modules and introduce a useful generalization.

We begin with the following definition.

**Definition 2.1.** We call a module  $M$  *retractable* if, for every  $0 \neq N \leq M$ ,  $\exists 0 \neq \varphi \in \text{End}(M)$  with  $\varphi(M) \subseteq N$ , i.e.  $\text{Hom}(M, N) \neq 0$ .

Examples include free modules, generators and semisimple modules. Torsionless modules over semiprime rings are also retractable. For a retractable module  $M$ , the converse of Theorem 4.1 in [21] does hold true (for more results concerning retractable modules, see [17]).

**Theorem 2.2** (Theorem 4.1 and Proposition 4.6, [21]). *If  $M$  is a Baer module, then  $S$  is a Baer ring. Furthermore, if  $M$  is retractable and if  $S$  is a Baer ring, then  $M$  is a Baer module.*

Next, we define a more general form of retractability in order to obtain a full characterization of a Baer module. We show that this general retractability is already an inherent property of every Baer module.

**Definition 2.3.** A module  $M$  is called *quasi-retractable* if  $\text{Hom}(M, r_M(I)) \neq 0$  for every  $I \leq {}_S S$  with  $r_M(I) \neq 0$  (or, equivalently, if  $r_S(I) \neq 0$  for every  $I \leq {}_S S$  with  $r_M(I) \neq 0$ ).

This new concept is a generalization of retractability as follows.

**Lemma 2.4.** *Every retractable module is quasi-retractable.*

**Proof.** Let  $M$  be a retractable module. Let  $I \leq {}_S S$  be such that  $r_M(I) \neq 0$ . By retractability,  $\exists \varphi \in S$  so that  $0 \neq \varphi M \subseteq r_M(I)$ . But in this case  $I(\varphi(M)) = 0 \Rightarrow (I\varphi)M = 0 \Rightarrow I\varphi = 0 \Rightarrow 0 \neq \varphi \in r_S(I)$ . Hence  $M$  is quasi-retractable.  $\square$

In the next result we drop the assumption of retractability from Theorem 2.2 by noting that every Baer module is quasi-retractable. This enables us to provide a complete characterization of a Baer module in terms of its endomorphism ring.

**Theorem 2.5.** *A module  $M$  is a Baer module if and only if  $S = \text{End}(M)$  is a Baer ring and  $M$  is quasi-retractable.*

**Proof.** In view of Theorem 2.2, for the necessity we only need prove that  $M$  is quasi-retractable if  $M$  is Baer. As  $M$  is Baer,  $r_M(I) = eM$  for  $e^2 = e \in S$ . Assuming that  $eM \neq 0 \Rightarrow I(eM) = 0 \Rightarrow (Ie)M = 0 \Rightarrow Ie = 0$ . Thus  $0 \neq e \in r_S(I)$ .

For the sufficiency, take  $I \leq {}_S S$ , then  $r_S(I) = eS$  where  $e^2 = e \in S$ , since  $S$  is a Baer ring. This implies that  $I \subseteq I_S(r_S(I)) = S(1 - e)$ . Hence  $eM \subseteq r_M(I)$ , since  $\varphi e = 0 \Rightarrow \varphi(eM) = 0$ , for every  $\varphi \in I$ . Assume that  $\exists 0 \neq m_0 = (1 - e)m_0 \in r_M(I)$ . Taking now the left ideal  $J = I + Se \leq {}_S S$ , since  $S$  is Baer we have  $r_S(J) = r_S(I) \cap r_S(e) = eS \cap (1 - e)S = 0$ . But  $m_0 \in r_M(J)$ , since  $Im_0 = 0$  and  $em_0 = e(1 - e)m_0 = 0$ , a contradiction since  $M$  is quasi-retractable.  $\square$

Each of the following examples exhibits an  $R$ -module  $M$  which is quasi-retractable but not retractable, showing that the class of retractable modules is a proper subclass of the class of quasi-retractable modules. The first example is due to Chatters.

**Example 2.6** (Example 3.4, [16]). Let  $K$  be a subfield of complex numbers  $\mathbb{C}$ . Let  $R$  be the ring  $\begin{bmatrix} K & \mathbb{C} \\ 0 & \mathbb{C} \end{bmatrix}$ . Then  $R$  is a right nonsingular right extending ring. Consider the module  $M = eR$  where  $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ . Then  $M$  is projective, extending and nonsingular (as it is a direct summand of  $R$ ), hence is Baer by Theorem 1.11. Thus  $M$  is quasi-retractable, by Theorem 2.5. But  $M$  is not retractable, since the endomorphism ring of  $M$ , which is isomorphic to  $K$ , consists of isomorphisms and the zero endomorphism; on the other hand,  $M$  is not simple, and retractability implies that there exist nonzero endomorphisms which are not onto. See Theorem 3.8 in [23].

**Example 2.7** (Example 3.3, [8]). Let

$$R = \begin{bmatrix} \mathbb{C} & \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{R} & \mathbb{C} \\ 0 & 0 & \mathbb{C} \end{bmatrix}.$$

Let  $M = fR$  where

$$f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then  $M$  is a nonsingular, projective extending right  $R$ -module, thus by Theorem 1.11 a Baer module. By Theorem 2.5,  $M$  is quasi-retractable. However,  $\text{End}(M) = fRf$  is not a right extending ring, and since  $M$  is nonsingular,  $M$  is not retractable, because otherwise  $\text{End}(M)$  would be right extending.

The next result shows that the property of retractability passes to arbitrary direct sums of copies of a retractable module.

**Lemma 2.8.** Let  $(M_i)_{i \in \mathcal{I}}$  be a class of retractable modules. Then  $\bigoplus_{i \in \mathcal{I}} M_i$  is retractable.

**Proof.** It is easy to see that retractability of a module  $M$  is equivalent to: for every  $0 \neq n \in M$ ,  $\exists 0 \neq \varphi \in \text{End}(M)$  with  $\varphi(M) \subseteq nR$ . Let  $0 \neq n \in \bigoplus_{i \in \mathcal{I}} M_i$ . There exists a finite  $\mathcal{J} \subseteq \mathcal{I}$  such that  $n \in \bigoplus_{i \in \mathcal{J}} M_i$ , therefore  $nR \subseteq \bigoplus_{i \in \mathcal{J}} M_i$ . Hence it suffices to show that any finite direct sum  $\bigoplus_{i \in \mathcal{J}} M_i$  of retractable modules is retractable. By induction, it suffices to show that the direct sum of two retractable modules is retractable.

Assume  $M_1, M_2$  retractable and  $0 \neq N \leq M_1 \oplus M_2$ . If  $N \cap M_1 \neq 0$ , the conclusion is immediate. If  $N \cap M_1 = 0$ , let  $\pi_2 : M_1 \oplus M_2 \rightarrow M_2$  denote the canonical projection. Then  $\pi_2(N) \cong N$ , so that  $0 \neq \pi_2(N) \leq M_2$ . Hence there is a nonzero morphism of  $M_2$  into its nonzero submodule  $\pi_2(N)$ . Thus there is a nonzero morphism  $M_2 \rightarrow N$ . This shows that there is a nonzero morphism  $M_1 \oplus M_2 \rightarrow N$ .  $\square$

A direct summand of a retractable module may not be retractable, as the following example will exhibit.

