



Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra



Bimodule herds

Tomasz Brzeziński^{a,*}, Joost Vercruysse^b^a Department of Mathematics, Swansea University, Singleton Park, Swansea SA2 8PP, UK^b Faculty of Engineering, Vrije Universiteit Brussel (VUB), B-1050 Brussels, Belgium

ARTICLE INFO

Article history:

Received 4 August 2008

Available online 14 February 2009

Communicated by Corrado de Concini

Keywords:

Bimodule herd

Torsor

Coring

Galois comodule

Entwining structure

ABSTRACT

The notion of a *bimodule herd* is introduced and studied. A bimodule herd consists of a B - A bimodule, its formal dual, called a *pen*, and a map, called a *shepherd*, which satisfies unitality and coassociativity conditions. It is shown that every bimodule herd gives rise to a pair of corings and coactions. If, in addition, a bimodule herd is *tame* i.e. it is faithfully flat and a progenerator, or if it is a progenerator and the underlying ring extensions are split, then these corings are associated to entwining structures; the bimodule herd is a Galois comodule of these corings. The notion of a *bicomodule coherd* is introduced as a formal dualisation of the definition of a bimodule herd. Every bicomodule coherd defines a pair of (non-unital) rings. It is shown that a tame B - A bimodule herd defines a bicomodule coherd, and sufficient conditions for the derived rings to be isomorphic to A and B are discussed. The composition of bimodule herds via the tensor product is outlined. The notion of a bimodule herd is illustrated by the example of Galois co-objects of a commutative, faithfully flat Hopf algebra.

© 2009 Elsevier Inc. All rights reserved.

1. Introduction

In classical geometry a torsor or a principal homogenous space is a G -set X on which the group G acts transitively and freely. Equivalently, torsors can be defined as sets, termed *herds* or *abstract cosets* (also called torsors), X with a structure mapping $X \times X \times X \rightarrow X$ satisfying some axioms; see [21, p. 170], [2, p. 202, footnote] or [12, Definition 2]. In this formulation, the group G is derived rather than given from the onset. This reconstruction of a group G from the axioms of herds is standard and well-known. More generally, a G -principal bundle X can be given a partially defined ternary operation

* Corresponding author.

E-mail addresses: t.brzezinski@swansea.ac.uk (T. Brzeziński), jvercruy@vub.ac.be (J. Vercruysse).

that makes it into a *pregroupoid* [19] or *herdoid* [18]. To a pregroupoid one can associate its *enveloping groupoid* [20] that in the case of a principal bundle coincides with the *Ehresmann* or *gauge groupoid*. For a recent review of these topics see [20].

All these points of view on herds and torsors together with the reconstructions of groups and groupoids are present in non-commutative geometry. On one hand non-commutative principal homogeneous spaces and principal bundles are represented by (faithfully flat) Hopf–Galois objects and extensions or, more generally, coalgebra–Galois extensions. On the other hand the Hopf algebra-free notion of a *quantum torsor* was introduced by Grunspan in [16]. That a faithfully flat quantum torsor is the same as a faithfully flat Galois object was observed in [24]. Independently, the notion of a *quantum heap* was proposed by Škoda [27], and it has been shown that the category of copointed quantum heaps (i.e. quantum heaps with a specified character) is isomorphic to the category of Hopf algebras; this gives the way of reconstructing a Hopf algebra from a quantum heap. The gauge groupoid associated to a Hopf–Galois extension or the *Ehresmann–Schauenburg bialgebroid* was constructed and led to the development of bi-Galois theory in [23]. The Ehresmann coring for coalgebra–Galois extensions is described in [7, pp. 392–393]. The need to describe Hopf–Galois extensions led to introduction of *B-torsors* in [24] (cf. [25, Section 2.8]), while the fully symmetric Hopf–bi-Galois theory necessitated studies of *A-B torsors* in [17, Chapter 5] (these can be understood as non-commutative counterparts of pregroupoids or herdoids). The most recent step in the approach to describing Galois-type extensions in terms of torsors was made in [3], where (faithfully flat) bi-Galois objects for coring extensions were described in terms of (faithfully flat) *pre-torsors*.

In all these algebraic approaches to torsors, a non-commutative torsor or a non-commutative principal bundle or a Galois-type extension is assumed to be an algebra with additional structure. Yet, *Galois comodules* for corings [13] have been recently shown to be an effective, general and unifying framework for the Hopf- and coalgebra–Galois theory; see [5,8,29]. The aim of the present paper is to introduce and study *bimodule herds*, i.e. torsor-like objects that are not assumed to be algebras, and to show their close relationship to Galois comodules. By using the terminology which refers to the older notion of herds (or flocks) on sets,¹ we want to stress that objects we study are no longer algebras (and hence are more general than previously studied torsors). At the same time we avoid a term *torsor* which might have been used in too many different contexts. On the other hand, as we will mention later and show in the last section of this paper, our notion in an abstract sense unifies the classical notion of herds or torsors with the non-commutative torsors.

We begin in Section 2 by defining what a bimodule herd is. The definition of a bimodule herd involves a bimodule and its dual. To keep the situation completely symmetric rather than considering one-sided duals with apriori no clear reason which side should be preferred, we consider a *formal dual* as given by evaluation and coevaluation maps in Definition 2.1. This is very reminiscent of Morita contexts and we discuss this relationship, by relating surjectivity of (co)evaluation maps with progenitor properties. Next we define a (tame) bimodule herd as a bimodule with a formal dual, called a *pen*, and a unital and coassociative structure map, called a *shepherd*, in Definition 2.4. It is shown that in this very general setup one can associate two corings to a bimodule herd; see Corollary 2.8. These corings can be understood as “gauge corings” associated to a bimodule herd. The first main result of Section 2, Theorem 2.16, reveals that in the tame case, each of these corings comes from an *entwining structure*. Thus, although a bimodule herd is defined by purely module-theoretic means and its definition makes no use of coalgebraic notions, tame bimodule herds are a source of corings and entwining structures. In fact Theorem 2.18 and Remark 2.19 show that tame bimodule herds can be identified with finite Galois comodules of corings associated to entwining structures. The approach to Galois theory through bimodule herds is fully left-right symmetric and hence lays foundations for the theory of *bi-Galois comodules*.

In Section 3 we formally dualise the notion of a bimodule herd, and define bicomodule coherds. These are bicomodules of two corings with a counital and associative structure map. Although bicomodule coherds might seem at first as a mere dualisation of bimodule herds, the main reason for

¹ The term *herd* is the English translation of the German *Schar* of Prüfer [21] and Baer [2], advocated by Johnstone in [18].

their introduction is revealed in Theorem 3.4, where it is shown how a coherd can be associated to a tame bimodule herd.

Section 4 is devoted to the description of ways in which two bimodule herds can be composed. It turns out that the composition via the tensor product is possible whenever the associated corings can form a smash coproduct. Various facets of bimodule herds discussed in this paper are illustrated in Section 5 by *Galois co-objects* for (commutative) Hopf algebras. As these objects are not algebras, even in this simple case, the use of bimodule herds (rather than previously studied torsor and pre-torsor algebras) becomes inevitable. The composition of Galois co-objects is shown to coincide with the composition of corresponding bimodule herds.

The paper is supplemented with an appendix, in which we describe a categorical formulation of bimodule herds. This is in-line with recent resurgence of interest in categorical aspects of module and comodule theory, and also indicates a categorical framework which unifies bimodule herds with standard geometric (or set-theoretic) herds. A different approach to the categorical formulation of pre-torsors is given in [4].

Notation. Throughout the paper, \mathbf{M}_R denotes the category of right R -modules and right R -linear maps where R is a unital associative ring. Similarly, we use notations ${}_R\mathbf{M}$ for the category of left R -modules and \mathbf{M}^C for the category of right comodules over an R -coring C . For an R -coring C , Δ_C denotes the coproduct and ε_C denotes the counit. We refer to [1] and [9] for comprehensive introductions. If X is an object in a category, then X is also used to denote the identity morphism on X . Simple tensors often represent a finite sum of simple tensors.

In Sections 2–4, R and S are unital associative rings, and

$$\alpha : R \rightarrow A, \quad \beta : S \rightarrow B,$$

are maps of associative unital rings. All A -, respectively B -modules are understood as R -, respectively S -modules via α , respectively β .

2. Bimodule herds

In this section the definition and fundamental properties of bimodule herds, including their relationship with Galois comodules, are given.

Formal duals. In the definition of a bimodule herd one needs to use a bimodule and its dual. We formalise this by introducing the notion of a *formal dual*. This might be well known to ring and module theorists; the definition and basic properties of formal duals are included for completeness and for fixing the notation.

2.1. Definition. Let T be a B - A bimodule. An A - B bimodule \widehat{T} is said to be a *formal dual* of T if there exist an A -bimodule map

$$\text{ev} : \widehat{T} \otimes_S T \rightarrow A,$$

and a B -bimodule map

$$\widehat{\text{ev}} : T \otimes_R \widehat{T} \rightarrow B,$$

rendering commutative the following diagrams:

$$\begin{array}{ccc}
 T \otimes_R \hat{T} \otimes_S T & \xrightarrow{\hat{\text{ev}} \otimes_S T} & B \otimes_S T \\
 T \otimes_R \text{ev} \downarrow & & \downarrow \\
 T \otimes_R A & \xrightarrow{\quad} & T,
 \end{array} \quad (2.1)$$

$$\begin{array}{ccc}
 \hat{T} \otimes_S T \otimes_R \hat{T} & \xrightarrow{\hat{T} \otimes_S \hat{\text{ev}}} & \hat{T} \otimes_S B \\
 \text{ev} \otimes_R \hat{T} \downarrow & & \downarrow \\
 A \otimes_R \hat{T} & \xrightarrow{\quad} & \hat{T}.
 \end{array} \quad (2.2)$$

Here the unlabeled arrows correspond to A - and B -multiplications on T and \hat{T} .

If T and \hat{T} are bimodules forming a Morita context, then \hat{T} is a formal dual of T . Also, if $B = \text{End}_A(T)$, then $T^* = \text{Hom}_A(T, A)$ is a formal dual of T (with ev the evaluation map, and $\hat{\text{ev}}$ the coevaluation map).

2.2. Lemma. Let T be a B - A bimodule and \hat{T} an A - B bimodule.

(1) Suppose that \hat{T} is a formal dual of T . Write T^* for $\text{Hom}_A(T, A)$ and *T for $\text{Hom}_B(T, B)$.

(a) The map $\hat{\text{ev}}$ is surjective if and only if T is a finitely generated and projective right A -module, a faithful left B -module and the map

$$\lambda : \hat{T} \rightarrow T^*, \quad \hat{x} \mapsto [x \mapsto \text{ev}(\hat{x} \otimes_S x)],$$

is an isomorphism of A - S bimodules. If this happens, then \hat{T} is a generator as a right B -module, and the map

$$T \otimes_A \hat{T} \rightarrow B, \quad x \otimes_A \hat{x} \mapsto \hat{\text{ev}}(x \otimes_R \hat{x}),$$

is an isomorphism of B -bimodules.

(b) The map ev is surjective if and only if T is a finitely generated and projective left B -module, a faithful right A -module and the map

$$\hat{\lambda} : \hat{T} \rightarrow {}^*T, \quad \hat{x} \mapsto [x \mapsto \hat{\text{ev}}(x \otimes_R \hat{x})],$$

is an isomorphism of R - B bimodules. If this happens, then \hat{T} is a generator as a left A -module, and the map

$$\hat{T} \otimes_B T \rightarrow A, \quad \hat{x} \otimes_B x \mapsto \text{ev}(\hat{x} \otimes_S x),$$

is an isomorphism of A -bimodules.

(2) The functors $- \otimes_B T : \mathbf{M}_B \rightarrow \mathbf{M}_A$ and $- \otimes_A \hat{T} : \mathbf{M}_A \rightarrow \mathbf{M}_B$ determine a pair of inverse equivalences (i.e. T is a progenerator as right A -module and $B = \text{End}_A(T)$) if and only if \hat{T} is a formal dual of T such that both maps ev and $\hat{\text{ev}}$ are surjective.

Proof. (1)(a) Assume that the map $\hat{\text{ev}}$ is surjective and let $e_i \in T$, $\hat{e}_i \in \hat{T}$ be such that $\hat{\text{ev}}(\sum_i e_i \otimes_R \hat{e}_i) = 1_B$. Then, for all $x \in T$,

$$\sum_i e_i \lambda(\widehat{e}_i)(x) = \sum_i e_i \operatorname{ev}(\widehat{e}_i \otimes_S x) = \sum_i \widehat{\operatorname{ev}}(e_i \otimes_R \widehat{e}_i)x = x,$$

where the penultimate equality follows by (2.1). This means that $\{e_i \in T, \lambda(\widehat{e}_i) \in T^*\}$ is a finite dual basis for the right A -module T . Using the above calculation, right A -linearity of ev and (2.2) one easily verifies that the map

$$\lambda^{-1} : T^* \rightarrow \widehat{T}, \quad f \mapsto \sum_i f(e_i) \widehat{e}_i,$$

is the inverse of λ .

Let $\ell : B \rightarrow \operatorname{End}_A(T)$ be defined by $b \mapsto [x \mapsto bx]$. Using diagrams (2.1) and (2.2) and the definition of e_i, \widehat{e}_i one can verify that

$$\ell^{-1} : \operatorname{End}_A(T) \rightarrow B, \quad s \mapsto \sum_i \widehat{\operatorname{ev}}(s(e_i) \otimes_R \widehat{e}_i),$$

is the inverse of ℓ . Thus, in particular, ℓ is injective, i.e. T is a faithful left B -module. Combining ℓ with λ and the fact that T is a finitely generated and projective right A -module, the map $T \otimes_A \widehat{T} \rightarrow B$, $x \otimes_A \widehat{x} \mapsto \widehat{\operatorname{ev}}(x \otimes_R \widehat{x})$ is recovered as a chain of isomorphisms

$$T \otimes_A \widehat{T} \cong T \otimes_A T^* \cong \operatorname{End}_A(T) \cong B.$$

Finally, take any right B -module map $f : M \rightarrow N$ such that $\operatorname{Hom}_B(\widehat{T}, f) = 0$. Then, in the view of just proven isomorphism,

$$\operatorname{Hom}_A(T, \operatorname{Hom}_B(\widehat{T}, f)) = 0 \Leftrightarrow \operatorname{Hom}_B(T \otimes_A \widehat{T}, f) = 0 \Leftrightarrow \operatorname{Hom}_B(B, f) = 0 \Leftrightarrow f = 0.$$

Therefore, \widehat{T} is a generator of right B -modules.

In the converse direction, assume that T is a finitely generated and projective right A -module, λ is an isomorphism and that the map $\ell : B \rightarrow \operatorname{End}_A(T)$ is a monomorphism (i.e. T is a faithful left B -module). Let $\{e_i \in T, e_i^* \in T^*\}$ be a dual basis for right A -module T . The commutativity of diagram (2.1) implies that the following diagram

$$\begin{array}{ccc} T \otimes_R \widehat{T} & \xrightarrow{\quad \quad \quad} & T \otimes_A \widehat{T} \\ \widehat{\operatorname{ev}} \downarrow & & \downarrow T \otimes_A \lambda \\ B & \xrightarrow{\quad \ell \quad} \operatorname{End}_A(T) \xleftarrow{\quad \quad \quad} & T \otimes_A T^* \end{array}$$

is commutative. The unmarked arrow in the top row is the canonical surjection, while the unmarked arrow in the bottom row is the standard coevaluation map. Apply the clockwise composition to $\sum_i e_i \otimes_R \lambda^{-1}(e_i^*)$ to obtain an endomorphism of T ,

$$x \mapsto \sum_i e_i \lambda(\lambda^{-1}(e_i^*))(x) = \sum_i e_i e_i^*(x) = x.$$

Since ℓ is a monomorphism, the preimage of this map is the unit element of B , i.e. $1_B = \widehat{\operatorname{ev}}(\sum_i e_i \otimes_R \lambda^{-1}(e_i^*))$. Since $\widehat{\operatorname{ev}}$ is a B -bimodule map, the above equality implies that $\widehat{\operatorname{ev}}$ is a surjective map.

The assertions (1)(b) are proven in a symmetric way.

(2) Suppose first that $\widehat{\operatorname{ev}}$ and ev are surjective. Then we know by part (1), that $\widehat{\operatorname{ev}}$ and ev induce well-defined bijective maps $T \otimes_A \widehat{T} \rightarrow B$ and $\widehat{T} \otimes_B T \rightarrow A$. One can easily check that these induced

maps form a Morita context between A and B , which is strict by construction and hence the categories \mathbf{M}_A and \mathbf{M}_B are equivalent.

Conversely, if the functors $-\otimes_B T$ and $-\otimes_A \hat{T}$ induce an equivalence between the categories \mathbf{M}_B and \mathbf{M}_A , then this equivalence is induced by a Morita context $(B, A, T, \hat{T}, \mu, \tau)$. By putting $\text{ev}: \hat{T} \otimes_S T \rightarrow \hat{T} \otimes_B T \rightarrow A$, where the first map is the canonical projection and the second map is the Morita map, and similarly $\hat{\text{ev}}: T \otimes_R \hat{T} \rightarrow T \otimes_A \hat{T} \rightarrow B$, we find that \hat{T} is a formal dual of T such that ev and $\hat{\text{ev}}$ are surjective. \square

2.3. Corollary. Let T be a B - A bimodule with a formal dual \hat{T} . If both ev and $\hat{\text{ev}}$ are surjective, then

- (1) T is faithfully flat as left S -module if and only if B is faithfully flat as left S -module.
- (2) T is faithfully flat as right R -module if and only if A is faithfully flat as right R -module.

