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Endoisomorphisms yield module and character correspondences

Alexandre Turull [1]

Department of Mathematics, University of Florida, Gainesville, FL 32611, USA

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ABSTRACT

In an earlier paper (Turull, 2012 [12]) the author introduced the concept of endoisomorphisms and showed its natural connection with Clifford theory of finite groups. Associated with Clifford theory are module and character correspondences. In the present paper the author shows that each endoisomorphism produces a *unique* module and character correspondence with excellent compatibility results.

1. Introduction

Clifford theory of finite groups is a central topic in representation theory. When we only consider representations over the complex numbers, we classify the possible Clifford theories with the help of a second cohomology group. If two Clifford theories are equivalent, then there exist well-behaved character correspondences between them, but these correspondences are not uniquely determined. When we consider representations over arbitrary (small) fields we can use the Brauer–Clifford group instead of the second cohomology group. In an earlier paper [12], we introduced the concept of *endoisomorphism*, and we showed that two Clifford theories for different groups yield the *same* element of the Brauer–Clifford group if and only if there is an endoisomorphism between modules associated with the Clifford theories.

In the present paper, we show that each endoisomorphism yields a *unique* correspondence among modules defined over many different fields and among their corresponding characters. We prove that

E-mail address: turull@ufl.edu.

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this correspondence has many useful properties. In particular, we will use these in a forthcoming paper to prove a strengthened version of the Alperin–McKay Conjecture for all p-solvable finite groups. Since from [12] we know that the existence of non-trivial endoisomorphisms can be deduced from equalities in the Brauer–Clifford group, the study of the module and character correspondences in this case is particularly useful in the applications. In Corollary 9.9 and Corollary 11.7 below we state some consequences for the correspondences that arise from certain equalities in the Brauer–Clifford group. The rest of the paper, however, mostly discusses the consequences of the existence of endoisomorphisms which may or may not arise in this manner. We show that from each endoisomorphism one can construct two large families of modules over many different fields and groups and a uniquely defined correspondence between them with excellent properties. From the module correspondence, one also gets a corresponding unique character correspondence, and even a corresponding unique Brauer character correspondence in many cases.

The concept of an endoisomorphism is a natural one. An endoisomorphism from a module M_1 to a module M_2 is simply an isomorphism of the $\overline{G}$-algebras obtained from the two modules M_1 and M_2 as $\overline{G}$-algebras of endomorphisms. We give a precise definition of endoisomorphism in Section 2 below. The original definition used in [12] assumes that the modules are defined over fields, and that they are finitely generated. This is sufficient for that paper since these are the only type of modules discussed there. In the present paper, we extend the family of modules discussed in two ways. First, we allow the modules to be defined over commutative rings instead of over fields. Second, we allow the modules to be infinitely generated.

In [12], we saw that endoisomorphisms for finitely generated modules over fields arise naturally from equality of elements of the Brauer–Clifford group, or from equality between families of elements of the Brauer–Clifford group. In Section 11 below, we see that the existence of an endoisomorphism over a field yields the existence of an endoisomorphism over a corresponding ring for finitely generated modules under certain conditions. These conditions are such as to make a consistent transition from modules in characteristic zero to modules in positive characteristic. We also show in the paper how endoisomorphisms of finitely generated modules naturally give rise to endoisomorphisms of certain infinitely generated modules, and use this to set up our module correspondence.

The main goal of the present paper is to show that fixing an endoisomorphism for two modules yields a *unique* module and character *correspondence* with excellent properties. The uniqueness of the correspondence for modules is, to a certain extent, up to module isomorphism. The correspondence at the character level is unique. We prove that each endoisomorphism produces a coherent family of module correspondences for a multitude of module categories.

In Section 4, we describe a natural correspondence between the sections of a given modules and certain objects obtained from the $\overline{G}$-algebra of endomorphisms. In order to be able to describe all finitely generated modules using these sections, it is natural to extend the original finitely generated module by taking the direct sum of a countably infinite set of copies of the original module. We are interested only in the finitely generated submodules of this direct sum. It is convenient to consider this direct sum as a topological module and to redefine the endomorphism $\overline{G}$-algebra for the direct sum as consisting of those endomorphisms whose kernel is an open set of the topology. This makes our results flow smoothly.

In Section 2, we define endoisomorphisms in our most general context, namely for topological modules over finite group algebras over commutative rings. In Section 3, we see that under our definitions the finitely generated submodules are in one-to-one correspondence with the finitely generated right ideals of the $\overline{G}$-algebra of endomorphisms. This then allows us to describe the sections of the module in Section 4. We also show that this naturally makes each endoisomorphism induce a unique correspondence of modules with good properties. This unique correspondence on sections of two related modules which only depends on an endoisomorphism ϵ is denoted κ_ϵ^0 below. We then show how one can naturally extend the original modules to their *completions* and then generate two much bigger families of modules. The completions are generally designed to handle all homomorphic images of finite direct sums of the original modules. One can also extend the original endoisomorphism in a uniquely prescribed way to a new one for the extended modules. From this extended endoisomorphism we obtain the unique correspondence, denoted κ_ϵ below. The correspondence κ_ϵ can be applied to a large collection of modules over subgroups over various fields in different characteristics.

We also prove a number of excellent compatibility properties for κ_ϵ including about reduction modulo a prime p, and even p-blocks.

Our results make unique, greatly generalize, and prove additional properties for the correspondences studied in [8,9]. These earlier results were used in [10] to prove a strengthening [7] of the McKay Conjecture for all p-solvable finite groups. Some of the additional properties that we prove in the present paper have to do with reduction modulo p and p-blocks. In a forthcoming paper, we will use the results of the present paper to prove a strengthening [7] of the Alperin–McKay Conjecture. We also plan to explore in forthcoming papers how the results of the present paper yield simplified and strengthened results for the character and module theory of solvable and p-solvable groups.

We use the notation and conventions of [11]. Note that we systematically write all functions on the left, and compose them from right to left. This allows us, in particular, to compose characters with elements of Galois groups. We also use left exponential notation (i.e. ${}^g a$ for the action of a group element g on an algebra element a). We also note that $\mathbf{N}$ means the set of natural numbers that is the set of cardinalities of the finite sets so that $\mathbf{N} = \{0, 1, 2, \ldots\}$, i.e. 0 is a natural number.

2. Endomorphism algebras and endoisomorphisms

While endoisomorphisms were defined in [12] in the context of finite dimensional modules defined over fields, it is convenient to extend the definition to a wider class of modules. We now extend our modules to include infinitely generated ones defined over commutative rings. Furthermore, it is also convenient for our purposes to allow our modules to be topological. We start with the definition of the endomorphism algebra of one such module.

Definition 2.1. Let G and $\overline{G}$ be finite groups, and suppose we are given a surjective homomorphism $\pi : G \to \overline{G}$ whose kernel is H. Let R be a commutative ring with identity, and let RG be the group ring, viewed with the discrete topology. Let M be a topological RG-module, that is a topological space and a unital RG-module where the operations are continuous. Then we let

$$\underline{\mathrm{End}}(M) = \left\{\phi \in \mathrm{End}_{RH}(M)\colon\ \ker(\phi) \text{ is open}\right\}.$$

In view of Proposition 2.2, we will call $\underline{\mathrm{End}}(M)$ the *endomorphism $\overline{G}$-algebra* of M, and, when we say that some map $f : M \to M$ is an *endomorphism* of M we will mean $f \in \underline{\mathrm{End}}(M)$ unless we explicitly say otherwise.

$\underline{\mathrm{End}}(M)$ has an algebraic structure, as we see in the next proposition. However, we will not assign to $\underline{\mathrm{End}}(M)$ any topological structure.

Proposition 2.2. *Assume the hypotheses of Definition 2.1. Then $\underline{\mathrm{End}}(M)$ is a $\overline{G}$-algebra over R. Furthermore, if M is discrete then $\underline{\mathrm{End}}(M) = \mathrm{End}_{RH}(M)$.*

Proof. In the topological abelian group M, a subgroup is open if and only if it contains a non-empty open subset. It follows that $\underline{\mathrm{End}}(M)$ is closed under addition, subtraction, scalar multiplication and composition. Furthermore, multiplication by any element of G induces on M a homeomorphism, so that $\underline{\mathrm{End}}(M)$ is closed under conjugation by G. Since obviously H acts trivially by conjugation on $\underline{\mathrm{End}}(M)$, we have that $\underline{\mathrm{End}}(M)$ is a $\overline{G}$-algebra over R, as desired. The second statement follows immediately from the definitions. $\square$

Remark 2.3. $\underline{\mathrm{End}}(M)$ need not be finitely generated over R, and $\underline{\mathrm{End}}(M)$ need not have an identity.

Remark 2.4. If a module M is not described as being topological, we will assume that it has the discrete topology, and, in particular, $\underline{\mathrm{End}}(M) = \mathrm{End}_{RH}(M)$.

For convenience, we label the following hypotheses.

Hypotheses 2.5. Let G, G' and $\overline{G}$ be finite groups, and suppose we are given surjective homomorphisms $\pi : G \to \overline{G}$ and $\pi' : G' \to \overline{G}$ whose kernels are, respectively, H and H'.

The key to an *isomorphism* between two Clifford theories is the definition below of an *endoisomorphism*. In [12], we saw that Clifford theories that yield the same element of the Brauer–Clifford group yield endoisomorphisms. The type of module considered in [12] is finite dimensional modules over fields, but we find it convenient to extend our allowed modules here.

Definition 2.6. Assume Hypotheses 2.5, and let R be a commutative ring with identity. Let M be a topological RG-module, and let M' be a topological RG'-module. An *endoisomorphism* over R from M to M' is a map

$$\epsilon : \underline{\mathrm{End}}(M) \to \underline{\mathrm{End}}\bigl(M'\bigr)$$

which is an isomorphism of $\overline{G}$-algebras over R. We will write

$$\epsilon : M \rightsquigarrow M'$$

to mean that ϵ is an endoisomorphism from M to M'.

3. Endomorphisms for large direct sums of modules

We are mainly interested in correspondences of finitely generated modules. In order to study these, it is convenient to define larger (non-finitely generated) modules, and to consider their finitely generated submodules. It is convenient to use topological language for these modules.

Hypotheses 3.1. Let G and $\overline{G}$ be finite groups, and suppose we are given a surjective homomorphism $\pi : G \to \overline{G}$ whose kernel is H. Suppose F is a field. We view F and FG as discrete topological spaces. Let C be a (possibly infinite) set of finitely generated FG-modules N such that $\mathrm{Res}^G_H(N)$ is completely reducible. We set M to be the direct sum of all elements of C. We consider

$$\mathcal{TB}(C) = \left\{x + \sum_{N \in D} N\colon\ x \in M \text{ and } D \subseteq C \text{ and } |C \backslash D| < +\infty\right\}.$$

We consider M as a topological space with basis of open sets given by $\mathcal{TB}(C)$.

Then M is a topological FG-module. When C is finite M is discrete, but M need not be discrete in general. In general, we can think of the topology on M to be that induced on the direct sum M by the product topology of the discrete topological modules in C.

Lemma 3.2. *Assume Hypotheses 3.1, and let V be an F-subspace of M. Let*

$$D = \{N \in C\colon\ N \subseteq V\}.$$

Then V is open if and only if $|C \backslash D| < +\infty$.

Proof. Suppose first that V is open. Then, since $0 \in V$, there exists some $x \in M$, and some $E \subseteq C$ such that $|C \backslash E| < +\infty$, and

$$0 \in x + \sum_{N \in E} N \subseteq V.$$

Hence, $E \subseteq D$, and the implication in the forward direction holds. Next suppose that $|C \backslash D| < +\infty$. Then, for each $x \in V$, we have that

$$x + \sum_{N \in D} N \subseteq V,$$

so that V is a union of elements of $\mathcal{TB}(C)$, and so it is open. □

Set $A = \underline{\mathrm{End}}(M)$. It follows from Proposition 2.2 that A is a $\overline{G}$-algebra over F, possibly of infinite dimension and possibly without identity. Furthermore, for each $a \in A$, we have that a is continuous and $a(M)$ has finite dimension over F. Finally, note that we will not need to assign a topology to A.

We are primarily interested in the finite dimensional submodules of M. Let V be any finite dimensional subspace of M. Then for every $m \in M$ we let $\langle V, m\rangle$ be the subspace generated by V and m, and there is an open FG-submodule O of M such that $O \cap \langle V, m\rangle = \{0\}$. If $m \notin V$, we have that $m + O$ is open and disjoint from V, so that V is closed. If $m \in V$, then $(m + O) \cap V = \{m\}$, so that V is discrete under the relative topology from M. For this reason, while the topology of M is important, the topology of the subspaces V themselves does not play an important role in our development.

Remark 3.3. Assume Hypotheses 3.1, and that C is finite. Then M has the discrete topology and we have $\underline{\mathrm{End}}(M) = \mathrm{End}_{FH}(M)$.

The algebra $\underline{\mathrm{End}}(M)$ can be described directly from ordinary full endomorphism algebras via suitable direct sums.

Proposition 3.4. *Assume Hypotheses 3.1, and set $A = \underline{\mathrm{End}}(M)$. Then, for each $N_1, N_2 \in C$, we may identify naturally $\mathrm{Hom}_{FH}(N_1, N_2)$ with a vector F-subspace of A. In this way A may be viewed as a $\overline{G}$-algebra over F as the direct sum of all the $\mathrm{Hom}_{FH}(N_1, N_2)$ as $N_1, N_2 \in C$ with the product induced by composition.*

Proof. We may view the elements of $\mathrm{Hom}_{FH}(N_1, N_2)$ as FH-module homomorphism $M \to M$ whose kernel contains the direct sum of all the elements of C except N_1 and whose image is in N_2. This identification preserves the F-vector space structure, as well as the $\overline{G}$-action. In addition, the product is induced from composition under these identifications. Let S be the sum of all the $\mathrm{Hom}_{FH}(N_1, N_2)$ for $N_1, N_2 \in C$. Then this sum is direct. Let $\phi \in A$. Then, since $\ker(\phi)$ is open, there is some finite subset $C_1 \subseteq C$ such that $\ker(\phi)$ contains all the elements of C not in C_1. It follows that the image of ϕ is finite dimensional, so that there exists a finite subset $C_2 \subseteq C$ such that $\phi(M) \subseteq \sum_{N \in C_2} N$. Then ϕ is in the sum of the $\mathrm{Hom}_{FH}(N_1, N_2)$ for $N_1 \in C_1$ and $N_2 \in C_2$. It follows that $\phi \in S$. Hence, $S = A$, and the proposition holds. □

Proposition 3.5. *Assume Hypotheses 3.1, and set $A = \underline{\mathrm{End}}(M)$. Then I is a finitely generated right ideal of A if and only if there exists some idempotent $e \in A$ such that $I = eA$.*

Proof. If there exists some idempotent $e \in A$ such that $I = eA$, then I is a finitely generated (in fact generated by e) right ideal of A. Hence, we now suppose that I is any finitely generated right ideal of A. We first show that if $a \in I$, $b \in A$ and $b(M) \subseteq a(M)$ then $b \in I$. We set $N = a(M)$ and $N_1 = b(M)$ so that $N_1 \subseteq N$. We set $K = \ker(a)$ and $K_1 = \ker(b)$. Since $\mathrm{Res}^G_H(M)$ is completely reducible, there exist FH-submodules $\tilde{N}$ and $\tilde{N}_1$ such that $M = \tilde{N} \oplus K$ and $M = \tilde{N}_1 \oplus K_1$. We restrict a to a map $\alpha : \tilde{N} \to N$ and we restrict b to a map $\beta : \tilde{N}_1 \to N_1$. Both α and β are FH-module isomorphisms. We define $c \in A$ by $c(n + k) = \alpha^{-1}(\beta(n))$ for all $n \in \tilde{N}_1$ and $k \in K_1$. Then $ac = b$ and it follows, since I is a right ideal, that $b \in I$, as desired.

Now let

$$\mathcal{C} = \{a(M) \colon a \in I\}.$$

Let $a_1, \dots, a_n$ be a finite set of generators of I. Then, if $N \in \mathcal{C}$, we have

$$\dim_F(N) \leqslant \sum_{i=1}^{n} \dim_F\bigl(a_i(M)\bigr).$$

It follows that $\mathcal{C}$ has a maximal element N. Suppose $N' \in \mathcal{C}$. Then since $\mathrm{Res}_H^G(M)$ is completely reducible, we can find an FH-submodule K of N' such that

$$N + N' = N \oplus K.$$

Note that these subspaces are all finite dimensional over F. Hence, there exists a subset $C_0 \subseteq C$ consisting of almost all the elements of C such that the direct sum D of all the elements of C_0 is such that $D \cap (N + N') = \{0\}$. Then, by the above there exist $a, b \in I$ such that a is the identity on N and contains $K + D$ in its kernel, and b is the identity on K and contains $N + D$ in its kernel. Hence, $a + b \in I$, and $(a + b)(M) \supseteq N + N'$. Since N is maximal, this implies that $N' \subseteq N$, so that N is the maximum of $\mathcal{C}$. Since $\mathrm{Res}_H^G(M)$ is completely reducible, there exists some FH-submodule R of M such that R contains almost all the elements of C and

$$M = N \oplus R.$$

The linear map $e : M \to M$ which is the identity on N and zero on R is an idempotent $e \in A$. Since $e(M) = N \in \mathcal{C}$, it follows from the above that $e \in I$. Therefore $eA \subseteq I$. Since N is the maximum of $\mathcal{C}$, for all $i \in I$ we have $ei = i$, so that $I \subseteq eA$. Hence, $I = eA$, as desired. □

The following definition is standard in this context.

Definition 3.6. Suppose $I \subseteq A$ and $N \subseteq M$. Then we denote by $I(N)$ the additive subgroup of M generated by all the elements of the form $i(n)$ for $i \in I$ and $n \in N$.

Theorem 3.7. *Assume Hypotheses 3.1, and set $A = \underline{\mathrm{End}}(M)$. Then the correspondence*

$$I \mapsto I(M)$$

is a one-to-one correspondence from the set of finitely generated right ideals of A to the set of finitely generated FH-submodules of M. Furthermore, the set of finitely generated right ideals of A and the set of finitely generated FH-submodules of M are ordered by inclusion and the correspondence is an isomorphism of partially ordered sets. Finally, if I is a finitely generated right ideal of A and $g \in G$, then

$$gI(M) = \bigl({}^{\pi(g)}I\bigr)(M),$$

in other words, multiplication by g of the FH-submodule corresponding to the right ideal I yields the FH-submodule corresponding to ${}^{\pi(g)}I$.

Proof. Let $\mathcal{I}$ be the set of finitely generated right ideals of A, and let $\mathcal{M}$ be the set of finitely generated FH-submodules of M. Let $I \in \mathcal{I}$. Then, for each $i \in I$, $i(M)$ is a finitely generated FH-submodule of M. Since I is finitely generated as a right ideal of A, it follows that $I(M)$ is a finitely generated FH-submodule of M, and $I(M) \in \mathcal{M}$. Therefore $I \mapsto I(M)$ is well defined. We define $f_1 : \mathcal{I} \to \mathcal{M}$ by $f_1(I) = I(M)$ for all $I \in \mathcal{I}$. Conversely, let $N \in \mathcal{M}$. Then we set

$$I_N = \bigl\{a \in A\colon a(M) \subseteq N\bigr\}.$$

There is an FH-submodule K of M such that K contains almost all the elements of C and $M = N \oplus K$. Let $e : M \to M$ be the linear map such that it is the identity on N and zero on K. Then $e \in A$ is an idempotent and the finitely generated right ideal $I_N = eA$ of A is such that $I_N(M) = eA(M) = N$. Therefore $I_N \in \mathcal{I}$ and the correspondence $N \mapsto I_N$ is well defined. We define $f_2 : \mathcal{M} \to \mathcal{I}$ by $f_2(N) = I_N$ for all $N \in \mathcal{M}$. Then f_2 is well defined and $f_1 f_2$ is the identity on $\mathcal{M}$. Let $I \in \mathcal{I}$. By Proposition 3.5, there exists some idempotent $e \in A$ such that $I = eA$. Then $f_1(I) = e(M)$, and $f_2 f_1(I) = eA = I$, so that $f_2 f_1$ is the identity on $\mathcal{I}$. Since it follows from their definition that both f_1 and f_2 preserve inclusion, we have that f_1 and f_2 are isomorphisms of partially ordered sets.

Suppose now that I is a finitely generated right ideal of A and $N = I(M)$, and $g \in G$. Then

$$gN = gIg^{-1}(M) = {}^{\pi(g)}I(M)$$

since the image of multiplication of M by g^{-1} is M itself, and by definition of the action of $\overline{G}$ on A. Hence, the lemma holds. $\square$

Corollary 3.8. *Assume Hypotheses 3.1, and set $A = \underline{\mathrm{End}}(M)$. Let S be a subgroup of G which contains H, and let $\overline{S} = \pi(S)$. Then the finitely generated FS-submodules of M are, under the correspondence of Theorem 3.7, in one-to-one correspondence with the $\overline{S}$-invariant finitely generated right ideals of A.*

Proof. This follows immediately from Theorem 3.7. $\square$

4. Ideal triples and sections of modules

The results of the previous section suggest that certain categories of modules may be described from the endomorphism algebras. More precisely, assume for a moment Hypotheses 3.1, and let S be a subgroup of G which contains H. The category of FS-modules which are quotients of finitely generated FS-submodules of M (sections of M for short) can be described purely in terms of the $\overline{G}$-algebra $\underline{\mathrm{End}}(M)$. In this section, we make this idea precise. We start by defining a category of ideal triples, and we show that it is isomorphic to the category of sections of M under appropriate conditions.

Definition 4.1. Let G be a finite group, let F be a field, and let A be a G-algebra over F. (We do not assume that A has finite dimension or that A has an identity.) We define the *category of ideal triples* of A to be the category $\mathcal{IT}(A)$ defined as follows.

