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Riemann surfaces with maximal real symmetry

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ABSTRACT

Let S be a compact Riemann surface of genus $g > 1$, and let $\tau : S \to S$ be any anti-conformal automorphism of S, of order 2. Such an anti-conformal involution is known as a *symmetry* of S, and the species of all conjugacy classes of all symmetries of S constitute what is known as the *symmetry type* of S. The surface S is said to have *maximal real symmetry* if it admits a symmetry $\tau : S \to S$ such that the compact Klein surface S/τ has maximal symmetry (which means that S/τ has the largest possible number of automorphisms with respect to its genus). If τ has fixed points, which is the only case we consider here, then the maximum number of automorphisms of S/τ is $12(g-1)$. In the first part of this paper, we develop a computational procedure to compute the symmetry type of every Riemann surface of genus g with maximal real symmetry, for given small values of $g > 1$. We have used this to find all of them for $1 < g \leq 101$, and give details for $1 < g \leq 25$ (in an appendix). In the second part, we determine the symmetry types of four infinite families of Riemann surfaces with maximal real symmetry. We also

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determine the full automorphism group of the Klein surface S/τ associated with each symmetry $\tau : S \to S$.

1. Introduction

Let S be a compact Riemann surface of genus $g > 1$, and let $\mathrm{Aut}^+(S)$ be the group of all conformal automorphisms of S, and $\mathrm{Aut}(S)$ be the full automorphism group of S, including both conformal and anti-conformal automorphisms (when the latter exist). An anti-conformal automorphism $\tau : S \to S$ of order 2 is known as a *symmetry* of S.

We may associate with each such τ a quantity known as the *species* of τ, defined as follows. Let k be the number of connected components (or *ovals*) of the fixed-point set $\mathrm{Fix}(\tau)$ of τ, and define $\varepsilon = +1$ if the orbit space S/τ of S under the action of $\langle\tau\rangle$ is orientable (or equivalently, if $S - \mathrm{Fix}(\tau)$ is not connected), and $\varepsilon = -1$ otherwise. Then the species of τ, denoted by $\mathrm{spc}(\tau)$, is given by $\mathrm{spc}(\tau) = \varepsilon k$. It is known that $\mathrm{spc}(\tau)$ determines τ up to homeomorphism. In particular, every conjugate of τ in the group $\mathrm{Aut}(S)$ has the same species as τ.

The *symmetry type* of S is the unordered list of species of representatives of all conjugacy classes of symmetries of S. This concept was introduced in [8].

There are very few families of Riemann surfaces for which the symmetry types are known. In this paper we address this issue, for a particular class of Riemann surfaces, namely those with maximal real symmetry.

If τ is any symmetry of the compact Riemann surface S, then the orbit space S/τ endowed with the dianalytic structure inherited naturally from S is known as a *Klein surface*. The *algebraic genus* of S/τ is defined to be the genus of S. Details are given in [1], where it is also shown that if S has genus $g > 1$, then since $\mathrm{Aut}(S)$ is finite, the same is true of the group $\mathrm{Aut}(S/\tau)$ of all automorphisms of S/τ, because the latter can be identified with the group of all conformal automorphisms of S that commute with τ (or in other words, the centraliser of τ in $\mathrm{Aut}^+(S)$).

A compact Riemann surface S of genus $g > 1$ is said to have *maximal real symmetry* if it admits a symmetry $\tau : S \to S$ such that the compact Klein surface S/τ has maximal symmetry (which means that S/τ has the largest possible number of automorphisms with respect to its genus). If S/τ has non-empty boundary, which is the only case we will consider here, then this maximum number is $12(g-1)$; see [15]. The automorphism groups of such bordered surfaces are called *M^*-groups*. These groups are smooth quotients of the extended modular group $\mathrm{PGL}(2,\mathbb{Z})$, and play a role for compact bordered Klein surfaces analogous to the one played by Hurwitz groups (smooth quotients of the ordinary $(2,3,7)$ triangle group) for compact Riemann surfaces. In contrast, however, relatively little is known about M^*-groups.

The contents of this paper can be summarised as follows. We give some further background in Section 2, and then in Section 3 we describe the structure of the full group

Aut (S) of automorphisms of a Riemann surface S with maximal real symmetry. In all cases except the Accola–Maclachlan surface of genus 2, the group Aut (S) is a direct product $G \times C_2$ of an M^*-group G and the cyclic group of order two. We address the issue of calculating the symmetry type of such a surface S in Sections 4 and 5. Specifically, in Section 4 we describe a procedure for doing this for a surface S of (up to) given genus g, and its implementation in the computational algebra system Magma [2]. We display the results for genus 2 to 25 in Appendix A. These results (and additional results for genus up to 101) were obtained with the help of the list of M^*-groups up to order 1200 used in [3]. We change the approach in Section 5, where we consider an infinite family $\{G_n\}$ of M^*-groups of order $12n^2$ (for $n \geq 1$), and determine the symmetry type of four infinite families of Riemann surfaces associated with these groups. As well as calculating the species of a representative τ of each conjugacy class of symmetries, we describe the automorphism group Aut (S/τ) of the compact Klein surface S/τ associated with every such τ.

2. Further background

The adjective "real" for the kind of symmetry we are considering on a Riemann surface comes from real algebraic geometry: if we view S as a complex algebraic curve, then the symmetry τ can be chosen to be complex conjugation, and then Aut (S/τ) can be identified with the group of birational transformations of S with real coefficients. This way, Riemann surfaces with maximal real symmetry correspond to complex algebraic curves with the largest possible number of real automorphisms.

The category of pairs (S, τ), where S is a compact Riemann surface and $\tau : S \to S$ is a symmetry, is co-equivalent to the category of algebraic function fields in one variable over $\mathbb{R}$; see [1, Chapter 2, Section 3]. Consequently, such a pair (S, τ) is usually called a real algebraic curve. Non-conjugate symmetries on a Riemann surface S correspond to non-isomorphic real curves whose complexifications are (isomorphic to) S. Hence the topological classification of these real curves is contained in the symmetry type of S.

Some topological and analytic features of the real curve (S, τ) can be obtained from the associated symmetry τ. For instance, the set of real points of the curve is homeomorphic to the fixed point set $\mathrm{Fix}(\tau)$ of the symmetry, so the number of ovals of the real curve equals the absolute value $|\mathrm{spc}(\tau)|$ of the species. In addition, $\mathrm{spc}(\tau) > 0$ if and only if the real curve disconnects its complexification. Because of this, symmetries with positive species are called *separating*, while symmetries with non-positive species are *non-separating*.

Numerous facts about symmetries of surfaces can be found in the literature, but there are few families of surfaces whose symmetry types have been completely determined. The symmetry types for surfaces of genus 0 and 1 were given in [1], for genus 2 in [8], and for genus 3 in [17]. Also the combinatorial theory of non-euclidean crystallographic groups was used to determine the symmetry type of the cyclic covers of the sphere branched

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over three points in [5], and a complete study of the symmetry types of hyperelliptic surfaces was carried out in [4].

In what follows, by a Riemann surface we will mean a compact Riemann surface S of genus $g > 1$, and we will say that S has *maximal real symmetry* if it admits a symmetry τ such that S/τ is a bordered Klein surface with $12(g-1)$ automorphisms.

We now recall the main ideas of Fuchsian and NEC groups in the setting of Riemann surfaces. We refer the reader to Chapters 0 and 1 in [7] for a general treatment.

An *NEC group* (which is short for *non-Euclidean crystallographic group*) is a discrete subgroup Γ of the group $\mathrm{PGL}(2,\mathbb{R})$ of all isometries of the hyperbolic plane $\mathbb{H}$ with the property that the orbit space $\mathbb{H}/\Gamma$ is compact. If Γ contains only orientation-preserving isometries, then it is a *Fuchsian group*, and otherwise it is a *proper NEC group*. The *canonical Fuchsian subgroup* of a proper NEC group Γ is the subgroup $\Gamma^+ = \Gamma \cap \mathrm{PSL}(2,\mathbb{R})$ consisting of its orientation-preserving elements.

A *surface Fuchsian group* is a torsion-free Fuchsian group. The term "surface" comes from the fact that every such group uniformises a compact Riemann surface of genus greater than one — that is, for any such surface S there exists a surface Fuchsian group Λ such $S = \mathbb{H}/\Lambda$. Two Riemann surfaces $\mathbb{H}/\Lambda_1$ and $\mathbb{H}/\Lambda_2$ are isomorphic if and only if Λ_1 and Λ_2 are conjugate subgroups within $\mathrm{PGL}(2,\mathbb{R})$.

A given finite group G is a group of conformal and anti-conformal automorphisms of $S = \mathbb{H}/\Lambda$ if and only if there exist an NEC group Γ and an epimorphism $\theta : \Gamma \to G$ which has Λ as its kernel. Any such epimorphism whose kernel is a surface Fuchsian group is called a *smooth epimorphism*. In that case, the order of G, the genus of $\mathbb{H}/\Lambda$ and the area of a fundamental domain for the action of Γ are related by the *Riemann–Hurwitz formula*.

The algebraic structure of every NEC group Γ and the topological features of the projection $\mathbb{H} \to \mathbb{H}/\Gamma$ are determined by the *signature* $\sigma(\Gamma)$ of Γ. Of special interest here are the NEC groups with signature

$$\sigma^* = (0; +; [-]; \{(2,2,2,3)\}).$$

Every NEC group Γ^* with this signature σ^* is isomorphic to the abstract group with presentation

$$\langle\, c_0, c_1, c_2, c_3 \mid c_0^2 = c_1^2 = c_2^2 = c_3^2 = (c_0c_1)^2 = (c_1c_2)^2 = (c_2c_3)^2 = (c_3c_0)^3 = 1 \,\rangle. \quad (1)$$

Here each c_i represents a hyperbolic reflection, and every other hyperbolic reflection $c \in \Gamma^*$ is conjugate in Γ^* to some c_i. The NEC groups with this presentation play a key role in this paper, for the following reason: if S is a Riemann surface with maximal real symmetry, then with the single exception of the Accola–Maclachlan surface of genus 2, its full automorphism group $\mathrm{Aut}\,(S)$ is described by means of a smooth epimorphism $\theta : \Gamma^* \to \mathrm{Aut}\,(S)$ from some NEC group Γ^* with presentation (1). In fact, the structure of $\mathrm{Aut}\,(S)$ is well known, and we will describe it in Theorem 3.1 below.