Proof. (1) Suppose that B is faithfully flat as a left S -module. Since ev and $\hat{\text{ev}}$ are surjective, $-\otimes_B T: \mathbf{M}_B \rightarrow \mathbf{M}_A$ is an equivalence of categories (see Lemma 2.2), hence T is faithfully flat as a left B -module. Therefore, ${}_S T \cong {}_S B \otimes_B T$ is faithfully flat as well.

Conversely, the surjectivity of ev and $\hat{\text{ev}}$ implies that $-\otimes_A \hat{T}: \mathbf{M}_A \rightarrow \mathbf{M}_B$ is an equivalence of categories, hence \hat{T} is faithfully flat as a left A -module, so ${}_S B \cong {}_S T \otimes_A \hat{T}$ is faithfully flat as well.

Part (2) is proven in a symmetric way. \square

B-A herds and associated corings. The main object of studies of this paper is given in the following

2.4. Definition. Let T be a B - A bimodule with a formal dual \hat{T} . T is called a *bimodule herd* or simply a *B-A herd* provided that there exists an S - R bimodule map $\gamma: T \rightarrow T \otimes_R \hat{T} \otimes_S T$ rendering commutative the following diagrams

$$\begin{array}{ccc} T & \xrightarrow{\gamma} & T \otimes_R \hat{T} \otimes_S T \\ \cong \downarrow & & \downarrow \hat{\text{ev}} \otimes_S T \\ S \otimes_S T & \xrightarrow{\beta \otimes_S T} & B \otimes_S T, \end{array} \quad (2.3)$$

$$\begin{array}{ccc} T & \xrightarrow{\gamma} & T \otimes_R \hat{T} \otimes_S T \\ \cong \downarrow & & \downarrow T \otimes_R \text{ev} \\ T \otimes_R R & \xrightarrow{T \otimes_R \alpha} & T \otimes_R A, \end{array} \quad (2.4)$$

$$\begin{array}{ccc} T & \xrightarrow{\gamma} & T \otimes_R \hat{T} \otimes_S T \\ \gamma \downarrow & & \downarrow \gamma \otimes_R \hat{T} \otimes_S T \\ T \otimes_R \hat{T} \otimes_S T & \xrightarrow{T \otimes_R \otimes_S \hat{T} \otimes_S \gamma} & T \otimes_R \hat{T} \otimes_S T \otimes_R \hat{T} \otimes_S T. \end{array} \quad (2.5)$$

The map γ is called the *shepherd*, and the formal dual \hat{T} is referred to as the *pen*.

The bimodule herd (T, γ) is said to be *tame* provided T satisfies conditions of Corollary 2.3, i.e. the maps ev and $\hat{\text{ev}}$ are surjective and T is faithfully flat as an R - and S -module.

2.5. Notation. Let \hat{T} be a formal dual of a B - A bimodule T , and let $\gamma: T \rightarrow T \otimes_R \hat{T} \otimes_S T$ be an S - R bimodule map. The application of γ to an element $x \in T$ is denoted by

$$\gamma(x) = x^{(1)} \otimes_R x^{(2)} \otimes_S x^{(3)},$$

(summation implicit). Given γ , define

$$\gamma_A : T \xrightarrow{\gamma} T \otimes_R \widehat{T} \otimes_S T \longrightarrow T \otimes_A \widehat{T} \otimes_S T,$$

and

$$\gamma_B : T \xrightarrow{\gamma} T \otimes_R \widehat{T} \otimes_S T \longrightarrow T \otimes_R \widehat{T} \otimes_B T,$$

where the second maps are the canonical surjections.

2.6. Lemma. Let \widehat{T} be a formal dual of a B - A bimodule T , and let $\gamma : T \rightarrow T \otimes_R \widehat{T} \otimes_S T$ be an S - R bimodule map. If γ satisfies property (2.3), then γ_A is a right A -module map. If γ makes (2.4) commute, then γ_B is a left B -module map.

Proof. The A -linearity of γ_A is proven by the explicit calculations that use diagram (2.3) in the second and the last equalities, diagram (2.2) in the third equality, and diagram (2.1) in the fifth equality. For all $a \in A$ and $x \in T$,

$$\begin{aligned} \gamma_A(x)a &= x^{(1)} \otimes_R x^{(2)} \otimes_S x^{(3)}a \\ &= x^{(1)} \otimes_A x^{(2)} \widehat{\text{ev}}((x^{(3)}a)^{(1)} \otimes_R (x^{(3)}a)^{(2)}) \otimes_S (x^{(3)}a)^{(3)} \\ &= x^{(1)} \otimes_A \text{ev}(x^{(2)} \otimes_S (x^{(3)}a)^{(1)}) (x^{(3)}a)^{(2)} \otimes_S (x^{(3)}a)^{(3)} \\ &= x^{(1)} \text{ev}(x^{(2)} \otimes_S (x^{(3)}a)^{(1)}) \otimes_A (x^{(3)}a)^{(2)} \otimes_S (x^{(3)}a)^{(3)} \\ &= \widehat{\text{ev}}(x^{(1)} \otimes_R x^{(2)}) (x^{(3)}a)^{(1)} \otimes_A (x^{(3)}a)^{(2)} \otimes_S (x^{(3)}a)^{(3)} \\ &= \gamma_A(xa). \end{aligned}$$

The second statement is proven by symmetric arguments. \square

2.7. Proposition. Let \widehat{T} be a formal dual of a B - A bimodule T , and let $\gamma : T \rightarrow T \otimes_R \widehat{T} \otimes_S T$ be an S - R bimodule map.

(1) Assume that the map γ makes diagram (2.3) commute. For all right A -modules N , the map

$$\Theta_N : \text{Hom}_A(T, N) \otimes_S T \rightarrow N \otimes_A \widehat{T} \otimes_S T, \quad f \otimes_S x \mapsto (f \otimes_A \widehat{T} \otimes_S T)(\gamma_A(x)),$$

is an isomorphism of right A -modules, natural in N . In particular, writing $T^* = \text{Hom}_A(T, A)$,

$$T^* \otimes_S T \cong \widehat{T} \otimes_S T,$$

as A -bimodules. Furthermore, the following diagram

$$\begin{array}{ccc} T^* \otimes_S T & \xrightarrow{\Theta_A} & \widehat{T} \otimes_S T \\ & \searrow & \swarrow \text{ev} \\ & A & \end{array}$$

in which the unmarked arrow is the evaluation map, is commutative. Finally, if T is completely faithful as a left S -module, then $\widehat{T} \cong T^*$.

(2) Assume that the map γ makes diagram (2.4) commute. For all left B -modules N , the map

$$\widehat{\Theta}_N : T \otimes_R \text{Hom}_B(T, N) \rightarrow T \otimes_R \widehat{T} \otimes_B N, \quad x \otimes_R f \mapsto (T \otimes_R \widehat{T} \otimes_B f)(\gamma_B(x)),$$

is an isomorphism of left B -modules, natural in N . In particular, writing ${}^*T = \text{Hom}_B(T, B)$,

$$T \otimes_R {}^*T \cong T \otimes_R \widehat{T},$$

as B -bimodules. Furthermore, the following diagram

$$\begin{array}{ccc} T \otimes_R {}^*T & \xrightarrow{\widehat{\Theta}_B} & T \otimes_R \widehat{T} \\ & \searrow & \swarrow \text{ev} \\ & B & \end{array}$$

in which the unmarked arrow is the evaluation map, is commutative. Finally, if T is completely faithful as a right R -module, then $\widehat{T} \cong {}^*T$.

Proof. (1) By Lemma 2.6, Θ_N is a right A -module map. The inverse of Θ_N is given by

$$\Theta_N^{-1} : N \otimes_A \widehat{T} \otimes_S T \rightarrow \text{Hom}_A(T, N) \otimes_S T, \quad n \otimes_A \widehat{x} \otimes_S x \mapsto n\lambda(\widehat{x}) \otimes_S x,$$

where $\lambda : \widehat{T} \rightarrow T^*$, $\widehat{x} \mapsto [x \mapsto \text{ev}(\widehat{x} \otimes_S x)]$ is the map described in Lemma 2.2. Indeed, first, for all $n \in N$, $\widehat{x} \in \widehat{T}$ and $x \in T$,

$$\begin{aligned} \Theta_N \circ \Theta_N^{-1}(n \otimes_A \widehat{x} \otimes_S x) &= n\lambda(\widehat{x})(x^{(1)}) \otimes_A x^{(2)} \otimes_S x^{(3)} \\ &= n \otimes_A \text{ev}(\widehat{x} \otimes_S x^{(1)}) x^{(2)} \otimes_S x^{(3)} \\ &= n \otimes_A \widehat{x} \widehat{\text{ev}}(x^{(1)} \otimes_R x^{(2)}) \otimes_S x^{(3)} \\ &= n \otimes_A \widehat{x} \otimes_S x, \end{aligned}$$

where the third equality follows by (2.2) and the final equality by (2.3). Second, for all $f \in \text{Hom}_A(T, N)$ and $x \in T$,

$$\begin{aligned} \Theta_N^{-1} \circ \Theta_N(f \otimes_S x) &= f(x^{(1)})\lambda(x^{(2)}) \otimes_S x^{(3)} \\ &= f(x^{(1)}) \text{ev}(x^{(2)} \otimes_S -) \otimes_S x^{(3)} \\ &= f(x^{(1)} \text{ev}(x^{(2)} \otimes_S -)) \otimes_S x^{(3)} \\ &= f(\widehat{\text{ev}}(x^{(1)} \otimes_R x^{(2)}) -) \otimes_S x^{(3)} \\ &= f \otimes_S x, \end{aligned}$$

where the third equality follows by the right A -linearity of f , the fourth equality is a consequence of (2.1), and the final equality follows by (2.3).

The forms of Θ_N and Θ_N^{-1} imply immediately that these maps are natural in N . The commutativity of the triangle diagram follows by the following direct calculation, for all $f \in \text{Hom}_A(T, A)$ and $x \in T$,

$$\text{ev}(\Theta_A(f \otimes_S x)) = \text{ev}(f(x^{(1)})x^{(2)} \otimes_S x^{(3)}) = f(x^{(1)} \text{ev}(x^{(2)} \otimes_S x^{(3)})) = f(\widehat{\text{ev}}(x^{(1)} \otimes_R x^{(2)})x^{(3)}) = f(x),$$

where the second equality follows by the right A -linearity of f and the left A -linearity of ev , and the last equality is a consequence of (2.3).

Finally, note that $\Theta_A^{-1} = \lambda \otimes_S T$. The tensor functor $-\otimes_S T$ of a completely faithful module reflects exact sequences (see [1, p. 233]), hence it also reflects isomorphisms. Thus, if T is a completely faithful left S -module, λ is the required isomorphism.

The statement (2) is proven by symmetric arguments. \square

Since, in the definition of a bimodule herd, the pen \widehat{T} appears only in the forms $\widehat{T} \otimes_S T$ and $T \otimes_R \widehat{T}$, Proposition 2.7 implies that *a posteriori* the definition of a bimodule herd does not depend on the choice of a formal dual of T .

2.8. Corollary. Let (T, γ) be a B - A herd.

(1) The A -bimodule $\mathcal{C} = \widehat{T} \otimes_S T \cong T^* \otimes_S T$ is an A -coring with coproduct

$$\Delta_{\mathcal{C}} : \widehat{x} \otimes_S x \mapsto \widehat{x} \otimes_S \gamma_A(x),$$

and the counit $\varepsilon_{\mathcal{C}} = \text{ev}$. T is a right \mathcal{C} -comodule with the coaction γ_A .

(2) The B -bimodule $\mathcal{D} = T \otimes_R \widehat{T} \cong T \otimes_R {}^*T$ is a B -coring with coproduct

$$\Delta_{\mathcal{D}} : x \otimes_R \widehat{x} \mapsto \gamma_B(x) \otimes_R \widehat{x},$$

and the counit $\varepsilon_{\mathcal{D}} = \widehat{\text{ev}}$. T is a left \mathcal{D} -comodule with the coaction γ_B .

Proof. (1) The maps Θ_N in Proposition 2.7 establish an isomorphism of functors $\Theta : \text{Hom}_A(T, -) \otimes_S T \rightarrow - \otimes_A \widehat{T} \otimes_S T$. The domain of Θ is a comonad on the category of right A -modules, hence so is the codomain of Θ . This implies that $\widehat{T} \otimes_S T$ is an A -coring with the described comultiplication and counit.

For a less categorical proof, one can use the following direct arguments. By the definition of the map ev and Lemma 2.6 both the coproduct and counit are A -bimodule maps. The coassociativity of $\Delta_{\mathcal{C}}$ follows immediately by diagram (2.5). The equality $(\varepsilon_{\mathcal{C}} \otimes_A \mathcal{C}) \circ \Delta_{\mathcal{C}} = \mathcal{C}$ is an immediate cosequence of diagram (2.4). The other counitality property, $(\mathcal{C} \otimes_A \varepsilon_{\mathcal{C}}) \circ \Delta_{\mathcal{C}} = \mathcal{C}$, is established by converting ev to $\widehat{\text{ev}}$ with the help of the diagram (2.2), and then by using (2.3).

By Lemma 2.6, γ_A is a right A -module map. It is coassociative by (2.5) and is counital by (2.4).

The assertion (2) is proven by symmetric arguments. \square

2.9. Example. Take a finitely generated projective right A -module T , and set $B = \text{End}_A(T)$ and $R = A$. Let $\widehat{T} = T^*$. Then T is a bimodule herd with

$$\gamma : T \rightarrow T \otimes_A \widehat{T} \otimes_S T, \quad x \mapsto \sum_i e_i \otimes_A f^i \otimes_S x,$$

where $e_i \in T$, $f^i \in \widehat{T}$ is (any) finite dual basis for T . The coring \mathcal{C} is simply the (finite) comatrix coring [13].

2.10. Example. Let \widehat{S} be a ring, possibly without a unit. We say that a right \widehat{S} -module M is *firm* if and only if the multiplication map induces an isomorphism $M \otimes_{\widehat{S}} \widehat{S} \rightarrow M$. A ring is called *firm* if it is firm as a left, or equivalently right, \widehat{S} -module. If \widehat{S} has a unit, then firm modules are exactly the unital modules. The category of all firm right modules of \widehat{S} and \widehat{S} -linear maps between them is denoted by $\mathbf{M}_{\widehat{S}}$.

A right A -module T is said to be \widehat{S} -firmly projective [28] if it is an \widehat{S} - A bimodule that is firm as a left \widehat{S} -module, and if the functor $-\otimes_{\widehat{S}} T : \mathbf{M}_{\widehat{S}} \rightarrow \mathbf{M}_A$ has a right adjoint of the form $-\otimes_A \widehat{T}$, where \widehat{T} is an A - \widehat{S} bimodule that is firm as a right \widehat{S} -module. Denote the unit of the adjunction by η and the counit by ϵ . Then $\eta_{\widehat{S}} : \widehat{S} \rightarrow T \otimes_A \widehat{T}$ and $\epsilon_A : \widehat{T} \otimes_{\widehat{S}} T \rightarrow A$. If \widehat{S} has a unit, then an \widehat{S} -firmly projective right A -module is precisely a finitely generated and projective right A -module.

Let T be an \widehat{S} -firmly projective right A -module and use notation as above. Let S be the Dorroh-extension of \widehat{S} , which is a ring with unit. One can easily observe that $M \otimes_{\widehat{S}} N \cong M \otimes_S N$ for $M \in \mathbf{M}_{\widehat{S}}$ and $N \in {}_{\widehat{S}}\mathbf{M}$. Furthermore, \widehat{T} is a formal dual of T , having $\text{ev} = \epsilon_A$, $R = A$, $B = \text{End}_A(T)$ and $\widehat{\text{ev}} : T \otimes_A \widehat{T} \rightarrow B$, $\widehat{\text{ev}}(x \otimes_A \widehat{x})(y) = x \epsilon_A(\widehat{x} \otimes_S y)$. Finally, T is a bimodule herd, where the shepherd $\gamma = \eta_T : T \rightarrow T \otimes_S \widehat{T} \otimes_A T$, is the unit of the adjunction on T . The associated A -coring \mathcal{C} is the comatrix coring associated to the firm bimodule T as defined in [15]. The associated B -coring \mathcal{D} coincides with the construction of a coring out of a firm ring that is an ideal in a unital ring (see [6, Theorem 1.6]).

2.11. Example. Let \mathcal{C} be an R -coring, and let $\psi : \mathcal{C} \otimes_R A \rightarrow A \otimes_R \mathcal{C}$ be an R bimodule map entwining \mathcal{C} with A . This means that ψ satisfies the following conditions

$$\begin{aligned} \psi \circ (\mathcal{C} \otimes_R \mu_A) &= (\mu_A \otimes_R \mathcal{C}) \circ (A \otimes_R \psi) \circ (\psi \otimes_R A), & \psi \circ (\mathcal{C} \otimes_R \alpha) &= \alpha \otimes_R \mathcal{C}, \\ (A \otimes_R \Delta_{\mathcal{C}}) \circ \psi &= (\psi \otimes_R \mathcal{C}) \circ (\mathcal{C} \otimes_R \psi) \circ (\Delta_{\mathcal{C}} \otimes_R A), & (A \otimes_R \varepsilon_{\mathcal{C}}) \circ \psi &= \varepsilon_{\mathcal{C}} \otimes_R A, \end{aligned}$$

where $\mu_A : A \otimes_R A \rightarrow A$ is the multiplication in the R -ring A (with unit $\alpha : R \rightarrow A$).