(1) The objects of $\mathcal{IT}(A)$ are the triples (I, J, S), where S is a subgroup of G, I and J are S-invariant right ideals of A such that $I \supseteq J$, and there exist idempotents e_I and e_J in A such that $I = e_I A$ and $J = e_J A$.
(2) Let $O_1 = (I_1, J_1, S_1)$ and $O_2 = (I_2, J_2, S_2)$ be objects of $\mathcal{IT}(A)$. Set

$$\Lambda(O_1, O_2) = \{a \in A\colon aI_1 \subseteq I_2 \text{ and } aJ_1 \subseteq J_2\},$$

$$\Xi(O_1, O_2) = \{a \in A\colon aI_1 \subseteq J_2\},$$

$$\mathrm{Hom}_{\mathcal{IT}(A)}(O_1, O_2) = \begin{cases} \emptyset & \text{if } S_1 \neq S_2; \\ (\Lambda(O_1, O_2)/\Xi(O_1, O_2))^S & \text{if } S_1 = S_2 = S. \end{cases}$$

(3) Let $O = (I, J, S)$ be an object of $\mathcal{IT}(A)$. Then we set $1_O = e_I + \Xi(O, O) \in \mathrm{Hom}_{\mathcal{IT}(A)}(O, O)$.
(4) Let $O_1 = (I_1, J_1, S_1)$, $O_2 = (I_2, J_2, S_2)$, and $O_3 = (I_3, J_3, S_3)$ be objects of $\mathcal{IT}(A)$, and let $a \in \Lambda(O_1, O_2)$ and $b \in \Lambda(O_2, O_3)$. Suppose that

$$a + \Xi(O_1, O_2) \in \mathrm{Hom}_{\mathcal{IT}(A)}(O_1, O_2)$$

and

$$b + \Xi(O_2, O_3) \in \mathrm{Hom}_{\mathcal{IT}(A)}(O_2, O_3).$$

Then,

$$ba + \Xi(O_1, O_3) \in \mathrm{Hom}_{\mathcal{IT}(A)}(O_1, O_3),$$

and we set

$$\big(b + \Xi(O_2, O_3)\big)\big(a + \Xi(O_1, O_2)\big) = ba + \Xi(O_1, O_3).$$

Theorem 4.2. *Definition 4.1 does define uniquely a (usually disconnected) F-linear small category $\mathcal{IT}(A)$.*

Proof. It is clear that the objects of $\mathcal{IT}(A)$ form a set. We note that for an object $O = (I, J, S)$ of $\mathcal{IT}(A)$ the idempotents e_I and e_J are assumed to exist, but they may not be uniquely determined by the object O. Assume the hypotheses of (2). Then, it is direct to check that $\Lambda(O_1, O_2)$ and $\Xi(O_1, O_2)$ are vector spaces over F, and, furthermore, $\Xi(O_1, O_2) \subseteq \Lambda(O_1, O_2)$. Hence, $\Lambda(O_1, O_2)/\Xi(O_1, O_2)$ is a vector space over F. Assume that $S_1 = S_2 = S$. Then, since I_1, I_2, J_1 and J_2 are all S-invariant, S acts on both $\Lambda(O_1, O_2)$ and $\Xi(O_1, O_2)$, and so S acts on $\Lambda(O_1, O_2)/\Xi(O_1, O_2)$ and, $\mathrm{Hom}_{\mathcal{IT}(A)}(O_1, O_2)$ is the set of S-fixed points of this action. In particular, in all cases, $\mathrm{Hom}_{\mathcal{IT}(A)}(O_1, O_2)$ is an F-vector space or empty.

Suppose now the hypotheses of (4). Then, by definition, we have $S_1 = S_2 = S_3 = S$. It is direct to show that

$$\Lambda(O_2, O_3)\Lambda(O_1, O_2) \subseteq \Lambda(O_1, O_3);$$

$$\Lambda(O_2, O_3)\Xi(O_1, O_2) \subseteq \Xi(O_1, O_3);$$

$$\Xi(O_2, O_3)\Lambda(O_1, O_2) \subseteq \Xi(O_1, O_3).$$

It then follows that the class of ba in $\Lambda(O_1, O_3)/\Xi(O_1, O_3)$ is uniquely determined by the class of a in $\Lambda(O_1, O_2)/\Xi(O_1, O_2)$ and the class of b in $\Lambda(O_2, O_3)/\Xi(O_2, O_3)$. Since the class of a in $\Lambda(O_1, O_2)/\Xi(O_1, O_2)$ and the class of b in $\Lambda(O_2, O_3)/\Xi(O_2, O_3)$ are both invariant under the action of S, it follows that

$$ba + \Xi(O_1, O_3) \in \big(\Lambda(O_1, O_3)/\Xi(O_1, O_3)\big)^S.$$

Hence, the product is well defined. Furthermore, it is bilinear, and associative. Let e be an idempotent of A such that $I_2 = eA$. Then, for all $c \in I_2$, we have $ec = c$, and it follows that $e \in \Lambda(O_2, O_2)$. Furthermore, $be - b \in \Xi(O_2, O_3)$, and $ea - a \in \Xi(O_1, O_2)$. In particular, $e + \Xi(O_2, O_2)$ is an identity of the set $\Lambda(O_2, O_2)/\Xi(O_2, O_2)$ under product. Since the identity is unique, it follows that it is uniquely determined by $\Lambda(O_2, O_2)/\Xi(O_2, O_2)$, and therefore it is fixed under the action of S_2. Hence, the identity of each object of $\mathcal{IT}(A)$ is uniquely defined, and satisfies the conditions of being an identity for the category. Hence, $\mathcal{IT}(A)$ is an F-linear small category, as claimed. □

The abstract category $\mathcal{IT}(A)$ depends only on the G-algebra A. When $A = \underline{\mathrm{End}}(M)$ for an appropriate module M, $\mathcal{IT}(A)$ is isomorphic to a category of sections of M. We now proceed to describe this category of sections. We first set up some notation. In view of our later applications, we work with ambient modules which are possibly infinitely generated.

Definition 4.3. Assume Hypotheses 3.1. Let $\bar{S}$ be a subgroup of $\overline{G}$, and let $S = \pi^{-1}(\bar{S})$. We define the *category of sections of* M over FS, $\mathcal{S}(M, FS)$, to be the following small category. Its objects are all the FS-modules which are quotients of a finitely generated FS-submodule of M by one of its FS-submodules. Its morphisms are all the FS-module homomorphisms. We define the *category of sections of* M, $\mathcal{S}(M)$, to be the disjoint union of the categories $\mathcal{S}(M, FS)$ as $\bar{S}$ runs through all the subgroups of $\overline{G}$.

Then $\mathcal{S}(M, FS)$ and $\mathcal{S}(M)$ are small F-linear categories.

Theorem 4.4. *Assume Hypotheses 3.1, and set $A = \underline{\mathrm{End}}(M)$. Then the categories $\mathcal{IT}(A)$ and $\mathcal{S}(M)$ are isomorphic under an isomorphism given as follows. For $O = (I, J, \bar{S})$ any object in $\mathcal{IT}(A)$, we set $S = \pi^{-1}(\bar{S})$, and we assign to O the FS-module $I(M)/J(M)$. For $O_1 = (I_1, J_1, \overline{S_1})$ and $O_2 = (I_2, J_2, \overline{S_2})$ objects in $\mathcal{IT}(A)$, and $a \in \Lambda(O_1, O_2)$ such that*

$$a + \Xi(O_1, O_2) \in \mathrm{Hom}_{\mathcal{IT}(A)}(O_1, O_2),$$

we assign the map $f_a : I_1(M)/J_1(M) \to I_2(M)/J_2(M)$ given by

$$f_a\big(n + J_1(M)\big) = a(n) + J_2(M)$$

for all $n \in I_1(M)$.

Proof. Let $O = (I, J, \bar{S})$ be any object in $\mathcal{IT}(A)$, and set $S = \pi^{-1}(\bar{S})$. Since $I \supseteq J$ are $\bar{S}$-stable, by Corollary 3.8, $I(M)$ and $J(M)$ are FS-submodules of M, and, $I(M) \supseteq J(M)$. Hence, $I(M)/J(M)$ can be viewed as an FS-module. Hence, the correspondence on objects is well defined. Furthermore, again by Corollary 3.8, we have a bijection from the objects of $\mathcal{IT}(A)$ to the objects of $\mathcal{S}(M)$.

Suppose now that $O_1 = (I_1, J_1, S_1)$ and $O_2 = (I_2, J_2, S_2)$ are objects in $\mathcal{IT}(A)$. We get the related objects $\tilde{O}_1 = (I_1, J_1, \bar{1})$ and $\tilde{O}_2 = (I_2, J_2, \bar{1})$ in $\mathcal{IT}(A)$. For each $a \in \Lambda(\tilde{O}_1, \tilde{O}_2)$, since $aI_1 \subseteq I_2$ and $aJ_1 \subseteq J_2$, the map f_a is well defined. Hence, we have a well-defined linear map

$$\mathcal{F} : \Lambda(\tilde{O}_1, \tilde{O}_2) \to \mathrm{Hom}_{FH}\big(I_1(M)/J_1(M), I_2(M)/J_2(M)\big)$$

by setting $\mathcal{F}(a) = f_a$ for all $a \in \Lambda(\tilde{O}_1, \tilde{O}_2)$. It follows from the definitions that the kernel of $\mathcal{F}$ is $\Xi(\tilde{O}_1, \tilde{O}_2)$. Let

$$\lambda \in \mathrm{Hom}_{FH}\big(I_1(M)/J_1(M), I_2(M)/J_2(M)\big).$$

Since M is completely reducible as an FH-module, we can find FH-submodules K_1, and K_2 of M such that K_1 is open, $M = K_1 \oplus I_1(M)$, and $I_2(M) = K_2 \oplus J_2(M)$. Then K_2 is isomorphic to $I_2(M)/J_2(M)$, and so it is of finite dimension over F. We can construct a map $\tilde{\lambda} : M \to M$, by saying that $\tilde{\lambda}$ is F-linear, it contains K_1 in its kernel, and on $I_1(M)$ it is just the projection $I_1(M) \to I_1(M)/J_1(M)$ followed by λ followed by the isomorphism $I_2(M)/J_2(M) \to K_2$ followed by the inclusion of K_2 into M. Now $\tilde{\lambda}$ is an FH-module homomorphism and its kernel is open, so that $\tilde{\lambda} \in A$. It follows from the definitions that $\tilde{\lambda} \in \Lambda(\tilde{O}_1, \tilde{O}_2)$ and $\mathcal{F}(\tilde{\lambda}) = \lambda$. Hence, $\mathcal{F}$ is surjective, and it provides an isomorphism

$$\overline{\mathcal{F}} : \Lambda(\tilde{O}_1, \tilde{O}_2)/\Xi(\tilde{O}_1, \tilde{O}_2) \to \mathrm{Hom}_{FH}\big(I_1(M)/J_1(M), I_2(M)/J_2(M)\big).$$

If $\overline{S_1} \neq \overline{S_2}$, then $\mathrm{Hom}_{\mathcal{IT}(A)}(O_1, O_2) = \emptyset$, and similarly there are no morphisms between the corresponding modules. Hence, we assume $\overline{S_1} = \overline{S_2} = \bar{S}$ and let $S = \pi^{-1}(\bar{S})$. Now I_1, J_1, I_2, and J_2 are all

S-invariant, and S acts via π on the domain of $\overline{\mathcal{F}}$, and S acts on the range of $\overline{\mathcal{F}}$, and a calculation shows that $\overline{\mathcal{F}}$ preserves this action. Hence, $\overline{\mathcal{F}}$ restricts to an isomorphism

$$\bigl(\Lambda(O_1, O_2)/\Xi(O_1, O_2)\bigr)^{\bar{S}} \to \mathrm{Hom}_{FS}\bigl(I_1(M)/J_1(M), I_2(M)/J_2(M)\bigr).$$

This proves that for all pairs of objects we have a linear isomorphism between the corresponding sets of homomorphism. A straightforward calculation shows that the defined correspondence of homomorphisms is compatible with composition, and this completes the proof of the theorem. $\square$

Proposition 4.5. *Assume the hypotheses and notation of Theorem 4.4. Then, the isomorphism of categories given in it from $\mathcal{IT}(A)$ to $\mathcal{S}(M)$ has the following properties.*

(1) *Let $\overline{S_1} \supseteq \overline{S_2}$ be subgroups of $\overline{G}$. The map*

$$(I, J, \overline{S_1}) \mapsto (I, J, \overline{S_2})$$

of objects of $\mathcal{IT}(A)$ corresponds in $\mathcal{S}(M)$ to restriction of modules from an FS_1-module to an FS_2-module, where $S_1 = \pi^{-1}(\overline{S_1})$ and $S_2 = \pi^{-1}(\overline{S_2})$.

(2) *Let $g \in G$. Let $(I, J, \overline{S})$ be an object of $\mathcal{IT}(A)$, and let N be the corresponding FS-module in $\mathcal{S}(M)$. Then*

$$\bigl({}^{\pi(g)}I, {}^{\pi(g)}J, {}^{\pi(g)}\overline{S}\bigr)$$

is an object of $\mathcal{IT}(A)$ and the module that corresponds to it is isomorphic to ${}^gN$ as $F{}^gS$-modules.

(3) *Let $O = (I, J, \overline{S})$ be an object of $\mathcal{IT}(A)$, and let N be the corresponding FS-module in $\mathcal{S}(M)$. Then,*

$$\mathrm{End}_{FS}(N) \simeq \mathrm{End}_{\mathcal{IT}(A)}(O)$$

as algebras over F.

(4) *Assume the hypotheses of (3). Assume that N is irreducible and let ϕ be the character afforded by some absolutely irreducible submodule of $\overline{F} \otimes_F N$ (where $\overline{F}$ is an algebraic closure of F and we assume that characters are functions $S \to \overline{F}$). Then,*

$$F(\phi) \simeq \mathrm{Z}\bigl(\mathrm{End}_{\mathcal{IT}(A)}(O)\bigr)$$

as algebras over F.

(5) *Assume the hypotheses and notation of (4). Then, the Schur index of ϕ over F is the Schur index of $\mathrm{End}_{\mathcal{IT}(A)}(O)$.*

(6) *Suppose that we are given a group Γ of field automorphisms of F, and a continuous action of Γ on M in such a way that Γ preserves the addition on M, for all $\gamma \in \Gamma$, $\lambda \in F$, and $v \in M$ we have $\gamma(\lambda v) = \gamma(\lambda)\gamma(v)$, and the action of each $\gamma \in \Gamma$ commutes with the action of each element of G on M. Then, for each $\gamma \in \Gamma$, conjugation determines an automorphism γ of the $\overline{G}$-ring A. Furthermore, let $(I, J, \overline{S})$ be an object of $\mathcal{IT}(A)$, and let N be the corresponding FS-module in $\mathcal{S}(M)$. Then*

$$\bigl(\gamma(I), \gamma(J), \overline{S}\bigr)$$

is an object of $\mathcal{IT}(A)$ and the module that corresponds to it is isomorphic to the γ-twist of N as an FS-module.

(7) *Assume the hypotheses of* (3)*. Assume, furthermore, that, for some algebraic closure $\overline{F}$, every irreducible $\overline{F}H$-submodule of $\overline{F} \otimes_F M$ has the same dimension d over F. Let $O' = (I, J, \overline{1})$ be the object of $\mathcal{IT}(A)$ obtained from O by replacing $\overline{S}$ by the trivial subgroup. Then $\overline{F} \otimes_F \operatorname{End}_{\mathcal{IT}(A)}(O')$ can be written uniquely as*

$$\overline{F} \otimes_F \operatorname{End}_{\mathcal{IT}(A)}\bigl(O'\bigr) = A_1 \oplus \cdots \oplus A_n$$

where each A_i is a central simple algebra over $\overline{F}$. Furthermore,

$$\dim_F(N) = d \sum_{i=1}^{n} \sqrt{\dim_{\overline{F}}(A_i)}.$$

Proof. (1) It follows from the definition of $\mathcal{IT}(A)$ that if $O_1 = (I, J, \overline{S_1})$ is an object of $\mathcal{IT}(A)$, then $O_2 = (I, J, \overline{S_2})$ is an object of $\mathcal{IT}(A)$. Furthermore, with $S_1 = \pi^{-1}(\overline{S_1})$ and $S_2 = \pi^{-1}(\overline{S_2})$, the module that corresponds to O_1 is $I(M)/J(M)$ viewed as an FS_1-module and the module that corresponds to O_2 is that same underlying abelian group $I(M)/J(M)$ viewed as an FS_2-module.

(2) Let $O = (I, J, \overline{S})$, and set

$$O' = \bigl({}^{\pi(g)}I, {}^{\pi(g)}J, {}^{\pi(g)}\overline{S}\bigr).$$

It follows from the definition of $\mathcal{IT}(A)$ that O' is an object of it. Now the FS-module $N = I(M)/J(M)$ is the module that corresponds to O. From the definition of the action of $\overline{G}$, we have that ${}^{\pi(g)}I(M) = gI(M)$ and ${}^{\pi(g)}J(M) = gJ(M)$. It follows that the module corresponding to O' is isomorphic to ${}^{g}N$.

(3) It follows from the isomorphism of F-linear categories of Theorem 4.4 that $\operatorname{End}_{\mathcal{IT}(A)}(O)$, the algebra of endomorphisms of the object O, is isomorphic to $\operatorname{End}_{FS}(N)$ as algebras over F.

(4) It is well known that the algebra $F(\phi)$ is isomorphic to the center of the endomorphism algebra $\operatorname{End}_{FS}(N)$, so the result follows immediately from (3).

(5) This follows from (3) in a similar way.

(6) If $a \in A$ and $\gamma \in \Gamma$, then $\gamma a \gamma^{-1} \in A$, and so we may define $\gamma(a) = \gamma a \gamma^{-1}$. This defines an action of Γ on the $\overline{G}$-ring A by automorphisms. Let $(I, J, \overline{S})$ be an object of $\mathcal{IT}(A)$, and let N be the corresponding FS-module in $\mathcal{S}(M)$. Then

$$\bigl(\gamma(I), \gamma(J), \overline{S}\bigr)$$

is an object of $\mathcal{IT}(A)$. The module that corresponds to it is

$$\gamma I \gamma^{-1}(M)/\gamma J \gamma^{-1}(M) = \gamma I(M)/\gamma J(M)$$

as an FS-module. This module is isomorphic to the γ-twist of N as an FS-module.

(7) By (1) the restriction of N to an FH-module corresponds to the object O'. It then follows from (3) that

$$\operatorname{End}_{FH}(N) \simeq \operatorname{End}_{\mathcal{IT}(A)}\bigl(O'\bigr)$$

as algebras over F. Hence,

$$\operatorname{End}_{\overline{F}H}(\overline{F} \otimes_F N) \simeq \overline{F} \otimes_F \operatorname{End}_{\mathcal{IT}(A)}\bigl(O'\bigr)$$

as algebras over $\overline{F}$. Since M is completely reducible as an FH-module, N is completely reducible as an FH-module, and $\overline{F} \otimes_F N$ is completely reducible as an $\overline{F}H$-module. Let $N_1, \dots, N_r$ be representatives for the distinct isomorphism classes of irreducible $\overline{F}H$-submodules of $\overline{F} \otimes_F N$, and let their multiplicities be respectively $\alpha_1, \dots, \alpha_n$. Then

$$\mathrm{End}_{\overline{F}H}(\overline{F} \otimes_F N)$$

is isomorphic to the direct sum of full matrix algebras over $\overline{F}$ of dimension $\alpha_1^2, \dots, \alpha_n^2$ respectively. Since

$$\dim_F(N) = \dim_{\overline{F}}(\overline{F} \otimes_F N) = \sum_{i=1}^{n} \alpha_i \dim_{\overline{F}}(N_i)$$

and the dimension of each N_i is d, the result then follows. □

Definition 4.6. Assume Hypotheses 2.5, and let F be a field. Let C be a (possibly infinite) set of finitely generated FG-modules N such that $\mathrm{Res}^G_H(N)$ is completely reducible, and let C' be a (possibly infinite) set of finitely generated FG'-modules N' such that $\mathrm{Res}^{G'}_{H'}(N')$ is completely reducible. We set M to be the direct sum of all elements of C, and we set M' to be the direct sum of all elements of C'. Suppose we have an endoisomorphism (Definition 2.6) from M to M'

$$\epsilon : \underline{\mathrm{End}}(M) \to \underline{\mathrm{End}}(M').$$

Then ϵ determines one isomorphism κ_ϵ^0 of small F-linear categories from $\mathcal{S}(M)$ to $\mathcal{S}(M')$, as follows. We set $A = \underline{\mathrm{End}}(M)$ and $A' = \underline{\mathrm{End}}(M')$. Then,

$$\epsilon : A \to A'$$

is an isomorphism of $\overline{G}$-algebras. By Theorem 4.4, the categories $\mathcal{IT}(A)$ and $\mathcal{S}(M)$ are isomorphic under a preferred isomorphism α, and similarly, the categories $\mathcal{IT}(A')$ and $\mathcal{S}(M')$ are isomorphic under a preferred isomorphism α'. Since ϵ is a $\overline{G}$-algebra isomorphism, it induces an isomorphism ι_ϵ of categories from $\mathcal{IT}(A)$ to $\mathcal{IT}(A')$. The isomorphism κ_ϵ^0 is the one that makes the following diagram of categories commutative:

$$\begin{tikzcd}
\mathcal{IT}(A) \arrow[r, "\iota_\epsilon"] \arrow[d, "\alpha"] & \mathcal{IT}(A') \arrow[d, "\alpha'"] \\
\mathcal{S}(M) \arrow[r, "\kappa_\epsilon^0"] & \mathcal{S}(M')
\end{tikzcd}$$

Given a category of modules, one can naturally consider the category of its isomorphism classes of modules.

Definition 4.7. Let $\mathcal{C}$ be a full category of modules. We denote by $\mathcal{C}^{\simeq}$ the category of all isomorphism classes of objects of $\mathcal{C}$. If $\mathcal{D}$ is also a full category of modules and I is an isomorphism of categories from $\mathcal{C}$ to $\mathcal{D}$, then there is a unique induced isomorphism of categories $I^{\simeq}$ from $\mathcal{C}^{\simeq}$ to $\mathcal{D}^{\simeq}$.

Remark 4.8. The following remarks follow from the above definitions.