The groups arising as the full automorphism group of a bordered Klein surface with maximal symmetry are called M^*-groups. It is well known that each M^*-group G is a factor group Γ^*/Λ, where Γ^* has presentation (1), and Λ is a bordered surface NEC group — that is, an NEC group containing a reflection, but no non-trivial orientation-preserving element of finite order.

In this case, at least one of $\{c_0, c_1, c_2, c_3\}$ must lie in the normal subgroup Λ. It is easy to see (by considering images of elements in the factor group Γ^*/Λ) that neither c_0 nor c_3 can lie in Λ, since c_0c_3 has order 3 (while c_0 and c_3 have order 2). Also Γ^* has an outer automorphism ξ that takes (c_0, c_1, c_2, c_3) to (c_3, c_2, c_1, c_0), since replacing each c_i by c_{3-i} in the presentation for Γ^* preserves the defining relations. Hence, after application of ξ if necessary, we may assume that $c_1 \in \Lambda$, in which case $c_2 \notin \Lambda$.

Then if we denote the images of c_0, c_2 and c_3 in Γ^*/Λ by a, c and d, respectively, we see that every M^*-group $G = \Gamma^*/\Lambda$ admits the following (partial) presentation:

$$\langle\, a, c, d \mid a^2 = c^2 = d^2 = (cd)^2 = (ad)^3 = \ldots = 1 \,\rangle, \tag{2}$$

where the remaining relators (indicated by '$\ldots$') determine the (finite) group G; indeed they are the images of a set of elements in Γ^* whose normal closure is Λ. The presentation without the extra relators can be re-written as $\langle\, x, y, t \mid x^2 = y^3 = t^2 = (xt)^2 = (yt)^2 = 1 \,\rangle$ in terms of the alternative generators $x = c$, $y = ad$ and $t = d$, and this is a presentation for the extended modular group $\mathrm{PGL}(2, \mathbb{Z})$, with the two elements x and y generating the modular group $\mathrm{PSL}(2, \mathbb{Z})$, a subgroup of index 2. Hence M^*-groups are smooth quotients of $\mathrm{PGL}(2, \mathbb{Z})$.

Presentations for all M^*-groups up to order 1200 can be found in [3, Section 5], and as we will explain in Section 4, we have used this list to compute the symmetry types of all Riemann surfaces with maximal real symmetry up to genus 101.

3. Riemann surfaces with maximal real symmetry

Recall that the symmetry type of a Riemann surface S is the unordered list of species of representatives of all conjugacy classes of symmetries of S. Clearly, a first step in finding the symmetry type of S is to know the full automorphism group $\mathrm{Aut}\,(S)$. It is difficult in general, however, to determine whether a given group G acting on S as a group of automorphisms is the full group $\mathrm{Aut}\,(S)$, or alternatively, whether S admits more automorphisms than those in G. In the case of Riemann surfaces with maximal real symmetry, this is easy, thanks to a helpful theorem established by May in [16]; see also [6]. Before stating it (as Theorem 3.1 below), we introduce some further notation which will also be helpful later.

Let S be a genus g Riemann surface with maximal real symmetry, and $\tau : S \to S$ a symmetry such that S/τ is a bordered Klein surface with full automorphism group $\mathrm{Aut}\,(S/\tau)$ of order $12(g-1)$. We have $S = \mathbb{H}/\Lambda^+$, where Λ is a proper NEC group and Λ^+ is its canonical Fuchsian subgroup, and also (with a slight abuse of notation) we can write

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$\tau = \Lambda/\Lambda^+$. Then $S/\tau \cong (\mathbb{H}/\Lambda^+)/(\Lambda/\Lambda^+) \cong \mathbb{H}/\Lambda$. Also we know that $\mathrm{Aut}\,(S/\tau) = \Gamma^*/\Lambda$ where Γ^* has presentation (1), so that $\mathrm{Aut}\,(S/\tau)$ is an M^*-group. Now observe that Γ^* normalises Λ^+, since conjugation preserves orientability, and so the quotient Γ^*/Λ^+ is a group of automorphisms of S. Moreover, Λ/Λ^+ is a normal subgroup of order 2 in Γ^*/Λ, and is therefore central in Γ^*/Λ, and also Λ/Λ^+ is complemented by the index 2 subgroup $(\Gamma^*)^+/\Lambda^+$, which is isomorphic to Γ^*/Λ, and hence to $\mathrm{Aut}\,(S/\tau)$. It follows that Γ^*/Λ^+ is isomorphic to the direct product $(\Gamma^*)^+/\Lambda^+ \times \Lambda/\Lambda^+ = \mathrm{Aut}\,(S/\tau) \times \Lambda/\Lambda^+$, and in particular, the full group $\mathrm{Aut}\,(S)$ has a subgroup isomorphic to $\mathrm{Aut}\,(S/\tau) \times C_2$.

Theorem 3.1 states that with one single exception, namely where S/τ is the regular pair of pants, this is the full group $\mathrm{Aut}\,(S)$.

The regular pair of pants is the surface S/τ where S is the Accola–Maclachlan curve of genus 2, given by $w^2 = z^6 - 1$, and $\tau : (z, w) \mapsto (1/\bar{z}, i\bar{w}/\bar{z}^3)$. Topologically, S/τ is a sphere with three holes, and therefore $\mathrm{spc}(\tau) = 3$. The full group $\mathrm{Aut}\,(S)$ has order 48 and presentation $\langle\, a, b, c \mid a^2 = b^2 = c^2 = (ab)^4 = (bc)^6 = (ac)^2 = (babc)^2 = 1\,\rangle$. The symmetry type of this surface is $\{-1, 0, 1, 3\}$; see [4, Section 3.4], where explicit algebraic equations for representatives of each conjugacy class of symmetries are also given.

Theorem 3.1. *(See [16].) With the above notation, suppose that S/τ is not homeomorphic to the regular pair of pants. Then the full group* $\mathrm{Aut}\,(S)$ *of all conformal and anti-conformal automorphisms of S is isomorphic to the direct product* $\mathrm{Aut}\,(S/\tau) \times C_2$*, and moreover,* $\mathrm{Aut}\,(S/\tau)$ *is isomorphic to the group* $\mathrm{Aut}^+(S)$ *of conformal automorphisms of S, while the factor C_2 is generated by the symmetry τ.*

For any given M^*-group G, Theorem 3.1 provides a way to obtain all Riemann surfaces with maximal real symmetry and with full group $\mathrm{Aut}\,(S)$ isomorphic to $G \times C_2$. To do this, we consider all possible smooth epimorphisms $\theta : \Gamma^* \to G \times C_2$ from an NEC group Γ^* with presentation (1), such that the index 2 subgroup $(\Gamma^*)^+ = \langle\, c_0c_1, c_1c_2, c_2c_3\,\rangle$ of all orientation-preserving elements in Γ^* is mapped onto G, and some hyperbolic reflection (with fixed points) in Γ^* is mapped to the non-trivial element of the C_2-factor, say τ. In every such case, $S = \mathbb{H}/\ker\theta$ is a compact Riemann surface, admitting a symmetry τ with the property that the full automorphism group $\mathrm{Aut}\,(S/\tau)$ of the bordered surface S/τ is isomorphic to G. Thus S is a Riemann surface with maximal real symmetry, and by Theorem 3.1, its full group $\mathrm{Aut}\,(S)$ is $G \times C_2$, unless S/τ is the regular pair of pants.

We will assume from now on that S/τ is not the regular pair of pants.

Because each hyperbolic reflection is conjugate to one of the canonical reflections c_0, c_1, c_2 and c_3 generating Γ^*, we may assume that $\tau = \theta(c_i)$ for some $i \in \{0, 1, 2, 3\}$. In fact, by the same argument as used in the observations about M^*-groups towards the end of the previous section, we may assume that $\tau = \theta(c_1)$.

The images of the other generating canonical reflections are of the form $\theta(c_0) = a\tau$, $\theta(c_2) = c\tau$ and $\theta(c_3) = d\tau$, where $a, c, d \in G$. Moreover, the elements a, c and d are involutions (since the $\theta(c_i)$ are involutions and τ is a central involution), and they must generate the given M^*-group G (the image of $\mathrm{Aut}\,(S) = G \times C_2$ when the C_2 is

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factored out). In fact, from the relations in the presentation (1) we know that this triple (a, c, d) of generators for G must satisfy the relations

$$a^2 = c^2 = d^2 = (cd)^2 = (ad)^3 = 1. \tag{3}$$

Accordingly, in order to find all Riemann surfaces S with maximal symmetry such that $\mathrm{Aut}\,(S) = G \times C_2$, we have to find all triples (a, c, d) of generators of G satisfying the relations (3). Technically, we only have to consider such triples up to conjugacy within $\mathrm{Aut}\,(G)$, since composing θ with an automorphism of $G \times C_2$ gives a smooth epimorphism from Γ^* to $G \times C_2$ with the same kernel in Γ^*. This, however, can all be taken care of in the determination of presentations for M^*-groups, as in [3].

Once the full automorphism group of S is known, the next step in computing the symmetry type of S is to find the conjugacy classes of symmetries in $\mathrm{Aut}\,(S) = G \times C_2$. From the above description of this group, see know that each symmetry is of the form $w\tau$ where $w \in G$ is a conformal automorphism of order 1 or 2. Hence this problem reduces to determining conjugacy classes of involutions in the M^*-group G.

