Journal of Algebra 468 (2016) 296–336

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Twistors of nonlocal vertex algebras

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A R T I C L E   I N F O

Article history:
Received 6 March 2016
Available online 30 August 2016
Communicated by Masaki Kashiwara

Keywords:
Nonlocal vertex algebra
Twisted tensor product
R-matrix
Twistor

A B S T R A C T

In this paper we introduce and study the concept of twistor for a nonlocal vertex algebra. This concept provides a unified framework for various constructions of nonlocal vertex algebras, such as R-matrices, twisted tensor products, iterated twisted tensor products and L-R-twisted tensor products of nonlocal vertex algebras. Among the main results, we find the relations among theses constructions and we also give one concrete example of a twistor. Furthermore, we study some properties of twistors.

1. Introduction

Vertex algebras are both analogues and generalizations of commutative and associative unital algebras, while nonlocal vertex algebras (or field algebras in the sense of [1]) are analogues and generalizations of associative unital algebras. In [6], one of an important series of papers, Etingof and Kazhdan developed a fundamental theory of quantum vertex operator algebras in the sense of formal deformation, where quantum vertex operator algebras are ($\hbar$-adic) nonlocal vertex algebras (over $\mathbb{C}[[\hbar]]$) which satisfy what was called $\mathcal{S}$-locality. Furthermore, a theory of (weak) quantum vertex algebras (see [10,11])

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http://dx.doi.org/10.1016/j.jalgebra.2016.07.038

was developed, where weak quantum vertex algebras are generalizations of vertex superalgebras, instead of formal deformations. Weak quantum vertex algebras in this sense are nonlocal vertex algebras that satisfy a variation of Etingof–Kazhdan's $\mathcal{S}$-locality. In this developing theory, constructing interesting examples of quantum vertex algebras is one of the most important problems. Previously, inspired by twisted tensor products theory of associative algebras (see [5,20,15,4]), we developed a theory of twisted tensor products of nonlocal vertex algebras and their modules in [14,17–19].

Let U and V be two nonlocal vertex algebras. Motivated by a recent study [14] on regular representations for Möbius quantum vertex algebras, a twisting operator in [13] was defined to be a linear map $R(x):\ V \otimes U \to U \otimes V \otimes \mathbb{C}((x))$ satisfying

$$R(x)(v \otimes \mathbf{1}) = \mathbf{1} \otimes v, \quad R(x)(\mathbf{1} \otimes u) = u \otimes \mathbf{1} \ \text{ for } u \in U,\, v \in V,$$
$$R(x_1)(1 \otimes Y(x_2)) = (Y(x_2) \otimes 1)R^{23}(x_1)R^{12}(x_1 + x_2),$$
$$R(x_1)(Y(x_2) \otimes 1) = (1 \otimes Y(x_2))R^{12}(x_1 - x_2)R^{23}(x_1).$$

These conditions are stringy analogues of those for a twisting map with associative algebras (see [5,20]). The underlying space of the twisted tensor product $U \otimes_R V$ associated to $R(x)$ is $U \otimes V$, while the vacuum vector is $\mathbf{1}_U \otimes \mathbf{1}_V$ and the vertex operator map, denoted by $Y_R(x)$, is given by

$$Y_R(u \otimes v, x)(u' \otimes v') = (Y_U(x) \otimes Y_V(x))(u \otimes R(x)(v \otimes u') \otimes v')$$

for $u, u' \in U$, $v, v' \in V$. It was proved in [13] that $U \otimes_R V$ is a nonlocal vertex algebra, containing U and V canonically as subalgebras which satisfy a certain commutation relation. (If both U and V are weak quantum vertex algebras, it was proved that $U \otimes_R V$ is a weak quantum vertex algebra.) On the other hand, it was proved that if a nonlocal vertex algebra K, which is non-degenerate in the sense of [6], contains subalgebras U and V satisfying a certain commutation relation, then K is isomorphic to the twisted tensor product $U \otimes_R V$ with respect to a twisting operator $R(x)$. Also established in that paper was a universal property for the twisted tensor product $U \otimes_R V$, similar to the one for the ordinary tensor product $U \otimes V$. The smash product $U \sharp V$, formulated in [12], was also slightly generalized and realized as the twisted tensor product with respect to a canonical twisting operator.