- κ_ϵ^0 is a well-defined isomorphism of categories, and in particular it will send isomorphic modules in $\mathcal{S}(M)$ to isomorphic modules in $\mathcal{S}(M')$. $\kappa_\epsilon^{0\simeq}$ is a well-defined isomorphism of categories from $\mathcal{S}(M)^{\simeq}$ to isomorphic modules in $\mathcal{S}(M')^{\simeq}$.

- If we can compose two endoisomorphisms say ϵ and ϵ', then it follows from the definition that the isomorphism of categories associated with $\epsilon'\epsilon$ is simply the composition of the two isomorphisms of categories:

$$\kappa^0_{\epsilon'\epsilon} = \kappa^0_{\epsilon'}\kappa^0_{\epsilon}.$$

- If $G = G'$ and $\pi = \pi'$, and $\phi : M \to M'$ is an isomorphism of topological FG-modules, then we can define

$$\epsilon : A \to A'$$

simply by setting

$$\epsilon(a) = \phi a \phi^{-1}$$

for all $a \in A$. Then ϵ is an endoisomorphism. Furthermore, it follows from the definitions that in this case, whenever $N = N_1/N_2$ is any element of $\mathcal{S}(M)$ then $\kappa^0_\epsilon(N) = \kappa^0_\epsilon(N_1/N_2) = \phi(N_1)/\phi(N_2)$. In particular,

$$\mathcal{S}(M)^{\simeq} = \mathcal{S}(M')^{\simeq}$$

and $I^{0\simeq}_\epsilon$ is the identity.

- It follows from the above comments that the correspondence that to ϵ assigns $\kappa^{0\simeq}_\epsilon$ is stable in the following sense. Suppose we precede $\epsilon : M \rightsquigarrow M'$ with an endoisomorphism $\epsilon_1 : M_1 \rightsquigarrow M$ that arose from a module isomorphism of M_1 to M, and follow it with an endoisomorphism $\epsilon_2 : M' \rightsquigarrow M'_1$ that arose from module isomorphism of M' to M'_1, then we get a new endoisomorphism $\epsilon_2\epsilon\epsilon_1 : M_1 \rightsquigarrow M'_1$, and

$$\kappa^{0\simeq}_\epsilon = \kappa^{0\simeq}_{\epsilon_2\epsilon\epsilon_1}.$$

Theorem 4.9. *Assume Definition 4.6. Then ϵ determines one isomorphism κ^0_ϵ of small F-linear categories from $\mathcal{S}(M)$ to $\mathcal{S}(M')$. Furthermore the isomorphism of categories satisfies the following:*

(1) *κ^0_ϵ gives bijections from the irreducible modules in $\mathcal{S}(M)$ to the irreducible modules in $\mathcal{S}(M')$, from the indecomposable modules to the indecomposable modules, and preserves direct sums of modules, and composition series.*
(2) *κ^0_ϵ commutes with restriction of modules. Furthermore, it commutes with induction of modules up to isomorphism.*
(3) *κ^0_ϵ commutes, up to module isomorphism, with conjugation by $\overline{G}$.*
(4) *κ^0_ϵ preserves the field of values of irreducible characters.*
(5) *κ^0_ϵ preserves the corresponding elements of the Brauer group and in particular the Schur indices.*
(6) *Suppose that we are given a group Γ of field automorphisms of F, and a continuous action of Γ on M and on M' in such a way that Γ preserves the addition on M and on M', for all $\gamma \in \Gamma$, $\lambda \in F$, $v \in M$ and $v' \in M'$ we have $\gamma(\lambda v) = \gamma(\lambda)\gamma(v)$, $\gamma(\lambda v') = \gamma(\lambda)\gamma(v')$, and the action of each $\gamma \in \Gamma$ commutes with the action of each element of G on M and of G' on M'. Then Γ acts as automorphisms of both $\overline{G}$-rings $\underline{\mathrm{End}}(M)$ and $\underline{\mathrm{End}}(M')$. We assume that ϵ preserves these actions. Then, whenever N is an FS-module and an object in $\mathcal{S}(M)$ and N' is the corresponding object under κ^0_ϵ, and $\gamma \in \Gamma$, then there is an object in $\mathcal{S}(M)$ which is isomorphic to the γ-twist of N, and κ^0_ϵ sends it to a module isomorphic to γ-twist of N'.*

(7) *Suppose all irreducible direct summands of* $\operatorname{Res}^G_H(\overline{F} \otimes_F M)$ *have the same dimension d, and all irreducible direct summands of* $\operatorname{Res}^{G'}_{H'}(\overline{F} \otimes_F M')$ *have the same dimension* d'*, where* $\overline{F}$ *is an algebraic closure of* F*. Then, whenever* N *is an object in* $\mathcal{S}(M)$ *and* N' *is the corresponding object under* κ^0_ϵ*, then* $\dim_F(N') = \frac{d'}{d}\dim_F(N)$.

Proof. By Definition 4.6, we have that κ_ϵ is an isomorphism of categories from $\mathcal{IT}(A)$ to $\mathcal{IT}(A')$. (1) then follows directly from this. Furthermore, the fact that κ^0_ϵ commutes with restriction follows from Proposition 4.5, and this implies that it will commute with induction up to isomorphism because induction is the adjoint of restriction. The rest of the properties follow directly from Proposition 4.5. □

In addition, the module correspondence described in the previous theorem has some good compatibility properties. We see that it commutes well with certain restrictions.

Proposition 4.10. *Assume that hypotheses of Theorem 4.9, and let* G_0 *be a subgroup of* G *that contains* H *and let* G'_0 *be the corresponding subgroup of* G' *that contains* H' *so that* $\pi(G_0) = \pi'(G'_0)$*. Then we can view* $\operatorname{Res}^G_{G_0}(M)$ *and* $\operatorname{Res}^{G'}_{G'_0}(M')$ *as the direct sums of the corresponding restriction modules, and, as algebras*

$$\underline{\mathrm{End}}\big(\operatorname{Res}^G_{G_0}(M)\big) = \underline{\mathrm{End}}(M) \quad \textit{and} \quad \underline{\mathrm{End}}\big(\operatorname{Res}^{G'}_{G'_0}(M')\big) = \underline{\mathrm{End}}\big(M'\big).$$

We let ϵ_0 *be the endoisomorphism from* $\operatorname{Res}^G_{G_0}(M)$ *to* $\operatorname{Res}^{G'}_{G'_0}(M')$ *which agrees with* ϵ *on every element. Then,* $\mathcal{S}(\operatorname{Res}^G_{G_0}(M))$ *is a full subcategory of* $\mathcal{S}(M)$ *and* $\mathcal{S}(\operatorname{Res}^{G'}_{G'_0}(M))$ *is a full subcategory of* $\mathcal{S}(M')$ *and the original correspondence* $\mathcal{I}^0_\epsilon$ *provides an isomorphism on these subcategories which is simply* $\mathcal{I}^0_{\epsilon_0}$.

Proof. The direct sum of the collection of restrictions of the modules in $\mathcal{C}$ and $\mathcal{C}'$ do provide, respectively, $\operatorname{Res}^G_{G_0}(M)$ and $\operatorname{Res}^{G'}_{G'_0}(M')$. Set $\overline{G}_0 = \pi(G_0) = \pi'(G'_0)$, $A = \underline{\mathrm{End}}(M)$, $B = \underline{\mathrm{End}}(M')$, $A_0 = \underline{\mathrm{End}}(\operatorname{Res}^G_{G_0}(M))$, and $B_0 = \underline{\mathrm{End}}(\operatorname{Res}^{G'}_{G'_0}(M'))$. As algebras over F, we have $A = A_0$ and $B = B_0$, but A and B are $\overline{G}$-algebras and A_0 and B_0 are $\overline{G}_0$-algebras. Now, ϵ does provide an endoisomorphism ϵ_0, as required. The modules in $\mathcal{S}(\operatorname{Res}^G_{G_0}(M))$ form a full subcategory of $\mathcal{S}(M)$ and those of $\mathcal{S}(\operatorname{Res}^{G'}_{G'_0}(M))$ form a full subcategory of $\mathcal{S}(M')$. Likewise the category $\mathcal{IT}(A_0)$ is a full subcategory of $\mathcal{IT}(A)$, and the category $\mathcal{IT}(B_0)$ is a full subcategory of $\mathcal{IT}(B)$. The proposition then follows from the definitions. □

5. The *F*-completion of *M*

The module correspondence that we have described in the previous section is limited to modules over the same field, and furthermore to modules which are section of a given module. In Clifford theory, it is often convenient to consider modules which are related to the original module, but are not necessarily a section of the original module (for example modules which require a large number of generators). Furthermore, we want to discuss simultaneously modules over different fields. In this section, we construct a larger module than the originally given one in order to control a larger class of modules. We assume the following hypotheses throughout the section.

Hypotheses 5.1. Let G and $\overline{G}$ be finite groups, and suppose we are given a surjective homomorphism $\pi : G \to \overline{G}$ whose kernel is H. Suppose R is an integral domain and M is a finitely generated RG-module such that M is free as an R-module.

The modules above M are to be defined over fields, and these fields are to be extensions of R in the following sense.

Definition 5.2. By a *field extension* F of R we mean a field F together with a unital ring homomorphism $\phi : R \to F$. In many cases, we will not name the ring homomorphism explicitly. Whenever F is a field extension of R, we will denote by $\mathrm{Gal}(F/R)$ the group of all field automorphisms of F which fix every element in the image of R in F.

Remark 5.3. In the situation of Definition 5.2, we get that $F \otimes_R M$ is a finite dimensional FG-module. Furthermore, $F \otimes_R \mathrm{End}_{RG}(M)$ is identified in a natural way with a subalgebra of $\mathrm{End}_{FH}(F \otimes_R M)$.

Among the field extensions of R, we identify the *good* ones.

Definition 5.4. We say that a field extension F of R is a *good extension* for M if $\mathrm{Res}^G_H(F \otimes_R M)$ is completely reducible and

$$F \otimes_R \mathrm{End}_{RH}(M) = \mathrm{End}_{FH}(F \otimes_R M).$$

Proposition 5.5. *Assume Hypotheses 5.1, and that F is a field extension of R of characteristic p (possibly equal to zero) such that p does not divide $|H|$. Then F is a good field extension of R for M.*

Proof. By Maschke's Theorem, $\mathrm{Res}^G_H(F \otimes_R M)$ is completely reducible. Furthermore,

$$F \otimes_R \mathrm{End}_{RH}(M) \subseteq \mathrm{End}_{FH}(F \otimes_R M).$$

Let $\phi \in \mathrm{End}_{FH}(F \otimes_R M)$. Since M is finitely generated and free as an R-module, we know that

$$\phi \in \mathrm{End}_F(F \otimes_R M) = F \otimes_R \mathrm{End}_R(M),$$

so that there exist $f_i \in F$ and $\theta_i \in \mathrm{End}_R(M)$ such that

$$\phi = \sum_{i=1}^n f_i \otimes_R \theta_i.$$

For $i \in \{1, \ldots, n\}$, we have

$$\zeta_i = \sum_{h \in H} {}^h\theta_i \in \mathrm{End}_{RH}(M).$$

Since ϕ is H-invariant and $|H|$ is invertible in F,

$$\phi = \sum_{i=1}^n \frac{f_i}{|H|} \otimes_R \zeta_i \in F \otimes_R \mathrm{End}_{RH}(M).$$

It follows that

$$F \otimes_R \mathrm{End}_{RH}(M) = \mathrm{End}_{FH}(F \otimes_R M).$$

Hence, Definition 5.4 holds as desired. □

While field extensions of a ring may or may not be good, all field extensions of a good field extension are good.

Proposition 5.6. *Suppose F is a good field extension of R for M, and K is any field extension of F. Then K is a good field extension of R for M. Furthermore, if R is a field and $\mathrm{Res}^G_H(M)$ is completely reducible, then every field extension of R is a good field extension for M.*

Proof. It follows directly from the definition that if F is a good field extension for M and K is a good field extension of F for $F \otimes_R M$, then K is a good field extension of R for M. Hence, it is enough to show that if R is a field and $\mathrm{Res}^G_H(M)$ is completely reducible, then every field extension F of R is a good field extension for M. However, in this case the dimension over R of $\mathrm{Hom}_{RH}(M)$ is the same as the dimension over F of $\mathrm{Hom}_{FH}(F \otimes_R M)$, and so

$$F \otimes_R \mathrm{End}_{RG}(M) = \mathrm{End}_{FH}(F \otimes_R M),$$

and the extension is good. $\square$

The *π-center algebra* was defined in [12], where it was proved that it is a simple $\overline{G}$-algebra in the relevant cases. Here, since we do not assume that our modules are quasi-homogeneous, it is convenient to relax the definition somewhat and allow for π-central $\overline{G}$-algebras which are not simple. The definition of π-center algebra in [12] then corresponds to the case when we have a π-center algebra in the sense of the current paper which is simple as a $\overline{G}$-ring. We denote, as is standard, by $J(R)$ the radical of a ring R. Of course, $J(R)$ acts trivially on any completely reducible R-module.

Definition 5.7. Let $\pi : G \to \overline{G}$ be a surjective group homomorphism of finite groups, let $H = \ker(\pi)$, and let F be a field. We say that Z is a *π-center algebra of FG* if it is a $\overline{G}$-algebra Z over F of the following form. We set $Z_0 = \mathrm{Z}(FH/J(FH))$, so that Z_0 is a commutative $\overline{G}$-algebra over F, and, for some idempotent e of $Z_0^{\overline{G}}$ we have $Z = eZ_0$.

Then Z is a $\overline{G}$-algebra over F. Standard arguments, which are given in more detail in [12], show how, if M is a finitely generated FG-module such that its restriction to H is completely reducible, we can view Z_0 as acting on M. The following definition uses this action.

Definition 5.8. Assume Hypotheses 5.1. Assume that F is a field extension of R good for M, and use the notation of Definition 5.7. Let e be the sum of all the primitive idempotents of $Z_0^{\overline{G}}$ which act non-trivially on $F \otimes_R M$. We say that eZ_0 is the *π-center algebra of FG associated with $F \otimes_R M$*, and we write $\mathrm{Z}(M, \pi, F) = eZ_0$.

Proposition 5.9. *Assume F is a good field extension of R for M, and let K be a field extension of F. The representation map induces a canonical isomorphism from $\mathrm{Z}(M, \pi, F)$ to $\mathrm{Z}(\mathrm{End}_{FH}(F \otimes_R M))$. Furthermore, the π-center $\overline{G}$-algebra $\mathrm{Z}(M, \pi, K)$ of KG associated with $K \otimes_R M$ can be identified with $K \otimes_F \mathrm{Z}(M, \pi, F)$.*

Proof. This follows from standard arguments. Some of these are given in more detail in [12]. $\square$

Definition 5.10. Assume F is a good field extension of R for M. Then $F\overline{G}$ is an FG-module in a natural way (with kernel H), and $M \otimes_R F\overline{G}$ is an FG-module. We consider a countably infinite number of copies of the FG-module $M \otimes_R F\overline{G}$, which we label with the elements of $\mathbf{N}$. The *Clifford F-completion* of M is the direct sum of this countably infinite collection of copies of $M \otimes_R F\overline{G}$. We denote the Clifford F-completion of M by $\widehat{M_F}$. If A is a subring of F which contains the image of R, then we also denote by $\widehat{M_A}$ the AG-submodule of $\widehat{M_F}$ corresponding to the direct sum of the countably infinite collection of copies of $M \otimes_R A\overline{G}$. By the *$\pi$-center algebra of FG associated with $\widehat{M_F}$*, we simply mean the $\overline{G}$-algebra $\mathrm{Z}(M, \pi, F)$ of Definition 5.8.

The algebra associated with the Clifford F-completion of M can be computed from the algebra associated with M.

Proposition 5.11. *Assume F is a good field extension of R for M. Then $\underline{\mathrm{End}}(\widehat{M_F})$ is the direct sum of copies of*

$$\mathrm{End}_{RH}(M) \otimes_R \mathrm{End}_F(F\overline{G})$$

labeled by pairs of elements of **N**, *and this determines its $\overline{G}$-algebra structure.*

Proof. Since F is a good field extension of R, the $\overline{G}$-algebra of endomorphisms of $M \otimes_R F$ is $\mathrm{End}_{RH}(M) \otimes_R F$. It follows that the $\overline{G}$-algebra of endomorphisms of $M \otimes_R F\overline{G}$ is

$$\mathrm{End}_{RH}(M) \otimes_R \mathrm{End}_F(F\overline{G}).$$

The proposition then follows from Proposition 3.4. □

Lemma 5.12. *Let F be a good field extension of R for M, and let K be a finite field extension of F. Then, there is a topological FG-module isomorphism*

$$\theta_{M,K,F} : \mathrm{Res}^{KG}_{FG}(\widehat{M_K}) \to \widehat{M_F}.$$

We denote by

$$\widehat{\theta_{M,K,F}} : \underline{\mathrm{End}}(\widehat{M_K}) \to \underline{\mathrm{End}}(\widehat{M_F})$$

the restriction of the corresponding endoisomorphism. Furthermore, whenever G_1, and G_2 are finite groups, and $\pi_1 : G_1 \to \overline{G}$ and $\pi_2 : G_2 \to \overline{G}$ are surjective homomorphisms, and M_1 is a finitely generated RG_1-module such that M_1 is free as an R-module, and M_2 is a finitely generated RG_2-module such that M_2 is free as an R-module, and $\epsilon : M_1 \rightsquigarrow M_2$ is an endoisomorphism, and F and K are good extensions of R for both M_1 and M_2 then the following diagram is commutative:

$$\begin{tikzcd}
\underline{\mathrm{End}}(\widehat{M_{1,K}}) \arrow[r, "\widehat{\epsilon_K}"] \arrow[d, mapsto, "\widehat{\theta_{M_1,K,F}}"'] & \underline{\mathrm{End}}(\widehat{M_{2,K}}) \arrow[d, mapsto, "\widehat{\theta_{M_2,K,F}}"] \\
\underline{\mathrm{End}}(\widehat{M_{1,F}}) \arrow[r, "\widehat{\epsilon_F}"] & \underline{\mathrm{End}}(\widehat{M_{2,F}})
\end{tikzcd}$$

Proof. It follows directly from the definition that

$$\underline{\mathrm{End}}(\widehat{M_K}) \subseteq \underline{\mathrm{End}}\big(\mathrm{Res}^{KG}_{FG}(\widehat{M_K})\big)$$

and this justifies the use of the term *restriction* in the statement of the lemma. We let N_F be the direct sum of a countable number of copies of $F\overline{G}$ and we let N_K be the direct sum of a countable number of copies of $K\overline{G}$. Since $[K : F]$ is finite, we have that $\mathrm{Res}^{KG}_{FG}(N_K)$ is isomorphic to N_F, and we pick some isomorphism

$$\theta : \mathrm{Res}^{KG}_{FG}(N_K) \to N_F$$

of topological modules. Now $\widehat{M_F} = M \otimes_R N_F$ and $\widehat{M_K} = M \otimes_R N_K$. Then we set

$$\theta_{M,K,F} = \mathrm{Id}_M \otimes_R \theta : \mathrm{Res}^{KG}_{FG}(\widehat{M_K}) \to \widehat{M_F},$$

and $\theta_{M,K,F}$ is an isomorphism of topological FG-modules.

Since all the maps in the diagram are additive, in order to prove that it is commutative it is enough to show that they have the same value on a set of additive generators for $\underline{\mathrm{End}}(\widehat{M_{1,K}})$. We pick some $\phi \in \mathrm{End}_{R\ker(\pi_1)}(M_1)$ and $\sigma \in \mathrm{End}_K(N_K)$ such that $\phi \otimes_R \sigma \in \underline{\mathrm{End}}(\widehat{M_{1,K}})$. Then

$$\widehat{\epsilon_F}\big(\widehat{\theta_{M_1,K,F}}(\phi \otimes_R \sigma)\big) = \widehat{\epsilon_F}\big(\phi \otimes_R \big(\theta\sigma\theta^{-1}\big)\big) = \epsilon(\phi) \otimes_R \big(\theta\sigma\theta^{-1}\big),$$

and

$$\widehat{\theta_{M_2,K,F}}\big(\widehat{\epsilon_K}(\phi \otimes_R \sigma)\big) = \widehat{\theta_{M_2,K,F}}\big(\epsilon(\phi) \otimes_R \sigma\big) = \epsilon(\phi) \otimes_R \big(\theta\sigma\theta^{-1}\big),$$

so that the diagram commutes as desired. □

6. Modules above *M*

We assume Hypotheses 5.1 throughout this section. We are ready to define when a module is *above* M. Roughly speaking, a module N is *above* M if it is a section of the module $\widehat{M_F}$, for F a good field extension of R.

Definition 6.1. Assume Hypotheses 5.1. We let S be a subgroup of G which contains H, and we let F be a good field extension of R. We denote by $\mathcal{A}(M,F)$ the small category $\mathcal{S}(\widehat{M_F})$ of Definition 4.3. The full subcategory of $\mathcal{A}(M,F)$ of those objects which are FS-modules is denoted $\mathcal{A}(M,FS)$. We say that an FS-module is *above* M if it is an object in the category $\mathcal{A}(M,FS)$. If $\mathcal{F}$ is a collection of good field extensions of R, we denote by $\mathcal{A}(M,\mathcal{F})$ the disjoint union of all the $\mathcal{A}(M,F)$ as we run through the F in $\mathcal{F}$.

We remark that the category $\mathcal{A}(M,\mathcal{F})$ is usually not connected. Furthermore, most often $\mathcal{A}(M,\mathcal{F})$ is not a small category.

In order for a module to be above M, we need it to be constructed in a specific way. However, if we are only interested in modules up to isomorphism, the following proposition describes the structure of all such modules.

Proposition 6.2. *Assume Hypotheses 5.1. We let S be a subgroup of G with $H \subseteq S$, we let F be a good field extension of R for M, and we let N be a finitely generated FS-module. Then the following are equivalent*:

(1) *N is isomorphic to some module above M.*
(2) $\mathrm{Res}^S_H(N)$ *is completely reducible, and for every irreducible submodule I of* $\mathrm{Res}^S_H(N)$*, there is some irreducible submodule I' of* $\mathrm{Res}^G_H(F \otimes_R M)$ *such that I is isomorphic to I'.*

Proof. Suppose first that (1) holds. Then, by Definition 6.1, N is isomorphic to a section of $\widehat{M_F}$. Since F is a good field extension of R for M, we have that $\mathrm{Res}^G_H(F \otimes_R M)$ is completely reducible. It follows that $\mathrm{Res}^G_H(\widehat{M_F})$ is completely reducible and all its irreducible submodules are isomorphic to some irreducible submodule of $\mathrm{Res}^G_H(F \otimes_R M)$, and (2) follows.