Finally, for a representative $w\tau$ of each conjugacy class of symmetries in $\mathrm{Aut}\,(S)$, we have to determine its species $\mathrm{spc}(w\tau)$. This splits into two problems: counting the number of ovals of $\mathrm{Fix}(w\tau)$, and determining the orientability of $S/w\tau$. We can solve these problems using the combinatorial theory of NEC groups in the following way.

Let Γ^* and $\theta : \Gamma^* \to \mathrm{Aut}\,(S)$ be as above. Then the images $\theta(c_i)$ of the canonical generators for Γ^* are symmetries, with fixed points (since each c_i is an involution with fixed points on $\mathbb{H}$). Moreover, since every reflection in Γ^* is conjugate to some c_i, it follows that every symmetry which is not conjugate to some $\theta(c_i)$ is fixed-point-free. Observe also that c_0 and c_3 generate a dihedral subgroup of order 6, and hence they are conjugates of each other, and the same is true of $\theta(c_0)$ and $\theta(c_3)$. This gives the following:

Lemma 3.2. *The number of conjugacy classes of symmetries in* $\mathrm{Aut}\,(S)$ *with fixed points is at most* 3. *In particular,* $\mathrm{spc}(\tau) = 0$ *for any symmetry* $\tau \in \mathrm{Aut}\,(S)$ *which is not conjugate to* $\theta(c_1)$ *or* $\theta(c_2)$ *or one of* $\theta(c_0)$ *and* $\theta(c_3)$.

On the other hand, we observe that the number of conjugacy classes of fixed-point-free symmetries can be arbitrarily large. For by theorems of the third author [9,10], all but finitely many alternating groups A_n and all but finitely many symmetric groups S_n are smooth quotients of the extended modular group $\mathrm{PGL}(2, \mathbb{Z})$, and hence are M^*-groups. Also the number of conjugacy classes of involutions in A_n and S_n increases with n (indeed is the integer part of $n/4$ and $n/2$ respectively), and the assertion follows.

In order to count the number of ovals of each symmetry $\theta(c_i)$, which we denote by $\|\theta(c_i)\|$, we use the following formula due to G. Gromadzki. Here $|A : B|$ denotes the index of a subgroup B in a group A, and $C_A(x)$ denotes the centraliser in a group A of an element x, as usual.

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Theorem 3.3. *(See [13].) The number of ovals of the symmetry $\theta(c_i)$ with fixed points is*

$$\|\theta(c_i)\| = \sum_{c} |\, C_{\mathrm{Aut}(S)}(\theta(c)) : \theta(C_{\Gamma^*}(c))\,|, \tag{4}$$

where c runs through the set of all non-conjugate canonical reflections of Γ^* *whose images under* θ *are conjugate in* $\mathrm{Aut}\,(S)$ *to* $\theta(c_i)$.

The centraliser in an NEC group of a canonical reflection was studied by Singerman [18]. It follows from his work that

$$C_{\Gamma^*}(c_0) = \langle\, c_0, c_1, (c_2c_3)^{c_0c_3}\,\rangle \;\cong\; \langle c_0\rangle \times \Big(\langle c_0c_1\rangle * \langle c_3c_0c_2c_3c_0c_3\rangle\Big), \tag{5}$$

$$C_{\Gamma^*}(c_1) = \langle\, c_0, c_1, c_2\,\rangle \;\cong\; \langle c_1\rangle \times \Big(\langle c_0c_1\rangle * \langle c_1c_2\rangle\Big), \tag{6}$$

$$C_{\Gamma^*}(c_2) = \langle\, c_1, c_2, c_3\,\rangle \;\cong\; \langle c_2\rangle \times \Big(\langle c_1c_2\rangle * \langle c_2c_3\rangle\Big), \quad \text{and} \tag{7}$$

$$C_{\Gamma^*}(c_3) = \langle\, c_3, c_2, (c_1c_0)^{c_3c_0}\,\rangle \;\cong\; \langle c_3\rangle \times \Big(\langle c_2c_3\rangle * \langle c_0c_3c_0c_1c_3c_0\rangle\Big), \tag{8}$$

where $A * B$ is the free product of the subgroups A and B. Note that we need only one of the expressions for $C_{\Gamma^*}(c_0)$ and $C_{\Gamma^*}(c_3)$, since c_3 is conjugate to c_0 in Γ^*. The other terms in formula (4) depend on the particular group $\mathrm{Aut}\,(S)$ and the epimorphism θ.

The separating character of each symmetry $\tau_i = \theta(c_i)$, or equivalently, the orientability of each Klein surface S/τ_i, can be computed by inspection of a Schreier coset graph for the group Γ^* with respect to the subgroup $(\theta)^{-1}(\langle\tau_i\rangle)$, with the reflection loops deleted, as explained by Hoare and Singerman in [14, Section 3]. Since all the generators of Γ^* are orientation-reversing, the symmetry τ_i is non-separating if and only if this graph is non-bipartite; see [14, Corollary 3].

4. Finding the symmetry types of Riemann surfaces of small genus with maximal real symmetry

In this section we give a brief description of a procedure for determining the Riemann surfaces of (up to) given small genus g that have maximal real symmetry, and for computing their symmetry types. This procedure is valid for all such surfaces except for the Accola–Maclachlan surface of genus 2.

As a first step, we can use the `LowIndexNormalSubgroups` procedure in the computational algebra system Magma [2] to find all normal subgroups of index up to $12(g-1)$ in the extended modular group

$$\langle\, a, c, d \mid a^2 = c^2 = d^2 = (cd)^2 = (ad)^3 = 1\,\rangle \cong \mathrm{PGL}(2,\mathbb{Z}),$$

as was done in [3] to find all M^*-groups of up to order 1200.

For each such normal subgroup K of index (say) n in $\Psi = \mathrm{PGL}(2, \mathbb{Z})$, we may consider the natural permutation representation of Ψ on the cosets of K, which gives an epimorphism ψ from Ψ to the quotient $G = \Psi/K$ of order n, with kernel K. We then check that the orders of the ψ-images of the three generators a, c and d and their products ad and cd are preserved. In that case, ψ is a smooth homomorphism, and so G is an M^*-group, which acts on a bordered Klein surface of algebraic genus $1 + n/12$. Then we can add an extra generator τ, plus relations that make it have order 2 and commute with the first three generators. This gives the group $G \times C_2$, and then also we can set up the epimorphism θ from the group with presentation (1), which we might as well call Γ^*, to $G \times C_2$.

The next step is to find the conjugacy classes of symmetries in $G \times C_2$. This can be achieved easily using the `ConjugacyClasses` command in Magma, although the group has to be converted into a suitable format, for example by taking its regular representation (as a permutation group of degree $|G \times C_2| = 2|G| = 2n$). We find all conjugacy classes of involutions in $G \times C_2$, and then it is also easy to find out which classes lie in G and which do not; the latter are the classes of symmetries of the corresponding surface S.

One slightly challenging aspect of this is to find representatives of the classes. It is easy to find out which class contains τ, because τ is a central involution and hence forms a class of size 1, and also it is easy to check which classes contain $a\tau$ (or $d\tau$) and $c\tau$. The problem occurs with the classes that do not contain one or more of these elements. But we can simply run through possibilities for a word w of increasing length in the generators a, c and d for G, until we find one such that w or $w\tau$ lies in the given class.

In contrast, it is easy to determine the automorphism group of the Klein surface S/τ_i associated with each representative symmetry τ_i, since we know that $\mathrm{Aut}\,(S/\tau_i)$ can be identified with the centraliser in $G = \mathrm{Aut}^+(S)$ of τ_i (by observations made in the Introduction). Note that when τ_i is the θ-image of one of the canonical generators of Γ^*, we also need the centraliser of τ_i in $G \times C_2 = \mathrm{Aut}\,(S)$ to compute the number of its ovals. These centralisers are easy to compute using Magma.

Similarly, finding the species of each representative symmetry is relatively easy. By Lemma 3.2, the species is 0 for every representative of a class that does not contain one or more of τ, $a\tau$, $c\tau$ and $d\tau$, so we need only consider the classes containing the latter. Also we know that $a\tau$ is always conjugate to $d\tau$ (since they are the θ-images of c_0 and c_3), so in fact there are at most three classes to consider.

Computation of the number of ovals of each representative symmetry is entirely straightforward, using Theorem 3.3 and the observations by Singerman following it. For example, the symmetry $\tau = \theta(c_1)$ is a central involution in $G \times C_2$ and so lies in a class of its own, containing no other images $\theta(c_i)$, and therefore the number of ovals of the symmetry τ is

$$\|\tau\| = |\,C_{G\times C_2}(\tau) : \theta(C_{\Gamma^*}(c_1))\,| = |\,G \times C_2 : \theta(\langle c_0, c_1, c_2\rangle)\,| = |\,G \times C_2 : \langle a, \tau, c\rangle)\,|.$$

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Now $\langle a, \tau, c\rangle$ is the direct product of $\langle a, c\rangle$ and $\langle \tau\rangle$, and hence isomorphic to $D_m \times C_2$, where m is the order of ac. In particular, the number of ovals for τ is equal to $|G|/2o(ac)$. This leaves only one or two representative symmetries to deal with, namely $a\tau$ and $c\tau$ (depending on whether they lie in the same class or not).

Finally, we find the separating character of each of τ, $a\tau$ (or $d\tau$) and $c\tau$, by considering the Schreier coset graph for the action of $G \times C_2$ via the generating set $\{a\tau, \tau, c\tau, d\tau\}$ on right cosets of the subgroup H of order 2 generated by the relevant symmetry. Again, if $c\tau$ is conjugate to $a\tau$ and $d\tau$, then we do this for τ and just one of the other three.