**Example 2.9.** Let  $M$  be an  $R$ -module that is not retractable. Let  $P = R \oplus M$ . The module  $P$  is retractable (for an arbitrary  $0 \neq N \leq P$ , let  $0 \neq x \in N$ ; construct a map from  $P$  to  $N$  by mapping  $1 \in R$  to  $x$ , and mapping elements from  $M$  to  $0$ ; the image of this well-defined, nonzero map is  $xR \subseteq N$ ).

The next result shows that this issue does not occur in a direct sum of copies of  $M$ .

**Proposition 2.10.** An arbitrary direct sum of copies of  $M$  is retractable if and only if  $M$  is retractable.

**Proof.** ( $\Rightarrow$ ): Let  $M^{(\mathcal{I})}$  be a retractable module, for a set  $\mathcal{I}$ . Let  $0 \neq N \leq M$ , and view  $M$  as one of the direct summands of  $M^{(\mathcal{I})}$ ; thus  $N \leq M \leq M^{(\mathcal{I})}$ . Then there exists  $0 \neq \varphi : M^{(\mathcal{I})} \rightarrow M^{(\mathcal{I})}$  so that  $\text{Im } \varphi \subseteq N$ . Since  $\varphi \neq 0$ ,  $\exists i \in \mathcal{I}$  so that  $\varphi \iota_i \neq 0$ , where  $\iota_i$  is the canonical inclusion of the  $i$ th coordinate in  $M^{(\mathcal{I})}$ . But  $0 \neq \varphi \iota_i : M \rightarrow N \leq M$ , thus  $M$  is retractable.

( $\Leftarrow$ ): It follows directly from Lemma 2.8.  $\square$

**Proposition 2.11.** *A module  $M$  is an indecomposable Baer module if and only if  $\text{End}(M)$  is a domain and  $M$  is quasi-retractable.*

**Proof.** Every domain is trivially a Baer ring. If  $M$  is quasi-retractable, then  $M$  is a Baer module, by Theorem 2.5. Since there are no idempotents other than 0 and 1 in a domain,  $M$  is also indecomposable.

For the converse, since  $M$  is an indecomposable Baer module, all nonzero endomorphisms have zero kernel; subsequently,  $\text{End}(M)$  is a domain. By Theorem 2.5,  $M$  is also quasi-retractable.  $\square$

We conclude this section with some results about indecomposable decompositions of a Baer module.

**Proposition 2.12.** *If  $M$  is a Baer module, with only countably many direct summands, then  $M$  is a finite direct sum of indecomposable summands.*

**Proof.** Since  $M$  is Baer,  $S$  is Baer by Theorem 2.2. Since  $M$  has countably many direct summands, then  $S$  has only countably many idempotents. By Theorems 2 and 3 in [18],  $S$  has no infinite sets of orthogonal idempotents, hence any direct sum decomposition of  $M$  must be finite, thus  $M$  is a finite direct sum of indecomposable submodules.  $\square$

**Remark 2.13.** In particular, if the base ring  $R$  is countable (e.g.  $R = \mathbb{Z}$ ), we obtain that a Baer  $R$ -module must be either finitely generated (in fact, it is a finite direct sum of indecomposable summands), or uncountably generated. Also, this implies that  $R^{(\mathbb{N})}$  is not a Baer module.

**Proposition 2.14.** *If a Baer module  $M$  can be decomposed into a finite direct sum of indecomposable summands, then every arbitrary direct sum decomposition of  $M$  is finite.*

**Proof.** Let  $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$  be a finite direct sum of indecomposable summands. Assume that  $M$  also decomposes as  $M = \bigoplus_{i \in \mathcal{I}} N_i$ . Let  $0 \neq m \in M_j$ , arbitrary, for a fixed  $j \in 1, \dots, n$ . Then,  $m = \sum_{i \in \mathcal{I}_j} n_i \in \bigoplus_{i \in \mathcal{I}_j} N_i$ , where  $|\mathcal{I}_j| \leq \infty$  and  $n_i \in N_i$ . Thus  $M_j \cap (\bigoplus_{i \in \mathcal{I}_j} N_i) \neq 0$  and, by Theorem 1.7,  $M_j \cap (\bigoplus_{i \in \mathcal{I}_j} N_i) \leq^\oplus M_j$ . Hence  $M_j \cap (\bigoplus_{i \in \mathcal{I}_j} N_i) = M_j$ , as  $M_j$  is indecomposable. Thus  $M_j \subseteq (\bigoplus_{i \in \mathcal{I}_j} N_i)$ .

Repeating this procedure for all of the indexes  $1 \leq j \leq n$ , we obtain that  $M = M_1 \oplus \cdots \oplus M_n \subseteq \bigoplus_{i \in \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_n} N_i \subseteq M$ , hence  $M$  is a finite direct sum of  $N_i$ . Since  $\mathcal{I}_1 \cup \cdots \cup \mathcal{I}_n$  is a finite set, only finitely many  $N_i$  are nonzero, for  $i \in \mathcal{I}$ .  $\square$

### 3. Baer modules and direct sums

In this main section of the paper our investigations focus on direct sums of Baer modules and related conditions. It is known that a direct sum of Baer modules need not be Baer in general [21] (e.g.  $M = \mathbb{Z} \oplus \mathbb{Z}_2$  is not Baer, while each of  $\mathbb{Z}$  and  $\mathbb{Z}_2$  is so). The general question of when is the direct sum of Baer modules also Baer remains open. Here we obtain conditions for free (and projective) modules over a Baer ring to be Baer. We show that the class of rings for which every free module is Baer is precisely that of semiprimary hereditary rings. For the finitely generated module case, we prove that every finitely generated free right  $R$ -module is Baer if and only if  $R$  is right semihereditary, left  $\Pi$ -coherent if and only if  $M_n(R)$  is Baer, for every  $n \in \mathbb{N}$ . Our result partially extends a result of

C. Faith on FGTF rings (Theorem 2.2, [12]) by dropping the condition of von Neumann regularity of the ring in the hypothesis. We also obtain necessary and sufficient conditions for a finite direct sum of copies of a retractable Baer module to be Baer in terms of its endomorphism ring. As a consequence we prove that a free module of finite rank, larger than 1, over a commutative domain  $R$  is Baer iff  $R$  is Prüfer. For an  $n$ -fir (free ideal ring)  $R$ , it is shown that  $R^n$  is always a Baer  $R$ -module. We show that there exists a module  $M$  such that  $M^n$  is a Baer module while  $M^{n+1}$  is not so, for a fixed  $n \in \mathbb{N}$ . The section concludes with some results for arbitrary direct sums of Baer modules to be Baer under some additional conditions.