Furthermore, in [17], we studied iterated twisted tensor products of nonlocal vertex algebras and of weak quantum vertex algebras. Let U, V and W be three nonlocal vertex algebras, let $R_1(x)$, $R_2(x)$ and $R_3(x)$ be twisting operators for the ordered pair (U, V), (V, W) and (U, W), respectively. We showed that U, V and W give rise to an iterated twisted tensor product with three factors, denoted by $U \otimes_{R_1} V \otimes_{R_2} W$, if $R_1(x)$, $R_2(x)$ and $R_3(x)$ satisfy the following compatibility condition

$$R_2^{23}(x_1 - x_2)R_3^{12}(x_1)R_1^{23}(x_2) = R_1^{12}(x_2)R_3^{23}(x_1)R_2^{12}(x_1 - x_2).$$

And we found conditions for twisting operators $T_1(x) : W \otimes (U \otimes_{R_1} V) \to (U \otimes_{R_1} V) \otimes W \otimes \mathbb{C}((x))$ and $T_2(x) : (V \otimes_{R_2} W) \otimes U \to U \otimes (V \otimes_{R_2} W) \otimes \mathbb{C}((x))$ could be split as a composition of two suitable twisting operators. If U, V and W are weak quantum vertex algebras, it was proved that $U \otimes_{R_1} V \otimes_{R_2} W$ is a weak quantum vertex algebra. Furthermore, we constructed some iterated twisted tensor product nonlocal vertex algebras, which are isomorphic to nonlocal vertex algebra $U \otimes_{R_1} V \otimes_{R_2} W$, with some suitable conditions. It was proved in [17] that the iterated twisted tensor product of three factors can be lifted to that of any number of factors with compatible twisting operators. Also established in that paper is a universal property for the iterated twisted tensor product $V_1 \otimes_{R_{12}} V_2 \otimes_{R_{23}} \cdots \otimes_{R_{n-1,n}} V_n$, similar to the one for the ordinary tensor product $V_1 \otimes V_2 \otimes \cdots \otimes V_n$. And it was shown that noncommutative $2l$-planes defined by Connes and Dubois-Violette in [3], can be realized as iterated twisted tensor product of some nonlocal vertex algebras with suitable twisting operators.

In the theory of associative algebras, there is a notion of a twistor (see [16]). Let A, B be algebras with multiplication μ_A, μ_B and let $A \otimes_R B$ be a twisted tensor product algebra with a twisting map R. The initial aim in [16] was to relate the multiplications $\mu_{A\otimes_R B}$ of $A \otimes_R B$ and $\mu_{A\otimes B}$ of $A \otimes B$. It can be readily seen that $\mu_{A\otimes_R B} = \mu_{A\otimes B}T$, where $\mu : (A \otimes B) \otimes (A \otimes B) \to (A \otimes B) \otimes (A \otimes B)$ is a map depending on R, and the problem is to find the abstract properties satisfied by this map T, which together with the associativity of $\mu_{A\otimes B}$ imply the associativity of $\mu_{A\otimes_R B}$. Then they were led to introduce the concept of *twistor* for an algebra D, as a linear map $T : D \otimes D \to D \otimes D$ satisfying a list of axioms which imply that the new multiplication $\mu_D T$ is an algebra structure on the vector space D. Explicitly, a twistor of D is a linear map $T : D \otimes D \to D \otimes D$ satisfying

$$T(1 \otimes d) = 1 \otimes d, \qquad T(d \otimes 1) = d \otimes 1 \qquad \text{for } d \in D,$$
$$\mu_{23}T_{13}T_{12} = T\mu_{23}, \qquad \mu_{12}T_{13}T_{23} = T\mu_{12}, \qquad T_{12}T_{23} = T_{23}T_{12}.$$

The multiplication μT gives another associative algebra structure on D, with the same unit 1.

Partly motivated by [16], in this paper, we study twistors of nonlocal vertex algebras and quantum vertex algebras. The main purpose is to build various tools for constructing new interesting quantum vertex algebras. Let V be a nonlocal vertex algebra. We define a twistor to be a linear map

$$T(x) : \; V \otimes V \to V \otimes V \otimes \mathbb{C}((x)),$$

satisfying a set of conditions which are stringy analogues of those listed before for a twistor of associative algebras. The underlying space of the nonlocal vertex algebra V^T associated to $T(x)$ is V, while the vacuum vector is $\mathbf{1}$ and the vertex operator, denoted by $Y_T(x)$, is given by

$$Y_T(u, x)v = Y(x)T(x)(u \otimes v)$$

for $u, v \in V$. It is proved that V^T is a nonlocal vertex algebra. The main results in this paper are the relations between twistors and twisted tensor products, and iterated twisted tensor products, and n-factor iterated twisted tensor products and L-R-twisted tensor products. Explicitly, as one of main results in this paper, let U, V and W be three nonlocal vertex algebras, let $R_1(x)$, $R_2(x)$ and $R_3(x)$ be twisting operators for the ordered pair (U, V), (V, W) and (U, W), respectively, and let σ be the flip operator. For nonlocal vertex algebra $K = U \otimes V \otimes W$, we show that the operator $T(x) : K \otimes K \to K \otimes K \otimes \mathbb{C}((x))$:

$$T(x) = \sigma^{34}\sigma^{45}\sigma^{23}R_1^{23}(-x)R_2^{45}(-x)R_3^{34}(-x)$$

is a twistor for K if and only if $(R_1(x), R_2(x), \sigma)$, $(R_1(x), \sigma, R_3(x))$ and $(\sigma, R_2(x), R_3(x))$ are compatible triples. Moreover, in this case it follows that $R_1(x)$, $R_2(x)$ and $R_3(x)$ are compatible twisting operators and $K^T = U \otimes_{R_1} V \otimes_{R_2} W$. Furthermore, we generalize this result to the case of any number of factors by induction.

Also established in the present paper is the relation between a twistor and an R-matrix. A concept of R-matrix for a ring was introduced by Borcherds in [2] to study quantum vertex algebras. Here we define an R-matrix for a nonlocal vertex algebra and using an R-matrix we construct a new nonlocal vertex algebra.

The concept of twistor gives a unifying framework for these deformed constructions of nonlocal vertex algebras. In particular, as an illustrating example we apply this theory to the nonlocal vertex algebra constructed from noncommutative $2l$-plane $V_{\mathbf{Q}}$. We prove that nonlocal vertex algebra $V_{\mathbf{Q}}$ is isomorphic to the nonlocal vertex algebra $(V_{11} \otimes V_{22} \otimes \cdots \otimes V_{ll})^T$ constructed from $V_{11} \otimes V_{22} \otimes \cdots \otimes V_{ll}$ by suitable twistor $T(x)$.

Motivated by [16], we may study twistors for modules of nonlocal vertex algebras and generalized twistors.

This paper is organized as follows: In Section 2, we present some basic notions. In Section 3, we study twistors of nonlocal vertex algebras. In Section 4, we present some examples of twistors.

2. Preliminaries

For an accessible introduction to the theory of vertex (operator) algebras and their representations, we refer the reader to [8]. We begin by recalling the notion of nonlocal vertex algebra. A *nonlocal vertex algebra* (see [9], cf. [1]) is a vector space V, equipped with a linear map

$$\begin{aligned} Y(\cdot, x): \quad & V \to \mathrm{Hom}(V, V((x))) \subset (\mathrm{End}V)[[x, x^{-1}]] \\ & v \mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \quad (\text{where } v_n \in \mathrm{End}V), \end{aligned}$$

and equipped with a vector $\mathbf{1} \in V$, satisfying the conditions that for $v \in V$,

$$Y(\mathbf{1}, x)v = v, \quad Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x \to 0} Y(v, x)\mathbf{1} = v, \tag{2.1}$$

and that for $u, v, w \in V$, there exists a nonnegative integer k such that

$$(x_0 + x_2)^k Y(u, x_0 + x_2)Y(v, x_2)w = (x_0 + x_2)^k Y(Y(u, x_0)v, x_2)w \tag{2.2}$$

(*weak associativity*).

We sometimes denote a nonlocal vertex algebra by a triple $(V, Y, \mathbf{1})$, to emphasize the *vertex operator map* Y and the *vacuum vector* $\mathbf{1}$.

We now recall from [10] the notion of quantum vertex algebra.

A *rational quantum Yang–Baxter operator* on a vector space U is a linear map

$$\mathcal{S}(x) : \ U \otimes U \to U \otimes U \otimes \mathbb{C}((x)),$$

satisfying

$$\mathcal{S}^{12}(x)\mathcal{S}^{13}(x + z)\mathcal{S}^{23}(z) = \mathcal{S}^{23}(z)\mathcal{S}^{13}(x + z)\mathcal{S}^{12}(x)$$

(the *quantum Yang–Baxter equation*), where for $1 \le i < j \le 3$,

$$\mathcal{S}^{ij}(x) : \ U \otimes U \otimes U \to U \otimes U \otimes U \otimes \mathbb{C}((x))$$

denotes the canonical extension of $\mathcal{S}(x)$. It is said to be *unitary* if

$$\mathcal{S}(x)\mathcal{S}^{21}(-x) = 1,$$

where $\mathcal{S}^{21}(x) = \sigma\mathcal{S}(x)\sigma$ with σ denoting the flip operator on $U \otimes U$.