In the case of the central symmetry τ, there is a loop for the effect of τ at every vertex, since τ is central in $G \times C_2$. When these loops are deleted, what remains is the Schreier coset graph for the regular representation of G via the generating set $\{a, c, d\}$. The latter graph is bipartite if and only if the subgroup of G generated by ac and cd (and $(ac)(cd) = ad$) has index 2 in G, or equivalently, there is no relation in the group G of odd length in the generators a, c and d. This is easy to check. In particular, the surface S/τ is orientable (and $\varepsilon = +1$ for the symmetry τ) if and only if none of the relators in the presentation for the M^*-group G has odd length.

For the other symmetries $a\tau$ (or $d\tau$) and $c\tau$, some more work is required. In these cases, a loop corresponding to the effect of the generator u ($= a\tau, \tau, c\tau$ or $d\tau$) occurs at a vertex Hx whenever $xux^{-1} \in H$, or equivalently, the generator u is a conjugate of the generator of H under the element x. This cannot happen when $u = \tau$ (again since τ is central), but it may happen in other cases. (For example, if $H = \langle d\tau\rangle$, then multiplication by $a\tau$ fixes the coset Had, since $(ad)a\tau(ad)^{-1} = adada\tau = d\tau \in H$.)

It is relatively easy to find out exactly where these loops occur, using knowledge of the centraliser of H. But it is even easier to construct the loop-less version of the graph directly, using the permutation representation of $G \times C_2$ on right cosets of H, and then to check whether or not this graph is bipartite. It is also not difficult to find an explicit cycle of odd length, when one exists, by tracing closed walks in the full coset graph corresponding to the defining relators for G or $G \times C_2$.

To summarise, we have the following algorithm, valid for all Riemann surfaces with maximal real symmetry except the Accola–Maclachlan surface of genus 2:

Algorithm 4.1. For every smooth epimorphism ψ from the extended modular group to an M^*-group G, of order $12(g-1)$, we can perform these steps to obtain the symmetry type of the corresponding surface S having maximal real symmetry, genus g, and automorphism group $G \times C_2$:

(1) Construct the direct product $G \times C_2$, by adjoining a central involution τ, and also construct the corresponding smooth epimorphism $\theta\colon \Gamma^* \to G \times C_2$;
(2) Find the conjugacy classes of symmetries in $G \times C_2$, and a representative of each class;
(3) Use centralisers and Theorem 3.3 to find the automorphism group of the Klein surface S/τ_i associated with each representative symmetry τ_i;

(4) For each class containing the θ-image of one of the canonical generators of Γ^*, use centralisers and Theorem 3.3 to find the number of ovals of each representative symmetry, and use a Schreier coset graph and the Hoare–Singerman method [14, Section 3] to determine its separating character, and hence its species.

By way of illustration, let us consider the M^*-group G of order 336 with presentation

$$\langle\, a, c, d \mid a^2 = c^2 = d^2 = (cd)^2 = (ad)^3 = (ac)^7 = (acd)^8 = 1 \,\rangle.$$

This is isomorphic to the group $\mathrm{PGL}(2, 7)$, and acts on a bordered surface of algebraic genus 29. As explained earlier, we adjoin a central involution τ to create $G\times C_2$, and then we can take the epimorphism $\theta\colon \Gamma^* \to G\times C_2$ under which $(c_0, c_1, c_2, c_3) \mapsto (a\tau, \tau, c\tau, d\tau)$. The resulting group $G\times C_2$ of order 672 has five conjugacy classes of involutions, of sizes 1, 21, 21, 28 and 28, with representatives τ, cd, $cd\tau$, $a\tau$ (or $c\tau$ or $d\tau$) and a (or c or d) respectively. Two of these lie in G (namely the ones containing a and cd), while the other three consist of symmetries, and we will take their representatives as τ, $c\tau$ and $cd\tau$.

The centralisers in $G\times C_2$ of these elements are respectively $G\times C_2$, $\langle c, d, (acdac)^3, \tau\rangle$ and $\langle c, d, (acd)^4, \tau\rangle$, and so their centralisers in G are respectively G, $\langle c, d, (acdac)^3\rangle$ and $\langle c, d, (acd)^4\rangle$. The latter centralisers, and hence also the automorphism groups of the Klein surfaces S/τ, $S/(c\tau)$ and $S/(cd\tau)$, have orders 336, 12 and 16.

Finally, we determine the species of each representative symmetry. We know the species of $cd\tau$ is 0 (since it contains none of the θ-images of the canonical generators c_i of Γ^*), and so we need only find the species of τ and $c\tau$.

In the special case of τ itself, we know by observations made above that the number of ovals of the symmetry τ is $\|\tau\| = |\, G : \langle a, c\rangle\,| = 336/14 = 24$, since ac has order 7.

The class containing $c\tau = \theta(c_2)$ contains also $a\tau = \theta(c_0)$ and $d\tau = \theta(c_3)$, and indeed c_0 and c_3 are conjugate in Γ^*, but c_0 and c_2 are not. By Theorem 3.3 and a Magma computation, it follows that the number of ovals for $c\tau$ is

$$\begin{aligned}\|c\tau\| &= |\, C_{G\times C_2}(a\tau) : \theta(C_{\Gamma^*}(c_0))\,| + |\, C_{G\times C_2}(c\tau) : \theta(C_{\Gamma^*}(c_2))\,| \\ &= |\, C_{G\times C_2}(a\tau) : \langle a, \tau, (cd)^{ad}\rangle\,| + |\, C_{G\times C_2}(c\tau) : \langle \tau, c, d\rangle\,| \\ &= 24/8 \;+\; 24/8 \;=\; 6,\end{aligned}$$

since both $\langle a, \tau, (cd)^{ad}\rangle$ and $\langle \tau, c, d\rangle$ have order 8 (and we know that the size of the conjugacy class of $G\times C_2$ containing $a\tau$ and $c\tau$ is 28, so the centralisers $C_{G\times C_2}(a\tau)$ and $C_{G\times C_2}(c\tau)$ both have order $672/28 = 24$).

To find the separating character of τ (and hence the orientability of S/τ), we note that every relator in the presentation we took for G has even length. It follows that when the loops for the effect of τ at each vertex of the Schreier coset graph for the action of $G\times C_2$ via the generating set $\{a\tau, \tau, c\tau, d\tau\}$ on cosets of $\langle\tau\rangle$ are deleted, we have a bipartite graph. Hence the species of the symmetry τ is $+24$.

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On the other hand, there are 36 loops in the Schreier coset graph for the action of $G \times C_2$ on the 336 cosets of the subgroup $\langle c\tau\rangle$, via the generating set $\{a\tau, \tau, c\tau, d\tau\}$, with 12 loops for the effect of each of $a\tau$, $c\tau$ and $d\tau$ (and none for $b\tau$). When these loops are deleted we have a non-bipartite graph, since the relation $(cda)^8 = 1$ in the group G implies a relation $((c\tau)(d\tau)(a\tau))^8 = 1$ in the group $G \times C_2$, and tracing this out in the coset graph gives a loop followed by a 23-cycle, and hence a cycle of odd length when the loops are deleted. (Note: the relations $(cd)^2 = 1$, $(ad)^3 = 1$ and $(ac)^7 = 1$ give closed walks of lengths 2, 6 and 12, respectively.) The cycle of odd length 23 shows that the quotient surface $S/\langle c\tau\rangle$ is non-orientable, and therefore the species of the symmetry $c\tau$ is -6.

To summarise, the species of τ, $c\tau$ and $cd\tau$ are 24, -6 and 0, it follows that the symmetry type of the corresponding surface S (of genus $1+336/12 = 29$) is $\{+24, -6, 0\}$.

We have implemented the above procedure in Magma, in order to compute the symmetry types of all compact Riemann surfaces of genus 2 to 101 with maximal real symmetry (except the Accola–Maclachlan surface of genus 2), with the help of the list of all M^*-groups of up to order 1200 found in [3]. The results for genus 2 to 25 are displayed in Appendix A, and those for higher genera are available from any one of the authors. For notational convenience, we have expressed the additional relators used to obtain each M^*-group G as a quotient of $\Psi = \mathrm{PGL}(2,\mathbb{Z})$ in terms of the two elements $u = ac$ and $v = (ac)^d$ $(= dacd = dadc = adac)$.

5. Infinite families of Riemann surfaces with maximal real symmetry

In this section we explicitly compute the symmetry type of some infinite families of Riemann surfaces with maximal real symmetry, leading to a proof of the following theorem:

Theorem 5.1. *For every $n > 3$, there are two compact Riemann surfaces of genus n^2+1 with maximal real symmetry, each having conformal automorphism group isomorphic to G_n and full automorphism group isomorphic to $G_n \times C_2$, where*

$$G_n = \langle\, a, c, d \mid a^2 = c^2 = d^2 = (cd)^2 = (ad)^3 = (ac)^6 = (acd)^{2n} = 1 \,\rangle,$$

which is isomorphic to the group $G^{3,6,2n}$ in the notation of Coxeter. Moreover, if b is the generator of the C_2 factor, and $u = acacad$, then the following hold:

(a) *If n is odd, these surfaces have symmetry types $\{n^2, -n, -n, 0\}$ and $\{3n, -n, -3, 0\}$. More specifically, each of the two surfaces has four conjugacy classes of symmetries, with representatives b, ab, cb and cdb, and their respective species are n^2, $-n$, $-n$ and 0 for one surface, and $3n$, $-n$, -3 and 0 for the other.*

(b) *If n is even, the two symmetry types are $\{n^2, -n, -n, 0, 0, 0\}$ and $\{3n, -n, -4, 0, 0, 0\}$.*

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More specifically, each of the two surfaces has six conjugacy classes of symmetries, with representatives b*,* ab*,* cb*,* cdb*,* $u^{n/2}b$ *and* $ucdb$*, and their respective species are* n^2*,* $-n$*,* $-n$*,* 0*,* 0 *and* 0 *for one surface, and* $3n$*,* $-n$*,* -4*,* 0*,* 0 *and* 0 *for the other.*