Suppose next that (2) holds. Then $\mathrm{Res}^S_H(N)$ is completely reducible, and for every irreducible submodule I of $\mathrm{Res}^S_H(N)$, there is some irreducible submodule I' of $\mathrm{Res}^G_H(F \otimes_R M)$ such that I is isomorphic to I'. We set $M_1 = \mathrm{Ind}^G_S(N)$. Hence, $\mathrm{Res}^G_H(M_1)$ is isomorphic to a direct summand of some finite multiple of $\mathrm{Res}^G_H(F \otimes_R M)$. It then follows that M_1 is a homomorphic image of some finite multiple of $M \otimes_R F\overline{G}$. Therefore M_1 is a homomorphic image of some finitely generated FG-submodule of $\widehat{M_F}$. Since N is isomorphic to a submodule of the restriction of M_1 to S, it follows that N is isomorphic to some module in $\mathcal{S}(\widehat{M_F})$ which means that N is isomorphic to some module above M. Hence, (1) holds. The proposition then follows. □

Remark 6.3. In the language of category theory, Proposition 6.2 shows that the category $\mathcal{A}(M, FS)$ is equivalent to the category of all finitely generated FS-modules which satisfy condition (2) of the proposition. However, the actual equivalence of categories is defined in terms of a choice of representatives, and as such it is not unique. We will use the category $\mathcal{A}(M, FS)$ and related ones to prove that endoisomorphisms define *unique* isomorphisms of categories. This uniqueness implies uniqueness of isomorphisms for corresponding categories of *isomorphism classes of* modules. Similar results do not directly extend to the categories of modules that include with any module in it all other modules isomorphic to it because of the need to use choice.

7. Module correspondences

We are now ready to show how each endoisomorphism provides a unique isomorphism of module categories.

Theorem 7.1. *Let G, G' and $\overline{G}$ be finite groups, and suppose we are given surjective homomorphisms $\pi : G \to \overline{G}$ and $\pi' : G' \to \overline{G}$ whose kernels are, respectively, H and H'. Let R be an integral domain. Suppose M is a finitely generated RG-module which is free as an R-module, and M' is a finitely generated RG'-module which is free as an R-module. Let*

$$\epsilon : \mathrm{End}_{RH}(M) \to \mathrm{End}_{RH'}\bigl(M'\bigr)$$

be an endoisomorphism from M to M'. Let $\mathcal{F}$ be a collection of field extensions of R which are good for M and good for M'. Then we may define isomorphisms of $\overline{G}$-algebras and an isomorphism κ_ϵ of categories from $\mathcal{A}(M, \mathcal{F})$ to $\mathcal{A}(M', \mathcal{F})$ as follows. For each field extension F of R which is in $\mathcal{F}$, ϵ determines uniquely an isomorphism of G-algebras over F

$$\overline{\epsilon_F} : \mathrm{Z}(M, \pi, F) \to \mathrm{Z}\bigl(M', \pi', F\bigr)$$

(Definition 5.8), and ϵ determines uniquely an endoisomorphism

$$\widehat{\epsilon_F} : \underline{\mathrm{End}}(\widehat{M_F}) \to \underline{\mathrm{End}}\bigl(\widehat{M'_F}\bigr).$$

Then $\widehat{\epsilon_F}$ determines uniquely an isomorphism of categories $\kappa^0_{\widehat{\epsilon_F}}$ from $\mathcal{A}(M, F)$ to $\mathcal{A}(M', F)$. The isomorphism κ_ϵ is obtained by putting together the isomorphisms $\kappa^0_{\widehat{\epsilon_F}}$ on their respective domains as F ranges over all the field extensions of R which are in $\mathcal{F}$. Furthermore, for each subgroup $\overline{S}$ of $\overline{G}$, if we set $S = \pi^{-1}(\overline{S})$ and $S' = (\pi')^{-1}(\overline{S})$, then κ_ϵ provides an isomorphism of F-linear categories from $\mathcal{A}(M, FS)$ to $\mathcal{A}(M', FS')$.

Proof. Let F be any field extension F of R which is good for both M and M'. The endoisomorphism ϵ yields a $\overline{G}$-algebra isomorphism

$$\mathrm{End}_{FH}(M \otimes_R F) \to \mathrm{End}_{FH'}\bigl(M' \otimes_R F\bigr),$$

which, by restriction, yields a $\overline{G}$-algebra isomorphism

$$\mathrm{Z}\bigl(\mathrm{End}_{FH}(M \otimes_R F)\bigr) \to \mathrm{Z}\bigl(\mathrm{End}_{FH'}\bigl(M' \otimes_R F\bigr)\bigr).$$

This, together with the isomorphisms of Proposition 5.9, yields the isomorphism of G-algebras over F

$$\overline{\epsilon_F} : \mathrm{Z}(M, \pi, F) \to \mathrm{Z}\bigl(M', \pi', F\bigr).$$

The endoisomorphism ϵ also yields a $\overline{G}$-algebra isomorphism

$$\epsilon \otimes_R \mathrm{Id} : \mathrm{End}_{RH}(M) \otimes_R \mathrm{End}_F(F\overline{G}) \to \mathrm{End}_{RH'}\big(M'\big) \otimes_R \mathrm{End}_F(F\overline{G}).$$

In view of Proposition 5.11, this defines uniquely an endoisomorphism $\widehat{\epsilon_F}$ from $\widehat{M_F}$ to $\widehat{M'_F}$:

$$\widehat{\epsilon_F} : \underline{\mathrm{End}}(\widehat{M_F}) \to \underline{\mathrm{End}}\big(\widehat{M'_F}\big).$$

It then follows from Theorem 4.9 that $\widehat{\epsilon_F}$ determines uniquely an isomorphism of categories $\kappa^0_{\widehat{\epsilon_F}}$ from $\mathcal{A}(M, F)$ to $\mathcal{A}(M', F)$. Furthermore, for each subgroup $\overline{S}$ of $\overline{G}$, if we set $S = \pi^{-1}(\overline{S})$ and $S' = (\pi')^{-1}(\overline{S})$, then $\kappa^0_{\widehat{\epsilon_F}}$ provides an isomorphism of F-linear categories from $\mathcal{A}(M, FS)$ to $\mathcal{A}(M', FS')$. The theorem then follows directly from this. $\square$

The isomorphism of $\overline{G}$-algebras $\overline{\epsilon_F}$ and the isomorphism of module categories of Theorem 7.1 are uniquely determined by ϵ. We now see that, up to isomorphism, these isomorphism are compatible with restriction of modules.

Proposition 7.2. *Assume the hypotheses and notation of Theorem 7.1. Let S be a subgroup of G which contains H and let S' be the corresponding subgroup of G' i.e. $G' \geqslant S' \geqslant H'$ and $\pi(S) = \pi'(S')$. Set $\overline{S} = \pi(S)$. Let $N = \mathrm{Res}^G_S(M)$ and $N' = \mathrm{Res}^{G'}_{S'}(M')$, and set $\pi_0 = \mathrm{Res}^G_S(\pi)$ and $\pi'_0 = \mathrm{Res}^{G'}_{S'}(\pi')$, so that $\pi_0 : S \to \overline{S}$ and $\pi'_0 : S' \to \overline{S}$ are surjective group homomorphisms. Note that the same ϵ can be viewed as $\epsilon : M \rightsquigarrow M'$ and $\epsilon : N \rightsquigarrow N'$. Then*

$$\mathrm{Z}(M, \pi, F) = \mathrm{Z}(N, \pi_0, F)$$

and

$$\mathrm{Z}\big(M', \pi, F\big) = \mathrm{Z}\big(N', \pi'_0, F\big)$$

and ϵ induces the same isomorphism

$$\overline{\epsilon_F} : \mathrm{Z}(M, \pi, F) \to \mathrm{Z}\big(M', \pi', F\big)$$

as $\epsilon : M \rightsquigarrow M'$ and as $\epsilon : N \rightsquigarrow N'$. In addition, $\mathcal{A}(N, \mathcal{F})^{\simeq}$ is a full subcategory of $\mathcal{A}(M, \mathcal{F})^{\simeq}$ and $\mathcal{A}(N', \mathcal{F})^{\simeq}$ is a full subcategory of $\mathcal{A}(M', \mathcal{F})^{\simeq}$, and the restriction of $\kappa^{\simeq}_{\epsilon}$ provides an isomorphism $\kappa^{1\simeq}_{\epsilon}$ from $\mathcal{A}(N, \mathcal{F})^{\simeq}$ to $\mathcal{A}(N', \mathcal{F})^{\simeq}$. Furthermore, ϵ is also an endoisomorphism from N to N', and, as such it provides an isomorphism κ^2_{ϵ} from $\mathcal{A}(N, \mathcal{F})$ to $\mathcal{A}(N', \mathcal{F})$. Finally, $\kappa^{1\simeq}_{\epsilon}$ and κ^2_{ϵ} coincide up to module isomorphisms, that is

$$\kappa^{1\simeq}_{\epsilon} = \kappa^{2\simeq}_{\epsilon}.$$

Proof. The statements about $\mathrm{Z}(M, \pi, F)$ and $\mathrm{Z}(M', \pi, F)$ and ϵ_F follow directly from the definitions. If F is a field in $\mathcal{F}$, then $\mathrm{Res}^G_S(F\overline{G})$ is isomorphic to the direct sum of a finite number of copies of $F\overline{S}$. It follows that

$$\mathrm{Res}^G_S(\widehat{M_F}) \simeq \widehat{N_F} \quad \text{and} \quad \mathrm{Res}^{G'}_{S'}\big(\widehat{M'_F}\big) \simeq \widehat{N'_F}$$

as topological FS-modules and FS'-modules respectively. Furthermore, the endoisomorphism from $\widehat{N_F}$ to $\widehat{N'_F}$ obtained from ϵ is obtained from $\widehat{\epsilon_F}$ by preceding it and following it by endoisomorphisms which arise from module isomorphisms. It then follows from Remark 4.8 that

$$\mathcal{S}\big(\mathrm{Res}^G_S(\widehat{M_F})\big)^{\simeq} = \mathcal{S}(\widehat{N_F})^{\simeq}$$

and

$$\mathcal{S}\big(\mathrm{Res}^{G'}_{S'}(\widehat{M'_F})\big)^{\simeq} = \mathcal{S}\big(\widehat{N'_F}\big)^{\simeq}.$$

It follows that $\mathcal{A}(N,\mathcal{F})^{\simeq}$ is the isomorphism classes of modules in $\mathcal{A}(M,\mathcal{F})$ which are modules for some subgroup of S that contains H, and that $\mathcal{A}(N',\mathcal{F})^{\simeq}$ is the isomorphism classes of modules in $\mathcal{A}(M',\mathcal{F})$ which are modules for some subgroup of S' that contains H'. Hence, $\mathcal{A}(N,\mathcal{F})^{\simeq}$ is a full subcategory of $\mathcal{A}(M,\mathcal{F})^{\simeq}$ and $\mathcal{A}(N',\mathcal{F})^{\simeq}$ is a full subcategory of $\mathcal{A}(M',\mathcal{F})^{\simeq}$, and the restriction of $\kappa_\epsilon^{\simeq}$ provides an isomorphism $\kappa_\epsilon^{1\simeq}$ from $\mathcal{A}(N,\mathcal{F})^{\simeq}$ to $\mathcal{A}(N',\mathcal{F})^{\simeq}$. It follows from Remark 4.8 and the fact that the endoisomorphism from $\widehat{N_F}$ to $\widehat{N'_F}$ that is obtained from ϵ is obtained from $\widehat{\epsilon_F}$ by preceding it and following it by endoisomorphisms which arise from module isomorphisms that $\kappa_\epsilon^{1\simeq}$ and κ_ϵ^2 coincide up to module isomorphisms, that is

$$\kappa_\epsilon^{1\simeq} = \kappa_\epsilon^{2\simeq}. \qquad \square$$

Under the hypotheses of Theorem 7.1 we have an isomorphism of module categories. This isomorphism has some excellent compatibility properties which we now investigate.

Lemma 7.3. *Assume the hypotheses of Theorem 7.1. Using Definition 5.2, we set $\Gamma = \mathrm{Gal}(F/R)$. Then Γ acts naturally on $F\overline{G}$, and so Γ acts on $\widehat{M_F}$ and on $\widehat{M'_F}$. Then these actions together with the $\overline{G}$-algebra isomorphism $\widehat{\epsilon_F}$ satisfy the hypotheses of Theorem 4.9 (6).*

Proof. Γ acts in a natural way on $F\overline{G}$, and so it acts in a natural way on $M \otimes_R F\overline{G}$ and on $M' \otimes_R F\overline{G}$, and it follows that Γ acts continuously on $\widehat{M_F}$ and $\widehat{M'_F}$. By the definition of $\widehat{\epsilon_F}$ in Theorem 7.1, we get that the $\overline{G}$-algebra isomorphism $\widehat{\epsilon_F}$ satisfies the hypotheses of Theorem 4.9 (6), as desired. $\square$

Notation 7.4 *(Subgroup correspondence).* Under the hypotheses of Theorem 7.1, we will denote by S an arbitrary subgroup of G which contains H, and we will denote by $\bar{S}$ its corresponding subgroup of $\overline{G}$, and by S' its corresponding subgroup of G', i.e. $\bar{S} = \pi(S)$, and $S' = (\pi')^{-1}(\bar{S})$.

The following theorem is related to [6, Theorem 3.5], and can be viewed as a modular version of it where the bijection is uniquely determined.

Theorem 7.5. *Assume the hypotheses and notation of Theorem 7.1. Then, the isomorphism of categories κ_ϵ from $\mathcal{A}(M)$ to $\mathcal{A}(M')$ has the following properties. We let F be any field extension of R which is good for M and for M'. To compare κ_ϵ with other module correspondences, we define*

$$\epsilon_F = \epsilon \otimes_R \mathrm{Id}_F : \mathrm{End}_{RH}(M) \otimes_R F \to \mathrm{End}_{RH'}\big(M'\big) \otimes_R F$$

so that $\epsilon_F : M \otimes_R F \rightsquigarrow M' \otimes_R F$ is an endoisomorphism, and ϵ_F defines a module correspondence $\kappa^0_{\epsilon_F}$ from $\mathcal{S}(M \otimes_R F)$ to $\mathcal{S}(M' \otimes_R F)$ as in Definition 4.6.

1. *κ_ϵ gives bijections from the irreducible modules in $\mathcal{A}(M, FS)$ to the irreducible modules in $\mathcal{A}(M', FS')$, from the indecomposable modules to the indecomposable modules, and preserves direct sums of modules, and composition series.*
2. *κ_ϵ commutes with restriction of modules. Furthermore, it commutes with induction of modules up to isomorphism.*
3. *Up to isomorphisms, κ_ϵ agrees with $\kappa^0_{\epsilon_F}$ on the domain of the latter. In particular, κ_ϵ sends modules which are isomorphic to sections $\mathrm{Res}^G_S(M \otimes_R F)$ to modules which are isomorphic to sections of $\mathrm{Res}^{G'}_{S'}(M' \otimes_R F)$.*

(4) *Let N be a module in $\mathcal{A}(M, FS)$ be sent to N' a module in $\mathcal{A}(M', FS')$ under κ_ϵ, and let $z \in \mathrm{Z}(M, \pi, F)$ and let $z' = \overline{\epsilon_F}(z)$ so that $z' \in \mathrm{Z}(M', \pi', F)$. Then zN is an FH-module, and $z'N'$ is an FH'-module, and, up to isomorphism, κ_ϵ sends zN to $z'N'$.*
(5) *κ_ϵ commutes, up to isomorphism, with field extensions and restrictions.*
(6) *κ_ϵ commutes up to isomorphisms with tensoring with $\overline{S}$-modules.*
(7) *κ_ϵ commutes up to isomorphisms with any Galois automorphism that fixes each element of (the image of) R.*
(8) *κ_ϵ commutes up to module isomorphisms with conjugation by $\overline{G}$.*
(9) *κ_ϵ preserves the field of values of irreducible characters.*
(10) *κ_ϵ preserves the corresponding elements of the Brauer group and in particular the Schur indices.*
(11) *Suppose for some algebraic closure $\overline{F}$ of F the irreducible $\overline{F}H$-submodules of $\overline{F} \otimes_R M$ are all of the same dimension and the irreducible $\overline{F}H'$-submodules of $\overline{F} \otimes_R M'$ are all of the same dimension. Then there is some rational constant d, such that, whenever N, a module over a field K, an extension of F, is an object in $\mathcal{A}(M, K)$ and N' is the corresponding object under κ_ϵ, then $\dim_K(N') = d \dim_K(N)$.*

Proof. In view of Lemma 7.3, we can apply all of Theorem 4.9, and we obtain all the claimed properties except (3), (4), (5) and (6).

We show (3). Let $N = N_1/N_2$ be an FS-module which is a section of $M \otimes_R F$, and let $N' = N'_1/N'_2$ be the FS'-module which is a section of $M' \otimes_R F$ and corresponds to N under $\kappa^0_{\epsilon_F}$. Set $v_0 = \sum_{\bar{g} \in \overline{G}} \bar{g}$. We may view $M \otimes_R Fv_0$ as an FG-submodule of $M \otimes_R F\overline{G}$, and $M' \otimes_R Fv_0$ as an FG'-submodule of $M' \otimes_R F\overline{G}$. We let

$$\phi : M \otimes_R F \to M \otimes_R Fv_0$$

be the natural FG-module isomorphism, and

$$\phi' : M' \otimes_R F \to M' \otimes_R Fv_0$$

be the natural FG'-module isomorphism. Now

$$N = N_1/N_2 \simeq \phi(N_1)/\phi(N_2)$$

as FS-modules, and

$$N' = N'_1/N'_2 \simeq \phi'\big(N'_1\big)/\phi\big(N'_2\big)$$

as FS'-modules. We let $\widehat{N_i}$ be the FS-submodule of $\widehat{M_F}$ which is $\phi(N_i)$ on the first coordinate and zero on all others. We let $\widehat{N'_i}$ to be the FS'-submodule of $\widehat{M'_F}$ which is $\phi'(N'_i)$ on the first coordinate and zero on all others. Then $\widehat{N_1}/\widehat{N_2}$ is in $\mathcal{A}(M, FS)$, and $\widehat{N'_1}/\widehat{N'_2}$ is in $\mathcal{A}(M', FS')$. Let $\widehat{N_1}/\widehat{N_2}$ correspond to the object $(I, J, \overline{S})$ in $\mathcal{IT}(\underline{\mathrm{End}}(\widehat{M_F}))$. Let $\widehat{N'_1}/\widehat{N'_2}$ correspond to the object $(I', J', \overline{S})$ in $\mathcal{IT}(\underline{\mathrm{End}}(\widehat{M'_F}))$. One can show that $I' = \widehat{\epsilon_F}(I)$ and $J' = \widehat{\epsilon_F}(J)$. Hence, κ_ϵ sends $\widehat{N_1}/\widehat{N_2}$ to $\widehat{N'_1}/\widehat{N'_2}$. It then follows that (3) holds.

Now we prove (4) and we assume its notation. Since the restriction of N to H is completely reducible and the restriction of N' to H' is completely reducible, we have that z acts on N, and z' acts on N', and the resulting module zN is at least an FH-module, and the resulting module $z'N'$ is at least an FH'-module. In a similar way z acts on $\widehat{M_F}$ and z' acts on $\widehat{M'_F}$. Let N correspond to the object $(I, J, \overline{S})$ in $\mathcal{IT}(\underline{\mathrm{End}}(\widehat{M_F}))$. Then N' corresponds to the object $(\epsilon_F(I), \epsilon_F(J), \overline{S})$ in $\mathcal{IT}(\underline{\mathrm{End}}(\widehat{M'_F}))$. The FH-module zN corresponds, up to isomorphism, to $(zI, zJ, \overline{H})$, and FH'-module $z'N'$ corresponds, up to isomorphism, to $(z'\epsilon_F(I), z'\epsilon(J), \overline{H'})$. Since it follows from the definitions that $\epsilon_F(zI) = z'\epsilon_F(I)$ and $\epsilon_F(zJ) = z'\epsilon_F(J)$, it follows that, up to isomorphism κ_ϵ sends zN to $z'N'$. So (4) as desired.

Let K be a field extension of F. Let N be an FS-module in $\mathcal{S}(\widehat{M_F})$. Let N correspond to the object $(I, J, \bar{S})$ in $\mathcal{IT}(\underline{\mathrm{End}}(\widehat{M_F}))$. It follows from the definitions that $\widehat{M_K}$ is canonically isomorphic to $K \otimes_F \widehat{M_F}$, and using this identification we have that $(K \otimes_F I, K \otimes_F J, \bar{S})$ is an object in $\mathcal{IT}(\underline{\mathrm{End}}(\widehat{M_K}))$ which corresponds to a module which is isomorphic to $K \otimes_F N$. An analogous situation holds over in the categories $\mathcal{IT}(\underline{\mathrm{End}}(\widehat{M'_F}))$ and $\mathcal{IT}(\underline{\mathrm{End}}(\widehat{M'_K}))$. Since $\widehat{\epsilon_F}$ and $\widehat{\epsilon_K}$ are compatible with these correspondences, it follows that κ_ϵ commutes up to isomorphism with field extensions.