In Theorem 2.20, [21] we characterized rings for which every (right)  $R$ -module is Baer as precisely the semisimple artinian ones. In the following we characterize the class of rings  $R$  for which every projective  $R$ -module is a Baer module. Our result utilizes some of the arguments of [28] and [27]. In particular:

**Lemma 3.1** (Lemma 3, [27]). *Every right perfect right (or left) Rickart ring is semiprimary, and in particular, left perfect.*

**Theorem 3.2** (Theorem 3.3, [7]). *For any ring  $R$  the following are equivalent:*

- (1) *the direct product of any family of projective right  $R$ -modules is projective;*
- (2) *the direct product of any family of copies of  $R$  is projective as a right  $R$ -module;*
- (3) *the ring  $R$  is right perfect, and every finitely generated left module is finitely presented.*

Our next result provides a characterization of rings  $R$  for which every free right  $R$ -module is Baer.

**Theorem 3.3.** *The following statements are equivalent for a ring  $R$ .*

- (1) *every free right  $R$ -module is a Baer module;*
- (2) *every projective right  $R$ -module is a Baer module;*
- (3)  *$R$  is a semiprimary, hereditary (Baer) ring.*

Since condition (3) is left–right symmetric, the left-handed versions of (1) and (2) also hold.

**Proof.** (1)  $\Leftrightarrow$  (2) follows from Proposition 1.8.

(1)  $\Rightarrow$  (3): Since  $R_R$  is free,  $R_R$  is a Baer module, hence  $R$  is a Baer ring.

By Theorem 1.7 for any Baer module  $M$ ,  $\text{Ker } \varphi \leq^\oplus M$  for every  $\varphi \in S = \text{End}(M)$ . In particular, every homomorphism from a free  $R$ -module  $M$  to  $R$  (viewed as an endomorphism of  $M$ ) will split, thus the image of such a homomorphism is a projective  $R$ -module. Since every right ideal of  $R$  (in fact, every right  $R$ -module) is a homomorphic image of a free right module, it implies that every right ideal of  $R$  is projective; thus,  $R$  is right hereditary.

Let  $M$  be an arbitrary direct product of copies of  $R$ ,  $M = R^{\mathcal{J}}$  ( $\mathcal{J}$  an index set). Then (using the observation above, that every  $R$ -module is the homomorphic image of a free module), there exists a set  $\mathcal{L}$  so that we can construct the following exact sequence:

$$0 \rightarrow K \rightarrow R^{(\mathcal{L})} \xrightarrow{\varphi} M \rightarrow 0.$$

We have that  $K = \bigcap_{j \in \mathcal{J}} \text{Ker } \pi_j \varphi$ , where  $\pi_j$  is the canonical projection of  $M$  onto its  $j$ th coordinate. But each  $\pi_j \varphi$  can be viewed as an endomorphism of  $R^{(\mathcal{L})}$ , hence its kernel is a direct summand of  $R^{(\mathcal{L})}$ ; moreover,  $R^{(\mathcal{L})}$  satisfies the strong summand intersection property (SSIP) by Theorem 1.7, hence  $K = \bigcap_{j \in \mathcal{J}} \text{Ker } \pi_j \varphi \leq^\oplus R^{(\mathcal{L})}$ . It follows that  $M$  is isomorphic to a direct summand of  $R^{(\mathcal{L})}$ , hence it is projective.

Since the set  $\mathcal{J}$  was arbitrarily chosen, we can apply Theorem 3.2; we therefore obtain that  $R$  is right perfect and that every finitely generated left ideal is finitely presented.

Since  $R$  is right hereditary (hence right Rickart) right perfect ring, by Lemma 3.1 we obtain that  $R$  is semiprimary. A semiprimary right hereditary ring is also left hereditary (Proposition 5.72, [19]), thus it follows that  $R$  is left hereditary.

(3)  $\Rightarrow$  (1): Since the ring  $R$  is hereditary, for every free right  $R$ -module  $M$  and for each  $\varphi \in \text{End}(M)$ ,  $\varphi$  splits, and thus  $\text{Ker } \varphi \leq^{\oplus} M$ . We only need to show that every free module satisfies the SSIP; then, using Theorem 1.7 we obtain that  $M$  is a Baer module.

Let  $I$  be a finitely generated left ideal of  $R$ ,  $I \cong_R R^{(n)}/K$  for some  ${}_R K \leq_R R$ . Since  $R$  is left hereditary, it follows that  ${}_R I$  is projective, so the following exact sequence:

$$0 \rightarrow {}_R K \rightarrow {}_R R^{(n)} \rightarrow {}_R I \rightarrow 0$$

splits. Hence  ${}_R K \leq^{\oplus} R^{(n)}$ , therefore  ${}_R K$  is finitely generated (it can be generated by  $\leq n$  elements). Hence every finitely generated left ideal in  $R$  is finitely presented.

Using Theorem 3.2, since  $R$  is right perfect and every finitely generated left ideal is finitely presented, we obtain that the direct product of any family of projective right  $R$ -modules is projective. Take now a free module  $M = R^{(\mathcal{K})}$  for an arbitrary index set  $\mathcal{K}$ . Let  $(N_j)_{j \in \mathcal{J}}$ , be an arbitrary family of summands, and let  $N'_j$  such that  $N_j \oplus N'_j = M$  for each  $j \in \mathcal{J}$ . We have the following exact sequence:

$$0 \rightarrow \bigcap_{j \in \mathcal{J}} N_j \rightarrow M \rightarrow \prod_{j \in \mathcal{J}} N'_j \rightarrow 0$$

where the map from  $M$  to  $\prod N'_j$  is  $\prod \pi_{N'_j}$ , the direct product of all canonical projections of  $M$  onto the  $N'_j$ s. Since  $\prod N'_j$  is a direct product of projective modules, it is also projective, hence the exact sequence splits, therefore  $\bigcap_{j \in \mathcal{J}} N_j \leq^{\oplus} M$ . Therefore  $M$  satisfies SSIP.

In conclusion, every free right  $R$ -module  $M$  satisfies SSIP and  $\text{Ker } \varphi \leq^{\oplus} M$ , for every  $\varphi \in S$ . Hence, by Theorem 1.7,  $M$  is a Baer module.  $\square$

Recall that every free module  $M$  is retractable ( $R$  is retractable, and use Lemma 2.8). Also, every direct sum of copies of an arbitrary retractable module  $M$  is retractable (by Lemma 2.10), and therefore quasi-retractable (by Lemma 2.4). In the next result we provide a characterization for an arbitrary direct sum of copies of a Baer module to be Baer, for the case when  $M$  is finitely generated and retractable.

**Theorem 3.4.** *Let  $M$  be a finitely generated, retractable module. Then an arbitrary direct sum of copies of  $M$  is a Baer module if and only if  $S = \text{End}(M)$  is semiprimary and (right) hereditary.*

**Proof.** We note that, for a finitely generated module  $M$  and  $S = \text{End}(M)$ , we have that  $\text{End}(M^{(\mathcal{J})}) \cong \text{End}(S^{(\mathcal{J})})$  as rings, where  $\mathcal{J}$  is an arbitrary set. Hence, if an arbitrary direct sum of copies of  $M$  is Baer, its endomorphism ring is a Baer ring (Theorem 2.2).  $S^{(\mathcal{J})}$  is a quasi-retractable  $S$ -module (it is, in fact, retractable, because it is a free  $S$ -module), by Theorem 2.5 we get that  $S^{(\mathcal{J})}$  is a Baer  $S$ -module. By Theorem 3.3,  $S$  is right semiprimary and right hereditary.