Furthermore, if $n = 1$ *then all the above holds, except that the symmetries* ab *and* cdb *for the second surface have positive species, giving symmetry type* $\{3, 1, 3, 0\}$*. Similarly, if* $n = 2$ *then all the above holds, except that the symmetries* ab *and* cdb *for the second surface have positive species, giving symmetry type* $\{6, 2, 4, 0, 0, 0\}$*. Finally, if* $n = 3$ *then all the above also holds, except that the two surfaces are the same, with symmetry type* $\{9, -3, -3, 0\}$*.*

All these surfaces can be constructed from an infinite family of quotients of the extended $(2, 3, 6)$ triangle group, which is the abstract group

$$\Delta^*(2, 3, 6) = \langle\, a, c, d \mid a^2 = c^2 = d^2 = (cd)^2 = (ad)^3 = (ac)^6 = 1 \,\rangle.$$

By the theory of this triangle group (as given in [11, Section 8.4], for instance), we know that the elements $v = [cd, ad]$ and $u = [cd, ad]^a$ generate a free abelian normal subgroup of rank 2, with index 12. In fact $v = [cd, ad] = (cd)^{-1}(ad)^{-1}cdad = cddacdad = cacada$, since $a^2 = c^2 = d^2 = (cd)^2 = 1$ and $ada = dad$ (because $(ad)^3 = 1$), and so $u = v^a = acacad$. Also $uv^{-1} = acacadadacac = acacdcac = acadac$, and it follows that conjugation by a swaps u and v, while conjugation by c fixes u (since $cacacadc = acacacdc = acacad$) and takes v to $acadac = uv^{-1}$, and conjugation by d takes u to $dacaca = u^{-1}$, and v to $dcacadad = dcacda = cdadca = cadaca = vu^{-1}$. Factoring out this normal subgroup gives a dihedral quotient of order 12, generated by the images of a and c, and it follows that the extended $(2, 3, 6)$ triangle group is isomorphic to an extension of $\mathbb{Z} \times \mathbb{Z}$ by D_6.

Now for each positive integer n, we can add the relation $u^n = 1$ and obtain the quotient

$$G_n = \langle\, a, c, d \mid a^2 = c^2 = d^2 = (cd)^2 = (ad)^3 = (ac)^6 = (acacad)^n = 1 \,\rangle.$$

Here we are abusing notation by using the same symbols a, c and d as generators for G_n. We take this further by letting

$$v = cacada = [cd, ad] \quad \text{and} \quad u = acacad = [cd, ad]^a$$

in this group G_n. In doing so, we see that G_n is obtained from $\Delta^*(2, 3, 6)$ by specifying the order of u (and its conjugate $v = u^a$) as n. Also the element $acacad$ is conjugate via ac to $acadac = acdadc = acdacd = (acd)^2$, so the final relation can be replaced by $(acd)^{2n} = 1$, which makes G_n isomorphic to the group $G^{3,6,2n}$ in the notation of [11]. Moreover, it follows from the above observations that u and v generate an abelian normal subgroup N isomorphic to $C_n \times C_n$, with

$$u^a = v, \quad v^a = u, \quad u^c = u, \quad v^c = uv^{-1}, \quad u^d = u^{-1} \;\text{ and }\; v^d = vu^{-1},$$

and quotient G_n/N isomorphic to D_6 (generated by the images of a and c). In particular, G_n has order $12n^2$, and is therefore an M^*-group of genus n^2+1.

When $n = 1$ we have $G_n \cong D_6$, which is the only M^*-group of genus 2, and when $n = 2$ we have the unique M^*-group of genus 5 (namely $S_4 \times C_2$); see [3]. The symmetry types of the Riemann surfaces with genus 2 or 5 with maximal real symmetry were considered in Section 4, and given (with a different definition for u and v) in Appendix A, so we can ignore these cases and assume that $n > 2$ from now on.

Next, we adjoin a central involution b to G_n, to create the direct product $G_n \times C_2$, generated by a, b, c and d. Note that the subgroup $N = \langle u, v\rangle$ remains normal in $G_n \times C_2$. We will consider all ways in which $G_n \times C_2$ is the automorphism group of a compact Riemann surface of genus $g = n^2 + 1$ with maximal real symmetry.

Before doing that, we exhibit some automorphisms of $G_n \times C_2$ that will be useful. Conjugation by any element $u^r v^s$ of N is an inner automorphism that has the following effect on the generators:

$$a^{u^r v^s} = u^{s-r} v^{r-s} a, \qquad b^{u^r v^s} = b, \qquad c^{u^r v^s} = u^s v^{-2s} c, \quad \text{and} \quad d^{u^r v^s} = u^{-2r-s} d.$$

Similarly, conjugation by elements of the dihedral subgroup $D = \langle a, c\rangle$ of order 12 gives a subgroup of inner automorphisms that transitively permutes the 12 pairs of involutions generating D, and transitively permutes the 12 pairs of involutions generating a dihedral subgroup of order 6 in D (and also the 6 pairs of involutions generating a subgroup of order 4 in D). Finally, for any unit α mod n, there exists an automorphism of $G_n \times C_2$ taking (a, b, c, d) to $(a, b, c, du^{\alpha-1}) = (a, b, c, (acaca)u^\alpha)$, since the latter four elements generate $G_n \times C_2$ and satisfy the same relations as (a, b, c, d). Note that this automorphism takes $u = (acaca)d$ to $(acaca)^2 u^\alpha = u^\alpha$.

Now we find the conjugacy classes of involutions in $G_n \times C_2$ that lie outside G_n. For obvious reasons, we will call any such involution x a *symmetry in* $G_n \times C_2$.

In each case, the image of x in the quotient $(G_n \times C_2)/N \cong D_6 \times C_2$ must be an involution lying outside $G_n/N \cong D_6$. There are four conjugacy classes of such elements in G_n/N, with representatives Nb, Nab, Ncb and $N(ac)^3 b$, of sizes 1, 3, 3 and 1, respectively. It follows that each x is conjugate to an element of the form wb, wab, wcb or $w(ac)^3 b$, where $w \in N$.

We can use this to prove the following:

Lemma 5.2. *If n is odd, then there are four conjugacy classes of symmetries in $G_n \times C_2$: one of size 1 with representative b, two of size $3n$ with representatives ab and cb, and one of size n^2 with representative dcb. On the other hand, if n is even, then there are six conjugacy classes of symmetries in $G_n \times C_2$: one of size 1 with representative b, one of size 3 with representative $u^{n/2}b$, two of size $3n$ with representatives ab and cb, one of size $n^2/4$ with representative $udcb$, and one of size $3n^2/4$ with representative dcb. In particular, b is the only central involution in $G_n \times C_2$ whenever $n > 2$.*

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Proof. If $x = wb$ then also $1 = (wb)^2 = w^2$ (since b is a central involution), and so $w = 1$, $u^{n/2}$, $v^{n/2}$, or $(uv)^{n/2}$ $(= u^{n/2}v^{n/2} = (uv^{-1})^{n/2})$, with the last three possibilities occurring only when n is even. Moreover, the last three involutions are mutually conjugate within G_n, by what we know about the effects on u and v by conjugation by each of a, c and d. Hence if n is odd we find one conjugacy class, of size 1, with representative $x = b$, while if n is even we have two classes, of sizes 1 and 3, with representatives $x = b$ and $x = u^{n/2}b$.

Next, suppose $x = wab$, where $w = u^iv^j$. Then $1 = (wab)^2 = u^iv^jau^iv^ja = u^{i+j}v^{j+i}$ and so $i + j \equiv 0 \mod n$, which implies that w is one of the n powers of uv^{-1}. On the other hand, the centraliser of ab in N consists of all elements u^rv^s for which $u^rv^s = (u^rv^s)^{ab} = v^ru^s$, that is, $r \equiv s \mod n$, and so ab has $|N : C_N(ab)| = n^2/n = n$ distinct conjugates under elements of N. These must be the n elements of the form $(uv^{-1})^iab$. Hence this gives one further conjugacy class in $G_n \times C_2$, of size $3n$ and with representative ab (irrespective of whether n is odd or even).

Applying an inner automorphism of $G_n \times C_2$ that takes a to c (within the dihedral subgroup $D = \langle a, c\rangle$), we find the same is true for the case $x = wcb$, namely that there is a single class of involutions, of size $3n$, with representative cb.

Finally, suppose $x = w(ac)^3b$, where $w = u^iv^j$. Note that conjugation by ac induces the 6-cycle $(u, uv^{-1}, v^{-1}, u^{-1}, u^{-1}v, v)$, and so conjugation by $(ac)^3b$ inverts each of u and v, and hence inverts every element of N. In particular, $(w(ac)^3b)^2 = ww^{-1}(ac)^6 = 1$ for all $w \in N$, so there are n^2 possibilities for w. Also G_n is generated by a, b, c and u $(= acacad)$, and conjugation by these generators has the following effect on $x = w(ac)^3b = u^iv^j(ac)^3b$:

$$\begin{aligned}
x^a &= (u^iv^j(ac)^3b)^a = v^iu^j(ca)^3b = u^jv^i(ac)^3b = (uv^{-1})^{j-i}x,\\
x^b &= (u^iv^j(ac)^3b)^b = u^iv^j(ac)^3b = x,\\
x^c &= (u^iv^j(ac)^3b)^c = u^iu^jv^{-j}(ca)^3b = u^{i+j}v^{-j}(ac)^3b = (uv^{-2})^jx, \text{ and}\\
x^u &= (u^iv^j(ac)^3b)^u = u^iv^ju^{-2}(ac)^3b = u^{i-2}v^j(ac)^3b = u^{-2}x.
\end{aligned}$$

When n is odd, these show there is just one conjugacy class of elements of the required form, of size n^2, and taking $(i, j) = (-1, 0)$ gives class representative $x = u^{-1}(ac)^3b = dcb$. On the other hand, when n is even there are two classes, of sizes $n^2/4$ and $3n^2/4$, with one consisting of the elements $u^rv^s(ac)^3b$ where r and s are both even, and the other of those where at least one of r and s is odd. In particular, taking $(r, s) = (0, 0)$ and $(-1, 0)$ gives representatives of these two classes as $(ac)^3b = udcb$ and $u^{-1}(ac)^3b = dcb$, respectively. □

Next, we find all smooth epimorphisms $\theta : \Gamma^* \to G_n \times C_2$, up to equivalence, where Γ^* is an NEC group with presentation given in (1), and θ takes each canonical generator c_i to a symmetry in $G_n \times C_2$, with $\theta(c_1) = b$. We will call such an epimorphism *admissible*. Two such epimorphisms θ and ν are equivalent if ν is the composite of θ with any

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automorphism of $G_n \times C_2$. Observe that in any such case, $\ker\theta = \ker\nu$, and then the Riemann surfaces $\mathbb{H}/\ker\theta$ and $\mathbb{H}/\ker\nu$ are the same.