To see that κ_ϵ commutes with field restrictions, since we are dealing with finitely generated modules, we assume that K is a field extension of F with finite degree. Let N be a KS-module in $\mathcal{S}(\widehat{M_K})$, and let N' be a KS'-module in $\mathcal{S}(\widehat{M'_K})$ corresponding to N under κ_ϵ. Let N correspond to the object $(I, J, \bar{S})$ in $\mathcal{IT}(\underline{\mathrm{End}}(\widehat{M_K}))$, so that N' corresponds to the object $(\epsilon_K(I), \epsilon_K(J), \bar{S})$ in $\mathcal{IT}(\underline{\mathrm{End}}(\widehat{M'_K}))$. Let N_r be the restriction of N to an FS-module, and let N'_r be the restriction of N' to an FS'-module. Let θ be as in Lemma 5.12. Then N_r is isomorphic to $\theta_{M,K,F}(N)$ and N'_r is isomorphic to $\theta_{M',K,F}(N')$. Then N_r corresponds to the object $((\widehat{\theta_{M,K,F}}(I)), (\widehat{\theta_{M,K,F}}(J)), \bar{S})$ in $\mathcal{IT}(\underline{\mathrm{End}}(\widehat{M_F}))$, and N'_r corresponds to the object

$$\bigl(\bigl(\widehat{\theta_{M',K,F}}\bigl(\epsilon_K(I)\bigr)\bigr), \bigl(\widehat{\theta_{M',K,F}}\bigl(\epsilon_K(J)\bigr)\bigr), \bar{S}\bigr)$$

in $\mathcal{IT}(\underline{\mathrm{End}}(\widehat{M'_F}))$, where we indicate the ideal generated by the various images of θ maps by parenthesis. It then follows from Lemma 5.12 that ϵ_F sends the first triple to the second one, so that κ_ϵ sends a module isomorphic to N_r to a module isomorphic to N'_r. Hence, κ_ϵ commutes up to isomorphism with field restrictions, and κ_ϵ satisfies (5).

It only remains to show (6). We need to consider modules over a subgroup S of G. However, in view of Proposition 7.2, it is enough to assume that our module is defined over $S = G$. Let $(I, J, \bar{G})$ be an object in $\mathcal{IT}(\widehat{M_F})$, and let N be its corresponding module in $\mathcal{S}(\widehat{M_F})$. Let $(\epsilon_F(I), \epsilon_F(J), \bar{G})$ be the corresponding object in $\mathcal{IT}(\widehat{M'_F})$, and let N' be its corresponding module in $\mathcal{S}(\widehat{M'_F})$. Then N is an FG-module and N' is an FG'-module. Let P be some finitely generated $F\bar{G}$-module. We need to show that κ_ϵ sends a module isomorphic to $N \otimes_F P$ to a module isomorphic to $N' \otimes_F P$. If $P = 0$ the result holds trivially, so we assume that $P \neq 0$. It follows that $F\bar{G} \otimes_F P$ is isomorphic to the direct sum of a finite positive number of copies of $F\bar{G}$ as $F\bar{G}$-modules. This induces naturally

$$\theta : \widehat{M_F} \otimes_F P \to \widehat{M_F} \quad \text{and} \quad \theta' : \widehat{M'_F} \otimes_F P \to \widehat{M'_F}$$

isomorphisms of topological FG-modules and topological FG'-modules. We denote by $\bar{\theta}$ and by $\overline{\theta'}$ the corresponding isomorphisms of endomorphism algebras:

$$\bar{\theta} : \underline{\mathrm{End}}(\widehat{M_F}) \otimes_F \mathrm{End}_F(P) \to \underline{\mathrm{End}}(\widehat{M_F})$$

and

$$\overline{\theta'} : \underline{\mathrm{End}}\bigl(\widehat{M'_F}\bigr) \otimes_F \mathrm{End}_F(P) \to \underline{\mathrm{End}}\bigl(\widehat{M'_F}\bigr).$$

Furthermore, the following diagram

$$\begin{array}{ccc}
\underline{\mathrm{End}}(\widehat{M_F}) \otimes_F \mathrm{End}_F(P) & \xrightarrow{\widehat{\epsilon_F} \otimes_F \mathrm{Id}} & \underline{\mathrm{End}}(\widehat{M'_F}) \otimes_F \mathrm{End}_F(P) \\
\downarrow{\scriptstyle \bar{\theta}} & & \downarrow{\scriptstyle \overline{\theta'}} \\
\underline{\mathrm{End}}(\widehat{M_F}) & \xrightarrow{\widehat{\epsilon_F}} & \underline{\mathrm{End}}(\widehat{M'_F})
\end{array}$$

is commutative. Set $I_p = I \otimes_F \mathrm{End}_F(P)$ and $J_p = J \otimes_F \mathrm{End}_F(P)$, and set $I'_p = (\widehat{\epsilon_F} \otimes_F \mathrm{Id})(I_p)$ and $J'_p = (\widehat{\epsilon_F} \otimes_F \mathrm{Id})(J_p)$. Let $N_1 = \bar{\theta}(I_p)(\widehat{M_F})$, $N_2 = \bar{\theta}(J_p)(\widehat{M_F})$, $N'_1 = \bar{\theta'}(I'_p)(\widehat{M'_F})$, and $N'_2 = \bar{\theta'}(J'_p)(\widehat{M'_F})$. Then, it follows from Remark 4.8 that $N \otimes_F P$ is isomorphic to N_1/N_2 as FG-modules and $N' \otimes_F P$ is isomorphic to N'_1/N'_2 as FG'-modules. Since κ_ϵ sends N_1/N_2 to N'_1/N'_2, it follows that κ_ϵ satisfies (6). □

Even without any assumption on M and M', it follows from the theorem that, in characteristic zero, the quotient of the degrees of corresponding irreducible modules is a quotient of divisors of $|H|$ and $|H'|$. In general, the quotient may be different for different irreducible modules.

Corollary 7.6. *Assume the hypotheses and notation of Theorem 7.5, and assume that F has characteristic which does not divide $|H||H'|$. Let N be a module which is an object in $\mathcal{A}(M, F)$ and let N' be its corresponding object under κ_ϵ. Set $d = \dim_F(N')/\dim_F(N)$. Then, if N is irreducible, then d is a rational number which is a quotient of a divisor of $|H'|$ by a divisor of $|H|$.*

Proof. We let S be the subgroup of G such that N is an irreducible FS-module, and we let S' be the subgroup of G' such that N' is an irreducible FS'-module. Let K be a finite Galois extension field of F which is a splitting field for all the subgroups of G and of G'. Let N_0 be an irreducible direct summand of the KS-module $K \otimes_F N$. Then if α is the Schur index of N_0 over F and β the number of non-isomorphic Galois conjugates of N_0 under $\mathrm{Gal}(K/F)$ then $\dim_F(N) = \alpha\beta \dim_K(N_0)$. Let N'_0 be the KS'-module corresponding to N_0 under κ_ϵ. By Theorem 7.5 (5), we also have $\dim_F(N') = \alpha\beta \dim_K(N'_0)$, with the same α and β. Hence, $d = \dim_K(N'_0)/\dim_K(N_0)$. Let I be an irreducible direct summand of $\mathrm{Res}^S_H(N_0)$. Now, by Clifford's Theorem, all the irreducible direct summands of $\mathrm{Res}^S_H(N_0)$ have the same degree, so that $\dim_K(N_0) = s \dim_K(I)$ where s is the composition length of the KH-module $\mathrm{Res}^S_H(N_0)$. Let I' be the module corresponding to I under κ_ϵ. By Theorem 7.5, N'_0 is irreducible, so that all irreducible direct summands of $\mathrm{Res}^{S'}_{H'}(N'_0)$ have the same dimension. By Theorem 7.5 (2), the composition length of $\mathrm{Res}^{S'}_{H'}(N')$ as a KH'-module is also s, and one of its direct summands is isomorphic to I'. Hence, $\dim_K(N'_0) = s \dim_K(I')$. Hence,

$$d = \frac{\dim_K(N'_0)}{\dim_K(N_0)} = \frac{\dim_K(I')}{\dim_K(I)}.$$

Since K has characteristic which does not divide $|H||H'|$ and K is a splitting field, we have $\dim_K(I')$ divides $|H'|$ and $\dim_K(I)$ divides $|H|$, and the corollary follows. □

A number of useful concepts are commonly used to describe the modular representation of finite groups. For any field F, and any finite group G, we will use the following notation (compare to [5,2]). We denote by $\mathrm{A}_F(G)$ the Green ring of G over F. It is the $\mathbf{Z}$-module generated by the isomorphism classes of finitely generated FG-modules, with direct sum inducing addition on them, and tensor product inducing multiplication, see [2, p. 92]. If M is any finitely generated FG-module, we denote by $[M]_a$ its (isomorphism) class in $\mathrm{A}_F(G)$, and we denote by $\mathrm{A}^+_F(G)$ the image of this map. We denote by $\mathrm{I}_F(G)$ a complete collection of isomorphism classes of finitely generated indecomposable FG-modules, so that the collection $([I]_a)_{I \in \mathrm{I}_F(G)}$ is a $\mathbf{Z}$-basis for $\mathrm{A}_F(G)$. We denote by $\mathrm{R}_F(G)$ the Grothendieck group of finitely generated FG-modules. $\mathrm{R}_F(G)$ is actually a ring and it is the quotient of $\mathrm{A}_F(G)$ by an ideal generated by $[U]_a - [V]_a + [W]_a$ for all modules U, V, W, for which there is a short exact sequence,

$$0 \to U \to V \to W \to 0,$$

see [2, p. 92], for example. If M is a finitely generated FG-module, we denote by $[M]_r$ its class in $\mathrm{R}_F(G)$, and we denote by $\mathrm{R}^+_F(G)$ the image of this map. We denote by $\mathrm{S}_F(G)$ a complete set of isomorphism classes of irreducible FG-modules, so that the collection $([S]_r)_{S \in \mathrm{S}_F(G)}$ is a $\mathbf{Z}$-basis for $\mathrm{R}_F(G)$. We denote by $\mathrm{P}_F^+(G)$ the subset of $\mathrm{R}_F(G)$ of the classes of finitely generated projective FG-modules, and by $\mathrm{P}_F(G)$ the subring of $\mathrm{R}_F(G)$ generated by $\mathrm{P}_F^+(G)$. We can think of $\mathrm{P}_F(G)$ as a subring of $\mathrm{A}_F(G)$ as well. Hence, we may use $\mathrm{A}_F(G)$ to describe modules up to isomorphism, $\mathrm{R}_F(G)$ to describe when modules have the same (Brauer) characters but without having to make any particular choice of modular system, and $\mathrm{P}_F(G)$ to describe the projective modules, which are uniquely determined by their (Brauer) characters.

These concepts have their corresponding version in the context of the module correspondence, as follows. For convenience, we assume for our definition the hypotheses of the theorem, even though these definitions involve only one group, rather than two.

Definition 7.7. Assume the hypotheses of Theorem 7.1, we let $\bar{S}$ be a subgroup of $\bar{G}$, and we set $S = \pi^{-1}(\bar{S})$, and we pick some F in $\mathcal{F}$. We denote by $\mathrm{A}_F^+(S, M)$ the image in $\mathrm{A}_F(S)$ of $\mathcal{A}(M, FS)$, and we denote by $\mathrm{A}_F(S, M)$ the subgroup of $\mathrm{A}_F(S)$ generated by $\mathrm{A}_F^+(S, M)$. We denote by $\mathrm{I}_F(S, M)$ a complete collection of isomorphism classes of finitely generated indecomposable FS-modules in $\mathcal{A}(M, FS)$, so that the collection $([I]_a)_{I \in \mathrm{I}_F(S,M)}$ is a $\mathbf{Z}$-basis for $\mathrm{A}_F(G, M)$. Likewise, we denote by $\mathrm{R}_F^+(S, M)$ the image in $\mathrm{R}_F(S)$ of $\mathcal{A}(M, FS)$, and we denote by $\mathrm{R}_F(S, M)$ the subgroup of $\mathrm{R}_F(S)$ generated by $\mathrm{R}_F^+(S, M)$. We denote by $\mathrm{S}_F(S, M)$ a complete collection of isomorphism classes of irreducible FS-modules in $\mathcal{A}(M, FS)$, so that the collection $([I']_r)_{I' \in \mathrm{S}_F(S,M)}$ is a $\mathbf{Z}$-basis for $\mathrm{R}_F(G, M)$. We denote by $\mathrm{P}_F^+(S, M)$ the set of $\mathrm{R}_F(S)$ of the classes of finitely generated projective FS-modules in $\mathcal{A}(M, FS)$, and we denote by $\mathrm{P}_F(S, M)$ the subring of $\mathrm{R}_F(S)$ generated by $\mathrm{P}_F^+(S, M)$. We can think of $\mathrm{P}_F(S, M)$ as a subring of $\mathrm{A}_F(S)$ as well.

We note that $\mathrm{A}_F(S, M)$ is naturally an $\mathrm{A}_F(\bar{S})$-module, and that $\mathrm{R}_F(S, M)$ is naturally an $\mathrm{R}_F(\bar{S})$-module, where, in each case, the module multiplication is induced from the ring structure of $\mathrm{A}_F(S)$ and $\mathrm{R}_F(S)$ respectively, which in turn derive from the tensor product of modules.

Corollary 7.8. *Assume the hypotheses and notation of Theorem 7.5. Then the following hold*:

(1) κ_ϵ *induces uniquely an* $\mathrm{A}_F(\bar{S})$*-module isomorphism*

$$\mathcal{H}_\epsilon(S, F) : \mathrm{A}_F(S, M) \to \mathrm{A}_F\big(S', M'\big).$$

(2) $\mathcal{H}_\epsilon(S, F)(\mathrm{A}_F^+(S, M)) = \mathrm{A}_F^+(S', M')$.
(3) $\mathcal{H}_\epsilon(S, F)(\mathrm{I}_F(S, M))$ *is in bijection with* $\mathrm{I}_F(S', M')$.
(4) κ_ϵ *induces uniquely an* $\mathrm{R}_F(\bar{S})$*-module isomorphism*

$$\mathcal{H}'_\epsilon(S, F) : \mathrm{R}_F(S, M) \to \mathrm{R}_F\big(S', M'\big).$$

(5) $\mathcal{H}'_\epsilon(S, F)(\mathrm{S}_F(S, M))$ *is in bijection with* $\mathrm{S}_F(S', M')$.
(6) *If both H and H' are p'-groups, and F has characteristic $p \neq 0$, then* $\mathcal{H}_\epsilon(S, F)(\mathrm{P}_K^+(S, M)) = \mathrm{P}_K^+(S', M')$, *and the restriction of* $\mathcal{H}_\epsilon(S, F)$ *provides an isomorphism* $\mathrm{P}_K(S, M) \to \mathrm{P}_K(S', M')$.

Proof. This follows directly from Theorem 7.5 and the definition of these groups and sets. □

8. Compatibility

The module correspondence κ_ϵ is determined by the endoisomorphism ϵ, and we saw in the previous section that each κ_ϵ has some excellent properties. In this section, we study how the module correspondence κ_ϵ varies naturally as we vary ϵ by some standard operations. The results are natural and follow directly from the definitions. We record them here so we can use them later. We note that Proposition 7.2 is a compatibility statement that we have already established.

Proposition 8.1. *Let G, G', G'' and $\overline{G}$ be finite groups, and suppose we are given surjective homomorphisms $\pi : G \to \overline{G}$, $\pi' : G' \to \overline{G}$ $\pi'' : G'' \to \overline{G}$ whose kernels are, respectively, H, H', and H''. Let R be an integral domain. Suppose M is a finitely generated RG-module which is free as an R-module, M' is a finitely generated RG'-module which is free as an R-module, and M'' is a finitely generated RG''-module which is free as an R-module. Let*

$$\epsilon : \mathrm{End}_{RH}(M) \to \mathrm{End}_{RH'}\big(M'\big)$$

be an endoisomorphism from M to M', and let

$$\epsilon' : \mathrm{End}_{RH'}\big(M'\big) \to \mathrm{End}_{RH''}\big(M''\big)$$

be an endoisomorphism from M' to M''. Let $\mathcal{F}$ be a collection of field extensions of R which are good for M, good for M', and good for M''. Then $\epsilon'\epsilon$ is an endoisomorphism, and for each F in $\mathcal{F}$, the isomorphism of $\overline{G}$-algebras

$$\overline{\epsilon'\epsilon_F} : \mathrm{Z}(M, \pi, F) \to \mathrm{Z}\big(M'', \pi'', F\big),$$

is simply the composition $\overline{\epsilon'_F}\,\overline{\epsilon_F}$, and the isomorphism $\kappa_{\epsilon'\epsilon}$ of categories from $\mathcal{A}(M, F)$ to $\mathcal{A}(M'', F)$ is the composition of $\kappa_{\epsilon'}$ and κ_ϵ.

Proof. This follows directly from the definitions. □

Proposition 8.2. *Let G, and $\overline{G}$ be finite groups, and suppose we are given a surjective homomorphism $\pi : G \to \overline{G}$, whose kernel is H. Let R be an integral domain. Suppose M and M' are finitely generated RG-modules which are free as R-modules. Let*

$$\phi : M \to M'$$

be a module isomorphism. Then ϕ induces

$$\epsilon : \mathrm{End}_{RH}(M) \to \mathrm{End}_{RH}\big(M'\big)$$

an endoisomorphism from M to M'. Let $\mathcal{F}$ be a collection of field extensions of R which are good for M and good for M'. Then for each F in $\mathcal{F}$, the isomorphism κ_ϵ of categories from $\mathcal{A}(M, F)$ to $\mathcal{A}(M', F)$ associates to each module a module isomorphic to it.

Proof. This follows directly from the definitions. □

Proposition 8.3. *Let G, G' and $\overline{G}$ be finite groups, and suppose we are given surjective homomorphisms $\pi : G \to \overline{G}$, and $\pi' : G' \to \overline{G}$ whose kernels are H and H'. Let R be an integral domain. Suppose M is a finitely generated RG-module which is free as an R-module, and M' is a finitely generated RG'-module which is free as an R-module. Let*

$$\epsilon : \mathrm{End}_{RH}(M) \to \mathrm{End}_{RH'}\big(M'\big)$$

be an endoisomorphism from M to M'. Let F be a good extension field of R for M and M'. Then we obtain an endoisomorphism

$$\epsilon_F : \mathrm{End}_{FH}(F \otimes_R M) \to \mathrm{End}_{FH'}\big(F \otimes_R M'\big)$$

and, up to module isomorphisms, $\mathcal{A}(F \otimes_R M)$ can be viewed as being contained in $\mathcal{A}(M)$, $\mathcal{A}(F \otimes_R M')$ can be viewed as being contained in $\mathcal{A}(M')$, and κ_{ϵ_F} agrees with the appropriate restriction of κ_ϵ.

Proof. This follows directly from the definitions. □

Proposition 8.4. *Let G, G' and $\overline{G}$ be finite groups, and suppose we are given surjective homomorphisms $\pi : G \to \overline{G}$, and $\pi' : G' \to \overline{G}$ whose kernels are H and H'. Let U be a normal subgroup of G such that $U \cap H = 1$ and let U' be a normal subgroup of G' such that $U' \cap H' = 1$ and suppose $\pi(U) = \pi'(U') = \overline{U}$. Let $\tau : G/U \to \overline{G}/\overline{U}$ be the surjective group homomorphism induced by π, and let $\tau' : G'/U' \to \overline{G}/\overline{U}$ be the surjective group homomorphism induced by π'. Let R be an integral domain. Suppose M is a finitely generated RG/U-module which is free as an R-module, and M' is a finitely generated RG'/U'-module which is free as an R-module. Let*

$$\epsilon : \mathrm{End}_{RHU/U}(M) \to \mathrm{End}_{RH'U'/U'}\big(M'\big)$$

be an endoisomorphism from M to M' with respect to τ and τ'. Let N be M as an RG-module and let N' be M' as an RG'-module. Then, the same map ϵ can be viewed as an endoisomorphism

$$\epsilon_0 : \mathrm{End}_{RH}(N) \to \mathrm{End}_{RH'}\big(N'\big).$$

Let $\mathcal{F}$ be a collection of field extensions of R which are good for M, and good for M'. Then for each F in $\mathcal{F}$, there are natural

$$\mathrm{Z}(N, \pi, F) \simeq \mathrm{Z}(M, \tau, F)$$

and

$$\mathrm{Z}\big(N', \pi', F\big) \simeq \mathrm{Z}\big(M', \tau', F\big)$$

and the isomorphism

$$\overline{\epsilon_{0F}} : \mathrm{Z}(N, \pi, F) \to \mathrm{Z}\big(N', \pi', F\big)$$

is induced from $\overline{\epsilon_F}$ and the isomorphisms, and the isomorphism κ_ϵ of categories from $\mathcal{A}(M, F)$ to $\mathcal{A}(M', F)$ agrees, on the appropriate modules after appropriate identifications, up to isomorphisms with the isomorphism κ_{ϵ_0} of categories from $\mathcal{A}(N, F)$ to $\mathcal{A}(N', F)$.

Proof. This follows directly from the definitions. □

9. Characteristic zero

When working over fields of characteristic zero, the results become considerably easier to describe. In this section, we describe this important special case. We recover, with the added benefit of the uniqueness of the correspondence, some of the results in [8,9]. Since to describe modules in characteristic zero it is enough to describe cyclic modules, some of the machinery set up earlier in the paper would be unnecessary if we were only interested in this case as we were in [8,9]. The fact that the character correspondence arises from the module correspondence provides useful additional information.

As is well known, the representations in characteristic zero can be efficiently studied with the help of characters. Ordinary characters take complex values, or more specifically, they take values in $\mathbf{Q}(e)$ the complex field of e-th roots of unity, where e is any positive integer multiple of the exponent of the group in question. In order to assign characters, one needs to fix some relationship between $\mathbf{Q}(e)$ and the field of definition of the module, and such relationship can be established for all fields of characteristic zero. For most purposes, however, it is sufficient to choose one field of characteristic zero K that contains a splitting field for $X^e - 1$, and to assign characters to modules over subfields of K. Choices of such K could be $\mathbf{C}$ or the field of all algebraic numbers in $\mathbf{C}$, or, more generally, any extension field of the field of all algebraic numbers in $\mathbf{C}$, but we do not need to have such large fields for our purposes here.

Hypotheses 9.1. Let K be a field, let e be a positive integer, and assume that K contains a subfield (identified with) $\mathbf{Q}(e)$. Further, we assume that R is subring of K which is a principal ideal domain, and F is the field of fractions of R.

Definition 9.2. Let G be a finite group whose exponent divides e and let M be a finitely generated FG-module. Then the *character* χ afforded by M is the map

$$\chi : G \to \mathbf{C}$$

such that $\chi(g)$ is the trace of the F-linear transformation induced by the action of g on M for all $g \in G$.