Conversely, for an arbitrary set  $\mathcal{J}$ , we obtain that  $S^{(\mathcal{J})}$  is a Baer  $S$ -module, hence  $\text{End}(S^{(\mathcal{J})})$  is a Baer ring, thus  $\text{End}(M^{(\mathcal{J})})$  is a Baer ring. Since  $M^{(\mathcal{J})}$  is also (quasi-) retractable, by Proposition 2.10,  $M^{(\mathcal{J})}$  is a Baer module, by Theorem 2.5.  $\square$

We recall that a module is called *torsionless* if it can be embedded in a direct product of copies of the base ring. In our next result we provide a characterization of rings  $R$  for which every finitely generated free right  $R$ -module is Baer. This result partially extends Theorem 2.2 in [12] by dropping von Neumann regularity of the ring  $R$ .



**Theorem 3.5.** *Let  $R$  be a ring. The following statements are equivalent:*

- (1) every finitely generated free right module over  $R$  is a Baer module;
- (2) every finitely generated projective right module over  $R$  is a Baer module;
- (3) every finitely generated torsionless right  $R$ -module is projective;
- (4) every finitely generated torsionless left  $R$ -module is projective;
- (5)  $R$  is left semihereditary and right  $\Pi$ -coherent (i.e. every finitely generated torsionless right  $R$ -module is finitely presented);
- (6)  $R$  is right semihereditary and left  $\Pi$ -coherent;
- (7)  $M_n(R)$  is Baer ring for every  $n \in \mathbb{N}$ .

In particular, a ring  $R$  satisfying these equivalent conditions is right and left semihereditary.

**Proof.** (1)  $\Leftrightarrow$  (2) is immediate, based on Proposition 1.8. (1)  $\Leftrightarrow$  (7) follows from Theorem 2.5.

(1)  $\Rightarrow$  (3): Let  $M$  be an infinite direct product of copies of  $R$ ,  $M = R^{\mathcal{J}}$ .

Let  $N$  be a finitely generated torsionless right  $R$ -module. It follows that there exists a finite set  $\mathcal{F}$  and a map  $\varphi: R^{(\mathcal{F})} \rightarrow M$  so that  $N = \text{Im } \varphi \subseteq M$ . We have the following exact sequence:

$$0 \rightarrow K \rightarrow R^{(\mathcal{F})} \xrightarrow{\varphi} N \rightarrow 0.$$

$K = \bigcap_{f \in \mathcal{F}} \text{Ker } \pi_f \varphi$ , where  $\pi_f$  is the canonical projection of  $M$  onto its  $f$ th coordinate. But each  $\pi_f \varphi$  can be viewed as an endomorphism of  $R^{(\mathcal{F})}$ , hence its kernel is a direct summand of  $R^{(\mathcal{F})}$ ; moreover,  $R^{(\mathcal{F})}$  satisfies SSIP (since it is Baer and by Theorem 1.7), hence  $K = \bigcap_{f \in \mathcal{F}} \text{Ker } \pi_f \varphi \leq^{\oplus} R^{(\mathcal{F})}$ . It follows that  $N$  is isomorphic to a direct summand of  $R^{(\mathcal{F})}$ , hence it is projective. In conclusion, since  $N$  was an arbitrarily chosen finitely generated torsionless right  $R$ -module, we obtain that every finitely generated torsionless right module is projective.

(3)  $\Rightarrow$  (1): Since  $R$  is right semihereditary, all endomorphisms of a finitely generated free right  $R$ -module  $M$  split, so in particular kernel of each endomorphism of  $M$  is a direct summand in  $M$ . We only need to show that every finitely generated free module also satisfies the SSIP, and the result will follow from Theorem 1.7.

Let  $\mathcal{F}$  be a finite set, and let  $\varphi_j \in \text{End}(R^{(\mathcal{F})})$  be idempotent endomorphisms, for every  $j \in \mathcal{J}$ , where  $\mathcal{J}$  is an arbitrary index set. Consider  $M = R^{\mathcal{J}}$ , direct product of copies of  $R$ . We assume  $\mathcal{J}$  to be infinite, and view  $R^{\mathcal{J}} = R^{\mathcal{F} \times \mathcal{J}}$ , for simplicity. We obtain the following exact sequence:

$$0 \rightarrow K \hookrightarrow R^{(\mathcal{F})} \xrightarrow{\prod_{j \in \mathcal{J}} \varphi_j} N \rightarrow 0$$

where  $N \subseteq M$  is the image of the morphism  $\prod_{j \in \mathcal{J}} \varphi_j: R^{(\mathcal{F})} \rightarrow M$ . This implies that  $N$  is a finitely generated torsionless module, hence  $N$  is projective, and the exact sequence splits. Hence  $K \leq^{\oplus} R^{(\mathcal{F})}$ . At the same time,  $K = \bigcap_{j \in \mathcal{J}} \text{Ker } \varphi_j$ . Since the choice of the index set  $\mathcal{J}$  as well as of the maps was arbitrary, and since any direct summand of  $R^{(\mathcal{F})}$  is the kernel of an idempotent endomorphism (in fact, of any endomorphism, in light of the statement above), we obtain that  $R^{(\mathcal{F})}$  has SSIP, as desired.

Since  $\mathcal{F}$  was an arbitrary finite set, we obtain that every finitely generated free  $R$ -modules is Baer.

(3)  $\Leftrightarrow$  (4): Assume  $R$  has the property that every finitely generated torsionless right  $R$ -module is projective. Let  $n \in \mathbb{N}$  be arbitrary. Since  $R^n$  is free, it is retractable, both as a right and as a left  $R$ -module. Since  $R^n$  is Baer as a right  $R$ -module, by equivalence between (3) and (2). Thus  $\text{End}(R^n_R) = M_n(R)$  is a Baer ring by Theorem 2.5. However,  $\text{End}({}_R R^n) = M_n(R)$ , thus  $\text{End}({}_R R^n)$  is also Baer. In conclusion,  ${}_R R^n$  is a Baer left  $R$ -module (Theorem 2.5). Using the equivalences (1)  $\Leftrightarrow$  (2)  $\Leftrightarrow$  (3), mirroring the proofs for left  $R$ -modules, we obtain that every finitely generated torsionless left  $R$  is projective.

The converse holds similarly.

(3)  $\Leftrightarrow$  (5): Since (4)  $\Leftrightarrow$  (3), condition (3) yields that  $R$  is left semihereditary. Moreover, since all finitely generated torsionless right modules are projective, they are also finitely presented; thus  $R$  is right  $\Pi$ -coherent. For the converse, since  $R$  is a left semihereditary ring, an application of Theorem 4.1

in [7] yields that all torsionless right  $R$ -modules are flat. By right  $\Pi$ -coherence of  $R$ , every finitely generated torsionless right module is finitely presented. Flatness of each finitely presented torsionless right  $R$ -module implies that each of them is also projective. Thus (3) holds.