Definition 2.1. A *quantum vertex algebra* is a nonlocal vertex algebra V equipped with a unitary rational quantum Yang–Baxter operator $\mathcal{S}(x)$ on V, satisfying the following conditions:

$$\mathcal{S}(x)(\mathbf{1} \otimes v) = \mathbf{1} \otimes v \quad \text{for } v \in V, \tag{2.3}$$

$$[\mathcal{D} \otimes 1, \mathcal{S}(x)] = -\frac{d}{dx}\mathcal{S}(x), \tag{2.4}$$

$$Y(u, x)v = e^{x\mathcal{D}}Y(-x)\mathcal{S}(-x)(v \otimes u) \quad \text{for } u, v \in V, \tag{2.5}$$

$$\mathcal{S}(x_1)(Y(x_2) \otimes 1) = (Y(x_2) \otimes 1)\mathcal{S}^{23}(x_1)\mathcal{S}^{13}(x_1 + x_2), \tag{2.6}$$

where $\mathcal{D}$ is the linear operator on V defined by $\mathcal{D}(v) = v_{-2}\mathbf{1}$ for $v \in V$. We denote a quantum vertex algebra by a pair $(V, \mathcal{S})$.

3. Twistors

In this section, firstly we define the notion of twistor and construct a new nonlocal vertex algebra structure from a nonlocal vertex algebra with a twistor. Then we study the inverse and composition of twistors.

Definition 3.1. Let V be a nonlocal vertex algebra. A *twistor* for V is a linear map

$$T(x):\ V\otimes V\to V\otimes V\otimes\mathbb{C}((x)),$$

satisfying the following conditions:

$$T(x)(\mathbf{1}\otimes v)=\mathbf{1}\otimes v\qquad\text{for } v\in V,\tag{3.1}$$

$$T(x)(v\otimes\mathbf{1})=v\otimes\mathbf{1}\qquad\text{for } v\in V,\tag{3.2}$$

$$T(x_1)(1\otimes Y(x_2))=(1\otimes Y(x_2))T^{13}(x_1)T^{12}(x_1-x_2),\tag{3.3}$$

$$T(x_1)(Y(x_2)\otimes 1)=(Y(x_2)\otimes 1)T^{13}(x_1+x_2)T^{23}(x_1),\tag{3.4}$$

$$T^{12}(x_1)T^{23}(x_2)=T^{23}(x_2)T^{12}(x_1).\tag{3.5}$$

We now present the general twisting of a nonlocal vertex algebra.

Theorem 3.2. *Let V be a nonlocal vertex algebra and let $T(x)$ be a twistor of V. Set*

$$Y_T(x)=Y(x)T(x):V\otimes V\to V((x)).$$

Then $(V,Y_T(x),\mathbf{1})$ carries the structure of a nonlocal vertex algebra, which is denoted by V^T.

Proof. For $u,v\in V$, by definition we have

$$Y_T(x)(u\otimes v)=\sum_{i=1}^{n}f_i(x)Y(u^{(i)},x)v^{(i)},$$

where

$$T(x)(u\otimes v)=\sum_{i=1}^{n}u^{(i)}\otimes v^{(i)}\otimes f_i(x)\in V\otimes V\otimes\mathbb{C}((x)).$$

As

$$f_i(x)\in\mathbb{C}((x)),\quad Y(u^{(i)},x)v^{(i)}\in V((x))$$

for $1\le i\le n$, we see that $Y_T(u,x)v$ exists in $V((x))$.

For $v\in V$, with (3.1), we have

$$Y_T(\mathbf{1},x)v=Y(x)T(x)(\mathbf{1}\otimes v)=Y(\mathbf{1},x)v=v.$$

On the other hand, for $v\in V$, from (3.2) we get

$$Y_T(v,x)\mathbf{1}=Y(x)T(x)(v\otimes\mathbf{1})=Y(v,x)\mathbf{1}\in V[[x]],$$

and

$$\lim_{x\to 0} Y_T(v,x)\mathbf{1} = \lim_{x\to 0} Y(x)T(x)(v\otimes \mathbf{1}) = \lim_{x\to 0} Y(v,x)\mathbf{1} = v.$$

To see weak associativity, let $u, v, w \in V$. Using (3.3) we have

$$\begin{aligned}
&Y_T(u, x_0+x_2)Y_T(v,x_2)w \\
&= Y(x_0+x_2)T(x_0+x_2)(1\otimes Y(x_2))T^{23}(x_2)(u\otimes v\otimes w) \\
&= Y(x_0+x_2)(1\otimes Y(x_2))T^{13}(x_0+x_2)T^{12}(x_0)T^{23}(x_2)(u\otimes v\otimes w).
\end{aligned}$$

On the other hand, using (3.4) we get

$$\begin{aligned}
&Y_T(Y_T(u,x_0)v,x_2)w \\
&= Y(x_2)T(x_2)(Y(x_0)\otimes 1)T^{12}(x_0)(u\otimes v\otimes w) \\
&= Y(x_2)(Y(x_0)\otimes 1)T^{13}(x_2+x_0)T^{23}(x_2)T^{12}(x_0)(u\otimes v\otimes w).
\end{aligned}$$

Then the desired weak associativity relation follows from (3.5). Thus $(V, Y_T(x), \mathbf{1})$ carries the structure of a nonlocal vertex algebra, as desired. □

The following result gives the general twisting of a quantum vertex algebra.