Of course the trace of the action of g on M is in $F \cap \mathbf{Q}(e)$, so we can think of it as a complex number.

Definition 9.3. A character χ of a finite group G is said to be *irreducible* if it is the character of some irreducible KG-module. The set of all irreducible characters of G is denoted $\mathrm{Irr}(G)$.

As is well known, we have a bijection between the isomorphism classes of irreducible KG-modules and $\mathrm{Irr}(G)$. Furthermore, $\mathrm{Irr}(G)$ are linearly independent over $\mathbf{C}$. In addition, the $\mathbf{Z}$-module $\mathbf{Z}\,\mathrm{Irr}(G)$ generated by $\mathrm{Irr}(G)$ is naturally isomorphic to $\mathrm{A}_K(G) = \mathrm{R}_K(G)$.

The concept of *good* field extension is unnecessary in the context of this section.

Proposition 9.4. *Assume Hypotheses 5.1 and Hypotheses 9.1. Then every characteristic 0 field extension of R is good for M.*

Proof. This follows immediately from Proposition 5.5. □

Assume Hypotheses 5.1, and assume that F is a field extension in characteristic zero of R. Then $J(FH) = 0$, and we take the π-center algebra of FG to be $\overline{G}$-algebra Z over F of the following form. We set $Z_0 = \mathrm{Z}(FH)$, so that Z_0 is a $\overline{G}$-algebra over F, and, for some idempotent e of $Z_0^{\overline{G}}$ we have $Z = eZ_0$. Let e be the sum of all the primitive idempotents of $Z_0^{\overline{G}}$ which act non-trivially on $F \otimes_R M$. We say that eZ_0 is the *π-center algebra of FG associated with $F \otimes_R M$*, and we write $\mathrm{Z}(M, \pi, F) = eZ_0$. This agrees with the earlier definition, except that, for the more general earlier definition, we need to take the quotient by the radical of FH.

Proposition 9.5. *Assume Hypotheses 5.1 and Hypotheses 9.1. Then the F-algebra homomorphisms from $\mathrm{Z}(M, \pi, F)$ to K are in natural one-to-one correspondence with the central characters of the irreducible characters which are summands of the character afforded by $\mathrm{Res}^G_H(K \otimes_R M)$.*

Proof. The central characters of FH are in one-to-one correspondence with the elements of $\mathrm{Irr}(H)$, and the ones which restrict non-trivially to $\mathrm{Z}(M, \pi, F)$ are exactly those which correspond to the irreducible summands of the character afforded by $\mathrm{Res}^G_H(K \otimes_R M)$. □

It is standard to define what it means for an irreducible character to be above some irreducible character of some normal subgroup. We will need a slight generalization of this standard definition, where we will not assume that the character of the normal subgroup is irreducible.

Definition 9.6. Let G be a finite group and let H be a normal subgroup of G, and let ζ be some ordinary character of H. We let $\mathrm{Irr}(G|\zeta)$ denote the set of all irreducible characters of G whose restriction to H contains some irreducible summand which is also a summand of ζ.

This concept is closely connected to the concept of modules *above* another one.

Proposition 9.7. *Assume Hypotheses 5.1 and Hypotheses 9.1, and assume that the exponent of G divides e. Let ζ be the sum of all the distinct irreducible characters contained in the character afforded by $\mathrm{Res}^G_H(K \otimes_R M)$. Let S be a subgroup of G that contains H. Let χ be a character of S. Then, χ is the character afforded by some module in $\mathcal{A}(M, KS)$ if and only if χ is an $\mathbf{N}$-linear combination of the elements of $\mathrm{Irr}(S|\zeta)$.*

Proof. This follows from Proposition 6.2. $\square$

In this context, the correspondences of modules described earlier yield unique correspondences of characters.

Theorem 9.8. *Assume Hypotheses 9.1. Let G, G' and $\overline{G}$ be finite groups, and suppose we are given surjective homomorphisms $\pi : G \to \overline{G}$ and $\pi' : G' \to \overline{G}$ whose kernels are, respectively, H and H'. Assume that the exponent of G and the exponent of G' both divide e. Suppose M is a finitely generated FG-module, and M' is a finitely generated FG'-module. Let*

$$\epsilon : \mathrm{End}_{FH}(M) \to \mathrm{End}_{FH'}\big(M'\big)$$

be an endoisomorphism from M to M'. Let ζ be the sum of all the distinct irreducible characters contained in the character afforded by $\mathrm{Res}^G_H(M)$, let ζ' be the sum of all the distinct irreducible characters contained in the character afforded by $\mathrm{Res}^{G'}_{H'}(M')$. For each subgroup $\overline{S}$ of $\overline{G}$, we set $S = \pi^{-1}(\overline{S})$ and $S' = (\pi')^{-1}(\overline{S})$. Then κ_ϵ provides isomorphisms of $\mathbf{Z}$-modules

$$\kappa_\epsilon : \mathbf{Z}\,\mathrm{Irr}(S|\zeta) \to \mathbf{Z}\,\mathrm{Irr}\big(S'\big|\zeta'\big).$$

Furthermore, these have the following properties.

(1) *κ_ϵ gives bijections*

$$\mathrm{Irr}(S|\zeta) \to \mathrm{Irr}\big(S'\big|\zeta'\big).$$

(2) *κ_ϵ commutes with restriction of characters and with induction of characters.*
(3) *κ_ϵ sends the characters which are summands of the character afforded by $K \otimes_F M$ to summands of the character afforded by $K \otimes_F M'$.*
(4) *The isomorphism of $\overline{G}$-algebras*

$$\overline{\epsilon_K} : \mathrm{Z}(M, \pi, K) \to \mathrm{Z}\big(M', \pi', K\big)$$

(Theorem 7.1) acts on some central characters (Proposition 9.5) and this action determines the bijection

$$\kappa_\epsilon : \mathrm{Irr}(H|\zeta) \to \mathrm{Irr}\big(H'\big|\zeta'\big).$$

(5) *Both* $\mathbf{Z}\,\mathrm{Irr}(S|\zeta)$ *and* $\mathbf{Z}\,\mathrm{Irr}(S'|\zeta')$ *are* $\mathbf{Z}\,\mathrm{Irr}(\bar{S})$*-modules, and the map*

$$\kappa_\epsilon : \mathbf{Z}\,\mathrm{Irr}(S|\zeta) \to \mathbf{Z}\,\mathrm{Irr}\bigl(S'\big|\zeta'\bigr)$$

is an isomorphism of $\mathbf{Z}\,\mathrm{Irr}(\bar{S})$*-modules.*

(6) *The isomorphism*

$$\kappa_\epsilon : \mathbf{Z}\,\mathrm{Irr}(S|\zeta) \to \mathbf{Z}\,\mathrm{Irr}\bigl(S'\big|\zeta'\bigr)$$

commutes with the action of $\mathrm{Gal}(K/F)$.

(7) κ_ϵ *commutes with conjugation by* $\bar{G}$.

(8) *For every* $\chi \in \mathbf{Z}\,\mathrm{Irr}(S|\zeta)$*, we have* $F(\kappa_\epsilon(\chi)) = F(\chi)$.

(9) *For every* $\chi \in \mathrm{Irr}(S|\zeta)$*, let* $[\chi]$ *denote the element of the Brauer group* $\mathrm{Br}(F(\chi))$ *associated with it. Then we have* $[\kappa_\epsilon(\chi)] = [\chi]$*. In particular, the Schur indices of the irreducible characters are preserved under* κ_ϵ.

(10) *Suppose that all the irreducible characters contained in* ζ *have the same degree, and all the irreducible characters contained in* ζ' *have the same degree. Then there is some rational constant d, not depending on* $\bar{S}$*, such that whenever* $\chi \in \mathbf{Z}\,\mathrm{Irr}(S|\zeta)$ *is sent to* $\kappa_\epsilon(\chi) = \chi' \in \mathbf{Z}\,\mathrm{Irr}(S'|\zeta')$*, then* $\chi'(1) = d\chi(1)$.

Proof. That the isomorphisms exist and are unique follows from Theorem 7.1 and Proposition 9.7. The rest of the theorem follows from Theorem 7.5. Perhaps it is worth giving a few more details about (4). Let $\chi \in \mathrm{Irr}(H|\zeta)$, and let $\chi' = \kappa_\epsilon(\chi) \in \mathrm{Irr}(H'|\zeta')$. Then χ corresponds to a primitive idempotent $e_\chi \in \mathrm{Z}(KH)$ and χ' corresponds to a primitive idempotent $e_{\chi'} \in \mathrm{Z}(KH)$. It follows from Theorem 7.5 (4) that $\overline{\epsilon_K}(e_\chi) = e_{\chi'}$. Hence, $\overline{\epsilon_K}$ alone determines $\kappa_\epsilon(\chi)$, and it does so on the basis of the central characters described in Proposition 9.5. $\square$

In view of applications, it is convenient to relate the results that we have obtained so far to known results about the Brauer–Clifford group. We refer the reader to [12] for unexplained definitions, notations and further details.

Corollary 9.9. *Assume Hypotheses 9.1. Let* G*,* G' *and* $\bar{G}$ *be finite groups, and suppose we are given surjective homomorphisms* $\pi : G \to \bar{G}$*, and* $\pi' : G' \to \bar{G}$ *whose kernels are* H *and* H'*. Assume that the exponent of* G *and the exponent of* G' *both divide e. Let* U *be a normal subgroup of* G *such that* $U \cap H = 1$ *and let* U' *be a normal subgroup of* G' *such that* $U' \cap H' = 1$ *and suppose* $\pi(U) = \pi'(U') = \bar{U}$*. Let* $\tau : G/U \to \bar{G}/\bar{U}$ *be the surjective group homomorphism induced by* π*, and let* $\tau' : G'/U' \to \bar{G}/\bar{U}$ *be the surjective group homomorphism induced by* π'*. Let* $\theta \in \mathrm{Irr}(H)$ *and let* $\theta' \in \mathrm{Irr}(H')$*. Let* θ_1 *be the character of* $\ker(\tau)$ *corresponding to* θ*, and let* θ'_1 *be the character of* $\ker(\tau')$ *corresponding to* θ'*. Set* $Z = \mathrm{Z}(\theta_1, F, \tau)$ *and* $Z' = \mathrm{Z}(\theta'_1, F, \tau')$ *be the respective center algebras, and let* $[[\theta_1]] \in \mathrm{BrClif}(\bar{G}/\bar{U}, Z)$ *and* $[[\theta'_1]] \in \mathrm{BrClif}(\bar{G}/\bar{U}, Z')$ *be the elements of the Brauer–Clifford group that correspond to the respective characters. Suppose there is a* $\bar{G}/\bar{U}$*-algebra isomorphism*

$$\alpha : Z \to Z'$$

such that α *sends the central character associated with* θ_1 *to the central character associated with* θ'_1*, and, with*

$$\bar{\alpha} : \mathrm{BrClif}(\bar{G}/\bar{U}, Z) \to \mathrm{BrClif}\bigl(\bar{G}/\bar{U}, Z'\bigr)$$

being the induced group isomorphism, we have

$$\bar{\alpha}\bigl([[\theta_1]]\bigr) = \bigl[[\theta'_1]\bigr].$$

Then, there exist M a finitely generated θ-quasi-homogeneous FG-module whose kernel contains U, M' a finitely generated θ'-quasi-homogeneous FG'-module whose kernel contains U', and an endoisomorphism

$$\epsilon : \mathrm{End}_{FH}(M) \to \mathrm{End}_{FH'}\big(M'\big)$$

such that:

(1) *κ_ϵ satisfies all the properties described in Theorem 9.8*;
(2) *In this theorem, ζ is the sum of the $G \times \mathrm{Gal}(K/F)$-orbit of θ, and ζ' is the sum of the $G' \times \mathrm{Gal}(K/F)$-orbit of θ'*;
(3) *For each subgroup $\bar{S}$ of $\bar{G}$, we set $S = \pi^{-1}(\bar{S})$ and $S' = (\pi')^{-1}(\bar{S})$. Then κ_ϵ provides isomorphisms of $\mathbf{Z}$-modules*

$$\kappa_\epsilon : \mathbf{Z}\,\mathrm{Irr}(S|\zeta) \to \mathbf{Z}\,\mathrm{Irr}\big(S'\big|\zeta'\big);$$

In addition, we set $f = \theta'(1)/\theta(1)$. Then for each $\chi \in \mathbf{Z}\,\mathrm{Irr}(S|\zeta)$, if we set $\chi' = \kappa_\epsilon(\chi)$ then $\chi'(1) = f\chi(1)$;
(4) *κ_ϵ sends θ to θ'*:

$$\kappa_\epsilon(\theta) = \theta'.$$

Proof. By [12, Proposition 4.7], there exist M a finitely generated θ_1-quasi-homogeneous FG/U-module, M' a finitely generated θ_1'-quasi-homogeneous FG'/U'-module, and an endoisomorphism

$$\epsilon : \mathrm{End}_{FH}(M) \to \mathrm{End}_{FH'}\big(M'\big).$$

Looking at the proof of [12, Proposition 4.7] we see that ϵ *induces* the isomorphism α, and sends the central character associated with θ_1 to the central character associated with θ_1'. By Proposition 8.4, ϵ is also an endoisomorphism from M to M' when viewed respectively as FG-module and as FG'-module. Hence, we obtain the properties of Theorem 9.8. Furthermore, by Proposition 8.4 and Property (4) of Theorem 9.8, it follows that

$$\kappa_\epsilon(\theta) = \theta'.$$

Finally, it follows from the definition of quasi-homogeneous, that the sum of the distinct irreducible summands of the character afforded by $\mathrm{Res}^G_H(M)$ is the sum of all the $G \times \mathrm{Gal}(K/F)$-conjugates of θ, and the sum of the distinct irreducible summands of the character afforded by $\mathrm{Res}^{G'}_{H'}(M')$ is the sum of all the $G' \times \mathrm{Gal}(K/F)$-conjugates of θ'. Since all the irreducible summands of ζ have the same degree, and all the irreducible summands of ζ' have the same degree, Property (10) of the theorem completes the proof of the corollary. Hence, the corollary holds. $\square$

10. Correspondences of characters and blocks

In the previous section, we used characters to describe the representations of groups in characteristic zero. It is also convenient and customary to study the blocks of finite groups with the help of characters. In this section, we study properties of blocks under the correspondences. We set up hypotheses to discuss both ordinary and Brauer characters for the relevant finite groups. Some results in this section are special cases of results of the previous section in a slightly different context: we state them here together with their corresponding results about Brauer characters for clarity.

Ordinary characters and Brauer characters take complex values, or more specifically, they take values in $\mathbf{Q}(e)$ the complex field of e-th roots of unity, where e is any positive integer multiple of the exponent of the group in question. In order to assign characters, one needs to fix some relationship between $\mathbf{Q}(e)$ and the field of definition of the module, and such relationship can be established for all fields. For most purposes, however, it is sufficient to choose, for each relevant characteristic, one field of this characteristic that contains a splitting field for $X^e - 1$, and to assign characters to modules over subfields of such fields.

Hypotheses 10.1. Let A be a principal ideal domain with field of fractions K. Let e be a positive integer, and assume that K contains a subfield (identified with) $\mathbf{Q}(e)$. We let p be any prime rational integer which is not a unit of A, and we let $\mathfrak{M}$ be any maximal ideal of A such that $p \in \mathfrak{M}$. We set $k = A/\mathfrak{M}$.

Under Hypotheses 10.1, A is integrally closed in K, and in particular, it contains all e-th roots of unity in $\mathbf{C}$. It follows that K is a field of characteristic zero which contains a splitting field for the polynomial $X^e - 1$ over the prime subfield, and k is a field of characteristic p which contains a splitting field for the polynomial $X^e - 1$ over the prime subfield. Furthermore, the projection homomorphism $A \to k$ restricts to a group isomorphism from the multiplicative group of $e_{p'}$-th roots of unity in $A^\times$ to the multiplicative group of $e_{p'}$-th roots of unity in $k^\times$.

In the previous section, we reminded the reader how to define ordinary characters in the present context. We can also define Brauer characters.

Definition 10.2. Let k_0 be some subfield of k, let G be a finite group whose exponent divides e and let M be a finitely generated k_0G-module. We let $G_{p'}$ be the subset of G of all elements of p' order, that is, the p-regular elements of G. Then the *Brauer character* χ afforded by M is the map

$$\chi : G_{p'} \to \mathbf{C}$$

defined as follows. Let $g \in G_{p'}$. Let $\overline{\epsilon_1}, \dots, \overline{\epsilon_n} \in k^\times$ be the eigenvalues with multiplicities of the k-linear action of g on $k \otimes_{k_0} M$, and let $\epsilon_1, \dots, \epsilon_n \in A^\times$ be the corresponding p'-roots of unity in $\mathbf{C}$ under the group isomorphism of induced by the projection $A \to k$. Then, we set

$$\chi(g) = \epsilon_1 + \dots + \epsilon_n.$$

Definition 10.3. A character χ of a finite group G is said to be *irreducible* if it is the character of some irreducible KG-module. The set of all irreducible characters of G is denoted $\mathrm{Irr}(G)$. Likewise, a Brauer character χ is said to be *irreducible* if it is the Brauer character of some irreducible kG-module, and the set of all irreducible Brauer characters of G is denoted $\mathrm{IBr}(G) = \mathrm{IBr}_p(G) = \mathrm{IBr}_{\mathfrak{M}}(G)$.

As is well known, we have a bijection between the isomorphism classes of irreducible KG-modules and $\mathrm{Irr}(G)$ and, likewise, we have a bijection between the isomorphism classes of irreducible kG-modules and $\mathrm{IBr}(G)$. Furthermore, both $\mathrm{Irr}(G)$ and $\mathrm{IBr}(G)$ are linearly independent over $\mathbf{C}$. In addition, the $\mathbf{Z}$-module $\mathbf{Z}\,\mathrm{Irr}(G)$ generated by $\mathrm{Irr}(G)$ is naturally isomorphic to $\mathrm{A}_K(G) = \mathrm{R}_K(G)$, and the $\mathbf{Z}$-module $\mathbf{Z}\,\mathrm{IBr}(G)$ generated by $\mathrm{IBr}(G)$ is naturally isomorphic to $\mathrm{R}_k(G)$.

In addition, there is a process of *reduction modulo* p of finitely generated KG-modules, and it induces the *decomposition homomorphism*. For each finite group G, the decomposition homomorphism $d_{G,p}$ is a group homomorphism

$$d_{G,p} : \mathrm{R}_K(G) \to \mathrm{R}_k(G)$$

from the Grothendieck group $\mathrm{R}_K(G)$ of representations over K to the Grothendieck group $\mathrm{R}_k(G)$ of representations over k. Given a finitely generated KG-module M, one may choose a *lattice* E (i.e. a finitely generated A-submodule of M which generates M as a K-module). One may pick E to be G-invariant by simply adding together all the images of E under the action of each element of G, and replacing E by this sum. Hence, we take E to be G-invariant lattice. This yields $E/\mathfrak{M}E$ which is a kG-module, and is called *a reduction modulo p* of M. This module is not unique even up to isomorphism. However, for all choices of E, the modules $E/\mathfrak{M}E$ have, up to isomorphism, the same composition factors. Then the map $d_{G,p}$ is the unique group homomorphism which sends, for all M, the class in $\mathrm{R}_K(G)$ of M to the class in $\mathrm{R}_k(G)$ of $E/\mathfrak{M}E$. See, for example, [5] for details. Since $\mathrm{R}_K(G)$ is naturally isomorphic to the $\mathbf{Z}$-module of generalized characters $\mathbf{Z}\,\mathrm{Irr}(G)$ and $\mathrm{R}_k(G)$ is naturally isomorphic to the $\mathbf{Z}$-module of generalized Brauer characters $\mathbf{Z}\,\mathrm{IBr}_p(G)$, we also get a group homomorphism with the same name

$$d_{G,p} : \mathbf{Z}\,\mathrm{Irr}(G) \to \mathbf{Z}\,\mathrm{IBr}_p(G).$$

In particular, for each $\chi \in \mathrm{Irr}(G)$, we can write

$$d_{G,p}(\chi) = \sum_{\psi \in \mathrm{IBr}(G)} d_{\chi,\psi}\psi,$$

for some unique $d_{\chi,\psi} \in \mathbf{N}$ called the *decomposition numbers*. The following proposition describes the map $d_{G,p}$ in terms of characters. It is well known, and follows easily from the definitions.

Proposition 10.4. *Let $\chi \in \mathbf{Z}\,\mathrm{Irr}(G)$, and set $\psi = d_{G,p}(\chi)$. Then ψ is simply the restriction to $G_{p'}$ of χ.*

It is standard to define what it means for an irreducible character to be above some irreducible character of some normal subgroup. We will need a slight generalization of this standard definition (and of Definition 9.6) where we will not assume that the character of the normal subgroup is irreducible.

Definition 10.5. Let G be a finite group and let H be a normal subgroup of G, and let ζ be some ordinary character of H. We let $\mathrm{Irr}(G|\zeta)$ denote the set of all irreducible characters of G whose restriction to H contains some irreducible summand which is also a summand of ζ. Similarly, if η is some Brauer character of H, we denote by $\mathrm{IBr}(G|\eta)$ the set of all irreducible Brauer characters of G whose restriction to H contains some irreducible summand which is also a summand of η.

This concept is closely connected to the concept of modules *above* another one.

Proposition 10.6. *Assume Hypotheses 5.1 and Hypotheses 10.1, and assume that the exponent of G divides e. Suppose that A contains the image of R in K. Assume furthermore that the extension of R obtained from the inclusion of R in A followed by the projection onto k is a good extension for M. Let ζ be the sum of all the distinct irreducible characters contained in the character afforded by $\mathrm{Res}^G_H(K \otimes_R M)$, and let η be the sum of all the distinct irreducible Brauer characters contained in the character afforded by $\mathrm{Res}^G_H(k \otimes_R M)$. Let S be a subgroup of G that contains H. Let χ be a character of S and let ψ be a Brauer character of S. Then, χ is the character afforded by some module in $\mathcal{A}(M, KS)$ if and only if χ is an $\mathbf{N}$-linear combination of the elements of $\mathrm{Irr}(S,\zeta)$. Furthermore, ψ is the Brauer character afforded by some module in $\mathcal{A}(M, kS)$ if and only if ψ is an $\mathbf{N}$-linear combination of the elements of $\mathrm{IBr}(S,\eta)$.*

Proof. By Proposition 5.5, K is a good extension of R for M. The result then follows from Proposition 6.2. $\square$

In this context, the correspondences of modules described in Section 7 yield unique correspondences of characters.