(4)  $\Leftrightarrow$  (6): Similar to the case of (3)  $\Leftrightarrow$  (5).  $\square$

**Remark 3.6.** Note that our Theorem 3.5 generalizes Theorem 2.2 in [12], which states that, for a von Neumann regular ring  $R$ , every finitely generated torsionless right  $R$ -module embeds in a free right  $R$ -module (FGTF property) iff  $M_n(R)$  is a Baer ring for every  $n \in \mathbb{N}$ . Our result in fact establishes that every finitely generated torsionless right module is projective iff  $M_n(R)$  is Baer for every  $n \in \mathbb{N}$ , even in the absence of von Neumann regularity of  $R$ .

As a consequence of Theorem 3.5, we can obtain the following result for finite direct sums of copies of an arbitrary retractable Baer module  $M$  (in this case, we do not require the modules to be finitely generated, in contrast to Theorem 3.4).

**Corollary 3.7.** *Let  $M$  be a retractable Baer module. Then a finite direct sum of copies of  $M$  is a Baer module if and only if  $S = \text{End}(M)$  is left semihereditary and right  $\Pi$ -coherent.*

**Proof.** As  $M$  is retractable, using the same technique as used in the proof of Theorem 3.4, we can replace  $M$  with  $S = \text{End}(M)$ . By Theorem 3.5, the endomorphism ring of a finite direct sum of copies of  $S$  is Baer iff  $S$  is left semihereditary and right  $\Pi$ -coherent. This gives us the desired result.  $\square$

Our next result illustrates an application to the case when the base ring  $R$  is commutative. Recall that a characterization for an  $n \times n$  matrix ring over a commutative integral domain to be Baer is well-known [15,26].

**Theorem 3.8** (Corollary 15, [26]). *If  $R$  is a commutative integral domain, then  $M_n(R)$  is a Baer ring (for some  $n > 1$ ) if and only if every finitely generated ideal of  $R$  is invertible, i.e., if  $R$  is a Prüfer domain.*

We can show the following for a finite rank free module over a commutative domain.

**Theorem 3.9.** *Let  $R$  be a commutative integral domain and  $M$  a free  $R$ -module of finite rank  $> 1$ . Then  $M$  is Baer if and only if  $R$  is a Prüfer domain.*

**Proof.** If  $R$  is a Prüfer domain, then the endomorphism ring of  $M$  is ring isomorphic to  $M_n(R)$ , and hence is Baer. A free module is retractable, so by Theorem 2.2 we obtain that  $M$  is a Baer module.

If  $M$  is a Baer module, then its endomorphism ring is Baer, by Theorem 2.2, hence  $M_n(R)$  for  $n > 1$  is a Baer ring, thus  $R$  must be a Prüfer domain.  $\square$

In Theorem 2.2, [21] we showed that a ring is semisimple artinian if and only if every  $R$ -module is Baer. For the commutative rings one can restrict the requirement of “every  $R$ -module” to “every free  $R$ -module” to obtain the same conclusion.

**Proposition 3.10.** *Let  $R$  be a commutative ring. Every free  $R$ -module is Baer if and only if  $R$  is semisimple artinian. In particular, every  $R$ -module is Baer if every free  $R$ -module is so.*

**Proof.** It follows from Theorem 6 in [28].  $\square$

If the endomorphism ring of a module is a PID, we obtain the following result, due to Wilson, which we have reformulated in our setting (Lemma 4, [25]).

**Proposition 3.11.** *Let  $M$  be a finite direct sum of copies of some finite rank, torsion-free module whose endomorphism ring is a PID. Then  $M$  is Baer module.*

**Proof.** By [25]  $M$  has SSIP and the kernel of any endomorphism of  $M$  is a direct summand of  $M$ . Hence, by our Theorem 1.7,  $M$  is Baer.  $\square$

For a fixed  $n \in \mathbb{N}$ , we obtain the following characterization for every  $n$ -generated free  $R$ -module to be Baer.

**Theorem 3.12.** *Let  $R$  be a ring, and  $n \in \mathbb{N}$ . The following statements are equivalent:*

- (1) every  $n$ -generated free right  $R$ -module is a Baer module;
- (2) every  $n$ -generated projective right  $R$ -module is a Baer module;
- (3) every  $n$ -generated torsionless right  $R$ -module is projective (therefore  $R$  is right  $n$ -hereditary).

**Proof.** The proof follows the same outline as in Theorem 3.5, where we replace “finite” with “ $n$  elements.”  $\square$

It is interesting to note that, as opposed to related notions (such as injectivity, quasi-injectivity, continuity and quasi-continuity), having  $M \oplus M$  Baer does not imply that  $M \oplus M \oplus M$  is also Baer.

We start with a lemma and by recalling the well-known concept of an  $n$ -fir.

**Lemma 3.13.** *Let  $(M_i)_{i \in \mathcal{I}}$  be a family of Baer modules, with  $M_j$  and  $M_k$  relatively Rickart ( $j, k \in \mathcal{I}$ ). Let  $N_1 \leq^\oplus \bigoplus_{i \in \mathcal{I}} M_i$ . Then  $N_1 \cap M_j \leq^\oplus M_j \leq^\oplus \bigoplus_{i \in \mathcal{I}} M_i$ , for every  $j \in \mathcal{I}$ .*

**Proof.** Let  $N_1 \oplus N_2 = \bigoplus_{i \in \mathcal{I}} M_i$ . Let  $\pi_j$  be the canonical projection of  $\bigoplus_{i \in \mathcal{I}} M_i$  onto  $M_j$ ,  $\mu_j$  canonical projection onto  $\bigoplus_{i \neq j} M_i$  ( $j \in \mathcal{I}$ ),  $p_1$  canonical projection onto  $N_1$  and  $p_2$  canonical projection onto  $N_2$ .

Fix  $j \in \mathcal{I}$  and consider the maps  $\varphi = \mu_j(p_1|_{M_j})$  and  $\psi = \pi_j(p_2|_{M_j})$ . Then  $\text{Ker } \varphi \leq^\oplus M_j$  ( $\text{Ker } \varphi = \bigcap_{i \neq j} \text{Ker } \pi_i(p_1|_{M_j})$ ;  $\text{Ker } \pi_i(p_1|_{M_j}) \leq^\oplus M_j$  since  $M_j$  and  $M_i$  are relatively Rickart, for every  $j \neq i \in \mathcal{I}$ , and  $\bigcap_{i \neq j} \text{Ker } \pi_i(p_1|_{M_j}) \leq^\oplus M_j$  since  $M_j$  is a Baer module and satisfies SSIP). Further,  $\text{Ker } \psi \leq^\oplus M_j$  (since  $M_j$  is Baer, and  $\pi_j(p_2|_{M_j})$  is an endomorphism of  $M_j$ ).

$K = \text{Ker } \varphi \cap \text{Ker } \psi \leq^\oplus M_j$ .  $K = \{m \in M_j \mid p_1(m) \in M_j \text{ and } p_2(m) \in \bigoplus_{i \neq j} M_i\}$ . But  $M_j \ni m = p_1(m) + p_2(m) \Rightarrow m - p_1(m) = p_2(m) \in M_j \cap \bigoplus_{i \neq j} M_i = 0 \Rightarrow p_1(m) = m$  and  $p_2(m) = 0$ . This implies that  $m \in M_j \cap N_1 \Rightarrow K \subseteq M_j \cap N_1$ . If we take now  $m \in M_j \cap N_1 \Rightarrow \varphi m = \mu_j p_1(m) = \mu_j(m) = 0$  and  $\psi m = \pi_j p_2(m) = \pi_j(0) = 0 \Rightarrow m \in K$ . Thus,  $M_j \cap N_1 = K \leq^\oplus M_j$ .