Proposition 3.3. *Let $(V, \mathcal{S})$ be a quantum vertex algebra and let $T(x)$ be a twistor of V satisfying*

$$T(x)\mathcal{S}(x)\sigma = \mathcal{S}(x)\sigma T(-x), \tag{3.6}$$

$$T^{12}(x_1)\mathcal{S}^{23}(x_2) = \mathcal{S}^{23}(x_2)T^{12}(x_1), \tag{3.7}$$

$$T^{12}(x_1)\mathcal{S}^{13}(x_2) = \mathcal{S}^{13}(x_2)T^{12}(x_1). \tag{3.8}$$

Then $(V^T, \mathcal{S})$ is also a quantum vertex algebra.

Proof. From Theorem 3.2, we just need to prove:

$$Y_T(x) = e^{x\mathcal{D}}Y_T(-x)\mathcal{S}(-x)\sigma, \tag{3.9}$$

$$\mathcal{S}(x_1)(Y_T(x_2)\otimes 1) = (Y_T(x_2)\otimes 1)\mathcal{S}^{23}(x_1)\mathcal{S}^{13}(x_1+x_2). \tag{3.10}$$

For (3.9), using (2.5) and (3.6), we have

$$\begin{aligned}
Y_T(x) &= Y(x)T(x) \\
&= e^{x\mathcal{D}}Y(-x)\mathcal{S}(-x)\sigma T(x) \\
&= e^{x\mathcal{D}}Y(-x)T(-x)\mathcal{S}(-x)\sigma \\
&= e^{x\mathcal{D}}Y_T(-x)\mathcal{S}(-x)\sigma.
\end{aligned}$$

Concerning (3.10), by (2.6), (3.8) and (3.7), we get

$$\begin{aligned}\mathcal{S}(x_1)(Y_T(x_2)\otimes 1) &= \mathcal{S}(x_1)(Y(x_2)\otimes 1)T^{12}(x_2)\\ &= (Y(x_2)\otimes 1)\mathcal{S}^{23}(x_1)\mathcal{S}^{13}(x_1+x_2)T^{12}(x_2)\\ &= (Y(x_2)\otimes 1)\mathcal{S}^{23}(x_1)T^{12}(x_2)\mathcal{S}^{13}(x_1+x_2)\\ &= (Y(x_2)\otimes 1)T^{12}(x_2)\mathcal{S}^{23}(x_1)\mathcal{S}^{13}(x_1+x_2)\\ &= (Y_T(x_2)\otimes 1)\mathcal{S}^{23}(x_1)\mathcal{S}^{13}(x_1+x_2).\end{aligned}$$

This concludes the proof. □

Now we consider the inverse of a twistor. We say that a twistor $T(x)$ is *invertible* if $T(x)$, viewed as a $\mathbb{C}((x))$-linear map from $V\otimes V\otimes\mathbb{C}((x))$ to $V\otimes V\otimes\mathbb{C}((x))$, is invertible. The inverse of an invertible $T(x)$ is a $\mathbb{C}((x))$-linear map $T^{-1}(x)$ from $V\otimes V\otimes\mathbb{C}((x))$ to $V\otimes V\otimes\mathbb{C}((x))$. We often consider $T^{-1}(x)$ as a $\mathbb{C}$-linear map

$$T^{-1}(x):\ V\otimes V\to V\otimes V\otimes\mathbb{C}((x)).$$

We can easily have:

Lemma 3.4. *Let V be a nonlocal vertex algebra and let $T(x)$ be an invertible twistor satisfying the following two conditions*

$$T^{12}(x_1)T^{13}(x_2) = T^{13}(x_2)T^{12}(x_1), \tag{3.11}$$

$$T^{13}(x_1)T^{23}(x_2) = T^{23}(x_2)T^{13}(x_1). \tag{3.12}$$

Then $T^{-1}(x)$ is also a twistor for V.