Theorem 10.7. *Let G, G' and $\overline{G}$ be finite groups, and suppose we are given surjective homomorphisms $\pi : G \to \overline{G}$ and $\pi' : G' \to \overline{G}$ whose kernels are, respectively, H and H'. Let R be an integral domain. Suppose*

M is a finitely generated RG-module which is free as an R-module, and M' is a finitely generated RG'-module which is free as an R-module. Let

$$\epsilon : \mathrm{End}_{RH}(M) \to \mathrm{End}_{RH'}\big(M'\big)$$

be an endoisomorphism from M to M'. Assume furthermore Hypotheses 10.1, and assume that the exponent of G and the exponent of G' both divide e. Suppose that A contains the image of R in K. We set F to be the smallest subfield of K containing the image of R. Assume furthermore that the extension of R obtained from the inclusion of R in A followed by the projection onto k is a good field extension for M and for M'. Let ζ be the sum of all the distinct irreducible characters contained in the character afforded by $\mathrm{Res}^G_H(K \otimes_R M)$, let ζ' be the sum of all the distinct irreducible characters contained in the character afforded by $\mathrm{Res}^{G'}_{H'}(K \otimes_R M')$, let η be the sum of all the distinct irreducible Brauer characters contained in the character afforded by $\mathrm{Res}^G_H(k \otimes_R M)$, and let η' be the sum of all the distinct irreducible Brauer characters contained in the character afforded by $\mathrm{Res}^{G'}_{H'}(k \otimes_R M')$. For each subgroup $\overline{S}$ of $\overline{G}$, we set $S = \pi^{-1}(\overline{S})$ and $S' = (\pi')^{-1}(\overline{S})$. Then κ_ϵ provides isomorphisms of $\mathbf{Z}$-modules

$$\kappa_\epsilon : \mathbf{Z}\,\mathrm{Irr}(S|\zeta) \to \mathbf{Z}\,\mathrm{Irr}\big(S'\big|\zeta'\big)$$

and

$$\kappa_\epsilon : \mathbf{Z}\,\mathrm{IBr}(S|\eta) \to \mathbf{Z}\,\mathrm{IBr}\big(S'\big|\eta'\big).$$

Furthermore, these have the following properties.

(1) *κ_ϵ gives bijections*

$$\mathrm{Irr}(S|\zeta) \to \mathrm{Irr}\big(S'\big|\zeta'\big)$$

and

$$\mathrm{IBr}(S|\eta) \to \mathrm{IBr}\big(S'\big|\eta'\big).$$

(2) *κ_ϵ commutes with restriction of characters and with induction of characters.*
(3) *κ_ϵ sends the characters which are summands of the character afforded by $K \otimes_R M$ to summands of the character afforded by $K \otimes_R M'$, and similarly for $k \otimes_R M$ and $k \otimes_R M'$.*
(4) *The isomorphism of $\overline{G}$-algebras*

$$\overline{\epsilon_K} : \mathrm{Z}(K \otimes_R M, \pi, K) \to \mathrm{Z}\big(K \otimes_R M', \pi', K\big)$$

(Theorem 7.1) acts on some central characters (Proposition 9.5) and this action determines the bijection

$$\kappa_\epsilon : \mathrm{Irr}(H|\zeta) \to \mathrm{Irr}\big(H'\big|\zeta'\big).$$

(5) *Both $\mathbf{Z}\,\mathrm{Irr}(S|\zeta)$ and $\mathbf{Z}\,\mathrm{Irr}(S'|\zeta')$ are $\mathbf{Z}\,\mathrm{Irr}(\overline{S})$-modules, and the map*

$$\kappa_\epsilon : \mathbf{Z}\,\mathrm{Irr}(S|\zeta) \to \mathbf{Z}\,\mathrm{Irr}\big(S'\big|\zeta'\big)$$

is an isomorphism of $\mathbf{Z}\,\mathrm{Irr}(\overline{S})$-modules. Similarly, the map

$$\kappa_\epsilon : \mathbf{Z}\,\mathrm{IBr}(S|\eta) \to \mathbf{Z}\,\mathrm{IBr}\big(S'\big|\eta'\big)$$

is an isomorphism of $\mathbf{Z}\,\mathrm{IBr}(\bar{S})$*-modules.*

(6) *The isomorphism*

$$\kappa_\epsilon : \mathbf{Z}\,\mathrm{Irr}(S|\zeta) \to \mathbf{Z}\,\mathrm{Irr}\big(S'\big|\zeta'\big)$$

commutes with the action of $\mathrm{Gal}(K/F)$.

(7) κ_ϵ *commutes with conjugation by* $\bar{G}$.

(8) *For every* $\chi \in \mathbf{Z}\,\mathrm{Irr}(S|\zeta)$*, we have* $F(\kappa_\epsilon(\chi)) = F(\chi)$.

(9) *For every* $\chi \in \mathrm{Irr}(S|\zeta)$*, let* $[\chi]$ *denote the element of the Brauer group* $\mathrm{Br}(F(\chi))$ *associated with it. Then we have* $[\kappa_\epsilon(\chi)] = [\chi]$*. In particular, the Schur indices of the irreducible characters are preserved under* κ_ϵ.

(10) *Suppose that all the irreducible characters contained in* ζ *have the same degree, and all the irreducible characters contained in* ζ' *have the same degree. Then there is some rational constant* d*, not depending on* $\bar{S}$*, such that whenever* $\chi \in \mathbf{Z}\,\mathrm{Irr}(S|\zeta)$ *is sent to* $\kappa_\epsilon(\chi) = \chi' \in \mathbf{Z}\,\mathrm{Irr}(S'|\zeta')$*, then* $\chi'(1) = d\chi(1)$.

Proof. By Proposition 5.5, K is a good extension of R for M and for M'. The result then follows either directly from Theorem 9.8 or from the results that proved that theorem. □

One can say more when the characters ζ and ζ' are sums of p-defect zero irreducible characters. We first set up some notation.

Definition 10.8. Let G be a finite group and let H be a normal subgroup of G. Suppose that ζ is a character of H which is a sum of p-defect zero irreducible characters. We denote by $\mathrm{Bl}(G, p) = \mathrm{Bl}(G, \mathfrak{M}) = \mathrm{Bl}(G)$ the set of all p-blocks of G. For convenience we consider blocks as sets of irreducible ordinary and Brauer characters. We denote by $\mathrm{Bl}(G, p|\zeta) = \mathrm{Bl}(G|\zeta)$ the set of all p-blocks of G which contain at least one irreducible character in $\mathrm{Irr}(G|\zeta)$.

Let η be the reduction modulo p of ζ. We note that if $B \in \mathrm{Bl}(G|\zeta)$ then

$$B \subseteq \mathrm{Irr}(G|\zeta) \cup \mathrm{IBr}(G|\eta),$$

and, in fact, $\mathrm{Bl}(G|\zeta)$ can be viewed as a partition of $\mathrm{Irr}(G|\zeta) \cup \mathrm{IBr}(G|\eta)$.

Lemma 10.9. *Assume the hypotheses of Theorem 10.7. Suppose that every irreducible summand of* ζ *has* p*-defect zero, and every irreducible summand of* ζ' *has* p*-defect zero. Suppose that* $\bar{S}$ *is a* p'*-group. Let* $\chi \in \mathrm{Irr}(S|\zeta)$*, let* $\chi_0 = d_{S,p}(\chi)$ *be its reduction modulo* p*, let* $\chi' = \kappa_\epsilon(\chi)$*, and let* $\chi_0' = d_{S',p}(\chi')$ *be the reduction modulo* p *of* χ'*. Then* χ *and* $\chi' \in \mathrm{Irr}(S'|\zeta')$ *are characters of* p*-defect zero,* $\chi_0 \in \mathrm{IBr}(S|\eta)$*,* $\chi_0' \in \mathrm{IBr}(S'|\eta')$*, and*

$$\kappa_\epsilon(\chi_0) = \chi_0'.$$

Proof. Notice that the p-part of the order of S equals the p-part of the order of H, and similarly, the p-part of the order of S' equals the p-part of the order of H'. Since a character of p-defect zero is one where the p-part of the degree is equal to the p-part of the order of the group, and the degree of χ is divisible by the degree of some irreducible summand of ζ, it follows that χ has p-defect zero, and, similarly, χ' has p-defect zero. In particular, $\chi_0 \in \mathrm{IBr}(S|\eta)$ and $\chi_0' \in \mathrm{IBr}(S'|\eta')$.

Set $M_1 = \mathrm{Res}^G_S(\widehat{M_K})$, and $M_2 = \mathrm{Res}^{G'}_{S'}(\widehat{M'_K})$. We also set $M_3 = \mathrm{Res}^G_S(\widehat{M_A})$, and $M_4 = \mathrm{Res}^{G'}_{S'}(\widehat{M'_A})$. In addition, we set $M_5 = k \otimes_A M_3$, and $M_6 = k \otimes_A M_4$. We set $A_1 = \underline{\mathrm{End}}(M_1)$ and $A_2 = \underline{\mathrm{End}}(M_2)$. We set

$$A_3 = \{f \in A_1\colon\ f(M_3) \subseteq M_3\}$$

and

$$A_4 = \{f \in A_2\colon\ f(M_4) \subseteq M_4\}.$$

We have natural identifications $A_3 = \underline{\mathrm{End}}(M_3)$ and $A_4 = \underline{\mathrm{End}}(M_4)$. We also set $A_5 = \underline{\mathrm{End}}(M_5)$ and $A_6 = \underline{\mathrm{End}}(M_6)$. Our assumptions now imply that we have an $\bar{S}$-algebra isomorphism

$$\widehat{\epsilon_K} : A_1 \to A_2,$$

that $\widehat{\epsilon_K}(A_3) \subseteq A_4$, and that we have an $\bar{S}$-algebra isomorphism

$$\widehat{\epsilon_k} : A_5 \to A_6.$$

Furthermore, for each $f \in A_3$ and $\alpha \in k$, we may identify $\alpha \otimes_A f$ with the element of A_5 such that $(\alpha \otimes_A f)(\beta \otimes_A m_3) = \alpha\beta \otimes_A f(m_3)$ for all $\beta \in k$, $m_3 \in M_3$, and we have

$$\widehat{\epsilon_k}(\alpha \otimes_A f) = \alpha \otimes_A \widehat{\epsilon_K}(f)$$

using a similar identification of $\alpha \otimes_A \widehat{\epsilon_K}(f)$ with an element of A_6.

There is a KS-submodule N_1 of M_1 which affords the character χ. Since M_1 is completely reducible and N_1 is finitely generated, there is an open KS-submodule of M_1 which is a complement to N_1. We denote by $a : M_1 \to M_1$ the projection KS-homomorphism whose image is N_1 and whose kernel is this complement. Now $a(M_3)$ is a non-zero finitely generated AS-submodule of N_1. There exists a K-multiple f_1 of a such that $f_1(M_3) \subseteq M_3$ but $f_1(M_3) \nsubseteq \mathfrak{M}M_3$. Now $f_1 \in A_3$. Set $I_1 = f_1A_1$, and set $L_1 = f_1(M_3)$. Then I_1 is a finitely generated right ideal of A_1, and the KS-module N_1 is (isomorphic to) the module that corresponds to the triple $(I_1, \{0\}, \bar{S})$. Furthermore, L_1 is a lattice in N_1, and a reduction modulo p of N_1 is given by $L_1/\mathfrak{M}L_1$. In particular, $L_1/\mathfrak{M}L_1$ affords the Brauer character χ_0. It follows that $L_1/\mathfrak{M}L_1$ is irreducible as a kS-module, and $L_1 \cap \mathfrak{M}M_3 = \mathfrak{M}L_1$. Set $L_3 = L_1 + \mathfrak{M}M_3$, and $L_5 = L_3/\mathfrak{M}M_3$. Then L_5 is a kS-module affording χ_0. Set $f_5 = 1_k \otimes_A f_1 \in A_5$, and set $I_5 = f_5A_5$. Then the module L_5 is (isomorphic to) the module that corresponds to the triple $(I_5, \{0\}, \bar{S})$. In particular, $\chi_0 \in \mathrm{IBr}(S, \eta)$.

Let $f_2 = \widehat{\epsilon_K}(f_1) \in A_2$, $N_2 = f_2(M_2)$, and $I_2 = f_2A_2$. Then f_2 is a KS'-homomorphism, I_2 is an $\bar{S}$-invariant finitely generated right ideal of A_2, and N_2 is a KS'-module which is isomorphic to the module that corresponds to the triple $(I_2, \{0\}, \bar{S})$. Since $(I_1, \{0\}, \bar{S})$ is mapped to $(I_2, \{0\}, \bar{S})$ under the map induced from $\widehat{\epsilon_K}$, in particular, N_2 affords the character χ'. By our choice, we further know that $f_1 \in A_3$, and it follows that $f_2 \in A_4$. We set $L_2 = f_2(M_4)$. Since the kernel of f_2 is open, we have that L_2 is a finitely generated AS'-submodule of M_4. Since $N_2 = f_2(M_2)$, we have that L_2 contains a K-basis for N_2, and it follows that L_2 is an S'-invariant lattice for N_2. Therefore $L_2/\mathfrak{M}L_2$ is an irreducible kS'-module and affords the Brauer character χ_0'. Then, $L_2 \cap \mathfrak{M}M_4 = \mathfrak{M}L_2$. Set $L_4 = L_2 + \mathfrak{M}M_4$, and $L_6 = L_4/\mathfrak{M}M_4$. Then L_6 is a kS'-module affording χ_0'. Set $f_6 = 1_k \otimes_A f_2 \in A_6$, and set $I_6 = f_6A_6$. Then the module L_6 is (isomorphic to) the module that corresponds to the triple $(I_6, \{0\}, \bar{S})$. In particular, $\chi_0' \in \mathrm{IBr}(S, \eta')$. Since $(I_5, \{0\}, \bar{S})$ is mapped to $(I_6, \{0\}, \bar{S})$ under the map induced from $\widehat{\epsilon_k}$, we have that $\kappa_\epsilon(\chi_0) = \chi_0'$. This completes the proof of the lemma. $\square$

Theorem 10.10. *Assume the hypotheses of Theorem 10.7. Suppose that every irreducible summand of ζ has p-defect zero, and every irreducible summand of ζ' has p-defect zero. Then, the isomorphisms induced by κ_ϵ have the following additional properties. We let S be any subgroup of G that contains H, and we let S' be the corresponding subgroup of G', that is $H' \subseteq S'$ and $\pi(S) = \pi'(S')$.*

(1) *κ_ϵ commutes with the decomposition map. More precisely, the following diagram commutes*:

$$\begin{tikzcd} \mathbf{Z}\,\mathrm{Irr}(S|\zeta) \arrow[r, "\kappa_\epsilon"] \arrow[d, "d_{S,p}"'] & \mathbf{Z}\,\mathrm{Irr}(S'|\zeta') \arrow[d, "d_{S',p}"'] \\ \mathbf{Z}\,\mathrm{IBr}(S|\eta) \arrow[r, "\kappa_\epsilon"] & \mathbf{Z}\,\mathrm{IBr}(S'|\eta') \end{tikzcd}$$

(2) *κ_ϵ sends p-blocks of S to p-blocks of S'. More precisely, the action of κ_ϵ on irreducible ordinary characters and irreducible Brauer characters induces a bijection*

$$\mathrm{Bl}(S|\zeta) \to \mathrm{Bl}\big(S'\big|\zeta'\big).$$

(3) *Assume the notation of* (2)*. Let $B \in \mathrm{Bl}(S|\zeta)$ and let $B' \in \mathrm{Bl}(S'|\zeta')$ be the corresponding block under κ_ϵ. Then κ_ϵ provides bijections*

$$\mathrm{Irr}(B) \to \mathrm{Irr}\big(B'\big)$$

and

$$\mathrm{IBr}(B) \to \mathrm{IBr}\big(B'\big).$$

(4) *Assume the notation of* (3)*. Order $\mathrm{Irr}(B)$ and $\mathrm{IBr}(B)$, and use the bijection of* (3) *to order $\mathrm{Irr}(B')$ and $\mathrm{IBr}(B')$. Then, the decomposition matrices and the Cartan matrices of B and of B' are equal.*

(5) *For each $B \in \mathrm{Bl}(S|\zeta)$ there exists a rational number f_B such that, whenever $\chi \in \mathrm{Irr}(B)$ is sent to $\chi' \in \mathrm{Irr}(B')$ under κ_ϵ, and $\psi \in \mathrm{IBr}(B)$ is sent to $\psi' \in \mathrm{IBr}(B')$ under κ_ϵ, then*

$$f_B = \frac{\chi'(1)}{\chi(1)} = \frac{\psi'(1)}{\psi(1)}.$$

Furthermore, the p-part of f_B is the p-part of $|H'|$ divided by the p-part of $|H|$.

(6) *Assume the notation of* (3)*. p-blocks also* contain *finitely generated indecomposable kS-modules. κ_ϵ provides a bijection between the isomorphism classes of finitely generated indecomposable kS-modules in B and the isomorphism classes of finitely generated indecomposable kS'-modules in B'. Furthermore, let N be a finitely generated indecomposable kS-module in B, and let V be a vertex for N. Let N' be the kS'-modules corresponding to N under κ_ϵ, and let V' be a vertex for N'. Then $\pi(V)$ and $\pi'(V')$ are conjugate in $\pi(S)$.*

(7) *Assume the notation of* (3)*. Let D be a defect group of B, and let D' be a defect group of B'. Then D, D', $\pi(D)$, and $\pi'(D')$ are all isomorphic, and $\pi(D)$ and $\pi'(D')$ are conjugate in $\pi(S)$.*

(8) *The bijection from* (3)*,*

$$\mathrm{Irr}(B) \to \mathrm{Irr}\big(B'\big)$$

given by κ_ϵ preserves the heights. In other words, if $\chi \in \mathrm{Irr}(B)$ is sent to $\kappa_\epsilon(\chi) = \chi' \in \mathrm{Irr}(B')$, then the p-height of χ equals the p-height of χ'.

(9) *Assume the notation of* (7) *and that $\pi(D) = \pi'(D')$. Assume in addition, that H and H' are p'-groups and that S_0 is a subgroup of G which contains H and is such that $D\mathrm{C}_S(D) \subseteq S_0 \subseteq S$. Let $b \in \mathrm{Bl}(S_0)$ be the Brauer correspondent to B. Furthermore, we let S'_0 be the subgroup of G' corresponding to S_0. Then $D'\mathrm{C}_{S'}(D') \subseteq S'_0$, the block $b' \in \mathrm{Bl}(S'_0)$ which is the Brauer correspondent to B' is defined, $b \in \mathrm{Bl}(S_0|\zeta)$, $b' \in \mathrm{Bl}(S'_0|\zeta')$ and κ_ϵ sends b to b' under* (2)*.*

Proof. Assume that (1) does not hold, and among all counterexamples choose one with $|\pi(S)|$ as small as possible. Since the maps in the diagram are all **Z**-module homomorphisms, it follows that there exists some $\chi \in \mathrm{Irr}(S|\zeta)$ for which the diagram does not commute. We set $\chi' = \kappa_\epsilon(\chi) \in \mathrm{Irr}(S'|\zeta')$, and we denote $\chi_0 = d_{S,p}(\chi) \in \mathbf{N}\,\mathrm{IBr}(S|\eta)$ and $\chi_0' = d_{S',p}(\chi') \in \mathbf{N}\,\mathrm{IBr}(S'|\eta')$. We also set $\xi = \kappa_\epsilon(\chi_0) \in \mathbf{N}\,\mathrm{IBr}(S'|\eta')$. We have $\xi \neq \chi_0'$. In particular, there is a p'-element $s' \in S'$ such that $\chi_0'(s') \neq \xi(s')$. By Theorem 10.7 the maps in the diagram all commute with restriction to subgroups that contain H (or H'), so, by our choice of a counterexample, we must have that $\pi(S) = \pi'(\langle s'\rangle)$. In particular, $\bar{S}$ is a p'-group. Then Lemma 10.9 yields a contradiction. Hence, (1) holds.

Since the irreducible characters and the irreducible Brauer characters in a block can be characterized from the decomposition map, it follows from (1) that (2), (3), and (4) all hold.

Let $B \in \mathrm{Bl}(S,\zeta)$ and $B' = \kappa_\epsilon(B)$. Assume that $\chi \in \mathrm{Irr}(B)$ is sent to $\chi' = \kappa_\epsilon(\chi) \in \mathrm{Irr}(B')$, and $\psi \in \mathrm{IBr}(B)$ is sent to $\psi' = \kappa_\epsilon(\psi) \in \mathrm{IBr}(B')$. Since ζ is a sum of characters of p-defect zero, the irreducible characters contained in the restriction of χ to H, which are all conjugate under the action of S, form a set which depends only on B. Let $\xi \in \mathrm{Irr}(H)$ be contained in the restriction $\mathrm{Res}^S_H(\chi)$, and let $d_B = \xi(1)$ be its degree. Of course, d_B depends only on B, and not on χ or our choice of ξ. Let $\xi_0 = d_{H,p}(\xi) \in \mathrm{IBr}(H)$, let $\xi' = \kappa_\epsilon(\xi) \in \mathrm{Irr}(H')$, and let $\xi_0' = \kappa_\epsilon(\xi_0) \in \mathrm{IBr}(H')$. We let $d_{B'} = \xi'(1)$. Then $d_{B'}$ only depends on B, and $\xi(1) = \xi_0(1) = d_B$ and $\xi'(1) = \xi_0'(1) = d_{B'}$. We set $f_B = d_{B'}/d_B$. Let α_χ be the number of irreducible characters contained in $\mathrm{Res}^S_H(\chi)$ counting multiplicities, and let β_ψ be the number of irreducible Brauer characters contained in $\mathrm{Res}^S_H(\psi)$ counting multiplicities. Since ξ_0 is one of the irreducible Brauer characters in $\mathrm{Res}^S_H(\psi)$, we have $\chi(1) = \alpha_\chi \xi(1) = \alpha_\chi d_B$ and $\psi(1) = \beta_\psi \xi_0(1) = \beta_\psi d_B$. By Theorem 10.7, we have that ξ' is contained in $\mathrm{Res}^{S'}_{H'}(\chi')$, that the number of irreducible characters contained in $\mathrm{Res}^{S'}_{H'}(\chi')$ counting multiplicities is α_χ, that ξ_0' is contained in $\mathrm{Res}^{S'}_{H'}(\psi')$, and that the number of irreducible Brauer characters contained in $\mathrm{Res}^{S'}_{H'}(\psi')$ counting multiplicities is β_ψ. It follows that $\chi'(1) = \alpha_\chi \xi'(1) = \alpha_\chi d_{B'}$ and $\psi'(1) = \beta_\psi \xi_0'(1) = \beta_\psi d_{B'}$. Hence,

$$\frac{\chi'(1)}{\chi(1)} = \frac{\alpha_\chi d_{B'}}{\alpha_\chi d_B} = f_B$$

and

$$\frac{\psi'(1)}{\psi(1)} = \frac{\beta_\psi d_{B'}}{\beta_\psi d_B} = f_B.$$

Since ξ is contained in ζ, ξ is of p-defect zero and the p-part of $\xi(1)$ is the p-part of the order of H, and similarly the p-part of $\xi'(1)$ is the p-part of the order of H'. Hence, (5) holds.