Lemma 5.3. *For every $n \neq 3$, up to equivalence there are exactly two admissible epimorphisms from Γ^* onto $G_n \times C_2$, namely the epimorphisms θ_1 and θ_2 under which*

$$\theta_1 \colon (c_0, c_1, c_2, c_3) \mapsto (ab, b, cb, db) \quad \text{and} \quad \theta_2 \colon (c_0, c_1, c_2, c_3) \mapsto (ab, b, cdb, db).$$

For $n = 3$, up to equivalence there is just one such epimorphism, and this is the same as θ_1 above.

Proof. First, we observe that each of θ_1 and θ_2 is an epimorphism, since the images of the generators c_0, c_1, c_2 and c_3 of Γ^* satisfy the relations in (1), and the orders of these generators and their products c_0c_1, c_1c_2, c_2c_3 and c_3c_0 are preserved. In particular, the restriction of θ_1 to the orientation-preserving subgroup $(\Gamma^*)^+ = \langle c_0c_1, c_1c_2, c_2c_3\rangle$ takes the triple (c_0c_1, c_2c_1, c_3c_1) to (a, c, d), which generates the M^*-group G_n. Similarly, the restriction of θ_2 to $(\Gamma^*)^+$ takes (c_0c_1, c_2c_1, c_3c_1) to (a, cd, d), which also generates G_n.

Now let $\theta : \Gamma^* \to G_n \times C_2$ be any admissible epimorphism. Then

$$\theta \colon (c_0, c_1, c_2, c_3) \mapsto (x_0b, b, x_2b, x_3b),$$

where x_0, x_2 and x_3 are involutions which generate G_n, and have the property that x_2x_3 has order 2 while x_3x_0 has order 3. Then in the quotient $G_n/N \cong D_6$, which is generated by the images of a and c, the image of the subgroup generated by x_0 and x_3 must be dihedral of order 6. By applying an inner automorphism if necessary, we may suppose that $Nx_0 = Na$ and $Nx_3 = Nacaca = Nc(ac)^3$, so that $x_0 = w_0a$ and $x_3 = w_3c(ac)^3$, where w_0 and w_3 lie in N. In particular, since x_0 and x_3 are involutions, we find that $x_0 = u^iv^{-i}a$ for some i, while $x_3 = u^kc(ac)^3$ for some k. Also the image of x_2 is an involution that commutes with the image of x_3, so $Nx_2 = Nc$ or $N(ac)^3$.

If $Nx_2 = Nc$ then $x_2 = w_2c$ for some $w_2 \in N$, and since x_2 is an involution, we find that $x_2 = u^jv^{-2j}c$ for some j. Now it is an easy exercise to see that conjugation by $u^{i-j}v^{-j}$ takes $x_0 = u^iv^{-i}a$ to a, and $x_2 = u^jv^{-2j}c$ to c, and $x_3 = u^kc(ac)^3$ to $u^{3j-2i+k}c(ac)^3 = u^\lambda c(ac)^3$, where $\lambda = 3j - 2i + k$. Also since x_0, x_2 and x_3 generate G_n, so do their conjugates a, c and $u^\lambda c(ac)^3$, and it follows that λ is a unit mod n, with inverse ζ, say. Next, we can apply the automorphism of $G_n \times C_2$ that takes (a, b, c, d) to $(a, b, c, du^{-\zeta-1}) = (a, b, c, (acaca)u^{-\zeta})$. This takes $u = (acaca)d$ to $(acaca)^2u^{-\zeta} = u^{-\zeta}$, so takes u^λ to $u^{-\lambda\zeta} = u^{-1}$, and therefore takes a, c and $u^\lambda c(ac)^3$ to a, c and $u^{-1}c(ac)^3 = u^{-1}acaca = d$. Hence the given epimorphism θ is equivalent to one for which $x_0 = a$, $x_2 = c$ and $x_3 = d$, namely the epimorphism θ_1.

On the other hand, if $Nx_2 = N(ac)^3$ then $x_2 = w_2(ac)^3$ for some $w_2 \in N$, say $w_2 = u^rv^s$. Here we note that since c_2c_3 has order 2, so must its θ-image $x_2x_3 = u^rv^s(ac)^3u^kc(ac)^3 = u^{r-k}v^sc$, and it follows that $u^{2r-2k+s} = 1$, so $s \equiv 2k - 2r \bmod n$.

Conjugation by $u^{k+i-r}v^{k-r}$ then takes x_0, x_2 and x_3 to a, $u^{\lambda}(ac)^3$ and $u^{\lambda}c(ac)^3$, respectively, where $\lambda = 3r-2i-2k$. Again λ must be a unit mod n, and this time if ζ is its inverse mod n then the automorphism of $G_n \times C_2$ that takes (a, b, c, d) to $(a, b, c, du^{-\zeta-1})$ and u to $u^{-\zeta}$ takes u^{λ} to u^{-1}, and therefore takes a, $u^{\lambda}(ac)^3$ and $u^{\lambda}c(ac)^3$ to a, $u^{-1}(ac)^3 = dc = cd$ and $u^{-1}c(ac)^3 = d(ac)^6 = d$. Hence in this case the given epimorphism θ is equivalent to θ_2.

Finally, we note that the images of c_0c_2 under each of θ_1 and θ_2 are ac and acd. The former has order 6, but the latter has order $2n$. Hence if $n \neq 3$, then the epimorphisms θ_1 and θ_2 are not equivalent. On the other hand, if $n = 3$ then there exists an automorphism of $G_n \times C_2$ taking (a, b, c, d) to (a, b, cd, d), and so θ_1 and θ_2 are equivalent in that case. □

We note that the inequivalence of the epimorphisms θ_1 and θ_2 will also be evident when we show below that the associated Riemann surfaces $\mathbb{H}/\ker\theta_1$ and $\mathbb{H}/\ker\theta_2$ have different symmetry types.

We now proceed to determine these symmetry types, and the automorphism groups of the corresponding Klein surfaces.

Recall that we need only compute the species of the images of c_0, c_2 and c_3 under each epimorphism θ_i, and to find the number of ovals of each $\theta_i(c_j)$, we need to consider the centraliser of each c_j in Γ^* and of each $\theta_i(c_j)$ in $G_n \times C_2$.

In $G_n \times C_2$ we know that b is centralised by all of $G_n \times C_2$, of order $24n^2$, and that each of the symmetries ab, $db = (ab)^{da}$ and cb lies in a class of size $3n$, so each has centraliser of order $8n$. For odd n, the symmetry dcb lies in a class of size n^2, so its centraliser has order 24, while for even n, it lies in a class of size $3n^2/4$, so its centraliser has order 32. Also if n is even, then $u^{n/2}b$ lies in a class of size 3, and its centraliser has order $8n^2$, and $udcb$ lies in a class of size $n^2/4$, and its centraliser has order 96.

The orders of the corresponding centralisers in G_n are precisely half of the above orders, because $C_{G_n\times C_2}(wb) = C_{G_n\times C_2}(w)$ is a direct product $C_{G_n}(w) \times \langle b\rangle$, for every $w \in G_n$. This observation and a little further analysis of the centralisers gives the following:

Proposition 5.4. *The full automorphism group* Aut (S/τ) *of the Klein surface* S/τ *associated with each representative symmetry* τ *in* $G_n \times C_2$ *is given by the following:*

(a) Aut $(S/b) \cong G_n$*, of order* $12n^2$*;*
(b) Aut $(S/ab) \cong \langle uv\rangle \times \langle a\rangle \times \langle (ac)^3\rangle \cong C_n \times C_2 \times C_2$*, of order* $4n$*;*
(c) Aut $(S/cb) \cong \langle u, (ac)^3\rangle \times \langle c\rangle \cong D_n \times C_2$*, of order* $4n$*;*
(d) Aut $(S/dcb) \cong \langle c, a(u^{-1}v)^{(n-1)/2}\rangle = D_6$*, of order* 12*, if* n *is odd,*
while Aut $(S/dcb) \cong \langle c, v^{n/2}\rangle \rtimes \langle d\rangle \cong D_4 \rtimes C_2$*, of order* 16*, if* n *is even;*
(e) Aut $(S/u^{n/2}b) \cong \langle u, v\rangle \rtimes \langle c, (ac)^3\rangle \cong (C_n \times C_n) \rtimes (C_2 \times C_2)$*, of order* $4n^2$*, if* n *is even;*
(f) Aut $(S/udcb) \cong \langle u^{n/2}, v^{n/2}\rangle \rtimes \langle a, c\rangle \cong (C_2 \times C_2) \rtimes D_6$*, of order* 48*, if* n *is even.*

We now return to the task of determining the species of the representative symmetries $\theta_i(c_j)$ for $j \in \{1,2,3\}$. For both values of i and for all n, these three symmetries lie in distinct conjugacy classes. Again we use Theorem 3.3 and the observations by Singerman following it, to compute the number of ovals in each case.