Next we discuss the composition of twistors. The composition of two twistors $T(x)$ and $S(x)$ is a $\mathbb{C}((x))$-linear map $T(x)S(x)$ from $V\otimes V$ to $V\otimes V\otimes\mathbb{C}((x))$. We have:

Proposition 3.5. *Let V be a nonlocal vertex algebra and let $T(x), S(x) : V\otimes V\to V\otimes V\otimes\mathbb{C}((x))$ be two twistors for V, satisfying (3.7), (3.8) and the following conditions:*

$$T^{23}(x_1)S^{12}(x_2) = S^{12}(x_2)T^{23}(x_1), \tag{3.13}$$

$$T^{23}(x_1)S^{13}(x_2) = S^{13}(x_2)T^{23}(x_1). \tag{3.14}$$

Then

(1) $S(x)$ is a twistor for V^T (resp. $T(x)$ is a twistor for V^S).
(2) $T(x)S(x)$ (resp. $S(x)T(x)$) is a twistor for V.
(3) $(V^T)^S = V^{TS}$ (resp. $(V^S)^T = V^{ST}$).

Proof. (1) We just need to prove:

$$S(x_1)(1 \otimes Y_T(x_2)) = (1 \otimes Y_T(x_2))S^{13}(x_1)S^{12}(x_1 - x_2), \tag{3.15}$$

$$S(x_1)(Y_T(x_2) \otimes 1) = (Y_T(x_2) \otimes 1)S^{13}(x_1 + x_2)S^{23}(x_1). \tag{3.16}$$

For (3.15), using (3.3), (3.13) and (3.14), we have

$$\begin{aligned} S(x_1)(1 \otimes Y_T(x_2)) &= S(x_1)(1 \otimes Y(x_2))T^{23}(x_2) \\ &= (1 \otimes Y(x_2))S^{13}(x_1)S^{12}(x_1 - x_2)T^{23}(x_2) \\ &= (1 \otimes Y(x_2))S^{13}(x_1)T^{23}(x_2)S^{12}(x_1 - x_2) \\ &= (1 \otimes Y(x_2))T^{23}(x_2)S^{13}(x_1)S^{12}(x_1 - x_2) \\ &= (1 \otimes Y_T(x_2))S^{13}(x_1)S^{12}(x_1 - x_2). \end{aligned}$$

Concerning (3.16), by (3.4), (3.7) and (3.8), we get

$$\begin{aligned} S(x_1)(Y_T(x_2) \otimes 1) &= S(x_1)(Y(x_2) \otimes 1)T^{12}(x_2) \\ &= (Y(x_2) \otimes 1)S^{13}(x_1 + x_2)S^{23}(x_1)T^{12}(x_2) \\ &= (Y(x_2) \otimes 1)S^{13}(x_1 + x_2)T^{12}(x_2)S^{23}(x_1) \\ &= (Y(x_2) \otimes 1)T^{12}(x_2)S^{13}(x_1 + x_2)S^{23}(x_1) \\ &= (Y_T(x_2) \otimes 1)S^{13}(x_1 + x_2)S^{23}(x_1). \end{aligned}$$

We can prove that $T(x)$ is a twistor for V^S similarly.

(2) Next we prove $T(x)S(x)$ is a twistor for V. In the first, for $v \in V$, using (3.1), we get

$$T(x)S(x)(\mathbf{1} \otimes v) = T(x)(\mathbf{1} \otimes v) = \mathbf{1} \otimes v,$$

and with (3.2), we have

$$T(x)S(x)(v \otimes \mathbf{1}) = T(x)(v \otimes \mathbf{1}) = v \otimes \mathbf{1}.$$

From (3.4) and (3.14), we have

$$\begin{aligned} T(x_1)S(x_1)(Y(x_2) \otimes 1) &= T(x_1)(Y(x_2) \otimes 1)S^{13}(x_1 + x_2)S^{23}(x_1) \\ &= (Y(x_2) \otimes 1)T^{13}(x_1 + x_2)T^{23}(x_1)S^{13}(x_1 + x_2)S^{23}(x_1) \\ &= (Y(x_2) \otimes 1)T^{13}(x_1 + x_2)S^{13}(x_1 + x_2)T^{23}(x_1)S^{23}(x_1), \end{aligned}$$

and using (3.3) and (3.8), we get

$$\begin{aligned}T(x_1)S(x_1)(1\otimes Y(x_2)) &= T(x_1)(1\otimes Y(x_2))S^{13}(x_1)S^{12}(x_1-x_2)\\ &= (1\otimes Y(x_2))T^{13}(x_1)T^{12}(x_1-x_2)S^{13}(x_1)S^{12}(x_1-x_2)\\ &= (1\otimes Y(x_2))T^{13}(x_1)S^{13}(x_1)T^{12}(x_1-x_2)S^{12}(x_1-x_2).\end{aligned}$$