Since  $j$  was arbitrarily chosen, the result is proved.  $\square$

**Remark 3.14.** Note that in Lemma 3.13, we get the result only by using the properties that  $M_j$  is Baer and that the kernels of all morphisms from  $M_j$  to  $M_i$  are direct summands in  $M_j$ , for every  $j \neq i \in \mathcal{I}$ .

**Definition 3.15.** A ring  $R$  is said to be a *right  $n$ -fir* if any right ideal that can be generated with  $\leq n$  elements is free of unique rank (i.e., for every  $I \leq R$ ,  $I \cong R^k$  for some  $k \leq n$ , and if  $I \cong R^l \Rightarrow k = l$ ) (for alternate definitions see Theorem 1.1, [10]).

The definition of (right)  $n$ -firs is left–right symmetric, thus we will call such rings simply  $n$ -firs. For more information on  $n$ -firs, see [10].

**Theorem 3.16.** *Let  $R$  be an  $n$ -fir. Then  $R^n$  is a Baer  $R$ -module. Consequently,  $M_n(R)$  is a Baer ring.*

**Proof.** Since  $R$  is an  $n$ -fir, it is in particular an integral domain (see page 45, [10]), thus trivially a Baer ring.

We will prove this result by induction on  $n$ . First, let  $R$  be a 2-fir, then we prove that the module  $R^2$  is Baer.

Take a collection of proper summands of  $R_R^2$ ,  $(N_i)_{i \in \mathcal{I}}$ , and fix an index  $i_0$ . Let  $N' \leq R_R^2$  such that  $N_{i_0} \oplus N' = R^2$ ; since  $N_{i_0}$  and  $N'$  must have lower rank than  $R^2$  (otherwise conflicting with the uniqueness of rank of  $R^2$ )  $N_{i_0} \cong R \cong N'$ , thus  $N_{i_0}$  and  $N'$  are Baer and relatively Rickart. Using Lemma 3.13,  $N_i \cap N_{i_0} \leq^\oplus N_{i_0}$  (more precisely, either 0 or  $N_{i_0}$ , due to the uniqueness of rank); then  $\bigcap_{i \in \mathcal{I}} N_i \leq^\oplus N_{i_0} \leq^\oplus R^2$ , thus  $R^2$  has SSIP. By properties of 2-firs, the image of every endomorphism of  $R_R^2$  is free, thus making the endomorphism split. Therefore, the kernel of every endomorphism of  $R_R^2$  is a direct summand in  $R_R^2$ . By Theorem 1.7,  $R_R^2$  is a Baer  $R$ -module.

Assume now that if a ring  $T$  is an  $(n-1)$ -fir, then  $T^{(n-1)}$  is a Baer module. Let  $R$  be an  $n$ -fir; we need to prove that  $R_R^n$  is Baer.

Since an  $n$ -fir is, in particular, an  $(n-1)$ -fir as well, we have that  $R^{(n-1)}$  is a Baer module. Moreover, the kernel of each homomorphism between  $R_R$  and  $R_R^{(n-1)}$  is a direct summand, since each can be extended to an endomorphism of  $R_R^{(n-1)}$ . In fact, every endomorphism of  $R_R^n$  also splits by properties of  $n$ -firs (images of such endomorphisms are  $n$ -generated, and thus free; this in turn makes the endomorphism split, and thus the kernel is a direct summand of  $R_R^n$ ). We only need to show that  $R_R^n$  has SSIP. Taking a collection of direct summands  $(N_i)_{i \in \mathcal{I}}$  (for an index set  $\mathcal{I}$ ) of  $R_R^n$ , and selecting one particular direct summand  $N_{i_0}$  (analogous to the proof for the  $n=2$  case),  $N_{i_0} \cong R_R^k$ , for some  $1 \leq k \leq n-1$  (since  $N_{i_0}$  is at most  $n$ -generated). Since  $R_R^k$  is a Baer module, and is relatively Rickart to  $R_R^{(n-k)}$  (because both  $k < n$  and  $n-k < n$ ,  $R_R^{\max(k, n-k)}$  is Baer by induction and Theorem 1.10). The intersection between  $N_{i_0}$  and any other direct summand of  $R_R^n$  will be a direct summand, by Lemma 3.13; in particular,  $N_{i_0} \cap N_i \leq^\oplus R_R^n$ , thus  $\bigcap_{i \in \mathcal{I}} N_i = \bigcap_{i \in \mathcal{I}} (N_{i_0} \cap N_i) \leq R_R^n$ . Therefore,  $R_R^n$  has SSIP. By Theorem 1.7,  $R_R^n$  is a Baer module.  $\square$

The example below proves the existence of a module  $M$  such that  $M^n$  is a Baer module, but  $M^{n+1}$  is not Baer.

**Example 3.17.** (See [14].) Let  $n$  be any natural number and let  $R$  be the  $K$ -algebra ( $K$  is any commutative field) on the  $2(n+1)$  generators  $X_i, Y_i$  ( $i = 1, \dots, n+1$ ) with the defining relation

$$\sum_{i=1}^{n+1} X_i Y_i = 0.$$

$R$  is an  $n$ -fir; however not all  $(n+1)$ -generated ideals are flat (see Theorem 2.3 in [14]).

In particular,  $R$  is not  $(n+1)$ -hereditary, since there exists an  $(n+1)$  generated ideal which is not flat, hence not projective.

Thus,  $R^n$  is a Baer module (due to  $R$  being an  $n$ -fir); however, since  $R$  is not  $(n+1)$ -hereditary,  $R^{n+1}$  is not Baer, by Theorem 3.12.

Given the connection provided by Theorem 1.11 between extending modules and Baer modules, we obtain the following result concerning  $(n, \text{finitely})$   $\Sigma$ -extending modules, i.e., modules  $M$  with the property that direct sums of  $(n, \text{finite number of})$  copies of  $M$  are extending. We generalize in this the results of Lemma 2.4 on polyform modules in [9] (recall that every polyform module is  $\mathcal{K}$ -nonsingular).

**Theorem 3.18.** Let  $M$  be a  $\mathcal{K}$ -nonsingular module, with  $S = \text{End}(M)$ .

- (1) If  $M^n$  is extending, then every  $n$ -generated right torsionless  $S$ -module is projective; it follows that  $S$  is a right  $n$ -hereditary ring.
- (2) If  $M^n$  is extending for every  $n \in \mathbb{N}$ , then  $S$  is right a semihereditary and left  $\Pi$ -coherent ring.
- (3) If  $M^{(\mathcal{I})}$  is extending for every index set  $\mathcal{I}$ , and  $M$  is finitely generated, then  $S$  is a semiprimary hereditary ring.