The epimorphism θ_1 takes $C_{\Gamma^*}(c_0) = \langle\, c_0, c_1, (c_2c_3)^{c_0c_3}\rangle$ to $\langle\, a, b, (cd)^{ad}\,\rangle$. The conjugate of this by da is $\langle d, b, cd\rangle \cong \langle d, cd\rangle \times \langle b\rangle \cong (C_2 \times C_2) \times C_2$, and so it has order 8. Also θ_1 takes $C_{\Gamma^*}(c_1) = \langle\, c_0, c_1, c_2\rangle$ to $\langle a, b, c\rangle \cong \langle a, c\rangle \times \langle b\rangle \cong D_6 \times C_2$, and so has order 24, while θ_1 takes $C_{\Gamma^*}(c_2) = \langle\, c_1, c_2, c_3\rangle$ to $\langle b, c, d\rangle$, which again has order 8. By Theorem 3.3, it follows that the number of ovals for $\theta_1(c_1) = b$ is $24n^2/24 = n^2$, while the number of ovals for each of $\theta_1(c_0) = ab$ and $\theta_1(c_2) = cb$ is $8n/8 = n$.

Similarly, the epimorphism θ_2 takes $C_{\Gamma^*}(c_0)$ to $\langle a, b, c^{ad}\,\rangle = \langle d, b, c\rangle^{ad}$, of order 8, and takes $C_{\Gamma^*}(c_2) = \langle\, c_1, c_2, c_3\rangle$ to $\langle b, cd, d\rangle$, also of order 8. On the other hand, θ_2 takes $C_{\Gamma^*}(c_1)$ to $\langle a, b, cd\rangle \cong \langle a, cd\rangle \times \langle b\rangle$, which is isomorphic to $D_{2n} \times C_2$, of order $8n$, because acd has order $2n$. By Theorem 3.3, it follows that the number of ovals for $\theta_2(c_0) = ab$ is $8n/8 = n$, while the number of ovals for $\theta_2(c_1) = b$ is $24n^2/(8n) = 3n$, and the number of ovals for $\theta_2(c_3) = cdb = dcb$ is $24/8 = 3$ when n is odd, or $32/8 = 4$ when n is even.

Remark 5.5. A consequence of Proposition 5.4 and what we have just shown about the numbers of ovals is that *for each $n > 3$ the group G_n is the automorphism group of two Klein surfaces with maximal symmetry, with algebraic genus $g = n^2 + 1$, and having n^2 and $3n$ boundary components respectively.* This was shown also by Etayo in [12, Prop. 1]. We will see below that both surfaces are orientable.

Next, we determine the separating character of each representative symmetry $\theta_i(c_j)$, again using the Hoare–Singerman method explained in [14, Section 3].

Dealing with $\theta_i(c_1) = b$ is easy. There is a loop for the effect of b at every vertex of the Schreier coset graph for the action of $G_n \times C_2$ on cosets of $\langle b\rangle$, and when these loops are deleted, what remains is the Schreier coset graph for the regular representation of G_n, via the generating set $\{a, c, d\}$ in the case of θ_1, or $\{a, cd, d\}$ in the case of θ_2. For θ_1, this graph is bipartite, since all the relators in the presentation defining G_n have even length (as words in $\{a, c, d\}$). Similarly, for θ_2, we can re-write the presentation for G_n in terms of the generators a, c' and d, where $c' = cd$, as

$$a^2 = (c'd)^2 = d^2 = (c')^2 = (ad)^3 = (ac'd)^6 = (ac')^{2n} = 1,$$

since the final relation $(acacad)^n$ is equivalent to $(acd)^{2n} = 1$; in particular, all the relators have even length as words in $\{a, c', d\}$, and so again the graph is bipartite. Hence in both cases, the quotient surface S/b is orientable, as claimed above.

In contrast, we will see that the graphs that occur for the images of other generators c_0 (or c_3) and c_2 are not bipartite, and so the corresponding surfaces are non-orientable, for all $n > 2$. When $n = 1$ or 2, the graphs for $\theta_2(c_2)$ and $\theta_2(c_3)$ are bipartite, while those for $\theta_1(c_2)$ and $\theta_1(c_3)$ are not.

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For θ_1, we consider the two Schreier coset graphs for the action of $G_n \times C_2$ on the cosets of $\langle cb \rangle$ and $\langle db \rangle$, respectively, via generators $x_0 = ab$, $x_1 = b$, $x_2 = cb$ and $x_3 = db$. We can find a cycle of odd length in the loop-free versions of these graphs, by taking $w = x_2x_3x_0 = cdab$. This element w has order $2n$, since $(cdab)^2 = cdacda = cdadca = cadaca = v^{ca}$, which has order n and lies in G_n, but $cdab \notin G_n$. It follows that $x_3x_0(x_2x_3x_0)^{2n-1} = x_2 \in \langle cb \rangle$, and similarly, $x_0x_2(x_3x_0x_2)^{2n-1} = x_3 \in \langle db \rangle$.

Applying the word $w = x_3x_0(x_2x_3x_0)^{2n-1}$ to the cosets of $H = \langle x_2 \rangle = \langle cb \rangle$ gives a closed walk of length $6n - 1$, namely $H—Hx_3—Hx_3x_0—Hx_3x_0x_2—\dots —Hx_0—H$. To see this, note that the walk can be re-written as $Hw_1—Hw_2—Hw_3— \dots —Hw_{6n-1}—Hw_1$, with coset representatives $w_1 = 1$, $w_2 = x_3 = db = u^{-1}acacab$, followed by

$$\begin{aligned} &w_{6j+3} = u^{-1}v^{-j}acac, && w_{6j+4} = u^{-1}v^{-j}acab, && w_{6j+5} = v^{-j-1}ca, \\ &w_{6j+6} = v^{-j-1}cb, && w_{6j+7} = v^{-j-1}, && w_{6j+8} = u^{-1}v^{-j-1}acacab, \end{aligned}$$

for $0 \le j \le n-2$, and then $w_{6n-3} = u^{-1}v(ac)^2$, $w_{6n-2} = u^{-1}vacab$, and $w_{6n-1} = ca$.

We now check that none of the edges of this closed walk is a loop. For multiplication by a generator x_k to create a loop at some coset of $H = \langle cb \rangle$, however, the generator x_k must be a conjugate of cb. Since none of x_0 $(= ab)$, x_1 $(= b)$ and x_3 $(= db)$ lies in the same conjugacy class as cb, this can happen only when $x_k = x_2 = cb$ itself. On the other hand, if x_2 fixes a coset Hw_i, then $x_2^{w_i} = x_2$ and so w_i commutes with $x_2 = cb$, and therefore with c. But it is easy to see that the only coset representative w_i that commutes with c is $w_1 = 1$, and so there is no such loop. Hence the loop-free version of the Schreier coset graph contains a closed walk of odd length $6n - 1$, and therefore cannot be bipartite.

A similar argument works for a closed walk of length $6n - 1$ in the other Schreier coset graph, traced out by the word $x_0x_2(x_3x_0x_2)^{2n-1}$ on cosets of $\langle x_3 \rangle = \langle db \rangle$. Hence the loop-free version of this graph is non-bipartite as well.

The case of θ_2 is a little easier to deal with than θ_1. Here we consider the two Schreier coset graphs for the action of $G_n \times C_2$ on the cosets of $\langle cdb \rangle$ and $\langle db \rangle$, respectively, via generators $y_0 = ab$, $y_1 = b$, $y_2 = cdb$ and $y_3 = db$. The element $y_2y_3y_0 = cdbdbab = cab$ has order 6, and it follows that $y_3y_0(y_2y_3y_0)^5 = y_2 \in \langle cdb \rangle$ and $y_0y_2(y_3y_0y_2)^5 = y_3 \in \langle db \rangle$. Applying the word $y_3y_0(y_2y_3y_0)^5$ to the cosets of $\langle cdb \rangle$ gives a circuit of length 17 in the first graph, and applying $y_0y_2(y_3y_0y_2)^5$ to the cosets of $\langle db \rangle$ gives a circuit of length 17 in the second graph. Again in both cases, the graph has a cycle of odd length and is therefore non-bipartite, as claimed.

(In fact, we believe the closed walks of length 17 and $6n - 1$ described above are the shortest closed walks of odd length in the respective graphs, but we will not even attempt to prove that here.)

This completes the proof Theorem 5.1.

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Remark 5.6. The surfaces in Theorem 5.1 are not unique for each n, but are members of an infinite family of surfaces with the same properties. The reason for this is that the dimension of the Teichmüller space of NEC groups with signature $(0; +; [-]; \{(2, 2, 2, 3)\})$ is positive, and so there are infinitely many such groups that are non-conjugate in $\mathrm{Aut}\,(\mathbb{H})$. On the other hand, a consequence of Lemma 5.3 is that if S is a Riemann surface with maximal real symmetry such that $\mathrm{Aut}\,(S) \cong G_n \times C_2$, then its symmetry type is one of those occurring in Theorem 5.1.

Appendix A. Symmetry types of surfaces of genus 2 to 25 with maximal real symmetry

[Note: In these tables, $u = ac$ and $v = (ac)^d$ $(= adac)$.]