The finitely generated indecomposable modules of B are those which afford as Brauer character some elements of $\mathbf{N}\,\mathrm{Irr}(B)$, and similarly for the finitely generated indecomposable modules of B'. Hence, κ_ϵ provides a bijection between the isomorphism classes of finitely generated indecomposable modules of B and the isomorphism classes of finitely generated indecomposable modules of B'. Let N be a finitely generated indecomposable kS-module in B and let V be a vertex for N. It follows that N is isomorphic to an object in $\mathcal{A}(M,kS)$, so we assume that N is an object in $\mathcal{A}(M,kS)$. We let N' be the element of $\mathcal{A}(M',kS')$ corresponding to N under κ_ϵ. Let V' be a vertex for N'. Since V is a vertex, V is minimal among the subgroups of S such that N is V-projective, and all such minimal subgroups are S-conjugate to V. It follows that VH is minimal among the subgroups of S that contain H and are such that N is VH-projective. By for example [2, Theorem I, 4.8], for any subgroup S_1 of S, N is S_1-projective if and only if N is isomorphic to a direct summand of $\mathrm{Ind}^S_{S_1}(\mathrm{Res}^S_{S_1}(N))$. Let $(VH)'$ be the subgroup of G' corresponding to VH, that is, $H' \subseteq (VH)'$ and $\pi'((VH)') = \pi(VH)$. It then follows from Theorem 7.5 that $(VH)'$ is minimal among all the subgroups of S' which contain H' and are such that N' is $(VH)'$-projective. Therefore, N' has some vertex $V_1 \subseteq (VH)'$. We then have $V_1H' = (VH)'$ by the minimality of $(VH)'$. Since V' and V_1 are conjugate in S', it follows that $(VH)'$ and $V'H'$ are conjugate in S', and, therefore, $\pi(V)$ and $\pi'(V')$ are conjugate in $\pi(S)$. Hence, (6) holds.

The defect groups of a block B are all conjugate to each other. By for example [1, Theorem 13.5] and [1, Corollary 14.5], the defect groups of a block B can be characterized as the maximal elements under inclusion among all the vertices of all the finitely generated indecomposable kS-modules in B. Assume the hypotheses of (7). Then it follows from (6) that $\pi(D)$ is conjugate to a subgroup of $\pi'(D')$ and conversely. Hence $\pi(D)$ and $\pi'(D')$ have the same order and they are conjugate to each other. Furthermore, it follows from Knörr's Theorem [3, Theorem (9.26)] that, since ζ is a sum of characters of defect zero, $D \cap H = 1$, and similarly, $D' \cap H' = 1$. Hence D and $\pi(D)$ are isomorphic and D' and $\pi'(D')$ are isomorphic, so that (7) holds.

Assume the hypotheses of (8), and let D be a defect group of B, and let D' be a defect group of B'. Let p^n be the p-part of $|\pi(S)|$, let p^{n_1} be the p-part of $|H|$, and let p^{n_2} be the p-part of $|H'|$. By (7), B and B' have the same defect, say d, and we have $|D| = |D'| = p^d$. Let χ have height h. Then the p-part of $\chi(1)$ is p^{n+n_1-d+h}. By (5), the p-part of $\chi'(1)$ is p^{n+n_2-d+h}. It follows that χ' has height h. Hence, (8) holds.

Finally, assume the hypotheses of (9). Since D and D' are p-groups and H and H' are p'-groups, it follows that $\pi(\mathrm{C}_S(D)) = \mathrm{C}_{\pi(S)}(\pi(D)) = \pi'(\mathrm{C}_{S'}(D'))$. Therefore $D'\mathrm{C}_{S'}(D') \subseteq S'_0 \subseteq S'$. We note that since $D\mathrm{C}_S(D) \subseteq S_0$ the existence of the block b is already guaranteed. Since $b \in \mathrm{Bl}(S_0)$ is the Brauer correspondent to B, we have that $b^S = B$. It follows that $b \in \mathrm{Bl}(S_0, \zeta)$. Let $b'_1 \in \mathrm{Bl}(S'_0, \zeta')$ be the block of S'_0 corresponding to b under κ_ϵ. Since $b'_1 \in \mathrm{Bl}(S'_0, \zeta')$ and it has defect D', we know that $(b'_1)^{S'}$ is defined, and $(b'_1)^{S'} \in \mathrm{Bl}(S', \zeta')$. It only remains to show that $(b'_1)^{S'} = B'$. Let $\theta \in \mathrm{Irr}(b)$ and set the induced character

$$\theta^S = \sum_{\chi \in \mathrm{Irr}(S)} a_\chi \chi$$

for suitable $a_\chi \in \mathbf{N}$. Recall that, for an integer n, we denote by n_p the p-part of n. Then, by a result of Brauer for example [3, Corollary (6.4)], for $B_1 \in \mathrm{Bl}(S)$, we have

$$\theta^S(1)_p = \Bigl(\sum_{\chi \in \mathrm{Irr}(B_1)} a_\chi \chi(1)\Bigr)_p \quad \text{if } B_1 = B,$$

$$\theta^S(1)_p < \Bigl(\sum_{\chi \in \mathrm{Irr}(B_1)} a_\chi \chi(1)\Bigr)_p \quad \text{if } B_1 \neq B.$$

Denoting by $\chi' \in \mathrm{Irr}(S'|\zeta')$ the character which corresponds to any $\chi \in \mathrm{Irr}(S|\zeta)$ under κ_ϵ, and using (5), it follows that

$$\theta'^S(1)_p = \Bigl(\sum_{\chi \in \mathrm{Irr}(B_1)} a_\chi \chi'(1)\Bigr)_p \quad \text{if } B_1 = B,$$

$$\theta'^S(1)_p < \Bigl(\sum_{\chi \in \mathrm{Irr}(B_1)} a_\chi \chi'(1)\Bigr)_p \quad \text{if } B_1 \neq B.$$

By the same result of Brauer and the properties of the correspondence κ_ϵ that we have already established, this implies that $(b'_1)^{S'} = B'$. Hence, (9) holds. This completes the proof of the theorem. □

Remark 10.11. The special case of the previous theorem which is most important for the applications is the case when both H and H' are p'-groups. In this case it is not necessary to use Knörr's Theorem.

11. Endoisomorphisms from fields to subrings

Our definition of endoisomorphism includes the possibility that our modules are defined over rings. We now see that, if we have an endoisomorphism over the field of fractions of some principal ideal domain, we will also have an endoisomorphism over the principal ideal domain itself. More details on orders, lattices and related concepts can be found in [4].

Definition 11.1. Let R be a principal ideal domain, and let F be the field of fractions of R. Let A be a finite dimensional algebra over F. Then an R-*order* in A is an R-algebra O contained in A, containing an F-basis for A, and such that every element of O is integral over R. (Integral over R means that it is a zero of some non-zero monic polynomial with coefficients in R.)

Lattices have already been mentioned earlier. Next, for clarity, we recall their definition in the present context.

Definition 11.2. Let R be a principal ideal domain, and let F be the field of fractions of R. Let M be a finite dimensional vector space over F. Then, a *lattice* of M over R is an R-submodule L of M which is finitely generated as an R-module, and contains an F-basis for M.

One way to obtain R-orders in our context is to start with some lattice.

Lemma 11.3. *Let G and $\overline{G}$ be finite groups, and suppose we are given a surjective homomorphism $\pi : G \to \overline{G}$ whose kernel is H. Let R be a principal ideal domain, and let F be the field of fractions of R. Suppose that M is a finitely generated FG-module, and we set $A = \mathrm{End}_{FH}(M)$. Let L be any G-invariant lattice of M. Then set*

$$O = \{a \in A\colon\ a(L) \subseteq L\}.$$

Then O is a $\overline{G}$-invariant R-order in A.

Proof. Since R is a principal ideal domain, and L is finitely generated there exists a free R-basis $e_1, \dots, e_n$ for L. This is also an F-basis for M. We have that O is a $\overline{G}$-invariant R-subalgebra of A. If $o \in O$, then the matrix of o has coefficients in R, and so its characteristic polynomial is with coefficients in R, and it follows that o is integral over R. Let $a \in A$. By considering the action of a on the basis for M, we see that there exists some non-zero $r \in R$ such that $ra \in O$. It follows that O contains an F-basis for A. Hence, O is an R-order in A, as desired. □

Theorem 11.4. *Let G and $\overline{G}$ be finite groups, and suppose we are given a surjective homomorphism $\pi : G \to \overline{G}$ whose kernel is H. Let R be a principal ideal domain, and let F be the field of fractions of R. Suppose that M is a finitely generated FG-module, and that the restriction $\mathrm{Res}^G_H(M)$ is completely reducible. We set $A = \mathrm{End}_{FH}(M)$. Then, any R-order in A is finitely generated as an R-module. Furthermore, A has some R-order O which is $\overline{G}$-invariant and which is maximal among the $\overline{G}$-invariant orders of A.*

Proof. We first show that every R-order O in A is finitely generated as an R-module. Suppose that $\mathrm{Res}^G_H(M)$ has n different homogeneous components,

$$M = N_1 \oplus \cdots \oplus N_n$$

and we set $B_i = \mathrm{End}_{FH}(N_i)$ for $i = 1, \dots, n$. Then B_i is a simple algebra, and

$$A = B_1 \oplus \cdots \oplus B_n.$$

We let $\pi_i : A \to B_i$ be the projection homomorphism, and e_i to be the identity of B_i. We let K_i be the center of B_i, and we let S_i be the integral closure of Re_i in K_i. Since K_i is a finite Galois extension of Fe_i, we have that S_i is Noetherian, and S_i is finitely generated as an R-module. Furthermore, $\pi_i(O)$ is an R-order of B_i. Let O_i be the S_i-span of $\pi_i(O)$. Since S_i-linear combinations of elements which are integral over R are also integral over S_i, we have that O_i is an S_i-order of B_i, where we view B_i as an algebra over K_i. By, for example, [4, Theorem 6.5], we know that O_i is finitely generated as an S_i-module. It follows that $\pi_i(O)$ is finitely generated as an R-module. This implies that O is finitely generated as an R-module, as desired.

Pick an F-basis for M, and let L be the R-submodule of M generated by this basis and all the images of the basis elements under the elements of G. Then L is a G-invariant lattice of M. By Lemma 11.3, it follows that A has some $\overline{G}$-invariant R-order O_1. By Zorn's Lemma, there exists a maximal element O among the orders which are $\overline{G}$-invariant and contain O_1. □

Lemma 11.5. *Assume the hypotheses of Theorem 11.4, and let O be a maximal $\overline{G}$-invariant R-order of A. Then, there is a G-invariant lattice L of M such that*

$$O = \{a \in A\colon a(L) \subseteq L\}.$$

Proof. Take an F-basis $e_1, \dots, e_n$ for M. By Theorem 11.4, O is finitely generated as an R-module. Let $b_1, \dots, b_m$ be a finite set of generators for O as an R-module. Let L be the R-submodule of M generated by all the gb_ie_j for all $g \in G$, $i = 1, \dots, m$ and $j = 1, \dots, n$. Then L is an O-module and also G-invariant. Furthermore, L is finitely generated as an R-module, so L is a lattice of M. Now, we have

$$O \subseteq \{a \in A\colon a(L) \subseteq L\}.$$

Since by Lemma 11.3 the right-hand side is a $\overline{G}$-invariant R-order of A, and O is a maximal $\overline{G}$-invariant R-order, this implies that we actually have equality. This completes the proof of the lemma. □

Theorem 11.6. *Assume Hypotheses 2.5. Suppose F is the field of fractions of a principal ideal domain R. Let M be a finitely generated FG-module, and let M' be a finitely generated FG'-module such that the restrictions $\mathrm{Res}^G_H(M)$ and $\mathrm{Res}^{G'}_{H'}(M')$ are completely reducible. Let*

$$\epsilon : \mathrm{End}_{FH}(M) \to \mathrm{End}_{FH'}(M')$$

be an endoisomorphism over F from M to M'. Then there exist a G-invariant R-lattice L of M, and a G'-invariant R-lattice L' of M' and an endoisomorphism

$$\nu : \mathrm{End}_{RH}(L) \to \mathrm{End}_{RH'}(L')$$

from L to L' such that, with proper identifications, we have

$$\mathrm{End}_{FH}(M) = F \otimes_R \mathrm{End}_{RH}(L),$$

$$\mathrm{End}_{FH'}(M') = F \otimes_R \mathrm{End}_{RH'}(L'),$$

$$\epsilon = \mathrm{Id}_F \otimes_R \nu.$$

Proof. Set $A = \mathrm{End}_{FH}(M)$ and $A' = \mathrm{End}_{FH'}(M')$. By Theorem 11.4, there is a maximal $\bar{G}$-invariant R-order O for A. We set $O' = \epsilon(O)$, so that O' is maximal $\bar{G}$-invariant R-order for A'. We let $\epsilon_0 : O \to O'$ be the restriction of ϵ, so that ϵ_0 is an isomorphism of $\bar{G}$-algebras over R. It follows from Lemma 11.5 that there exist a G-invariant lattice L of M, and a G'-invariant lattice L' of M' such that

$$O = \{a \in A\colon a(L) \subseteq L\},$$

and

$$O' = \{a \in A'\colon a(L') \subseteq L'\}.$$

Now restrictions provide an isomorphism $O \to \mathrm{End}_{RH}(L)$, and an isomorphism $O' \to \mathrm{End}_{RH'}(L')$. Hence, ϵ_0 induces an isomorphism

$$\nu : \mathrm{End}_{RH}(L) \to \mathrm{End}_{RH'}(L')$$

of $\bar{G}$-algebras over R, i.e. an endoisomorphism from L to L'. It is clear that $F \otimes_R O$ is naturally identified with $\mathrm{End}_{FH}(M)$, and $F \otimes_R O'$ is naturally identified with $\mathrm{End}_{FH'}(M')$. The identifications stated in the theorem follow naturally from this and the construction given above. $\square$

In view of applications, it is convenient to again relate the results that we have obtained so far to known results about the Brauer–Clifford group. Again, we refer the reader to [12] for unexplained definitions, notations and further details.

Corollary 11.7. *Assume Hypotheses 10.1. Let R be a principal ideal domain with field of fractions F, and assume that K is a field extension of F and that A contains R. Let G, G' and $\bar{G}$ be finite groups, and suppose we are given surjective homomorphisms $\pi : G \to \bar{G}$, and $\pi' : G' \to \bar{G}$ whose kernels are H and H'. Assume that the exponent of G and the exponent of G' both divide e, and that p does not divide $|H||H'|$. Let U be a normal subgroup of G such that $U \cap H = 1$ and let U' be a normal subgroup of G' such that $U' \cap H' = 1$ and suppose $\pi(U) = \pi'(U') = \bar{U}$. Let $\tau : G/U \to \bar{G}/\bar{U}$ be the surjective group homomorphism induced by π, and let $\tau' : G'/U' \to \bar{G}/\bar{U}$ be the surjective group homomorphism induced by π'. Let $\theta \in \mathrm{Irr}(H)$ and let $\theta' \in \mathrm{Irr}(H')$. Let θ_1 be the character of $\ker(\tau)$ corresponding to θ, and let θ_1' be the character of $\ker(\tau')$ corresponding to θ'. Set $Z = \mathrm{Z}(\theta_1, F, \tau)$ and $Z' = \mathrm{Z}(\theta_1', F, \tau')$ be the respective center algebras, and let $[[\theta_1]] \in \mathrm{BrClif}(\bar{G}/\bar{U}, Z)$ and $[[\theta_1']] \in \mathrm{BrClif}(\bar{G}/\bar{U}, Z')$ be the elements of the Brauer–Clifford group that correspond to the respective characters. Suppose there is a $\bar{G}/\bar{U}$-algebra isomorphism*

$$\alpha : Z \to Z'$$

such that α sends the central character associated with θ_1 to the central character associated with θ_1', and, with

$$\bar{\alpha} : \mathrm{BrClif}(\bar{G}/\bar{U}, Z) \to \mathrm{BrClif}(\bar{G}/\bar{U}, Z')$$

being the induced group isomorphism, we have

$$\bar{\alpha}([[\theta_1]]) = [[\theta_1']].$$

Then, there exist M a finitely generated RG-module whose kernel contains U, M' a finitely generated RG'-module whose kernel contains U', and an endoisomorphism

$$\epsilon : \mathrm{End}_{RH}(M) \to \mathrm{End}_{RH'}\big(M'\big)$$

such that all of the following are satisfied.

(1) *Both M and M' are free as R-modules.*
(2) *Both F and k are good extensions of R for both M and for M'.*
(3) *$F \otimes_R M$ is a finitely generated θ-quasi-homogeneous FG-module whose kernel contains U, and $F \otimes_R M'$ is a finitely generated θ'-quasi-homogeneous FG'-module whose kernel contains U'.*
(4) *The hypotheses and conclusions of Theorem 10.7 and Theorem 10.10 hold.*
(5) *In these theorems, ζ is the sum of the $G \times \mathrm{Gal}(K/F)$-orbit of θ, ζ' is the sum of the $G' \times \mathrm{Gal}(K/F)$-orbit of θ', and as maps the Brauer characters are $\eta = \zeta$ and $\eta' = \zeta'$.*
(6) *For each subgroup $\bar{S}$ of $\bar{G}$, we set $S = \pi^{-1}(\bar{S})$ and $S' = (\pi')^{-1}(\bar{S})$. Then κ_ϵ provides isomorphisms of $\mathbf{Z}$-modules*

$$\kappa_\epsilon : \mathbf{Z}\,\mathrm{Irr}(S|\zeta) \to \mathbf{Z}\,\mathrm{Irr}\big(S'\big|\zeta'\big)$$

and

$$\kappa_\epsilon : \mathbf{Z}\,\mathrm{IBr}(S|\eta) \to \mathbf{Z}\,\mathrm{IBr}\big(S'\big|\eta'\big).$$

In addition, we set $f = \theta'(1)/\theta(1)$. Then for each $\chi \in \mathbf{Z}\,\mathrm{Irr}(S|\zeta)$ and each $\psi \in \mathbf{Z}\,\mathrm{IBr}(S|\eta)$, if we set $\chi' = \kappa_\epsilon(\chi)$ and $\psi' = \kappa_\epsilon(\psi)$ then $\chi'(1) = f\chi(1)$ and $\psi'(1) = f\psi(1)$.
(7) *θ can be viewed both as an ordinary character or as a Brauer character, and viewed either way κ_ϵ sends it to θ':*

$$\kappa_\epsilon(\theta) = \theta'.$$

Proof. By Corollary 9.9, there exist M_1, M_1' and ϵ_1 with the following properties. M_1 is a finitely generated θ-quasi-homogeneous FG-module whose kernel contains U, M_1' is a finitely generated θ'-quasi-homogeneous FG'-module whose kernel contains U', and ϵ_1 is an endoisomorphism

$$\epsilon_1 : \mathrm{End}_{FH}(M_1) \to \mathrm{End}_{FH'}\big(M_1'\big)$$

such that ϵ_1 *induces* the isomorphism α, and sends the central character associated with θ to the central character associated with θ'. Furthermore, by Proposition 8.4 and Property (4) of Theorem 9.8, it follows that

$$\kappa_{\epsilon_1}(\theta) = \theta'.$$

Finally, the sum of the distinct irreducible summands of the character afforded by M_1 is the sum of all the $G \times \mathrm{Gal}(K/F)$-conjugates of θ, and the sum of the distinct irreducible summands of the character afforded by M_1' is the sum of all the $G' \times \mathrm{Gal}(K/F)$-conjugates of θ'.

By Theorem 11.6, there exist a G-invariant R-lattice M of M_1, and a G'-invariant R-lattice M' of M_1' and an endoisomorphism

$$\epsilon : \mathrm{End}_{RH}(M) \to \mathrm{End}_{RH'}\big(M'\big)$$

from M to M' such that, with proper identifications, we have

$$\mathrm{End}_{FH}(M_1) = F \otimes_R \mathrm{End}_{RH}(M),$$

$$\mathrm{End}_{FH'}\bigl(M_1'\bigr) = F \otimes_R \mathrm{End}_{RH'}\bigl(M'\bigr),$$

$$\epsilon_1 = \mathrm{Id}_F \otimes_R \epsilon.$$

Of course, the kernel of M contains U, and the kernel of M' contains U'. Furthermore, (1) holds. By Proposition 5.5, we know that (2) holds. It follows from our construction that (3) and (4) hold. By Theorem 10.10, κ_ϵ commutes with reduction modulo p, so that the effect of κ_ϵ of ordinary characters or Brauer characters of subgroups of order prime to p is the same. In view of Proposition 8.3, the other stated properties follow from Corollary 9.9. Hence, the corollary holds. □

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