Furthermore, using (3.13), (3.5) and (3.7), we have

$$\begin{aligned}T^{12}(x_1)S^{12}(x_1)T^{23}(x_2)S^{23}(x_2) &= T^{12}(x_1)T^{23}(x_2)S^{12}(x_1)S^{23}(x_2)\\ &= T^{23}(x_2)T^{12}(x_1)S^{23}(x_2)S^{12}(x_1)\\ &= T^{23}(x_2)S^{23}(x_2)T^{12}(x_1)S^{12}(x_1).\end{aligned}$$

(3) The vertex operator map of $(V^T)^S$ is $(Y_T)_S(x) = Y_T(x)S(x) = Y(x)T(x)S(x)$, while the vertex operator map of V^{TS} is $Y_{TS}(x) = Y(x)T(x)S(x)$. So it is immediate that $(V^T)^S = V^{TS}$. This completes the proof. □

4. Examples of twistors

In this section we present some examples of twistors, such as R-matrices, twisted tensor products, iterated twisted tensor products and L-R-twisted tensor products. Furthermore, the nonlocal vertex algebra constructed from noncommutative $2l$-plane is realized as a deformation of a nonlocal vertex algebra constructed from a polynomial algebra via a suitable twistor.

4.1. R-matrix

Motivated by the notion of an R-matrix of a ring in [2], we formulate the following notion:

Definition 4.1. Let V be a nonlocal vertex algebra. An *R-matrix* for V is a linear map

$$\mathcal{S}(x) : V\otimes V \to V\otimes V\otimes \mathbb{C}((x)),$$

satisfying the following conditions:

$$\mathcal{S}(x)(\mathbf{1}\otimes v) = \mathbf{1}\otimes v \qquad \text{for } v\in V, \tag{4.1}$$

$$\mathcal{S}(x)(v\otimes \mathbf{1}) = v\otimes \mathbf{1} \qquad \text{for } v\in V, \tag{4.2}$$

$$\mathcal{S}(x_1)(1\otimes Y(x_2)) = (1\otimes Y(x_2))\mathcal{S}^{12}(x_1-x_2)\mathcal{S}^{13}(x_1), \tag{4.3}$$

$$\mathcal{S}(x_1)(Y(x_2)\otimes 1) = (Y(x_2)\otimes 1)\mathcal{S}^{23}(x_1)\mathcal{S}^{13}(x_1+x_2), \tag{4.4}$$

$$\mathcal{S}^{12}(x_1)\mathcal{S}^{13}(x_1+x_2)\mathcal{S}^{23}(x_2) = \mathcal{S}^{23}(x_2)\mathcal{S}^{13}(x_1+x_2)\mathcal{S}^{12}(x_1). \tag{4.5}$$

The following is from [14]:

Lemma 4.2. *Let V be a nonlocal vertex algebra and let $\mathcal{S}(x)$ be a unitary rational quantum Yang–Baxter equation on V. Then (2.3) and (2.6) are equivalent to (4.2) and (4.3), respectively.*

Remark 4.3. Note that from Lemma 4.2 the notion of a unitary rational quantum Yang–Baxter operator $\mathcal{S}(x)$ in the definition of a quantum vertex algebra is a special case of an R-matrix.

We have the following construction of nonlocal vertex algebras from R-matrices:

Theorem 4.4. *Let V be a nonlocal vertex algebra and let $\mathcal{S}(x)$ be an R-matrix for V. Set*

$$Y_{\mathcal{S}}(x) = Y(x)\mathcal{S}(x) : V \otimes V \to V((x)).$$

Then $(V, Y_{\mathcal{S}}(x), \mathbf{1})$ carries the structure of a nonlocal vertex algebra.

Proof. For $u, v \in V$, by definition we have

$$Y_{\mathcal{S}}(x)(u \otimes v) = \sum_{j=1}^{m} f_j(x) Y(u^{(j)}, x) v^{(j)},$$

where

$$\mathcal{S}(x)(u \otimes v) = \sum_{j=1}^{m} u^{(j)} \otimes v^{(j)} \otimes f_j(x) \in V \otimes V \otimes \mathbb{C}((x)).$$

Since

$$f_j(x) \in \mathbb{C}((x)), \quad Y(u^{(j)}, x) v^{(j)} \in V((x))$$

for $1 \leq j \leq m$, we see that $Y_{\mathcal{S}}(u, x)v$ exists in $V((x))$.