**Proof.** We start by stating that, if  $M$  is  $\mathcal{K}$ -nonsingular, then  $M^{(\mathcal{I})}$  is  $\mathcal{K}$ -nonsingular for every index set  $\mathcal{I}$  (Theorem 2.17 in [23]). Moreover, if  $M^{(\mathcal{I})}$  is extending, by Theorem 1.11, we get that  $M^{(\mathcal{I})}$  is a Baer module. By Theorem 3.4 and Corollary 3.7, as well as Theorem 3.12 (with a proof similar to that of Corollary 3.7) we obtain the desired implications.  $\square$

A more detailed discussion on these necessary conditions, as well as completing sufficient conditions for a module to be  $\Sigma$ -extending will appear in a sequel to this paper.

For now, recall that a sufficient condition for a finite direct sum of extending modules to be extending is that each direct summand be relatively injective to all others (see [13] or Proposition 7.10 in [11]). We prove that an analogue holds true also for the case of Baer modules.

**Theorem 3.19.** *Let  $\{M_i\}_{1 \leq i \leq n}$  be a class of Baer modules, where  $n \in \mathbb{N}$ . Assume that, for any  $i \neq j$ ,  $M_i$  and  $M_j$  are relative Rickart and relative injective. Then  $\bigoplus_{i=1}^n M_i$  is a Baer module.*

**Proof.** We prove the result by induction on  $n$ .

Start with  $n = 2$ . Let  $\{\varphi_j\}_{j \in \mathcal{J}}$  be a class of endomorphisms of  $M_1 \oplus M_2$ , where  $\mathcal{J}$  is any index set. We want to prove that  $K = \bigcap_{j \in \mathcal{J}} \text{Ker}(\varphi_j) \leq^\oplus M_1 \oplus M_2$ .

We show that we can reduce the problem to the case when  $K$  has zero intersection with either  $M_1$  or  $M_2$ . Assume  $K \cap M_1 \neq 0$ . We have that  $\text{Ker}(\varphi_j) \cap M_1 = \text{Ker}(\pi_1 \varphi_j \iota_1) \cap \text{Ker}(\pi_2 \varphi_j \iota_1)$ , where  $\pi_1, \pi_2$  are the canonical projections of  $M_1 \oplus M_2$  onto, respectively,  $M_1$  and  $M_2$ , and  $\iota_1, \iota_2$  are the canonical inclusions of the same. By hypothesis, both  $\text{Ker}(\pi_1 \varphi_j \iota_1)$  and  $\text{Ker}(\pi_2 \varphi_j \iota_1)$  are direct summands of  $M_1$ , as the first is the kernel of the endomorphism  $\pi_1 \varphi_j \iota_1$  of  $M_1$ , and the second is the kernel of the morphism  $\pi_2 \varphi_j \iota_1$  from  $M_1$  to  $M_2$ . Since  $M_1$  has SSIP by Theorem 1.7,  $\text{Ker} \varphi_j \cap M_1 \leq^\oplus M_1$ . Hence  $K \cap M_1 = (\bigcap_{j \in \mathcal{J}} \text{Ker}(\varphi_j)) \cap M_1 = \bigcap_{j \in \mathcal{J}} (\text{Ker}(\varphi_j) \cap M_1) \leq^\oplus M_1$ . Therefore  $K = (K \cap M_1) \oplus K'$  and  $M_1 = (K \cap M_1) \oplus M'_1$ . Similarly, we obtain  $K' \cap M_2 \leq^\oplus M_2$  and that  $K' = (K' \cap M_2) \oplus K''$  and  $M_2 = (K' \cap M_2) \oplus M'_2$ . In that case,  $K''$  is the intersection of the kernels of all morphisms  $\varphi_j$  restricted to  $M'_1 \oplus M'_2$ . Being summands of  $M_1$  and  $M_2$  respectively,  $K'' \cap M'_1 = 0$  and  $K'' \cap M'_2 = 0$ ;  $M'_1, M'_2$  are Baer.  $M'_1$  and  $M'_2$  are relatively Rickart, and are relatively injective. Hence, without loss of generality, we may assume that  $K \cap M_1 = 0$  and  $K \cap M_2 = 0$  (to simplify our notation, we say that  $K'' = K$ ,  $M'_1 = M_1, M'_2 = M_2$ ).

Because of relative injectivity we can embed  $K$  into a direct summand  $N_2$  with the properties:  $K \subseteq N_2$  and  $M_1 \oplus N_2 = M_1 \oplus M_2$ .  $N_2 \cong M_2$  and so  $N_2$  is Baer, and relatively Rickart and relatively injective with  $M_1$ . Taking  $p_1$  and  $p_2$  the canonical projections onto  $M_1$  and  $N_2$ , and  $i_1, i_2$  the canonical inclusions into  $M_1$  and  $N_2$ , respectively, we obtain, similar to the above argument, that  $K = \bigcap_{j \in \mathcal{J}} (\text{Ker}(p_1 \varphi_j i_2) \cap \text{Ker}(p_2 \varphi_j i_2))$ . For each  $j$  both of these kernels are direct summands in  $N_2$  (by Baer and relative Rickart assumption on  $N_2$ ), and the intersection of arbitrary number of direct summands is again a direct summand (by Theorem 1.7). Thus  $K \leq^\oplus N_2 \leq^\oplus M_1 \oplus M_2$ , which is what we wanted to prove.

Similarly, we can prove that (in the settings of the above hypothesis)  $M_1 \oplus M_2$  and  $M_3$  are relatively Rickart. Take any  $\varphi : M_3 \rightarrow M_1 \oplus M_2$ ;  $\text{Ker}(\varphi) = \text{Ker}(\pi_1 \varphi) \cap \text{Ker}(\pi_2 \varphi) \leq^\oplus M_3$ . Take now  $\psi : M_1 \oplus M_2 \rightarrow M_3$ . If  $\text{Ker}(\psi) \cap M_1 \neq 0$ , then  $\text{Ker}(\psi) \cap M_1 \leq^\oplus M_1$  and  $\text{Ker}(\psi) \cap M_1 \leq^\oplus \text{Ker}(\psi)$  (since  $\psi|_{M_1}$  is a map from  $M_1$  to  $M_3$ , and since  $M_1$  is relative Rickart to  $M_3$ ). Hence we can reduce the problem (similar to the situation above) to the case when  $\text{Ker}(\psi) \cap M_1 = 0$ . But since  $M_1$  and  $M_2$  are relative injective, we can embed  $\text{Ker}(\psi)$  into a summand  $L$ ,  $\text{Ker}(\psi) \leq L$ ,  $M_1 \oplus L = M_1 \oplus M_2$  where  $L \cong M_2$ . From this it easily follows that  $\text{Ker}(\psi) \leq^\oplus L$  ( $L$  is Baer and relatively Rickart to  $M_1$ ), which, together with the Baer property of  $M_1 \oplus M_2$ , yields that  $M_1 \oplus M_2$  and  $M_3$  are relatively Rickart to each other.

Assuming now that a direct sum of  $n$  Baer modules  $M_i$ ,  $1 \leq i \leq n$ , which are both relative Rickart and relative injective to each other, is Baer, and that this direct sum is relative Rickart with respect to  $M_{n+1}$ , we go to step  $n + 1$ . Since  $\bigoplus_{i \leq n} M_i$  is relatively injective to  $M_{n+1}$ , we obtain that  $\bigoplus_{i \leq n} M_i$  and  $M_{n+1}$  are both Baer modules; they are relatively Rickart and relatively injective to each other. Hence  $\bigoplus_{i=1}^n M_i \oplus M_{n+1} = \bigoplus_{i=1}^{n+1} M_i$  is a Baer module.