Set $\mathcal{C} := A \otimes_R \mathcal{C}$ to be the A -coring associated to this entwining structure. Assume that T is a finite Galois (right) comodule of \mathcal{C} . This means that T is a right \mathcal{C} -comodule that is finitely generated and projective as a right A -module and that the canonical map

$$\text{can} : T^* \otimes_S T \rightarrow \mathcal{C} = A \otimes_R \mathcal{C}, \quad f \otimes_S x \mapsto f(x_{(0)}) \otimes_R x_{(1)},$$

where $S = \text{End}^{\mathcal{C}}(T)$, is bijective (an isomorphism of A -corings). Here $x \mapsto x_{(0)} \otimes_R x_{(1)}$ (summation implicit) denotes the coaction of \mathcal{C} on T (the \mathcal{C} -coaction is then $x \mapsto x_{(0)} \otimes_A 1_A \otimes_R x_{(1)}$). Set $B = \text{End}_A(T)$, $\widehat{T} = T^*$, $\widehat{\text{ev}} : T \otimes_R T^* \rightarrow T \otimes_A T^* \cong B$, and $\text{ev} : T^* \otimes_S T \rightarrow A$ the standard evaluation. Consider the translation map

$$\tau : \mathcal{C} \rightarrow T^* \otimes_S T, \quad c \mapsto \text{can}^{-1}(1_A \otimes_R c).$$

Then T is a bimodule herd with the shepherd

$$\gamma : T \rightarrow T \otimes_R T^* \otimes_S T, \quad x \mapsto x_{(0)} \otimes_R \tau(x_{(1)}).$$

Proof. Since γ is a composition of left S -module maps, it is a left S -module map. For all $a \in A$ and $c \in \mathcal{C}$, write

$$\psi(c \otimes_R a) = \sum_{\psi} a_{\psi} \otimes_R c^{\psi}.$$

The right A -linearity of can^{-1} implies that, for all $a \in A$ and $c \in \mathcal{C}$,

$$\tau(c)a = \sum_{\psi} a_{\psi} \tau(c^{\psi}). \quad (2.6)$$

In particular, for all $r \in R$, $\tau(c)\alpha(r) = \alpha(r)\tau(c)$, i.e. the image of τ is in the centraliser of R in $T^* \otimes_S T$. Since the coaction of $A \otimes_R \mathcal{C}$ on T is right A -linear, and the right A -multiplication in $A \otimes_R \mathcal{C}$ is given through ψ , the equality mentioned above yields, for all $x \in T$ and $r \in R$,

$$\gamma(xr) = (xr)_{(0)} \otimes_R \tau((xr)_{(0)}) = \sum_{\psi} x_{(0)} r_{\psi} \otimes_R \tau(x_{(1)}^{\psi}) = x_{(0)} r \otimes_R \tau(x_{(1)}) = x_{(0)} \otimes_R \tau(x_{(1)}) r = \gamma(x) r.$$

This proves that γ is a right R -module map.

Let $\{e_i \in T, e_i^* \in T^*\}$ be a dual basis. Identifying B with $T \otimes_A T^*$ we can identify 1_B with $\sum_i e_i \otimes_A e_i^*$. Take any $x \in T$ and apply the identity map $(T \otimes_A \text{can}^{-1}) \circ (T \otimes_A \text{can})$ to $\sum_i e_i \otimes_A e_i^* \otimes_S x$ to conclude that

$$x_{(0)} \otimes_A \tau(x_{(1)}) = \sum_i e_i \otimes_A e_i^* \otimes_S x.$$

This means that the map γ makes the diagram (2.3) commute. Next, take any $c \in C$, and evaluate the identity map $\text{can} \circ \text{can}^{-1}$ on $1_A \otimes_R c$ to obtain $\text{ev} \circ \tau = \alpha \circ \varepsilon_C$. This equality then yields, for all $x \in T$,

$$x_{(0)} \otimes_R \text{ev}(\tau(x_{(1)})) = x_{(0)} \otimes_R \alpha(\varepsilon_C(x_{(1)})) = x \otimes_R 1_A,$$

i.e. the diagram (2.4) is commutative. The commutativity of diagram (2.5) follows by the C -colinearity of τ . \square

2.12. Notation. Given a B - A herd (T, γ) with a pen \widehat{T} , define an R -bimodule C as the equaliser

$$C \longrightarrow \widehat{T} \otimes_S T \xrightleftharpoons[\alpha \otimes_R \widehat{T} \otimes_S T]{(\text{ev} \otimes_R \widehat{T} \otimes_S T) \circ (\widehat{T} \otimes_S \gamma)} A \otimes_R \widehat{T} \otimes_S T. \quad (2.7)$$

Symmetrically, define an S -bimodule D as the equaliser

$$D \longrightarrow T \otimes_R \widehat{T} \xrightleftharpoons[T \otimes_R \widehat{T} \otimes_S \beta]{(T \otimes_R \widehat{T} \otimes_S \text{ev}) \circ (\gamma \otimes_R \widehat{T})} T \otimes_R \widehat{T} \otimes_S B. \quad (2.8)$$

2.13. Proposition. Let (T, γ) be a B - A herd. Define C by the equaliser (2.7) and D by the equaliser (2.8).

(1) If the equaliser (2.7) is a T_R -pure equaliser (i.e. if it is preserved by the functor $T \otimes_R -$), then

$$C = \widehat{T} \otimes_S T \cong A \otimes_R C,$$

as A - R bimodules.

(2) If the equaliser (2.8) is a $_S T$ -pure equaliser, then

$$D = T \otimes_R \widehat{T} \cong D \otimes_S B,$$

as S - B bimodules.

Proof. (1) Set

$$\bar{\alpha} = T \otimes_R \alpha \otimes_R \widehat{T} \otimes_S T, \quad \kappa = T \otimes_R ((\text{ev} \otimes_R \widehat{T} \otimes_S T) \circ (\widehat{T} \otimes_S \gamma)),$$

and note that, for all $x \in T$,

$$\begin{aligned}
\kappa \circ \gamma(x) &= x^{(1)} \otimes_R \text{ev}(x^{(2)} \otimes_S x^{(3)(1)}) \otimes_R x^{(3)(2)} \otimes_S x^{(3)(3)} \\
&= x^{(1)(1)} \otimes_R \text{ev}(x^{(1)(2)} \otimes_S x^{(1)(3)}) \otimes_R x^{(2)} \otimes_S x^{(3)} \\
&= x^{(1)} \otimes_R 1_A \otimes_R x^{(2)} \otimes_S x^{(3)} \\
&= \bar{\alpha} \circ \gamma(x).
\end{aligned}$$

The second equality follows by (2.5), and the third one is a consequence of (2.4). Since the equaliser (2.7) is T_R -pure, and κ and $\bar{\alpha}$ are the equalised maps tensored with T_R , we conclude that, for all $x \in T$,

$$\gamma(x) \in T \otimes_R C.$$

Hence we can define

$$\theta : \widehat{T} \otimes_S T \rightarrow A \otimes_R C, \quad \theta = (\text{ev} \otimes_R \widehat{T} \otimes_S T) \circ (\widehat{T} \otimes_S \gamma).$$

The map θ is left A -linear, since ev is left A -linear, and it is right R -linear since γ is right R -linear. Furthermore, the map θ is bijective with the inverse

$$\theta^{-1} : A \otimes_R C \rightarrow \widehat{T} \otimes_S T, \quad a \otimes_R \sum_i \widehat{x}_i \otimes_S x_i \mapsto \sum_i a \widehat{x}_i \otimes_S x_i.$$

Indeed, for all $\widehat{x} \in \widehat{T}$ and $x \in T$,

$$\theta^{-1} \circ \theta(\widehat{x} \otimes_S x) = \text{ev}(\widehat{x} \otimes_S x^{(1)}) x^{(2)} \otimes_S x^{(3)} = \widehat{x} \text{ev}(x^{(1)} \otimes_R x^{(2)}) \otimes_S x^{(3)} = \widehat{x} \otimes_S x,$$

where the second equality follows by (2.2), while the last equality is a consequence of (2.3). Second, for all $a \in A$ and $\sum_i \widehat{x}_i \otimes_S x_i \in C$,

$$\begin{aligned}
\theta \circ \theta^{-1} \left(a \otimes_R \sum_i \widehat{x}_i \otimes_S x_i \right) &= \sum_i \text{ev}(a \widehat{x}_i \otimes_S x_i^{(1)}) \otimes_R x_i^{(2)} \otimes_S x_i^{(3)} \\
&= a \text{ev} \left(\sum_i \widehat{x}_i \otimes_S x_i^{(1)} \right) \otimes_R x_i^{(2)} \otimes_S x_i^{(3)} \\
&= a \otimes_R \sum_i \widehat{x}_i \otimes_S x_i,
\end{aligned}$$

where the A -linearity of ev is used in the first equality, and the last equality follows by the definition of C .

Statement (2) is proven by symmetric arguments. \square

2.14. Lemma. *Let (T, γ) be a B - A herd.*

- (1) *The equaliser (2.7) tensored with T_R is a split equaliser. Consequently, if T is a faithfully flat right R -module, then (2.7) is a pure equaliser in \mathbf{M}_R (i.e. tensoring it with any left R -module V gives an equaliser).*
- (2) *The equaliser (2.8) tensored with ${}_S T$ is a split equaliser. Consequently, if T is a faithfully flat left S -module, then (2.8) is a pure equaliser in ${}_S \mathbf{M}$.*

Proof. (1) Denote the equalised maps in (2.7) by ζ_C and ξ_C and set as before

$$\bar{\alpha} = T \otimes_R \zeta_C = T \otimes_R \alpha \otimes_R \widehat{T} \otimes_S T, \quad \kappa = T \otimes_R \xi_C = T \otimes_R ((\text{ev} \otimes_R \widehat{T} \otimes_S T) \circ (\widehat{T} \otimes_S \gamma)).$$

Define

$$\pi_C : T \otimes_R A \otimes_R \widehat{T} \otimes_S T \rightarrow T \otimes_R \widehat{T} \otimes_S T, \quad x \otimes_R a \otimes_R \widehat{x} \otimes_S y \mapsto xa \otimes_R \widehat{x} \otimes_S y.$$

Obviously, $\pi_C \circ \bar{\alpha} = T \otimes_R \widehat{T} \otimes_S T$. Furthermore, for all $x, y \in T$ and $\widehat{x} \in \widehat{T}$,

$$\begin{aligned} \kappa \circ \pi_C \circ \kappa(y \otimes_R \widehat{x} \otimes_S x) &= y \text{ ev}(\widehat{x} \otimes_S x^{(1)}) \otimes_R \text{ev}(x^{(2)} \otimes_S x^{(3)(1)}) \otimes_R x^{(3)(2)} \otimes_S x^{(3)(3)} \\ &= y \text{ ev}(\widehat{x} \otimes_S x^{(1)(1)}) \otimes_R \text{ev}(x^{(1)(2)} \otimes_S x^{(1)(3)}) \otimes_R x^{(2)} \otimes_S x^{(3)} \\ &= y \text{ ev}(\widehat{x} \otimes_S x^{(1)}) \otimes_R 1_A \otimes_R x^{(2)} \otimes_S x^{(3)} \\ &= \bar{\alpha} \circ \pi_C \circ \kappa(y \otimes_R \widehat{x} \otimes_S x), \end{aligned}$$

where the diagram (2.5) is used to derive the second equality. The third equality follows by (2.4). This proves that $T \otimes_R \zeta_C$ and $T \otimes_R \xi_C$ is a contractible pair, hence (2.7) tensored with T_R is a split equaliser.

Assume now that T is a faithfully flat right R -module. Since T_R is flat, $T \otimes_R C$ is the equaliser of $T \otimes_R \zeta_C$ and $T \otimes_R \xi_C$. The latter is a split, hence absolute, equaliser of right R -module maps, thus, for all left R -modules V , $T \otimes_R C \otimes_R V$ is the equaliser of $T \otimes_R \zeta_C \otimes_R V$ and $T \otimes_R \xi_C \otimes_R V$. Since faithfully flat modules reflect equalisers, we conclude that $C \otimes_R V$ is the equaliser of $\zeta_C \otimes_R V$ and $\xi_C \otimes_R V$. This means that (2.7) is a pure equaliser of right R -module maps.

(2) This is proven by symmetric arguments. In particular, the splitting morphism is

$$\pi_D : T \otimes_R \widehat{T} \otimes_S B \otimes_S T \rightarrow T \otimes_R \widehat{T} \otimes_S T, \quad x \otimes_R \widehat{x} \otimes_S b \otimes_S y \mapsto x \otimes_R \widehat{x} \otimes_S by. \quad \square$$

Recall that, for any (unital associative) rings K and L , a ring map $K \rightarrow L$ is called a *split extension* if it is a K -bimodule section.

2.15. Lemma. *Let (T, γ) be a B - A herd.*

- (1) *If α is a split extension, then the equaliser (2.7) is a split (hence pure) equaliser of R -bimodules.*
- (2) *If β is a split extension, then the equaliser (2.8) is a split (hence pure) equaliser of S -bimodules.*

Proof. This lemma is proven by calculations similar to that in the proof of Lemma 2.14. If $\pi_\alpha : A \rightarrow R$ is an R -bimodule map such that $\pi_\alpha \circ \alpha = R$, then the splitting morphism for the equaliser (2.7) is $\pi_C : A \otimes_R \widehat{T} \otimes_S T \rightarrow \widehat{T} \otimes_S T$, $a \otimes_R \widehat{x} \otimes_S x \mapsto \pi_\alpha(a) \widehat{x} \otimes_S x$. Symmetrically, if $\pi_\beta : B \rightarrow S$ is an S -bimodule map such that $\pi_\beta \circ \beta = S$, then the splitting morphism for the equaliser (2.7) is $\pi_D : T \otimes_R \widehat{T} \otimes_S B \rightarrow T \otimes_R \widehat{T}$, $x \otimes_R \widehat{x} \otimes_S b \mapsto x \otimes_R \widehat{x} \pi_\beta(b)$. \square

2.16. Theorem. *Let (T, γ) be a B - A herd, and let C be defined by the equaliser (2.7) and D by the equaliser (2.8).*

- (1) *Assume that*
 - (i) *T is a faithfully flat right R -module and A is a faithfully flat right (or left) R -module, or*
 - (ii) *α is a split extension.*

Then:

(a) C is an R -coring with coproduct

$$\Delta_C : C \rightarrow C \otimes_R C, \quad \sum_i \widehat{x}_i \otimes_S x_i \mapsto \widehat{x}_i \otimes_S \gamma(x_i),$$

and counit $\varepsilon_C = \text{ev}|_C$.

(b) C is entwined with A (over R) by the map $\psi : C \otimes_R A \rightarrow A \otimes_R C$,

$$\sum_i \widehat{x}_i \otimes_S x_i \otimes_R a \mapsto \sum_i \text{ev}(\widehat{x}_i \otimes_S (x_i a)^{(1)}) \otimes_R (x_i a)^{(2)} \otimes_S (x_i a)^{(3)}.$$

(c) T is a right $(A, C, \psi)_R$ -entwined module with the coaction γ .

(2) Assume that

- (i) T is a faithfully flat left S -module and B is a faithfully flat left (or right) S -module, or
- (ii) β is a split extension.

Then:

(a) D is an S -coring with coproduct

$$\Delta_D : D \rightarrow D \otimes_R D, \quad \sum_i x_i \otimes_R \widehat{x}_i \mapsto \gamma(x_i) \otimes_R \widehat{x}_i,$$

and counit $\varepsilon_D = \widehat{\text{ev}}|_D$.

(b) B is entwined with D (over S) by the map $\varphi : B \otimes_S D \rightarrow D \otimes_S B$,

$$\sum_i b \otimes_S x_i \otimes_R \widehat{x}_i \mapsto \sum_i (bx_i)^{(1)} \otimes_R (bx_i)^{(2)} \otimes_S \widehat{\text{ev}}((bx_i)^{(3)} \otimes_R \widehat{x}_i).$$

(c) T is a left $(B, D, \varphi)_S$ -entwined module with the coaction γ .

Proof. (1)(a) Under either of the hypotheses, the equaliser (2.7) is T_R -pure, thus, as explained in the proof of Proposition 2.13, $\gamma(T) \subseteq T \otimes_R C$. Consequently, $\Delta_C(C) \subseteq \widehat{T} \otimes_S T \otimes_R C$. Furthermore, writing as before ζ_C and ξ_C for the maps equalised in (2.7),

$$\begin{aligned} (\xi_C \otimes_R C) \circ \Delta_C &= (\text{ev} \otimes_R \widehat{T} \otimes_S T \otimes_R C) \circ (\widehat{T} \otimes_S \gamma \otimes_R C) \circ (\widehat{T} \otimes_S \gamma)|_C \\ &= (\text{ev} \otimes_R \widehat{T} \otimes_S T \otimes_R C) \circ (\widehat{T} \otimes_S T \otimes_R \widehat{T} \otimes_S \gamma) \circ (\widehat{T} \otimes_S \gamma)|_C \\ &= (A \otimes_R \widehat{T} \otimes_S \gamma) \circ (\text{ev} \otimes_R C) \circ (\widehat{T} \otimes_S \gamma)|_C \\ &= (A \otimes_R \widehat{T} \otimes_S \gamma) \circ (\alpha \otimes_R C) \\ &= (\zeta_C \otimes_R C) \circ \Delta_C, \end{aligned}$$

where the second equality follows by diagram (2.5), and the fourth equality is a consequence of the definition of C . In view of Lemma 2.14 (in the case of hypothesis (i)) or Lemma 2.15 (in the case of hypothesis (ii)), the equaliser of right R -module maps ζ_C and ξ_C , i.e. the equaliser defining C , is a pure equaliser, hence $\Delta_C(C) \subseteq C \otimes_R C$. Therefore, Δ_C is a well defined R -bimodule map $C \rightarrow C \otimes_R C$. It is coassociative by diagram (2.5).