Genus 2	The Accola–Maclachlan curve of genus 2 $\mathrm{Aut}(S) \cong \langle\, x, y, z \mid x^2 = y^2 = z^2 = (xy)^4 = (yz)^6 = (xz)^2 = (yxyz)^2 = 1\,\rangle$ $\lvert\mathrm{Aut}(S)\rvert = 48$ $\lvert\mathrm{Aut}(\mathrm{Aut}(S))\rvert = 96$ No. symmetry classes = 4		
	Representative symmetry τ_i	Species	$\lvert\mathrm{Aut}(S/\tau_i)\rvert$
	x	3	12
	y	−1	4
	z	1	8
	$(xy)^2z$	0	8

Genus 2	$\lvert G\rvert = 12$ $\lvert\mathrm{Aut}(G \times C_2)\rvert = 144$ Additional relators: u^2	No. symmetry classes = 4	
	Representative symmetry τ_i	Species	$\lvert\mathrm{Aut}(S/\tau_i)\rvert$
	τ	3	12
	$c\tau$	3	12
	$d\tau$	1	4
	$cd\tau$	0	4

Genus 2	$\lvert G\rvert = 12$ $\lvert\mathrm{Aut}(G \times C_2)\rvert = 144$ Additional relators: uv	No. symmetry classes = 4	
	Representative symmetry τ_i	Species	$\lvert\mathrm{Aut}(S/\tau_i)\rvert$
	τ	1	12
	$c\tau$	−1	4
	$d\tau$	−1	4
	$cd\tau$	0	12

Genus 3	$\lvert G\rvert = 24$ $\lvert\mathrm{Aut}(G \times C_2)\rvert = 48$ Additional relators: u^3	No. symmetry classes = 3	
	Representative symmetry τ_i	Species	$\lvert\mathrm{Aut}(S/\tau_i)\rvert$
	τ	4	24
	$c\tau$	2	4
	$cd\tau$	0	8

Genus 3	$\lvert G\rvert = 24$ $\lvert\mathrm{Aut}(G \times C_2)\rvert = 48$ Additional relators: $duvu$	No. symmetry classes = 3	
	Representative symmetry τ_i	Species	$\lvert\mathrm{Aut}(S/\tau_i)\rvert$
	τ	−3	24
	$c\tau$	−2	8
	$d\tau$	−1	4

Genus 4	$\lvert G\rvert = 36$ $\lvert\mathrm{Aut}(G \times C_2)\rvert = 288$ Additional relators: u^2v^2	No. symmetry classes = 4	
	Representative symmetry τ_i	Species	$\lvert\mathrm{Aut}(S/\tau_i)\rvert$
	τ	3	36
	$c\tau$	−3	12
	$d\tau$	−1	4
	$cd\tau$	0	12

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Genus 5	$\vert G\vert = 48$ $\vert\mathrm{Aut}(G \times C_2)\vert = 576$	No. symmetry classes = 6	
	Additional relators: u^4		
	Representative symmetry τ_i	Species	$\vert\mathrm{Aut}(S/\tau_i)\vert$
	τ	6	48
	$c\tau$	4	16
	$d\tau$	2	8
	$cd\tau$	0	8
	$(ca)^2\tau$	0	16
	$(cda)^3\tau$	0	48
Genus 5	$\vert G\vert = 48$ $\vert\mathrm{Aut}(G \times C_2)\vert = 576$	No. symmetry classes = 6	
	Additional relators: $(uv)^2$		
	Representative symmetry τ_i	Species	$\vert\mathrm{Aut}(S/\tau_i)\vert$
	τ	4	48
	$c\tau$	−2	8
	$d\tau$	−2	8
	$cd\tau$	0	16
	$(cda)^2\tau$	0	16
	$(ca)^3\tau$	0	48
Genus 6	$\vert G\vert = 60$ $\vert\mathrm{Aut}(G \times C_2)\vert = 120$	No. symmetry classes = 2	
	Additional relators: $u^5, duvuvu$		
	Representative symmetry τ_i	Species	$\vert\mathrm{Aut}(S/\tau_i)\vert$
	τ	−6	60
	$c\tau$	−2	4
Genus 9	$\vert G\vert = 96$ $\vert\mathrm{Aut}(G \times C_2)\vert = 1536$	No. symmetry classes = 5	
	Additional relators: u^3v^3		
	Representative symmetry τ_i	Species	$\vert\mathrm{Aut}(S/\tau_i)\vert$
	τ	4	96
	$c\tau$	−2	8
	$d\tau$	−2	8
	$cd\tau$	0	16
	$(cda)^4\tau$	0	96
Genus 9	$\vert G\vert = 96$ $\vert\mathrm{Aut}(G \times C_2)\vert = 1536$	No. symmetry classes = 5	
	Additional relators: $(u^2v)^2$		
	Representative symmetry τ_i	Species	$\vert\mathrm{Aut}(S/\tau_i)\vert$
	τ	6	96
	$c\tau$	−4	16
	$d\tau$	−2	8
	$cd\tau$	0	8
	$(ca)^4\tau$	0	96
Genus 10	$\vert G\vert = 108$ $\vert\mathrm{Aut}(G \times C_2)\vert = 864$	No. symmetry classes = 4	
	Additional relators: $u^6, (uv)^3$		
	Representative symmetry τ_i	Species	$\vert\mathrm{Aut}(S/\tau_i)\vert$
	τ	9	108
	$c\tau$	−3	12
	$d\tau$	−3	12
	$cd\tau$	0	12
Genus 11	$\vert G\vert = 120$ $\vert\mathrm{Aut}(G \times C_2)\vert = 720$	No. symmetry classes = 4	
	Additional relators: u^5		
	Representative symmetry τ_i	Species	$\vert\mathrm{Aut}(S/\tau_i)\vert$
	τ	12	120
	$c\tau$	4	8
	$cd\tau$	0	8
	$(cda)^5\tau$	0	120

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Genus 11	$\lvert G\rvert = 120$ $\lvert\mathrm{Aut}(G \times C_2)\rvert = 720$	No. symmetry classes = 4	
	Additional relators: $duvuvu$		
	Representative symmetry τ_i	Species	$\lvert\mathrm{Aut}(S/\tau_i)\rvert$
	τ	-6	120
	$c\tau$	-2	8
	$d\tau$	-2	8
	$(ca)^5\tau$	0	120
Genus 11	$\lvert G\rvert = 120$ $\lvert\mathrm{Aut}(G \times C_2)\rvert = 720$	No. symmetry classes = 4	
	Additional relators: $du^2v^2u^2$		
	Representative symmetry τ_i	Species	$\lvert\mathrm{Aut}(S/\tau_i)\rvert$
	τ	-6	120
	$c\tau$	-2	8
	$d\tau$	-2	8
	$(ca)^5\tau$	0	120
Genus 13	$\lvert G\rvert = 144$ $\lvert\mathrm{Aut}(G \times C_2)\rvert = 576$	No. symmetry classes = 6	
	Additional relators: u^6, u^2vuv^2uv		
	Representative symmetry τ_i	Species	$\lvert\mathrm{Aut}(S/\tau_i)\rvert$
	τ	12	144
	$c\tau$	-6	24
	$d\tau$	-2	8
	$cd\tau$	0	16
	$(ca)^3\tau$	0	48
	$((ca)^2d)^2\tau$	0	48
Genus 13	$\lvert G\rvert = 144$ $\lvert\mathrm{Aut}(G \times C_2)\rvert = 576$	No. symmetry classes = 6	
	Additional relators: $(uv)^3, u^4v^4$		
	Representative symmetry τ_i	Species	$\lvert\mathrm{Aut}(S/\tau_i)\rvert$
	τ	6	144
	$c\tau$	-4	16
	$d\tau$	-2	8
	$cd\tau$	0	24
	$(cda)^3\tau$	0	48
	$((ca)^2d)^2\tau$	0	48
Genus 17	$\lvert G\rvert = 192$ $\lvert\mathrm{Aut}(G \times C_2)\rvert = 1536$	No. symmetry classes = 6	
	Additional relators: $u^6, (uv)^4$		
	Representative symmetry τ_i	Species	$\lvert\mathrm{Aut}(S/\tau_i)\rvert$
	τ	16	192
	$c\tau$	-4	16
	$d\tau$	-4	16
	$cd\tau$	0	16
	$(ca)^3\tau$	0	48
	$(cda)^4\tau$	0	64
Genus 17	$\lvert G\rvert = 192$ $\lvert\mathrm{Aut}(G \times C_2)\rvert = 1536$	No. symmetry classes = 6	
	Additional relators: $(uv)^3, u^8$		
	Representative symmetry τ_i	Species	$\lvert\mathrm{Aut}(S/\tau_i)\rvert$
	τ	12	192
	$c\tau$	-4	16
	$d\tau$	-4	16
	$cd\tau$	0	16
	$(ca)^4\tau$	0	64
	$(cda)^3\tau$	0	48
Genus 21	$\lvert G\rvert = 240$ $\lvert\mathrm{Aut}(G \times C_2)\rvert = 20\,160$	No. symmetry classes = 8	
	Additional relators: $(u^3v)^2$		
	Representative symmetry τ_i	Species	$\lvert\mathrm{Aut}(S/\tau_i)\rvert$
	τ	12	240
	$c\tau$	-4	16
	$d\tau$	-4	16
	$cd\tau$	0	16
	$(ca)^4da\tau$	0	16
	$(ca)^5\tau$	0	240
	$(cda)^5\tau$	0	240
	$((ca)^2d)^3\tau$	0	240

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Genus 25	$\vert G\vert = 288$ $\vert \mathrm{Aut}(G \times C_2)\vert = 4608$	No. symmetry classes = 5	
	Additional relators: u^4v^4		
	Representative symmetry τ_i	Species	$\vert \mathrm{Aut}(S/\tau_i)\vert$
	τ	6	288
	$c\tau$	−4	16
	$d\tau$	−2	8
	$cd\tau$	0	24
	$(cda)^6\tau$	0	288
Genus 25	$\vert G\vert = 288$ $\vert \mathrm{Aut}(G \times C_2)\vert = 4608$	No. symmetry classes = 5	
	Additional relators: u^2vuv^2uv		
	Representative symmetry τ_i	Species	$\vert \mathrm{Aut}(S/\tau_i)\vert$
	τ	12	288
	$c\tau$	−6	24
	$d\tau$	−2	8
	$cd\tau$	0	16
	$(ca)^6\tau$	0	288

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