Lastly, we need to show that  $\bigoplus_{i \leq (n+1)} M_i$  and  $M_{n+2}$  are relatively Rickart. Take any  $\varphi : M_{n+2} \rightarrow \bigoplus_{i \leq (n+1)} M_i$ ;  $\text{Ker}(\varphi) = \bigcap_{1 \leq i \leq (n+1)} \text{Ker}(\pi_i \varphi) \leq^\oplus M_{n+2}$ , since  $M_{n+2}$  is relatively Rickart to  $M_i$ , and  $M_{n+2}$  is Baer ( $1 \leq i \leq (n+1)$ ). Take now  $\psi : \bigoplus_{i \leq (n+1)} M_i \rightarrow M_{n+2}$ . If  $\text{Ker}(\psi) \cap M_1 \neq 0$ , then  $\text{Ker}(\psi) \cap M_1 \leq^\oplus M_1$  and  $\text{Ker}(\psi) \cap M_1 \leq^\oplus \text{Ker}(\psi)$ . We reduce again the problem to the when  $\text{Ker}(\psi) \cap M_1 = 0$ . But since  $M_1$  and  $\bigoplus_{2 \leq i \leq (n+1)} M_i$  are relative injective, we can embed  $\text{Ker}(\psi)$  into a direct summand  $L$ ,  $\text{Ker}(\psi) \leq L$ ,  $M_1 \oplus L = M_1 \oplus \bigoplus_{2 \leq i \leq (n+1)} M_i$  where  $L \cong \bigoplus_{2 \leq i \leq (n+1)} M_i$ . From this, it easily follows that  $\text{Ker}(\psi) \leq^\oplus L$  ( $L$  is Baer, relatively Rickart to  $M_i$ ), which, together with the Baer property of  $\bigoplus_{2 \leq i \leq (n+1)} M_i$ , gives us relative Rickart property of  $\bigoplus_{2 \leq i \leq (n+1)} M_i$  and  $M_{n+2}$ .  $\square$

Next, we provide a complete characterization for an arbitrary direct sum of Baer modules to be Baer, provided that each module is fully invariant in the direct sum (see Proposition 2.4.15 in [24]).

**Proposition 3.20.** *Let  $M = \bigoplus_{i \in \mathcal{I}} M_i$  ( $\mathcal{I}$  an index set) be such that  $\text{Hom}(M_i, M_j) = 0$  for every  $i \neq j \in \mathcal{I}$  (i.e.,  $M_i \trianglelefteq M$ , for every  $i \in \mathcal{I}$ ). Then  $M$  is a Baer module if and only if  $M_i$  is a Baer module for every  $i \in \mathcal{I}$ .*

**Proof.** The necessity is clear, by Theorem 1.8.

To prove sufficiency, note that in the endomorphism ring of  $M = \bigoplus_{i \in \mathcal{I}} M_i$ , viewed as a matrix ring, each endomorphism is represented with only elements on the ‘diagonal’. Let  $I \leq {}_S S$ . Hence  $r_M(I) = \bigoplus_{i \in \mathcal{I}} r_{M_i}(I \cap S_i)$ , where  $S_i = \text{End}_R(M_i)$ . Since on each component, the right annihilator is a summand in  $M_i$  (since each  $M_i$  is Baer) it follows that  $r_M(I) = \bigoplus_{i \in \mathcal{I}} r_{M_i}(I \cap S_i) \leq^\oplus \bigoplus_{i \in \mathcal{I}} M_i = M$ , hence  $M$  is a Baer module.  $\square$

We end this section with some results on indecomposable Baer modules and their direct sums. First, a lemma.

**Lemma 3.21.** *Let  $M$  be an indecomposable Baer module and  $0 \neq m \in M$ . Then, for any  $\varphi \in \text{End}(M)$ ,  $\varphi$  is uniquely determined by the image under  $\varphi$  of  $m$ . Consequently,  $\text{End}(M)$  embeds in the set  $\{m \in M \mid r_R(m) \supseteq r_R(m_0)\}$ , for a fixed arbitrary nonzero element  $0 \neq m_0 \in M$ .*

**Proof.** Let  $0 \neq m_0 \in M$ . Assume there exist  $\varphi_1, \varphi_2 \in \text{End}(M)$ ,  $\varphi_1(m_0) = \varphi_2(m_0)$ . Then  $m_0 \in \text{Ker}(\varphi_1 - \varphi_2) \neq 0$ , hence, by Theorem 1.9,  $\varphi_1 - \varphi_2 = 0$ . Hence, any morphism  $\varphi$  is uniquely defined by the image at  $m_0$ . Since  $m_0$  can only be mapped into an element with a larger right annihilator in  $R$ , the last part of the conclusion follows easily.  $\square$

**Proposition 3.22.** *Let  $M = \bigoplus_{i \in \mathcal{I}} M_i$ , where each  $M_i$  is an indecomposable Baer module and relatively Rickart to  $M_j$ , for every  $i, j \in \mathcal{I}$ , where  $\mathcal{I}$  an index set. If  $N \leq^\oplus M$  then for every  $i \in \mathcal{I}$  either  $M_i \subseteq N$  or  $N \cap M_i = 0$ .*

**Proof.** Let  $S$  be the endomorphism ring of  $M$ . Let  $e_i^2 = e_i$  be the idempotents in  $S$  corresponding to the decomposition  $M = \bigoplus_{i \in \mathcal{I}} M_i$ . For any  $i \in \mathcal{I}$ ,  $e_i S e_i \cong S_i = \text{End}(M_i)$ . Let  $N = fM$ , for some  $f^2 = f \in S$ .

Assume  $0 \neq m \in N \cap M_i$ , for a certain  $i \in \mathcal{I}$ . Then  $e_i m = m$ ;  $f m = m$ ; so,  $e_i f e_i m = m$ . Since  $M_i$  is indecomposable Baer, by Lemma 3.21 the endomorphism  $e_i f e_i$  is uniquely defined by its value at  $m$ , hence  $e_i f e_i = e_i$ . Similarly, taking  $(1 - e_i) f e_i m = 0$ , we obtain that  $\text{Ker}(1 - e_i) f e_i \neq 0$ , yet, by relative Rickart property,  $\text{Ker}(1 - e_i) f e_i \leq^\oplus M_i$ , hence  $\text{Ker}(1 - e_i) f e_i = M_i$ .

Consequently,  $f e_i = e_i f e_i + (1 - e_i) f e_i = e_i$ , hence  $M_i \subseteq N$ .  $\square$

**Corollary 3.23.** *Let  $M$  be an indecomposable Baer module, and let  $M_i \cong M$ , for  $i \in \mathcal{I}$ ,  $\mathcal{I}$  an index set. Then for every  $N \leq^\oplus \bigoplus_{i \in \mathcal{I}} M_i$  we have either  $M_i \cap N = 0$  or  $M_i \subseteq N$ .*

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