For any $c \in C$,

$$(\text{ev} \otimes_R \widehat{T} \otimes_S T) \circ (\widehat{T} \otimes_S \gamma)(c) = 1_A \otimes_R c.$$

Applying $A \otimes_R \text{ev}$ to this equality and using (2.4) we immediately obtain

$$1_A \otimes_R \text{ev}(c) = \text{ev}(c) \otimes_R 1_A.$$

If A is a faithfully flat right or left R -module (hypothesis (i)), the above equality implies that, for all $c \in C$, $\text{ev}(c) \in R$. On the other hand, if there is an R -bimodule map $\pi_\alpha : A \rightarrow R$ such that $\pi_\alpha \circ \alpha = R$, then applying it to both sides of the above equality one concludes that $\text{ev}(c) = (\alpha \circ \pi_\alpha \circ \text{ev})(c)$, i.e. $\text{ev}(c) \in R$ as needed. That $\varepsilon_C = \text{ev}|_C$ is a counit for Δ_C follows by the definition of C and diagram (2.4).

(1)(b) and (1)(c). By either of the hypotheses, C is defined by a T_R -pure equaliser. Thus, by Proposition 2.13(1), $A \otimes_R C \cong \widehat{T} \otimes_S T$ as A - R -bimodules. Using the explicit form of this isomorphism in the proof of Proposition 2.13(1), one easily finds that the induced right A -module structure on $A \otimes_R C$ is, for all $a \in A$, $c \in C$,

$$(1_A \otimes_R c)a := \theta(\theta^{-1}(1_A \otimes_R c)a) = \psi(c \otimes_R a).$$

Furthermore, the induced (i.e. compatible with the isomorphism θ) A -coring structure on $A \otimes_R C$ comes out as $A \otimes_R \Delta_C$ and $A \otimes_R \varepsilon_C$. This implies that C is entwined with A by ψ (cf. [7, Proposition 2.1]). Since T is a right $\widehat{T} \otimes_S T \cong A \otimes_R C$ -comodule with the coaction γ_A , it is a right entwined module. The induced C -coaction $(T \otimes_A \theta) \circ \gamma_A$ comes out as γ .

The assertions (2) are proven by symmetric arguments. \square

2.17. Remark. The observations of Theorem 2.16 under the hypotheses (ii) are a bimodule version of the construction of a Hopf algebra from a (copointed) quantum heap in [27]. More specifically, let H be a quantum heap (over a commutative ring k) with the structure map $\gamma : H \rightarrow H \otimes_k H \otimes_k H$, and let $\pi_\alpha : H \rightarrow k$ be an algebra character. Then H is a k - k bimodule herd, and let C be the associated k -coring (coalgebra). Then the map $\pi_\alpha \otimes_k H|_C : C \rightarrow H$ is an isomorphism of coalgebras with the inverse $(\pi_\alpha \otimes_k H \otimes_k H) \circ \gamma$.

Herds and Galois comodules. The following theorem, which is the main result of this section, establishes tame B - A -herds as a way of describing finite Galois comodules.

2.18. Theorem. Let T be a B - A bimodule that is a progenerator as a right A -module, $B = \text{End}_A(T)$, and assume that

- (i) A is a faithfully flat right R -module and B is a faithfully flat left S -module, or
- (ii) α and β are split extensions.

Then the following statements are equivalent:

- (a) T is a (tame) bimodule herd.
- (b) There exists a right entwining $\psi : C \otimes_R A \rightarrow A \otimes_R C$ over R such that T is a right Galois comodule over $C = A \otimes_R C$ with $S = \text{End}^C(T)$.
- (c) There exists a left entwining $\varphi : B \otimes_S D \rightarrow D \otimes_S B$ over S such that T is a left Galois comodule over $D = D \otimes_S B$ with $R = \text{End}^D(T)$.

Proof. (a) \Rightarrow (b) Theorem 2.16 implies that there is an entwining as stated and that T is a right entwined module (i.e. a right comodule of the coring $C = A \otimes_R C$) with coaction γ (note that by Corollary 2.3, T_R is faithfully flat under condition (i)). By construction, $T^* \otimes_S T \cong A \otimes_R C$, with the isomorphism described in the proof of Proposition 2.13 which, with the choice of the coaction on T ,

coincides with the map can . Thus it only remains to identify S with the endomorphism ring $\text{End}^C(T)$. Since $B = \text{End}_A(T)$, $\text{End}^C(T)$ is a subalgebra of B consisting of all $s \in B$ such that, for all $x \in T$,

$$s\gamma(x) = \gamma(sx).$$

Obviously, $S \subseteq \text{End}^C(T)$. Apply the map $\hat{e}\nu \otimes_S T$ to this equality and use diagram (2.3) to find that

$$s \otimes_S x = 1_B \otimes_S sx. \quad (2.9)$$

If hypothesis (i) holds, then B is faithfully flat as a left S -module and—by the fact that B is an endomorphism ring of a progenerator— T is a progenerator of left B -modules, T is also faithfully flat as a left S -module. Thus the equality $s \otimes_S x = 1_B \otimes_S sx$, for all $x \in T$, implies that $s \in S$, hence $S = \text{End}^C(T)$.

On the other hand, suppose that there is an S -bimodule map $\pi_\beta : B \rightarrow S$, such that $\pi_\beta \circ \beta = S$. Combining π_β with the inclusion $\text{End}^C(T) \subseteq \text{End}_A(T) = B$, we obtain a map $\pi : \text{End}^C(T) \rightarrow S$. Clearly, π is a retraction for the inclusion $S \subseteq \text{End}^C(T)$. If we apply $\pi_\beta \otimes_S T$ to (2.9), then we find $\pi(s)x = sx$, which means exactly that $S = \text{End}^C(T)$.

(b) \Rightarrow (a) Follows by Example 2.11.

The equivalence of (a) and (c) is proven by symmetric arguments. \square

2.19. Remark. With the assumptions of Theorem 2.18, there is a bijective correspondence between the following sets:

- (a) the set of shepherds $\gamma : T \rightarrow T \otimes_R T^* \otimes_S T$;
- (b) the set of right entwining structures $\psi : C \otimes_R A \rightarrow A \otimes_R C$ over R such that T is a right Galois comodule over $C = A \otimes_R C$ with $S = \text{End}^C(T)$;
- (c) the set of left entwining structures $\varphi : B \otimes_S D \rightarrow D \otimes_S B$ over S such that T is a left Galois comodule over $D = D \otimes_S B$ with $R = \text{End}^D(T)$.

Starting with γ one constructs the R -coring $C \subseteq T^* \otimes_S T$ and entwining ψ as in Theorem 2.16. The translation map $\tau : C \rightarrow T^* \otimes_S T$ (cf. Example 2.11) is simply the obvious inclusion, and since the C -coaction on T is given by γ , the procedure of obtaining a shepherd from τ described in Example 2.11 reproduces γ .

Starting with an entwining map ψ and the translation map $\tau : C \rightarrow T^* \otimes_S T$, one defines γ as in Example 2.11. Using the fact that $\tau(c) = \text{can}^{-1}(1_A \otimes_R c)$ one easily finds that the image of τ is in the R -coring \bar{C} defined by equaliser (2.7). The corestriction of τ establishes then an isomorphism of C with \bar{C} . Explicitly, the inverse of τ is $\bar{C} \ni \sum_i f^i \otimes_S x^i \mapsto \sum_i f^i(x_{(0)}^i)x_{(1)}^i$. By Theorem 2.16 there is an entwining map $\bar{\psi} : \bar{C} \otimes_R A \rightarrow \bar{C} \otimes_R A$. Using the A -linearity of τ , (2.6), one finds that the composition

$$C \otimes_R A \xrightarrow{\tau \otimes_R A} \bar{C} \otimes_R A \xrightarrow{\bar{\psi}} A \otimes_R \bar{C} \xrightarrow{A \otimes_R \tau^{-1}} A \otimes_R C,$$

equals ψ .

2.20. Remark. In some interesting situations, condition (ii) of Theorem 2.18 implies already condition (i). This can be seen as follows. Let T be a B - A bimodule that is a progenerator as right A -module. Then, by applying the Hom-tensor relations, we obtain the following natural isomorphisms

$${}_S \text{Hom}(T, -) \cong {}_S \text{Hom}(B \otimes_B T, -) \cong {}_S \text{Hom}(B, \text{Hom}_A(T, -))$$

and

$${}_S \text{Hom}(B, -) \cong {}_S \text{Hom}(T \otimes_A \widehat{T}, -) \cong {}_S \text{Hom}(T, \text{Hom}_B(\widehat{T}, -)).$$

Therefore, ${}_S B$ is projective if and only if ${}_S T$ is projective. Under this projectivity condition, ${}_S B$ is faithfully flat if and only if β is a split monomorphism of left S -modules; see [22, 2.11.29].

Similarly one proves that A_R is projective if and only if T_R is projective. Under this condition, A_R is faithfully flat if and only if α is a split monomorphism of right R -modules.

In particular, if ${}_S B$ and A_R are projective and, α and β are split extensions (condition (ii) of Theorem 2.18), then ${}_S B$ and A_R are faithfully flat (condition (i)).

3. Herds versus coherds

By formally dualising the definition of bimodule herds, the notion of a *bicomodule coherd* is introduced. It is shown that a tame bimodule herd is also a bicomodule coherd of corresponding corings.

Bicomodule coherds.

3.1. Definition. Let C be an R -coring and D an S -coring. Consider a bicomodule $(X, \rho^{D,X}, \rho^{X,C}) \in {}^D \mathbf{M}^C$. A bicomodule $(\bar{X}, \rho^{C,\bar{X}}, \rho^{\bar{X},D}) \in {}^C \mathbf{M}^D$ is called a *companion* of X if there exist a C -bicomodule map

$$\text{cov} : C \rightarrow \bar{X} \otimes_S X,$$

and a D -bicomodule map

$$\overline{\text{cov}} : D \rightarrow X \otimes_R \bar{X},$$

such that the following diagrams commute,

$$\begin{array}{ccc} X & \xrightarrow{\rho^{X,C}} & X \otimes_R C \\ \rho^{D,X} \downarrow & & \downarrow X \otimes_R \text{cov} \\ D \otimes_S X & \xrightarrow{\overline{\text{cov}} \otimes_S X} & X \otimes_R \bar{X} \otimes_S X, \end{array} \quad (3.1)$$

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\rho^{C,\bar{X}}} & C \otimes_R \bar{X} \\ \rho^{\bar{X},D} \downarrow & & \downarrow \text{cov} \otimes_R \bar{X} \\ \bar{X} \otimes_S D & \xrightarrow{\bar{X} \otimes_S \overline{\text{cov}}} & \bar{X} \otimes_S X \otimes_R \bar{X}. \end{array} \quad (3.2)$$

Furthermore, a D - C bicomodule X with a companion \bar{X} is called a *bicomodule coherd* if there exists an S - R bimodule map

$$\chi : X \otimes_R \bar{X} \otimes_S X \rightarrow X,$$

rendering commutative the following diagrams,

$$\begin{array}{ccc}
 X \otimes_R C & \xrightarrow{X \otimes_R \text{cov}} & X \otimes_R \bar{X} \otimes_S X \\
 \downarrow X \otimes_R \varepsilon_C & & \downarrow \chi \\
 X \otimes_R R & \xrightarrow{\cong} & X,
 \end{array} \quad (3.3)$$

$$\begin{array}{ccc}
 D \otimes_S X & \xrightarrow{\overline{\text{cov}} \otimes_S X} & X \otimes_R \bar{X} \otimes_S X \\
 \downarrow \varepsilon_D \otimes_S X & & \downarrow \chi \\
 S \otimes_S X & \xrightarrow{\cong} & X,
 \end{array} \quad (3.4)$$

$$\begin{array}{ccc}
 X \otimes_R \bar{X} \otimes_S X \otimes_R \bar{X} \otimes_S X & \xrightarrow{X \otimes_R \bar{X} \otimes_S \chi} & X \otimes_R \bar{X} \otimes_S X \\
 \downarrow \chi \otimes_R \bar{X} \otimes_S X & & \downarrow \chi \\
 X \otimes_R \bar{X} \otimes_S X & \xrightarrow{\chi} & X.
 \end{array} \quad (3.5)$$

The theory of herds as developed in Section 2 can now be formally dualised. In particular, given a bicomodule coherd X , the C -bicomodule $\bar{X} \otimes_S X$ is a non-unital ring (over R) with multiplication

$$\mu_X := \bar{X} \otimes_S \chi : \bar{X} \otimes_S X \otimes_R \bar{X} \otimes_S X \rightarrow \bar{X} \otimes_S X.$$

The map $\text{cov} : C \rightarrow \bar{X} \otimes_S X$ is a C -unit for $\bar{X} \otimes_S X$, i.e. the following diagram is commutative

$$\begin{array}{ccccc}
 C \otimes_R \bar{X} \otimes_S X & \xleftarrow{\rho^C \cdot \bar{X} \otimes_S X} & \bar{X} \otimes_S X & \xrightarrow{\bar{X} \otimes_S \rho^{X,C}} & \bar{X} \otimes_S X \otimes_R C \\
 \downarrow \text{cov} \otimes_R \bar{X} \otimes_S X & & \downarrow \bar{X} \otimes_S X & & \downarrow \bar{X} \otimes_S X \otimes_R \text{cov} \\
 \bar{X} \otimes_S X \otimes_R \bar{X} \otimes_S X & \xrightarrow{\mu_X} & \bar{X} \otimes_S X & \xleftarrow{\mu_X} & \bar{X} \otimes_S X \otimes_R \bar{X} \otimes_S X.
 \end{array}$$

Symmetrically, $X \otimes_R \bar{X}$ is a ring with product $\chi \otimes_R \bar{X}$ and with a D -unit $\overline{\text{cov}}$. Furthermore, one can define an R -bimodule A' as the following split coequaliser

$$C \otimes_R \bar{X} \otimes_S X \xrightarrow[\varepsilon_C \otimes_R \bar{X} \otimes_S X]{(\bar{X} \otimes_S \chi) \circ (\text{cov} \otimes_R \bar{X} \otimes_S X)} \bar{X} \otimes_S X \xrightarrow{\pi_A} A'.$$

Since the tensor functor preserves coequalisers, the map μ_X descends to the associative product $\mu_{A'} : A' \otimes_R A' \rightarrow A'$, by the formula

$$\mu_{A'} \circ (\pi_A \otimes_R \pi_A) = \pi_A \circ \mu_X.$$

Suppose C is faithfully flat as a left R -module, then by a (dual) descent argument, we can construct a unit for the R -ring A' as follows. Consider the following split coequaliser of R -bimodules

$$\begin{array}{ccccc}
 & & C \otimes_R \varepsilon_C \otimes_R C & & \\
 & & \xrightarrow{\quad} & & \\
 C \otimes_R C \otimes_R C & \xrightarrow{\quad \varepsilon_C \otimes_R C \otimes_R C \quad} & C \otimes_R C & \xleftarrow{\quad \varepsilon_C \otimes_R C \quad} & C \\
 & \xleftarrow{\quad C \otimes_R \Delta_C \quad} & & \xleftarrow{\quad \Delta_C \quad} & \\
 & & & &
 \end{array}$$

Since tensoring with a faithfully flat module reflects coequalisers, one obtains the following coequaliser of R -bimodules

$$C \otimes_R C \xrightarrow[\varepsilon_C \otimes_R C]{C \otimes_R \varepsilon_C} C \xrightarrow{\varepsilon_C} R.$$

By the universal property of coequalisers there exists a unique R -bimodule map $\alpha' : R \rightarrow A'$ such that

$$\pi_A \circ \text{cov} = \alpha' \circ \varepsilon_C.$$

One easily checks that, for all $a \in A'$, $\mu_{A'}(a \otimes_R \alpha'(1_R)) = \mu_{A'}(\alpha'(1_R) \otimes_R a) = a$, i.e. that A' is a (unital) R -ring with the unit map α' .

In a symmetric way, if D is a faithfully flat left or right S -module one obtains the (unital) S -ring B' as the coequaliser

$$X \otimes_R \bar{X} \otimes_S D \xrightarrow[X \otimes_R \bar{X} \otimes_S \varepsilon_D]{(\chi \otimes_R \bar{\chi}) \circ (X \otimes_R \bar{X} \otimes_S \overline{\text{cov}})} X \otimes_R \bar{X} \xrightarrow{\pi_B} B'.$$

The unit map in B' is a unique morphism $\beta' : S \rightarrow B'$ such that $\pi_B \circ \overline{\text{cov}} = \beta' \circ \varepsilon_D$.

Construction of coherds. Given an A -coring C and a right C -comodule T , set $S = \text{End}^C(T)$. By the strong structure theorem for T is meant that the functor $- \otimes_S T$ is an equivalence of the categories \mathbf{M}_S and \mathbf{M}^C .

3.2. Lemma. Let C be an A -coring, T a right C -comodule for which the strong structure theorem holds. Then for all $N \in \mathbf{M}_A$ and $M \in {}_A \mathbf{M}^C$, the canonical morphism

$$N \otimes_A \text{Hom}^C(T, M) \rightarrow \text{Hom}^C(T, N \otimes_A M)$$

is an isomorphism.

Proof. This follows by a double application of the equivalence of categories between \mathbf{M}_S and \mathbf{M}^C through the functors $- \otimes_S T$ and $\text{Hom}^C(T, -)$,

$$N \otimes_A \text{Hom}^C(T, M) \otimes_S T \cong N \otimes_A M \cong \text{Hom}^C(T, N \otimes_A M) \otimes_S T.$$

Since T is faithfully flat as a left S -module, the claim follows immediately. \square

Let (T, γ) be a tame B - A herd. Then we can consider the R -coring C , which is entwined with the R -ring A by ψ and the S -coring D which is entwined with the S -ring B by ϕ as in Theorem 2.16. Denote as before $\mathcal{C} = A \otimes_R C$ and $\mathcal{D} = D \otimes_S B$ for the associated A -coring and B -coring. Recall from [9, 32.8(2)] that $C \otimes_R A$ is a right C -module (i.e. a right entwined module); the right A -module structure is given by $C \otimes_R \mu_A$, where μ_A is the multiplication on A , and the right C -coaction is given by $(C \otimes_R \psi) \circ (\Delta_C \otimes_R A)$. For an element $\widehat{x} \otimes_S x \otimes_R a \in C \otimes_R A$ (representing a finite sum of simple tensors), the right C -coaction reads explicitly as

$$\rho^{C \otimes_R A}(\widehat{x} \otimes_S x \otimes_R a) = \widehat{x} \otimes_S x^{(1)} \otimes_R \text{ev}(x^{(2)} \otimes_S (x^{(3)} a)^{(1)}) \otimes_R (x^{(3)} a)^{(2)} \otimes_S (x^{(3)} a)^{(3)}. \quad (3.6)$$

Symmetrically, $B \otimes_S D$ is a left \mathcal{D} -comodule.

3.3. Theorem. Let (T, γ) be a tame B - A herd. Consider the R - S bimodule $\bar{T} = (C \otimes_R \widehat{T}) \cap (\widehat{T} \otimes_S D)$. Then

- (1) $h_1 : \bar{T} \rightarrow \text{Hom}^C(T, C \otimes_R A)$, $h_1(\bar{x})(y) = (\widehat{T} \otimes_S T \otimes_R \text{ev})(\bar{x} \otimes_S y)$, is an isomorphism of R - S bimodules;
- (2) $h_2 : \bar{T} \rightarrow {}^{\mathcal{D}}\text{Hom}(T, B \otimes_S D)$, $h_2(\bar{x})(y) = (\widehat{\text{ev}} \otimes_S T \otimes_R \widehat{T})(y \otimes_R \bar{x})$ is an isomorphism of R - S bimodules;
- (3) $h : \bar{T} \rightarrow \widehat{T}$, $h = \text{ev} \otimes_A \widehat{T}|_{\bar{T}} = \widehat{T} \otimes_B \widehat{\text{ev}}|_{\bar{T}}$, is an R - S bimodule map;
- (4) \bar{T} is a C - D bicomodule.

Proof. (1) Elements of $\text{Hom}^C(T, C \otimes_R A)$ are exactly right A -linear and right C -colinear morphisms $T \rightarrow C \otimes_R A$. Since ev is right A -linear, for any $\bar{x} \in \bar{T}$, $h_1(\bar{x})$ is right A -linear as well. To check that $h_1(\bar{x})$ is right C -colinear, write $\bar{x} = \widehat{x} \otimes_S x \otimes_R \widehat{y} \in \bar{T}$ (summation implicit), and calculate,

$$\begin{aligned} \rho^{C \otimes_R A}(h_1(\widehat{x} \otimes_S x \otimes_R \widehat{y})(y)) \\ &= \widehat{x} \otimes_S x^{(1)} \otimes_R \text{ev}(x^{(2)} \otimes_S (x^{(3)} \text{ev}(\widehat{y} \otimes_S y))^{(1)}) \otimes_R (x^{(3)} \text{ev}(\widehat{y} \otimes_S y))^{(2)} \otimes_S (x^{(3)} \text{ev}(\widehat{y} \otimes_S y))^{(3)} \\ &= \widehat{x} \otimes_S x^{(1)} \otimes_R \text{ev}(x^{(2)} \otimes_S (\widehat{\text{ev}}(x^{(3)} \otimes_R \widehat{y})y)^{(1)}) \otimes_R (\widehat{\text{ev}}(x^{(3)} \otimes_R \widehat{y})y)^{(2)} \otimes_S (\widehat{\text{ev}}(x^{(3)} \otimes_R \widehat{y})y)^{(3)} \\ &= \widehat{x} \otimes_S x \otimes_R \text{ev}(\widehat{y} \otimes_S y^{(1)}) \otimes_R y^{(2)} \otimes_S y^{(3)} \\ &= h_1(\widehat{x} \otimes_S x \otimes_R \widehat{y})(y^{(1)}) \otimes_R y^{(2)} \otimes_S y^{(3)} \\ &= (h_1(\widehat{x} \otimes_S x \otimes_R \widehat{y}) \otimes_R \widehat{T} \otimes_S T)(\gamma(y)), \end{aligned}$$

where we used (3.6) in the first equation, diagram (2.1) in the second equality and the defining property of D applied on the element $\widehat{x} \otimes_S x \otimes_R \widehat{y} \in \bar{T} \subset \widehat{T} \otimes_S D$ in the third equality. Therefore, $h_1(\bar{x})$ is right C -colinear. Since $\widehat{\text{ev}}$ is surjective, there are $e_i \in T$ and $\widehat{e}_i \in \widehat{T} \cong T^*$, such that $\widehat{\text{ev}}(\sum_i e_i \otimes_R \widehat{e}_i) = 1_B$. Hence it is possible to define a map

$$k_1 : \text{Hom}^C(T, C \otimes_R A) \rightarrow \bar{T}, \quad \varphi \mapsto (\varphi \otimes_A \widehat{T})\left(\sum_i e_i \otimes_A \widehat{e}_i\right).$$

Diagram (2.2) and the property $\widehat{\text{ev}}(\sum_i e_i \otimes_R \widehat{e}_i) = 1_B$ immediately imply that $k_1 \circ h_1 = \bar{T}$. In the other direction,

$$h_1 \circ k_1(\varphi)(x) = \sum_i \varphi(e_i) \text{ev}(\widehat{e}_i \otimes_S x) = \sum_i \varphi(e_i \text{ev}(\widehat{e}_i \otimes_S x)) = \varphi(x),$$

by the right A -linearity of φ , diagram (2.1) and $\widehat{\text{ev}}(\sum_i e_i \otimes_R \widehat{e}_i) = 1_B$.

(2) This is proven by symmetric arguments.

(3) Obvious.

(4) We first prove that \bar{T} is a left C -comodule. By Theorem 2.16, C is an R -coring with comultiplication $T \otimes_R \gamma|_C$, hence $(T \otimes_R \gamma)(C) \subset C \otimes_R C$. Similarly, D is an S -coring with comultiplication $\gamma \otimes_R T|_D$ hence $(\gamma \otimes_S T)(D) \subset D \otimes_S D$. Therefore, it follows that $(T \otimes_R \gamma \otimes_S T)(\bar{T}) \subset (C \otimes_R C \otimes \widehat{T}) \cap (\widehat{T} \otimes_S D \otimes_S D)$. Consider $C \otimes_R C \otimes_R A$ as a right \mathcal{C} -comodule with coaction $C \otimes_R \rho^{C \otimes_R A}$. By a similar computation as for h_1 , we find that the map

$$\begin{aligned} \bar{h} : (T \otimes_R \gamma \otimes_S T)(\bar{T}) &\subset (C \otimes_R C \otimes_R \widehat{T}) \cap (\widehat{T} \otimes_S D \otimes_S D) \rightarrow \text{Hom}^C(T, C \otimes_R C \otimes_R A), \\ \bar{h}(c \otimes_R c' \otimes_R \widehat{x})(x) &= c \otimes_R c' \otimes_R \text{ev}(\widehat{x} \otimes_S x), \end{aligned}$$

is well defined. Applying Lemma 3.2, we therefore find a well-defined map

$$\bar{h} \circ (T \otimes_R \gamma \otimes_R T) : \bar{T} \rightarrow \text{Hom}^C(T, C \otimes_R C \otimes_R A) \cong C \otimes_R \text{Hom}^C(T, C \otimes_R A).$$

Hence $(C \otimes_R k_1) \circ \bar{h} \circ (T \otimes_R \gamma \otimes_R T) : \bar{T} \rightarrow C \otimes_R \bar{T}$ defines a comultiplication on \bar{T} . Up to an isomorphism this is just the restriction of the map $T \otimes_R \gamma \otimes_R T$. Coassociativity and counitality now follow immediately from the diagrams (2.5) and (2.4). By symmetric arguments one shows that \bar{T} is a right D -comodule. The coassociativity between left C - and right D -coaction follows from diagram (2.5). \square

3.4. Theorem. *Let (T, γ) be a tame B - A herd. Consider corings C and D of Theorem 2.16. Then the R - S bimodule $\bar{T} = (C \otimes_R \hat{T}) \cap (\hat{T} \otimes_S D)$ of Theorem 3.3 is a companion of T and (T, χ) is a D - C coherd, where $\chi : T \otimes_R \bar{T} \otimes_S T \rightarrow T$ is given by*

$$\chi(x \otimes_R \hat{x} \otimes_S y \otimes_R \hat{y} \otimes_S z) = \widehat{\text{ev}}(x \otimes_R \hat{x})y \text{ev}(\hat{y} \otimes_S z), \quad (3.7)$$

for all $x, z \in T$ and $\hat{x} \otimes_S y \otimes_R \hat{y} \in \bar{T} \subset \hat{T} \otimes_S T \otimes_R \hat{T}$ (summation implicit).

Proof. We first prove that \bar{T} is a companion for T . To this end, we must define a C -bicomodule map $\text{cov} : C \rightarrow \bar{T} \otimes_S T$. By means of the canonical inclusion $\iota : C \rightarrow \hat{T} \otimes_S T$, from the definition of C as an equaliser, we can consider Δ_C as a map $\Delta_C : C \rightarrow C \otimes_R \hat{T} \otimes_S T$. Furthermore, as $\gamma(T) \subset D \otimes_S T$, we know that $\Delta_C(C) = (\hat{T} \otimes_S \gamma)(C) \subset \hat{T} \otimes_S D \otimes_S T$. Since T is flat as a left S -module the functor $-\otimes_S T$ preserves all limits, so in particular intersections. Therefore

$$(C \otimes_R \hat{T} \otimes_S T) \cap (\hat{T} \otimes_S D \otimes_S T) = ((C \otimes_R \hat{T}) \cap (\hat{T} \otimes_S D)) \otimes_S T = \bar{T} \otimes_S T.$$

This defines a C -bilinear map

$$\text{cov} = \Delta_C : C \rightarrow \bar{T} \otimes_S T.$$

By similar arguments,

$$\overline{\text{cov}} = \Delta_D : D \rightarrow T \otimes_R \bar{T}$$

is well defined and is clearly D -bilinear. Diagram (3.1) is now exactly diagram (2.5). Diagram (3.2) follows from (2.5) tensored on the left with $\hat{T} \otimes_R -$ and on the right with $-\otimes_S \hat{T}$.

Now take any $x \otimes_R \hat{x} \otimes_S y \in T \otimes_R C$ (summation implicit). The condition of diagram (3.3) comes out as

$$x \text{ev}(\hat{x} \otimes_S y) = \widehat{\text{ev}}(x \otimes_R \hat{x})y^{(1)} \text{ev}(y^{(2)} \otimes_S y^{(3)}).$$

Similarly, for all $x \otimes_R \hat{x} \otimes_S y \in D \otimes_R T$ (summation implicit), diagram (3.4) commutes since

$$\widehat{\text{ev}}(x \otimes_S \hat{x})y = \widehat{\text{ev}}(x^{(1)} \otimes_R x^{(2)})x^{(3)} \text{ev}(\hat{x} \otimes_S y).$$

Finally, diagram (3.5) commutes because of the bilinearity of ev and $\widehat{\text{ev}}$ and by diagrams (2.1) and (2.2). \square

Reconstruction of the herd. Let (T, γ) be a tame A - B herd. By Theorem 3.4, $\bar{T} = (C \otimes_R \hat{T}) \cap (\hat{T} \otimes_S D)$ is a companion of T and (T, χ) is a D - C coherd, where C and D are corings of Theorem 2.16, and $\chi : T \otimes_R \bar{T} \otimes_S T \rightarrow T$ is given by (3.7). Furthermore, we know from the first part of this section that we can construct the R -bimodule A' as the following coequaliser

$$C \otimes_R \bar{T} \otimes_S T \xrightarrow[\varepsilon_C \otimes_R \bar{T} \otimes_S T]{(\bar{T} \otimes_S \chi) \circ (\text{cov} \otimes_R \bar{T} \otimes_S T)} \bar{T} \otimes_S T \xrightarrow{\pi_A} A'.$$

Recall that A' is in general a non-unital R -ring, but if C is faithfully flat as a left or right R -module (i.e., if \hat{T} is faithfully flat as a left R - or right S -module), then A' has a unit.

Put $\omega = (\bar{T} \otimes_S \chi) \circ (\text{cov} \otimes_R \bar{T} \otimes_S T) - \varepsilon_C \otimes_R \bar{T} \otimes_S T$. Then $A' = \bar{T} \otimes_S T / \text{Im } \omega$ consists of classes that satisfy

$$[\text{ev}(\hat{x} \otimes_S x) \hat{y} \otimes_S y \otimes_R \hat{z} \otimes_S z] = [\hat{x} \otimes_S x \otimes_R \hat{y} \otimes_S y \text{ev}(\hat{z} \otimes_S z)] \quad (3.8)$$

(summation implicit). This follows by the defining property of D .

Similarly, there is a (non-unital) S -ring B' given by the coequaliser

$$T \otimes_R \bar{T} \otimes_S D \xrightarrow[T \otimes_R \bar{T} \otimes_S \varepsilon_D]{(\chi \otimes_R \bar{T}) \circ (T \otimes_R \bar{T} \otimes_S \text{cov})} T \otimes_R \bar{T} \xrightarrow{\pi_B} B'.$$

3.5. Theorem. Let (T, γ) be a tame A - B herd and let A', B' be rings constructed above.

- (1) There are ring morphisms $\nu_A : A' \rightarrow A$ and $\nu_B : B' \rightarrow B$.
- (2) If the map $h : \bar{T} \rightarrow \hat{T}$ of Theorem 3.3(3) is an isomorphism, then ν_A and ν_B are isomorphisms, in particular A' and B' are unital rings.
- (3) If \hat{T} is flat as a left R - and right S -module and $\bar{T} = \hat{T} \otimes_S D = C \otimes_R \hat{T}$, then maps ν_A and ν_B are isomorphisms of rings.

Proof. We only prove the statements for rings A and A' . The statements for B and B' are verified by symmetric arguments.

- (1) Consider the map $\bar{\text{ev}} : \bar{T} \otimes_S T \rightarrow A$, given by

$$\bar{\text{ev}}(\hat{x} \otimes_S x \otimes_R \hat{y} \otimes_S y) = \text{ev}(\hat{x} \otimes_S x) \text{ev}(\hat{y} \otimes_S y),$$

then obviously, $\bar{\text{ev}} \circ \omega = 0$. Hence by the universal property of coequalisers, there is a map $\nu_A : A' \rightarrow A$. Using the properties of the evaluation maps, it is easily checked that ν_A is a ring morphism.

- (2) Consider the following diagram with coequalisers as rows:

$$\begin{array}{ccccc} C \otimes_R \bar{T} \otimes_S T & \xrightarrow[\varepsilon_C \otimes_R \bar{T} \otimes_S T]{(\bar{T} \otimes_S \chi) \circ (\text{cov} \otimes_R \bar{T} \otimes_S T)} & \bar{T} \otimes_S T & \xrightarrow{\pi_A} & A' \\ \downarrow Q \otimes_R h \otimes_S T & & \downarrow h \otimes_S T & & \downarrow \nu_Q \\ C \otimes_R \hat{T} \otimes_S T & \xrightarrow[\varepsilon_C \otimes_R \hat{T} \otimes_S T]{(\hat{T} \otimes_S \mu_{T,A}) \circ (C \otimes_R \text{ev})} & \hat{T} \otimes_S T & \xrightarrow{\pi_Q} & Q \end{array}$$

Here $\mu_{T,A} : T \otimes_R A \rightarrow T$ denotes the action of A on T . One can check that the diagram is commutative, hence the map ν_Q exists by the universal property of coequalisers. Since h is an isomorphism,

ν_Q is an isomorphism as well. We claim that $A \cong Q$. Put $\varpi = (\widehat{T} \otimes_S \mu_{T,A}) \circ (C \otimes_R \text{ev}) - \varepsilon_C \otimes_R \widehat{T} \otimes_S T$. Then $Q = \widehat{T} \otimes_S T / \text{Im } \varpi$ consists of classes of elements that satisfy

$$[\text{ev}(\widehat{x} \otimes_S x) \widehat{y} \otimes_S y] = [\widehat{x} \otimes_S x \text{ev}(\widehat{y} \otimes_S y)]. \quad (3.9)$$

Obviously $\text{ev} : \widehat{T} \otimes_S T \rightarrow A$ satisfies $\text{ev} \circ \varpi = 0$, hence the universal property of coequalisers yields a map $\nu : Q \rightarrow A$. Conversely, define a map $A \rightarrow Q$ as follows. By assumption, the map ev is surjective, therefore, for all $a \in A$, there exists a (not necessarily unique) element $\widehat{x}_a \otimes_S x_a \in \widehat{T} \otimes_S T$ (summation implicit) such that $a = \text{ev}(\widehat{x}_a \otimes_S x_a)$. Define $\nu' : A \rightarrow Q$, $\nu'(a) = [\widehat{x}_a \otimes_S x_a]$. Take another element $\widehat{y}_a \otimes_S y_a \in \widehat{T} \otimes_S T$ such that $\text{ev}(\widehat{y}_a \otimes_S y_a) = a$, and use the defining property of Q (3.9) to compute

$$\begin{aligned} [\widehat{x}_a \otimes_S x_a] &= [\widehat{x}_a \otimes_S x_a \cdot 1] \\ &= [\widehat{x}_a \otimes_S x_a \text{ev}(\widehat{x}_1 \otimes_S x_1)] \\ &= [\text{ev}(\widehat{x}_a \otimes_S x_a) \widehat{x}_1 \otimes_S x_1] \\ &= [\text{ev}(\widehat{y}_a \otimes_S y_a) \widehat{x}_1 \otimes_S x_1] \\ &= [\widehat{y}_a \otimes_S y_a]. \end{aligned}$$

Thus the map ν' is well defined. Obviously $\nu \circ \nu' = A$, and a similar computation to the one above shows that $\nu' \circ \nu = Q$. Hence $A \cong Q \cong A'$.

(3) Under these conditions, the defining coequaliser diagram for A' reduces to

$$C \otimes_R \widehat{T} \otimes_S D \otimes_S T \xrightarrow[\varepsilon_C \otimes_R \widehat{T} \otimes_S D \otimes_S T]{(\widehat{T} \otimes_S \chi) \circ (\text{cov} \otimes_R \widehat{T} \otimes_S D \otimes_S T)} \widehat{T} \otimes_S D \otimes_S T \xrightarrow{\pi_A} A'.$$

Since $\gamma(T) \subset D \otimes_S T$, the flatness of \widehat{T} as a right S -module implies that $\widehat{T} \otimes_S \gamma(\widehat{T} \otimes_S T) \subset \widehat{T} \otimes_S D \otimes_S T$. Using a similar notation as in the proof of part (2), we define a map $\nu'_A : A \rightarrow A'$ by $\nu'_A(a) = [\widehat{x}_a \otimes_S \gamma(x_a)]$. This map is well defined since,

$$\begin{aligned} [\widehat{x}_a \otimes_S x_a^{(1)} \otimes_R x_a^{(2)} \otimes_S x_a^{(3)}] &= [\widehat{x}_a \otimes_S x_a^{(1)} \otimes_R x_a^{(2)} \otimes_S x_a^{(3)} \text{ev}(\widehat{x}_1 \otimes_S x_1)] \\ &= [\text{ev}(\widehat{x}_a \otimes_S x_a^{(1)}) x_a^{(2)} \otimes_S x_a^{(3)} \otimes_R \widehat{x}_1 \otimes_S x_1^{(1)} \text{ev}(x_1^{(2)} \otimes_S x_1^{(3)})] \\ &= [\text{ev}(\widehat{x}_a \otimes_S x_a^{(1)}) \text{ev}(x_a^{(2)} \otimes_S x_a^{(3)}) \widehat{x}_1 \otimes_S x_1^{(1)} \otimes_R x_1^{(2)} \otimes_S x_1^{(3)}] \\ &= [\text{ev}(\widehat{y}_a \otimes_S y_a) \widehat{x}_1 \otimes_S x_1^{(1)} \otimes_R x_1^{(2)} \otimes_S x_1^{(3)}] \\ &= [\widehat{y}_a \otimes_S \gamma(y_a)], \end{aligned}$$

where $\widehat{y}_a \otimes_S y_a \in \widehat{T} \otimes_S T$ is any element such that $\text{ev}(\widehat{y}_a \otimes_S y_a) = \text{ev}(\widehat{x}_a \otimes_S x_a) = a$. It is easily checked that ν'_A is the inverse of ν_A . \square

3.6. Remark. Theorem 3.5 shows that there are (at least) two situations in which the original base rings A and B of the tame bimodule herd T can be reconstructed from the associated coherd. Both cases have non-empty sets of examples.

The situation of Theorem 3.5(2) occurs when $T = A = B$ is a ring and the associated entwining maps ψ and ϕ of Theorem 2.16 are bijective. This is the case described in [3, Theorem 4.9].

An explicit example of the situation of Theorem 3.5(3) will be discussed in Section 5, where we consider Galois co-objects. As in this situation $R = S = k$ is a commutative ring, and $T \cong \widehat{T}$ as k -module, the flatness conditions on \widehat{T} are already contained in the flatness of T .

4. Composition of Herds

The aim of this section is to describe a way in which two bimodule herds can be composed by means of the tensor product.

Consider in addition to the ring morphisms $\alpha : R \rightarrow A$ and $\beta : S \rightarrow B$ a third ring morphism $\kappa : Z \rightarrow K$.

4.1. Lemma. *Let T be a B - A bimodule with a formal dual \widehat{T} , and let P be an A - K bimodule with a formal dual \widehat{P} . Then $V = T \otimes_A P$ is a B - K bimodule with formal dual $\widehat{V} = \widehat{P} \otimes_A \widehat{T}$.*

Proof. We define ev_V out of ev_P and ev_T as follows

$$\text{ev}_V = \text{ev}_P \circ (\widehat{P} \otimes_A \text{ev}_T \otimes_A T) : \widehat{P} \otimes_A \widehat{T} \otimes_S T \otimes_A P \rightarrow K.$$

Similarly, we define $\widehat{\text{ev}}_V$ out of $\widehat{\text{ev}}_P$ and $\widehat{\text{ev}}_T$ by

$$\widehat{\text{ev}}_V = \widehat{\text{ev}}_T \circ (T \otimes_A \widehat{\text{ev}}_P \otimes_A \widehat{T}) : T \otimes_A P \otimes_Z \widehat{P} \otimes_A \widehat{T} \rightarrow B.$$

An easy computation shows that the commutativity of the diagrams (2.1) and (2.2) applied on P and T forces the same diagrams for V to be commutative. \square

Let \mathcal{E} and \mathcal{C} be two A -corings and consider an A -bimodule map

$$\sigma : \mathcal{C} \otimes_A \mathcal{E} \rightarrow \mathcal{E} \otimes_A \mathcal{C},$$

which renders commutative the following diagrams,

$$\begin{array}{ccccc}
 & \mathcal{C} \otimes_A \mathcal{C} \otimes_A \mathcal{E} & \xleftarrow{\Delta_{\mathcal{C}} \otimes_A \mathcal{E}} & \mathcal{C} \otimes_A \mathcal{E} & \xrightarrow{\mathcal{C} \otimes_A \Delta_{\mathcal{E}}} & \mathcal{C} \otimes_A \mathcal{E} \otimes_A \mathcal{E} \\
 \swarrow \mathcal{C} \otimes_A \sigma & & & \downarrow \sigma & & \searrow \sigma \otimes_A \mathcal{E} \\
 \mathcal{C} \otimes_A \mathcal{E} \otimes_A \mathcal{C} & & & & & \mathcal{E} \otimes_A \mathcal{C} \otimes_A \mathcal{E} \\
 \searrow \sigma \otimes_A \mathcal{C} & & & & & \swarrow \mathcal{E} \otimes_A \sigma \\
 & \mathcal{E} \otimes_A \mathcal{C} \otimes_A \mathcal{C} & \xrightarrow{\mathcal{E} \otimes_A \Delta_{\mathcal{C}}} & \mathcal{E} \otimes_A \mathcal{C} & \xrightarrow{\Delta_{\mathcal{E}} \otimes_A \mathcal{C}} & \mathcal{E} \otimes_A \mathcal{E} \otimes_A \mathcal{C}
 \end{array} \quad (4.1)$$

$$\begin{array}{ccc}
 \mathcal{C} \otimes_A \mathcal{E} & \xrightarrow{\mathcal{C} \otimes_A \varepsilon_{\mathcal{E}}} & \mathcal{C} \otimes_A A \\
 \sigma \downarrow & & \downarrow \cong \\
 \mathcal{E} \otimes_A \mathcal{C} & \xrightarrow{\varepsilon_{\mathcal{E}} \otimes_A \mathcal{C}} & A \otimes_A \mathcal{C}
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathcal{C} \otimes_A \mathcal{E} & \xrightarrow{\varepsilon_{\mathcal{C}} \otimes_A \mathcal{E}} & A \otimes_A \mathcal{E} \\
 \sigma \downarrow & & \downarrow \cong \\
 \mathcal{E} \otimes_A \mathcal{C} & \xrightarrow{\mathcal{E} \otimes_A \varepsilon_{\mathcal{C}}} & \mathcal{E} \otimes_A A.
 \end{array} \quad (4.2)$$

These conditions hold if and only if the A -bimodule $\mathcal{E} \otimes_A \mathcal{C}$ is an A -coring with coproduct $(\mathcal{E} \otimes_A \sigma \otimes_A \mathcal{C}) \circ (\Delta_{\mathcal{E}} \otimes_A \Delta_{\mathcal{C}})$ and counit $\varepsilon_{\mathcal{E}} \otimes_A \varepsilon_{\mathcal{C}}$ (see [11] for the case of a commutative base). In this case, the A -coring structure on $\mathcal{E} \otimes_A \mathcal{C}$ is called the *smash coproduct* of \mathcal{E} and \mathcal{C} and denoted by $\mathcal{E} \otimes_{\sigma} \mathcal{C}$.

4.2. Theorem. Let (T, γ_T) be a B - A herd and (P, γ_P) an A - K herd. Denote by $\mathcal{C} = \widehat{T} \otimes_S T$ the A -coring associated to T as in Corollary 2.8 and $\mathcal{E} = P \otimes_Z \widehat{P}$ the A -coring associated to P . Then $V = T \otimes_A P$ is a B - K herd with shepherd γ_V satisfying

$$\begin{aligned}\gamma_{T,A} \otimes_A P &= (T \otimes_A \widehat{\text{ev}}_P \otimes_A \widehat{T} \otimes_S T \otimes_A P) \circ \gamma_V, \\ T \otimes_A \gamma_{P,A} &= (T \otimes_A P \otimes_Z \widehat{P} \otimes_A \text{ev}_T \otimes_A P) \circ \gamma_V,\end{aligned}\quad (4.3)$$

if and only if there exists a map $\sigma : \mathcal{C} \otimes_A \mathcal{E} \rightarrow \mathcal{E} \otimes_A \mathcal{C}$, which defines a smash coproduct $\mathcal{E} \otimes_\sigma \mathcal{C}$. (Hence $\mathcal{C} \otimes_A \mathcal{E}$ is an A -coring.)

Proof. By Lemma 4.1, $\widehat{V} = \widehat{P} \otimes_A \widehat{T}$ is a formal dual of $T \otimes_A P$. Suppose first that $\mathcal{C} \otimes_\sigma \mathcal{E}$ is a smash coproduct. Denote by $\gamma_{T,A} : T \rightarrow T \otimes_A \widehat{T} \otimes_S T$ and $\gamma_{P,A} : P \rightarrow P \otimes_Z \widehat{P} \otimes_A P$ the projections of γ_T and γ_P respectively, constructed as in Notation 2.5. Define $\gamma_V : V \rightarrow V \otimes_K \widehat{V} \otimes_S V$ as the following composition:

$$\begin{array}{ccc} V = T \otimes_A P & \xrightarrow{\gamma_V} & V \otimes_K \widehat{V} \otimes_S V \\ \downarrow \gamma_{T,A} \otimes_A \gamma_{P,A} & & \parallel \\ T \otimes_A \widehat{T} \otimes_S T \otimes_A P \otimes_Z \widehat{P} \otimes_A P & \xrightarrow{T \otimes_A \sigma \otimes_A P} & T \otimes_A P \otimes_Z \widehat{P} \otimes_A \widehat{T} \otimes_S T \otimes_A P. \end{array}$$

We need to check that γ_V satisfies diagrams (2.3)–(2.5). First note that $\widehat{\text{ev}}_P = \varepsilon_{\mathcal{E}}$. Diagram (2.3) for V then comes out as (unadorned tensor product is over A)

$$\begin{array}{ccccc} T \otimes P & \xrightarrow{\gamma_{T,A} \otimes \gamma_{P,A}} & T \otimes \widehat{T} \otimes_S T \otimes P \otimes_Z \widehat{P} \otimes P & \xrightarrow{T \otimes \sigma \otimes P} & T \otimes P \otimes_Z \widehat{P} \otimes \widehat{T} \otimes_S T \otimes P \\ \downarrow \cong & & \downarrow T \otimes \widehat{T} \otimes_S T \otimes \widehat{\text{ev}}_P \otimes P & & \downarrow T \otimes \widehat{\text{ev}}_P \otimes \widehat{T} \otimes_S T \otimes P \\ & & T \otimes \widehat{T} \otimes_S T \otimes A \otimes P & \xrightarrow{\cong} & T \otimes \widehat{T} \otimes_S T \otimes P \\ & & & & \downarrow \widehat{\text{ev}}_T \otimes_S T \otimes P \\ R \otimes_R T \otimes P & \xrightarrow{\beta \otimes_S T \otimes P} & & & B \otimes_S T \otimes P. \end{array}$$

The small square in this diagram commutes because of the left diagram in (4.2), the other part of the diagram commutes by diagram (2.3) applied to T and P . In the same way, one proves that V satisfies the condition of diagram (2.4). Diagram (2.5) for V looks as follows (unadorned tensor product is over A)

$$\begin{array}{ccccc} T \otimes P & \xrightarrow{\gamma_{T,A} \otimes \gamma_{P,A}} & T \otimes \mathcal{C} \otimes \mathcal{E} \otimes P & \xrightarrow{T \otimes \sigma \otimes P} & T \otimes \mathcal{E} \otimes \mathcal{C} \otimes P \\ \downarrow \gamma_{T,A} \otimes \gamma_{P,A} & & \downarrow \gamma_{T,A} \otimes \mathcal{C} \otimes \gamma_{P,A} \otimes \widehat{P} \otimes P & & \downarrow \gamma_{T,A} \otimes \gamma_{P,A} \otimes \widehat{P} \otimes \mathcal{C} \otimes P \\ T \otimes \mathcal{C} \otimes \mathcal{E} \otimes P & \xrightarrow{T \otimes \widehat{T} \otimes_S \gamma_{T,A} \otimes \mathcal{E} \otimes \gamma_{P,A}} & T \otimes \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{E} \otimes \mathcal{E} \otimes P & \xrightarrow{T \otimes \mathcal{C} \otimes \sigma_2 \otimes P} & T \otimes \mathcal{C} \otimes \mathcal{E} \otimes \mathcal{E} \otimes \mathcal{C} \otimes P \\ \downarrow T \otimes \sigma \otimes P & & \downarrow T \otimes \sigma^2 \otimes \mathcal{E} \otimes P & & \downarrow T \otimes \sigma \otimes \mathcal{E} \otimes \mathcal{C} \otimes P \\ T \otimes \mathcal{E} \otimes \mathcal{C} \otimes P & \xrightarrow{T \otimes \mathcal{E} \otimes \widehat{T} \otimes_S \gamma_{T,A} \otimes \gamma_{P,A}} & T \otimes_R \mathcal{E} \otimes \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{E} \otimes P & \xrightarrow{T \otimes \mathcal{E} \otimes \mathcal{C} \otimes \sigma \otimes P} & T \otimes \mathcal{E} \otimes \mathcal{C} \otimes \mathcal{E} \otimes \mathcal{C} \otimes P, \end{array}$$

where $\sigma^2 = (\sigma \otimes_A C) \circ (C \otimes_A \sigma)$ and $\sigma_2 = (\mathcal{E} \otimes_A \sigma) \circ (\sigma \otimes_A \mathcal{E})$. The upper left square in this diagram commutes by (2.5) for T and P , the upper right diagram commutes by the right pentagon in diagram (4.1), the lower left square commutes by the left pentagon in (4.1), and the lower right square commutes trivially. Finally, Eq. (4.3) can be easily verified.

Conversely, suppose that $V = T \otimes_A P$ is a herd with the shepherd γ_V . Then we define $\sigma : C \otimes_A \mathcal{E} \rightarrow \mathcal{E} \otimes_A C$ as follows,

$$\begin{array}{ccc}
 C \otimes_A \mathcal{E} & \xrightarrow{\sigma} & \mathcal{E} \otimes_A C \\
 \parallel & & \parallel \\
 \widehat{T} \otimes_S T \otimes_A P \otimes_Z \widehat{P} & \xrightarrow{\widehat{T} \otimes_S \gamma_V \otimes_Z \widehat{P}} & P \otimes_Z \widehat{P} \otimes_A \widehat{T} \otimes_S T \\
 & \searrow \widehat{T} \otimes_S T \otimes_A P \otimes_Z \widehat{P} \otimes_A \widehat{T} \otimes_S T \otimes_A P \otimes_Z \widehat{P} \nearrow & \\
 & \xrightarrow{\text{ev}_T \otimes_A \mathcal{E} \otimes_A C \otimes_A \text{ev}_P} &
 \end{array}$$

By similar diagram chasing arguments, one proves that σ indeed defines a smash coproduct on $\mathcal{E} \otimes_A C$, provided that Eq. (4.3) are satisfied. \square

5. Galois co-objects

The aim of this section is to show how Galois co-objects for a commutative Hopf algebra and their composition can be interpreted in terms of bimodule herds. In this section we fix a commutative ground ring k , and do not deal with k -rings and k -corings, but with k -algebras and k -coalgebras. Throughout this section H is a Hopf algebra (with coproduct Δ_H , counit ε_H and the unit map $\eta_H : k \rightarrow H, x \mapsto x1_H$) that is faithfully flat over its commutative base ring k , with a bijective antipode. The symbol S denotes the antipode of a Hopf algebra H . The unadorned tensor product is over k . We use the Sweedler notation for coproduct, i.e. $\Delta_H(h) = h_{(1)} \otimes h_{(2)}$, $\Delta_H^2(h) = h_{(1)} \otimes h_{(2)} \otimes h_{(3)}$, etc.

Galois co-objects as Galois comodules. Let C be a right H -module coalgebra, that is, C is a k -coalgebra, with coproduct Δ_C and counit ε_C , and a right H -module such that, for all $c \in C$ and $h \in H$,

$$\Delta_C(ch) = c_{(1)}h_{(1)} \otimes c_{(2)}h_{(2)}, \quad \varepsilon_C(ch) = \varepsilon_C(c)\varepsilon_H(h).$$

A right (H, C) -Hopf module M is a right k -module that has a right H -module structure and a right C -comodule structure $Q^M : M \rightarrow M \otimes C$ with the following compatibility condition

$$Q^M(mh) = m_{(0)}h_{(1)} \otimes m_{(1)}h_{(2)},$$

for all $m \in M$ and $h \in H$, where $Q^M(m) = m_{(0)} \otimes m_{(1)}$ is the Sweedler notation for a coaction. The category of all (H, C) -Hopf modules with H -linear C -colinear maps between them is denoted by $\mathbf{M}_H^C(H)$. It is known that out of these data one can construct an H -coring $\mathcal{C} = H \otimes C$, with H -bimodule structure

$$g(h \otimes c)g' = ghg'_{(1)} \otimes cg'_{(2)},$$

for all $h, g, g' \in H$ and $c \in C$, coproduct $H \otimes \Delta_C$ and counit $H \otimes \varepsilon_C$. In this way the category of (H, C) -Hopf modules is isomorphic to the category of right C -comodules, $\mathbf{M}_H^C(H) \cong \mathbf{M}^C$. Furthermore, C is a right (H, C) -Hopf module with the regular H -module and C -comodule structures. Hence there is a functor

$$\mathcal{G} = - \otimes C : \mathbf{M}_k \rightarrow \mathbf{M}_H^C(H).$$

This functor has both a left adjoint \mathcal{F} and a right adjoint \mathcal{H} given by

$$\mathcal{F} = - \otimes_H k : \mathbf{M}_H^C(H) \rightarrow \mathbf{M}_k,$$

$$\mathcal{H} = \text{Hom}_H^C(C, -) : \mathbf{M}_H^C(H) \rightarrow \mathbf{M}_k.$$

The adjointness of $(\mathcal{F}, \mathcal{G})$ follows by [10, Proposition 8.7.1], the adjointness of $(\mathcal{G}, \mathcal{H})$ is a general Hom-tensor relation.

An H -module coalgebra C is called a *Hopf-Galois co-object* if and only if the pair $(\mathcal{F}, \mathcal{G})$ is an inverse equivalence. The uniqueness of adjoints implies that C is a Galois co-object if and only if $(\mathcal{G}, \mathcal{H})$ is a pair of inverse equivalences, which means in particular that C is a right Galois comodule for the H -coring $\mathcal{C} = H \otimes C$. By [28, 3.4 and 3.7], every Galois co-object C is therefore finitely generated and projective as a right H -module. Furthermore, Example 2.11 shows that out of this Galois co-object we can construct a k - H herd which describes exactly the Galois properties of C as a right \mathcal{C} -comodule (see Theorem 2.18).

The group of Galois co-objects and the composition of herds. Recall from [10, Theorem 8.7.4] that if C is a Galois co-object for H , then the map

$$\delta : C \otimes H \rightarrow C \otimes C, \quad \delta(c \otimes h) = c_{(1)} \otimes c_{(2)}h,$$

is bijective. Define \hat{C} as the left H -module, which is isomorphic to C as a k -module and with H -action given by

$$h \rightharpoonup \hat{c} = \hat{c}S^{-1}(h).$$

5.1. Lemma. *Let C be a Galois co-object for H , then*

$$\text{ev} : \hat{C} \otimes C \rightarrow H, \quad \text{ev} = (\varepsilon_C \otimes H) \circ \delta^{-1}$$

is an H -bilinear map.

Proof. The map ev is the *cotranslation map*, and the H -bilinearity property is a dualisation of the H -bilinearity property of the *translation map*; see [26, Remark 3.4(d), (e)]. We include the direct proof for completeness.

Since δ is bijective, we can write an element in $C \otimes C$ uniquely as a finite sum of elements of the form $d_{(1)} \otimes d_{(2)}h$, where $d \in C$, $h \in H$. By definition, $\text{ev}(d_{(1)} \otimes d_{(2)}h) = \varepsilon_C(d)h$. The map δ is a right H -module map, hence the right H -linearity of ev is clear. The left H -linearity is proven as follows:

$$\begin{aligned} \text{ev}(h' \rightharpoonup d_{(1)} \otimes d_{(2)}h) &= \text{ev}(d_{(1)}S^{-1}(h') \otimes d_{(2)}h) \\ &= \text{ev}(d_{(1)}S^{-1}(h')_{(1)} \otimes d_{(2)}\varepsilon_C(S^{-1}(h')_{(2)})h) \\ &= \text{ev}(d_{(1)}S^{-1}(h')_{(1)} \otimes d_{(2)}S^{-1}(h')_{(2)}S(S^{-1}(h')_{(3)})h) \\ &= \varepsilon_C(dS^{-1}(h')_{(1)})S(S^{-1}(h')_{(2)})h = \varepsilon_C(d)h'h \\ &= h' \text{ev}(d_{(1)} \otimes d_{(2)}h), \end{aligned}$$

where we used the antipode property in the third equality. \square

Recall that the antipode S of a commutative Hopf algebra H is always involutive, that is S is bijective and $S^{-1} = S$. Furthermore, if H is a commutative Hopf algebra, then the set of Galois co-objects forms a group with the tensor product over H as the composition. Our next aim is to show that this composition can be obtained from the composition of herds as described in Section 4. For this purpose, we need to associate to a Galois co-object C a bimodule herd different from the one described in Example 2.11. For the rest of the section we assume that H is commutative as k -algebra. A right H -module coalgebra C is now understood as an H -bimodule with the same left and right action, and \widehat{C} is an H -bimodule with the same left and right action \rightarrow defined in the preamble to Lemma 5.1.

5.2. Theorem. *Let C be a Galois co-object for a commutative Hopf algebra H . With notation as above, \widehat{C} is a formal dual of the H -bimodule C , with $\text{ev} = (\varepsilon_C \otimes H) \circ \delta^{-1}$ and $\widehat{\text{ev}} = S \circ \text{ev}$. Furthermore, C is an H - H herd with shepherd*

$$\Delta_C^2 = (\Delta_C \otimes C) \circ \Delta_C = (C \otimes \Delta_C) \circ \Delta_C : C \rightarrow C \otimes \widehat{C} \otimes C,$$

and pen \widehat{C} .

Proof. The maps ev and $\widehat{\text{ev}}$ are H -bilinear by Lemma 5.1 and by the fact that $S^{-1} = S$. Since δ is bijective, there is an isomorphism

$$\vartheta = (C \otimes \delta) \circ (\delta \otimes H) : C \otimes H \otimes H \rightarrow C \otimes C \otimes C.$$

Thus to check the commutativity of diagrams (2.1) and (2.2) suffices it to evaluate them on elements $\vartheta(c \otimes h \otimes h') = c_{(1)} \otimes c_{(2)}h_{(1)} \otimes c_{(3)}h_{(2)}h'$, where $c \in C$ and $h, h' \in H$. For (2.1),

$$\begin{aligned} c_{(1)} \text{ev}(c_{(2)}h_{(1)} \otimes c_{(3)}h_{(2)}h') &= c_{(1)}\varepsilon_C(c_{(2)})\varepsilon_H(h)h' = c\varepsilon_H(h)h' \\ &= \varepsilon_C(c_{(1)})S(h_{(1)})c_{(2)}h_{(2)}h' \\ &= \widehat{\text{ev}}(c_{(1)} \otimes c_{(2)}h_{(1)})c_{(3)}h_{(2)}h', \end{aligned}$$

where we used the antipode property in the penultimate equality. The commutativity of (2.2) can be checked in a similar way:

$$\begin{aligned} \text{ev}(c_{(1)} \otimes c_{(2)}h_{(1)})c_{(3)} &\rightarrow h_{(2)}h' = \varepsilon_C(c_{(1)})h_{(1)} \rightarrow c_{(2)}h_{(2)}h' \\ &= cS(h_{(1)})h_{(2)}h' \\ &= c\varepsilon_H(h)h' = c\varepsilon_H(h)S^2(h') \\ &= c_{(1)} \leftarrow \varepsilon_C(c_{(2)})\varepsilon_H(h)S(h') \\ &= c_{(1)} \leftarrow \widehat{\text{ev}}(c_{(2)}h_{(1)} \otimes c_{(3)}h_{(2)}h'). \end{aligned}$$

Finally, we need to check that Δ_C^2 is a shepherd. Clearly, the map satisfies the coassociativity condition. Since $\delta(c \otimes 1_H) = c_{(1)} \otimes c_{(2)}$, $\text{ev}(\Delta_C(c)) = \varepsilon_C(c)1_H = \widehat{\text{ev}}(\Delta_C(c))$. Hence diagrams (2.3) and (2.4) commute as a consequence of the counit condition of C . \square

5.3. Theorem. *Let C be a Galois co-object for a commutative Hopf algebra H . Consider C as an H - H herd with a pen \widehat{C} and shepherd Δ_C^2 as in Theorem 5.2. Then the coalgebra E defined by the equaliser*

$$E \xrightarrow{e} \widehat{C} \otimes C \xrightarrow[\eta_H \otimes \widehat{C} \otimes C]{(\text{ev} \otimes \widehat{C} \otimes C) \circ (\widehat{C} \otimes \Delta_C^2)} H \otimes \widehat{C} \otimes C$$

is isomorphic to C .

Symmetrically, the coalgebra F defined by the equaliser of $(C \otimes \widehat{C} \otimes \widehat{\text{ev}}) \circ (\Delta_C^2 \otimes \widehat{C})$ and $(C \otimes \widehat{C} \otimes \eta_H)$ is also isomorphic to C .

Consequently, if C is a flat k -module (and hence the herd C is tame), rings A and B constructed from the coherd associated to C in Theorem 3.5 are isomorphic to H .

Proof. Set $\omega = (\text{ev} \otimes \widehat{C} \otimes C) \circ (\widehat{C} \otimes \Delta_C^2) - (\eta_H \otimes \widehat{C} \otimes C)$. For any $\delta(c \otimes h) = c_{(1)} \otimes c_{(2)}h \in \widehat{C} \otimes C$,

$$\omega(c_{(1)} \otimes c_{(2)}h) = h_{(1)} \otimes c_{(1)}h_{(2)} \otimes c_{(2)}h_{(3)} - 1_H \otimes c_{(1)} \otimes c_{(2)}h.$$

This implies that

$$\omega \circ \Delta_C = 0, \quad (\varepsilon_H \otimes C \otimes C) \circ \omega \circ \delta = \Delta_C \circ \mu_{C,H} - \delta, \quad (5.1)$$

where $\mu_{C,H} : C \otimes H \rightarrow C$ is the multiplication of H on C . By the first of Eqs. (5.1) and the universal property of equalisers, there is a map $v_C : C \rightarrow E$ such that $\Delta_C = e \circ v_C$. Since Δ_C is injective, so is v_C . The second of Eqs. (5.1) implies that $e \circ v_C \circ \mu_{C,H} \circ \delta^{-1} \circ e$ is the identity map on E . Hence v_C is surjective.

The statement about the coalgebra F follows by symmetric arguments. The statement about the rings A and B follows by the fact that $\bar{C} \cong \widehat{C} \otimes E \cong F \otimes \widehat{C} \cong C \otimes C$ as k -modules and then by Theorem 3.5. \square

Consider two H -module coalgebras C and D over a commutative Hopf algebra H . Then $C \otimes_H D$ is again an H -module coalgebra with the H -module structure given by

$$(c \otimes_H d)h = c \otimes_H (dh) = (ch) \otimes_H d,$$

and comultiplication

$$\Delta_{C \otimes_H D}(c \otimes_H d) = c_{(1)} \otimes_H d_{(1)} \otimes c_{(2)} \otimes_H d_{(2)}.$$

Moreover, if C and D are Galois co-objects, then $C \otimes_H D$ is again a Galois co-object (see [10, Section 10.1]). In particular, the map

$$\delta_{C \otimes_H D} : C \otimes_H D \otimes H \rightarrow C \otimes_H D \otimes C \otimes_H D, \quad c \otimes_H d \otimes h \mapsto c_{(1)} \otimes_H d_{(1)} \otimes c_{(2)} \otimes_H d_{(2)}h,$$

is an isomorphism as the composite of isomorphisms

$$\begin{aligned} C \otimes_H D \otimes H &\cong C \otimes_H D \otimes H \otimes_H H \cong (C \otimes H) \otimes_{H \otimes H} (D \otimes H) \\ &\cong (C \otimes C) \otimes_{H \otimes H} (D \otimes D) \cong (C \otimes_H D) \otimes (C \otimes_H D), \end{aligned}$$

where $\delta_C \otimes_{H \otimes H} \delta_D$ is the penultimate isomorphism.

There are two ways of constructing a formal dual $\widehat{C \otimes_H D}$ of $C \otimes_H D$: one as in Theorem 5.2 (with evaluation maps denoted by ev and $\widehat{\text{ev}}$) the other as in Lemma 4.1. The latter construction gives a formal dual of the form $\widehat{D} \otimes_H \widehat{C}$. The following lemma asserts that both constructions are mutually equivalent.

5.4. Lemma. *With notation as above, the twist map*

$$\tau : \widehat{C \otimes_H D} \rightarrow \widehat{D} \otimes_H \widehat{C},$$

is an isomorphism of H -modules. Furthermore, the following diagrams commute,

$$\begin{array}{ccc} \widehat{C \otimes_H D} \otimes C \otimes_H D & \xrightarrow{\text{ev}} & H \\ \downarrow \tau \otimes C \otimes_H D & & \parallel \\ \widehat{D} \otimes_H \widehat{C} \otimes C \otimes_H D & \xrightarrow{\widehat{D} \otimes_H \text{ev}_C \otimes_H D} \widehat{D} \otimes_H D \xrightarrow{\text{ev}_D} & H \end{array}$$

$$\begin{array}{ccc} C \otimes_H D \otimes \widehat{C \otimes_H D} & \xrightarrow{\widehat{\text{ev}}} & H \\ \downarrow C \otimes_H D \otimes \tau & & \parallel \\ C \otimes_H D \otimes \widehat{D} \otimes_H \widehat{C} & \xrightarrow{C \otimes_H \widehat{\text{ev}}_D \otimes_H \widehat{C}} C \otimes_H \widehat{C} \xrightarrow{\widehat{\text{ev}}_C} & H \end{array}$$

Proof. We only check the commutativity of the first diagram, the commutativity of the second diagram follows by similar arguments. By bijectivity of $\delta_{C \otimes_H D}$, an element of $\widehat{C \otimes_H D} \otimes C \otimes_H D$ is a k -linear combination of $c_{(1)} \otimes_H d_{(1)} \otimes c_{(2)} \otimes_H d_{(2)}h$, with $c \in C$, $d \in D$ and $h \in H$. Note that $\text{ev}(c_{(1)} \otimes_H d_{(1)} \otimes c_{(2)} \otimes_H d_{(2)}h) = \varepsilon_C(c)\varepsilon_D(d)h$. On the other hand

$$\begin{aligned} \text{ev}_D \circ (\widehat{D} \otimes_H \text{ev}_C \otimes_H D)(\tau(c_{(1)} \otimes_H d_{(1)}) \otimes c_{(2)} \otimes_H d_{(2)}h) &= \text{ev}_D(d_{(1)} \otimes_H \text{ev}_C(c_{(1)} \otimes c_{(2)})d_{(2)}h) \\ &= \text{ev}_D(d_{(1)} \otimes_H \varepsilon_C(c)d_{(2)}h) \\ &= \varepsilon_C(c)\varepsilon_D(d)h. \end{aligned}$$

This completes the proof. \square

5.5. Lemma. *Given H -Galois co-objects C and D , write $\Delta_{C,H}^2$ for the projection of Δ_C^2 to $C \otimes_H \widehat{C} \otimes C$ and $\Delta_{D,H}^2$ for the projection of Δ_D^2 to $D \otimes \widehat{D} \otimes_H D$. Consider the H -coring $\mathcal{C} = \widehat{C} \otimes C$ with comultiplication*

$$\widehat{C} \otimes \Delta_{C,H}^2 : \widehat{C} \otimes C \rightarrow \widehat{C} \otimes C \otimes_H \widehat{C} \otimes C,$$

and counit ev_C , and the H -coring $\mathcal{D} = D \otimes \widehat{D}$ with comultiplication

$$\Delta_{D,H}^2 \otimes \widehat{D} : D \otimes \widehat{D} \rightarrow D \otimes \widehat{D} \otimes_H D \otimes \widehat{D},$$

and counit $\widehat{\text{ev}}_D$ (see Corollary 2.8). Then the map $\sigma : \mathcal{C} \otimes_H \mathcal{D} \rightarrow \mathcal{D} \otimes_H \mathcal{C}$, for all $x \in C$, $d \in D$, $h \in H$, given by

$$\sigma(c_{(1)} \otimes c_{(2)} \otimes_H d_{(1)} \otimes d_{(2)}h) = d_{(1)} \otimes d_{(2)} \otimes_H c_{(1)} \otimes c_{(2)}S(h),$$

defines a smash coproduct between \mathcal{C} and \mathcal{D} .

Proof. Note that σ is well defined, since the combined isomorphism

$$C \otimes D \otimes H \cong C \otimes H \otimes_H D \otimes H \xrightarrow{\delta_C \otimes_H \delta_D} C \otimes C \otimes_H D \otimes D,$$

means that any element of $C \otimes_H D$ is a k -linear combination of $c_{(1)} \otimes c_{(2)} \otimes_H d_{(1)} \otimes d_{(2)}h$, for $c \in C$, $d \in D$, $h \in H$. The involutivity of S and the definition of σ immediately imply that σ is a right H -linear map. For the left H -linearity, first note that the repeated application of the antipode and counit axioms yields, for all $c \in C$, $d \in D$, $g, h \in H$

$$\begin{aligned} g &\rightharpoonup c_{(1)} \otimes c_{(2)} \otimes_H d_{(1)} \otimes d_{(2)}h \\ &= c_{(1)}S(g)_{(1)} \otimes c_{(2)}S(g)_{(2)} \otimes_H d_{(1)}S(S(g)_{(3)})_{(1)} \otimes d_{(2)}S(S(g)_{(3)})_{(2)}S(S(g)_{(3)})_{(3)}h. \end{aligned}$$

Applying σ to the above equation, and using the properties of the antipode, including $S^{-1} = S$, we obtain

$$\begin{aligned} &d_{(1)}S(S(g)_{(3)})_{(1)} \otimes d_{(2)}S(S(g)_{(3)})_{(2)} \otimes_H c_{(1)}S(g)_{(1)} \otimes c_{(2)}S(g)_{(2)}S(S(g)_{(3)})_{(3)}S(h) \\ &= d_{(1)}S^2(g_{(1)}) \otimes d_{(2)}S^2(g_{(2)}) \otimes_H c_{(1)}S(g_{(5)}) \otimes c_{(2)}S(g_{(4)})S^2(g_{(3)})S(h) \\ &= d_{(1)}g_{(1)} \otimes d_{(2)}g_{(2)} \otimes_H c_{(1)}S(g_{(4)}) \otimes c_{(2)}\varepsilon_H(g_{(3)})S(h) \\ &= d_{(1)}g \otimes d_{(2)} \otimes_H c_{(1)} \otimes c_{(2)}S(h). \end{aligned}$$

This proves that σ is also left H -linear. To check the left pentagon in (4.1), for any $c_{(1)} \otimes c_{(2)} \otimes_H d_{(1)} \otimes d_{(2)}h \in C \otimes_H D$, compute

$$\begin{aligned} &(\sigma \otimes_H C) \circ (C \otimes_H \sigma)(c_{(1)} \otimes c_{(2)} \otimes_H c_{(3)} \otimes c_{(4)} \otimes_H d_{(1)} \otimes d_{(2)}h) \\ &= d_{(1)} \otimes d_{(2)} \otimes_H c_{(1)} \otimes c_{(2)} \otimes_H c_{(3)} \otimes c_{(4)}S(h). \end{aligned}$$

On the other hand

$$\begin{aligned} &(\mathcal{D} \otimes_H \Delta_C) \circ \sigma(c_{(1)} \otimes c_{(2)} \otimes_H d_{(1)} \otimes d_{(2)}h) \\ &= d_{(1)} \otimes d_{(2)} \otimes_H c_{(1)} \otimes c_{(2)}S(h)_{(1)} \otimes_H c_{(3)}S(h)_{(2)} \otimes c_{(4)}S(h)_{(3)} \\ &= d_{(1)} \otimes d_{(2)} \otimes_H c_{(1)} \otimes c_{(2)} \otimes_H c_{(3)}S(S(h)_{(1)})S(h)_{(2)} \otimes c_{(4)}S(h)_{(3)} \\ &= d_{(1)} \otimes d_{(2)} \otimes_H c_{(1)} \otimes c_{(2)} \otimes_H c_{(3)} \otimes c_{(4)}S(h). \end{aligned}$$

The commutativity of the right pentagon (4.1) is easy. To check the right diagram in (4.2) take again $c_{(1)} \otimes c_{(2)} \otimes_H d_{(1)} \otimes d_{(2)}h \in C \otimes_H D$, and compute

$$\begin{aligned} d_{(1)} \otimes d_{(2)} &\leftarrow \text{ev}_C(c_{(1)} \otimes c_{(2)}S(h)) = d_{(1)} \otimes d_{(2)} \leftarrow \varepsilon_C(c)S(h) \\ &= \varepsilon_C(c)d_{(1)} \otimes d_{(2)}h \\ &= \text{ev}_C(c_{(1)} \otimes c_{(2)})d_{(1)} \otimes d_{(2)}h. \end{aligned}$$

Similarly,

$$\widehat{\text{ev}}_D(d_{(1)} \otimes d_{(2)}) \rightharpoonup c_{(1)} \otimes c_{(2)}S(h) = \varepsilon_D(d)c_{(1)} \otimes c_{(2)}S(h) = c_{(1)} \otimes c_{(2)}\widehat{\text{ev}}_D(d_{(1)} \otimes d_{(2)}h),$$

which expresses the commutativity of the left diagram in (4.2). So we conclude that σ is a smash coproduct map as required. \square

5.6. Theorem. Let H be a commutative Hopf algebra. Consider two Galois co-objects C and D . Then the herd associated to the composed Galois co-object $C \otimes_H D$ coincides with the herd obtained by composing the herd associated to C and the herd associated to D , using the smash coproduct described in Lemma 5.5.

Proof. By Theorem 5.2, the shepherd of $C \otimes_H D$ is given by

$$\Delta_{C \otimes_H D}^2 : C \otimes_H D \rightarrow C \otimes_H D \otimes \widehat{C \otimes_H D} \otimes C \otimes_H D.$$

On the other hand, the shepherd of the composed herd constructed by Theorem 4.2 is given by

$$\gamma_{C \otimes_H D} = (C \otimes_H \sigma \otimes_H D) \circ (\Delta_{C,H}^2 \otimes_H \Delta_{D,H}^2) : C \otimes_H D \rightarrow C \otimes_H D \otimes \widehat{D} \otimes_H \widehat{C} \otimes C \otimes_H D.$$

These two herd structures can be mutually identified by the commutativity of the following diagram

$$\begin{array}{ccc} C \otimes_H D & \xrightarrow{\Delta_{C \otimes_H D}^2} & C \otimes_H D \otimes \widehat{C \otimes_H D} \otimes C \otimes_H D \\ \Delta_{C,H}^2 \otimes_H \Delta_{D,H}^2 \downarrow & & \downarrow C \otimes_H D \otimes \tau \otimes C \otimes_H D \\ C \otimes_H \widehat{C} \otimes C \otimes_H D \otimes \widehat{D} \otimes_H D & \xrightarrow{C \otimes_H \sigma \otimes_H D} & C \otimes_H D \otimes \widehat{D} \otimes_H \widehat{C} \otimes C \otimes_H D. \quad \square \end{array}$$

5.7. Remarks. Dualising the results of this section, it is possible to construct bicomodule coherds out of Galois objects over a cocommutative Hopf algebra H . These Galois objects are known to form a group under the cotensor product. This composition would be then related to a composition of bicomodule coherds by means of smash products.

In [14, Chapter 10], the group of Galois co-objects for a commutative Hopf-algebroid is computed. The results of this section can be extended to this more general framework.

Acknowledgments

We are grateful to Zoran Škoda for inspiring comments. The first author would like to thank Gabriella Böhm for a discussion. The second author would like to thank members of the Mathematics Department at Swansea University for the warm hospitality during his stay in Spring 2008. He also thanks the Fund for Scientific Research – Flanders (Belgium) (F.W.O. – Vlaanderen) for a Postdoctoral Fellowship.

Appendix A. The categorical formulation of (co)herd bi(co)modules

Let \mathfrak{X} and \mathfrak{S} be categories, and take a monad $\mathbf{A} = (A, m_A, \eta_A)$ on \mathfrak{X} (A is an endofunctor: $\mathfrak{X} \rightarrow \mathfrak{X}$, m_A is a multiplication and η_A is a unit) and a monad $\mathbf{B} = (B, m_B, \eta_B)$ on \mathfrak{S} . Let $F : \mathfrak{S} \rightarrow \mathfrak{X}$ be an A - B bialgebra (or bimodule) functor. This means that F comes equipped with natural transformations $\varrho : FB \rightarrow F$ and $\lambda : AF \rightarrow F$ such that

$$\begin{array}{ccc} FBB & \xrightarrow{Fm_B} & FB \\ \varrho B \downarrow & & \downarrow \varrho \\ FB & \xrightarrow{\varrho} & F, \end{array} \quad \begin{array}{ccc} FB & \xrightarrow{\varrho} & F \\ & \nwarrow F\eta_B & \uparrow = \\ & & F, \end{array}$$

$$\begin{array}{ccccc}
 AAF & \xrightarrow{m_A F} & AF & & AF & \xrightarrow{\lambda} & F & & AFB & \xrightarrow{\lambda B} & FB \\
 A\lambda \downarrow & & \downarrow \lambda & & \nwarrow \eta_A F & & \uparrow = & & A_Q \downarrow & & \downarrow \varrho \\
 AF & \xrightarrow{\lambda} & F & , & F & & & & AF & \xrightarrow{\lambda} & F .
 \end{array}$$

Consider a B - A bialgebra functor $\widehat{F} : \mathfrak{A} \rightarrow \mathfrak{S}$ with structure maps $\widehat{\varrho}, \widehat{\lambda}$. Then $F\widehat{F}$ is an A - A bialgebra functor with structure natural transformations $\lambda\widehat{F}$ and $F\widehat{\varrho}$. Similarly $\widehat{F}F$ is a B - B bialgebra functor with structure natural transformations $\widehat{\lambda}F$ and $\widehat{F}\varrho$. With these data \widehat{F} together with bialgebra natural transformations

$$ev : F\widehat{F} \rightarrow A, \quad \widehat{ev} : \widehat{F}F \rightarrow B,$$

such that

$$\begin{array}{ccc}
 F\widehat{F}F & \xrightarrow{F\widehat{ev}} & FB \\
 \downarrow ev F & & \downarrow \varrho \\
 AF & \xrightarrow{\lambda} & F ,
 \end{array}
 \quad
 \begin{array}{ccc}
 \widehat{F}F\widehat{F} & \xrightarrow{\widehat{ev}\widehat{F}} & B\widehat{F} \\
 \downarrow \widehat{F}ev & & \downarrow \widehat{\lambda} \\
 \widehat{F}A & \xrightarrow{\widehat{\varrho}} & \widehat{F}
 \end{array}$$

is called a *formal dual* of F .² An A - B bialgebra F together with a formal dual \widehat{F} is called a *herd functor* if there exists a natural transformation, the *shepherd*,

$$\gamma : F \rightarrow F\widehat{F}F,$$

such that

$$\begin{array}{ccc}
 & F & \\
 F\eta_B \swarrow & & \searrow \eta_A F \\
 FB & & AF \\
 F\widehat{ev} \swarrow & \gamma \downarrow & \searrow ev F \\
 & F\widehat{F}F &
 \end{array}
 ,
 \quad
 \begin{array}{ccc}
 F & \xrightarrow{\gamma} & F\widehat{F}F \\
 \gamma \downarrow & & \downarrow F\widehat{F}\gamma \\
 F\widehat{F}F & \xrightarrow{\gamma\widehat{F}F} & F\widehat{F}F\widehat{F}F .
 \end{array}$$

Dually, one defines a *coherd functor* as a C - D bicoalgebra functor $F : \mathfrak{S} \rightarrow \mathfrak{A}$ of two comonads $C : \mathfrak{A} \rightarrow \mathfrak{A}$ and $D : \mathfrak{S} \rightarrow \mathfrak{S}$ with a companion D - C bialgebra functor $\widehat{F} : \mathfrak{A} \rightarrow \mathfrak{S}$ together with a natural transformation $\chi : F\widehat{F}F \rightarrow F$ satisfying axioms dual to the ones for a herd functor.

Take $\mathfrak{A} = \mathfrak{S} = \mathbf{Set}$, fix a set X and consider the X -representable functor $A = B = F = \widehat{F} = \text{Map}(X, -)$. Since all the functors appearing in this example are representable, by the Yoneda lemma all the natural transformations between them are determined by suitable functions (elements of $\text{Map}(Y, Z)$ for suitable sets Y and Z). For example, the multiplication $m_A : \text{Map}(X, \text{Map}(X, -)) \cong \text{Map}(X \times X, -) \rightarrow \text{Map}(X, -)$ is determined by a mapping $\delta : X \rightarrow X \times X$ so that $m_A = \text{Map}(\delta, -)$.

² In view of the similarity of the axioms of a formally dual functor to that of a Morita context, the sextuple $(A, B, F, \widehat{F}, ev, \widehat{ev})$ can be termed a *pre-Morita context*.

Similarly, the unit is determined by the (only possible) function $X \rightarrow \{*\}$. The only choice for δ that makes $\text{Map}(X, -)$ a monad is the diagonal mapping $\delta : x \mapsto (x, x)$. Set

$$\lambda = \varrho = \hat{\lambda} = \hat{\varrho} = \text{ev} = \widehat{\text{ev}} = \text{Map}(\delta, -).$$

A shepherd γ is determined by a function $\chi : X \times X \times X \rightarrow X$. In terms of the mapping χ , the triangle and square diagrams for γ read, for all $x_i \in X$, $i = 1, \dots, 5$,

$$\begin{aligned}\chi(x_1, x_1, x_2) &= x_2 = \chi(x_2, x_1, x_1), \\ \chi(\chi(x_1, x_2, x_3), x_4, x_5) &= \chi(x_1, x_2, \chi(x_3, x_4, x_5)).\end{aligned}$$

Thus $\text{Map}(X, -)$ is a herd functor on **Set** (with the formal dual and bicoalgebra structures described above) if and only if X is a herd; see [21, p. 170], [2, p. 202, footnote] or [12, Definition 2]. This example justifies the choice of terminology.

Next take $\mathfrak{R} = \mathbf{M}_R$ and $\mathfrak{S} = \mathbf{M}_S$, the categories of right modules over rings R and S respectively, and consider monads $- \otimes_R A$, $- \otimes_S B$, for an R -ring A and an S -ring B . Take F to be the tensor functor $- \otimes_S T$ (for an S - R bimodule T). Then F is a bialgebra over the above monads if and only if T is a B - A bimodule. Furthermore, a functor $\widehat{F} = - \otimes_R \widehat{T}$ is a formal dual of F if and only if \widehat{T} is a formal dual of T . Finally, T is a herd B - A bimodule if and only if $- \otimes_S T$ is a herd functor (with the formal dual, monads, etc. as specified above).

References

- [1] F.W. Anderson, K.R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, Berlin, 1974.
- [2] R. Baer, Zur Einführung des Scharbegriffs, *J. Reine Angew. Math.* 160 (1929) 199–207.
- [3] G. Böhm, T. Brzeziński, Pre-torsors and equivalences, *J. Algebra* 317 (2007) 544–580; *J. Algebra* 319 (2007) 1339–1340, erratum.
- [4] G. Böhm, C. Menini, Pre-torsors and Galois comodules over mixed distributive laws, *Appl. Categ. Structures*, in press, arXiv:0806.1212, 2008.
- [5] G. Böhm, J. Vercruysse, Morita theory for coring extensions and cleft bicomodules, *Adv. Math.* 209 (2007) 611–648.
- [6] G. Böhm, J. Vercruysse, Morita theory of comodules, *Comm. Algebra*, in press, arXiv:0710.1017, 2007.
- [7] T. Brzeziński, The structure of corings. Induction functors, Maschke-type theorem, and Frobenius and Galois-type properties, *Algebr. Represent. Theory* 5 (2002) 389–410.
- [8] T. Brzeziński, Galois comodules, *J. Algebra* 290 (2005) 503–537.
- [9] T. Brzeziński, R. Wisbauer, *Corings and Comodules*, Cambridge University Press, Cambridge, 2003; <http://www-maths.swan.ac.uk/staff/tb/Corings.htm>, erratum.
- [10] S. Caenepeel, Brauer Groups, Hopf Algebras and Galois Theory, *K-Monogr. Math.*, vol. 4, Kluwer Academic, Dordrecht, 1998.
- [11] S. Caenepeel, G. Militaru, Shenglin Zhu, Frobenius and Separable Functors for Generalized Module Categories and Nonlinear Equations, *Lecture Notes in Math.*, Springer-Verlag, Berlin, 2002.
- [12] J. Cartan, The ternary operation $(abc) = ab^{-1}c$ of a group, *Bull. Amer. Math. Soc.* 49 (1943) 869–877.
- [13] L. El Kaoutit, J. Gómez-Torrecillas, Comatrix corings: Galois corings, descent theory, and a structure theorem for cosemisimple corings, *Math. Z.* 244 (2003) 887–906.
- [14] B. Femić, Azumaya corings, braided Hopf–Galois theory and Brauer groups, PhD thesis, Universidad de Almería and Vrije Universiteit Brussel, 2008.
- [15] J. Gómez-Torrecillas, J. Vercruysse, Comatrix corings and Galois comodules over firm rings, *Algebr. Represent. Theory* 10 (2007) 271–306.
- [16] C. Grunspan, Quantum torsors, *J. Pure Appl. Algebra* 184 (2003) 229–255.
- [17] D. Hobst, Antipodes in the theory of noncommutative torsors, PhD thesis, Ludwig-Maximilians Universität München, 2004.
- [18] P.T. Johnstone, The ‘closed subgroup theorem’ for localic herds and pregroupoids, *J. Pure Appl. Algebra* 70 (1991) 97–106.
- [19] A. Kock, Generalized fibre bundles, in: *Categorical Algebra and Its Applications*, Louvain-La-Neuve, 1987, in: *Lecture Notes in Math.*, vol. 1348, Springer-Verlag, Berlin, 1988, pp. 194–207.
- [20] A. Kock, Principal bundles, groupoids, and connections, in: *Geometry and Topology of Manifolds*, in: *Banach Center Publ.*, vol. 76, Polish Acad. Sci., Warsaw, 2007, pp. 185–200.
- [21] H. Prüfer, Theorie der Abelschen Gruppen. I. Grundeigenschaften, *Math. Z.* 20 (1924) 165–187.
- [22] L.H. Rowen, *Ring Theory*, vol. I, Academic Press, Boston, 1988.
- [23] P. Schauenburg, Hopf bi-Galois extensions, *Comm. Algebra* 24 (1996) 3797–3825.
- [24] P. Schauenburg, Quantum torsors with fewer axioms, *math.QA/0302003*, 2003, preprint.
- [25] P. Schauenburg, Hopf–Galois and bi-Galois extensions, in: *Galois Theory, Hopf Algebras, and Semiabelian Categories*, in: *Fields Inst. Commun.*, vol. 43, Amer. Math. Soc., Providence, RI, 2004, pp. 469–515.

- [26] H.-J. Schneider, Representation theory of Hopf–Galois extensions, *Israel J. Math.* 72 (1990) 196–231.
- [27] Z. Škoda, Quantum heaps, cops and heapy categories, *Math. Commun.* 12 (2007) 1–9, arXiv:math.QA/0701749.
- [28] J. Vercruysse, Equivalences between categories of modules and categories of comodules, *Acta Math. Sin. (Engl. Ser.)* 24 (2008) 1655–1674.
- [29] R. Wisbauer, On Galois comodules, *Comm. Algebra* 34 (2006) 2683–2711.