For $v \in V$, by (4.1) we get

$$Y_{\mathcal{S}}(\mathbf{1}, x)v = Y(x)\mathcal{S}(x)(\mathbf{1} \otimes v) = Y(\mathbf{1}, x)v = v.$$

On the other hand, for $v \in V$, with (4.2), we have

$$Y_{\mathcal{S}}(v, x)\mathbf{1} = Y(x)\mathcal{S}(x)(v \otimes \mathbf{1}) = Y(v, x)\mathbf{1} \in V[[x]],$$

and

$$\lim_{x \to 0} Y_{\mathcal{S}}(v, x)\mathbf{1} = \lim_{x \to 0} Y(x)\mathcal{S}(x)(v \otimes \mathbf{1}) = \lim_{x \to 0} Y(v, x)\mathbf{1} = v.$$

Concerning weak associativity, let $u, v, w \in V$. Using (4.3) we obtain

$$\begin{aligned}
&Y_{\mathcal{S}}(u, x_0 + x_2)Y_{\mathcal{S}}(v, x_2)w \\
&= Y(x_0 + x_2)\mathcal{S}(x_0 + x_2)(1 \otimes Y(x_2))\mathcal{S}^{23}(x_2)(u \otimes v \otimes w) \\
&= Y(x_0 + x_2)(1 \otimes Y(x_2))\mathcal{S}^{12}(x_0)\mathcal{S}^{13}(x_0 + x_2)\mathcal{S}^{23}(x_2)(u \otimes v \otimes w).
\end{aligned}$$

On the other hand, from (4.4) we have

$$\begin{aligned}
&Y_{\mathcal{S}}(Y_{\mathcal{S}}(u, x_0)v, x_2)w \\
&= Y(x_2)\mathcal{S}(x_2)(Y(x_0) \otimes 1)\mathcal{S}^{12}(x_0)(u \otimes v \otimes w) \\
&= Y(x_2)(Y(x_0) \otimes 1)\mathcal{S}^{23}(x_2)\mathcal{S}^{13}(x_2 + x_0)\mathcal{S}^{12}(x_0)(u \otimes v \otimes w).
\end{aligned}$$

Then the desired weak associativity relation follows from (4.5). Then $(V, Y_{\mathcal{S}}(x), \mathbf{1})$ carries the structure of a nonlocal vertex algebra. This completes the proof. □

The following two lemmas give the relations between a twistor and an R-matrix. First of all, we have:

Lemma 4.5. *Let $T(x)$ be a twistor satisfying (3.11) and (3.12). Then $T(x)$ is an R-matrix.*

Proof. Using (3.3) and (3.11), we have

$$\begin{aligned}
T(x_1)(1 \otimes Y(x_2)) &= (1 \otimes Y(x_2))T^{13}(x_1)T^{12}(x_1 - x_2) \\
&= (1 \otimes Y(x_2))T^{12}(x_1 - x_2)T^{13}(x_1).
\end{aligned}$$

By (3.4) and (3.12), we get

$$\begin{aligned}
T(x_1)(Y(x_2) \otimes 1) &= (Y(x_2) \otimes 1)T^{13}(x_1 + x_2)T^{23}(x_1) \\
&= (Y(x_2) \otimes 1)T^{23}(x_1)T^{13}(x_1 + x_2).
\end{aligned}$$

From (3.11), (3.5) and (3.12), we obtain

$$\begin{aligned}
T^{12}(x)T^{13}(x+z)T^{23}(z) &= T^{13}(x+z)T^{12}(x)T^{23}(z) \\
&= T^{13}(x+z)T^{23}(z)T^{12}(x) \\
&= T^{23}(z)T^{13}(x+z)T^{12}(x).
\end{aligned}$$

This completes the proof. □

Conversely, we have

Lemma 4.6. *Each invertible R-matrix $\mathcal{S}(x)$ satisfying (3.11) and (3.12) is a twistor.*

Proof. Using (4.3) and (3.11), we have

$$\begin{aligned}\mathcal{S}(x_1)(1\otimes Y(x_2)) &= (1\otimes Y(x_2))\mathcal{S}^{12}(x_1-x_2)\mathcal{S}^{13}(x_1)\\ &= (1\otimes Y(x_2))\mathcal{S}^{13}(x_1)\mathcal{S}^{12}(x_1-x_2).\end{aligned}$$

By (4.4) and (3.12), we get

$$\begin{aligned}\mathcal{S}(x_1)(Y(x_2)\otimes 1) &= (Y(x_2)\otimes 1)\mathcal{S}^{23}(x_1)\mathcal{S}^{13}(x_1+x_2)\\ &= (Y(x_2)\otimes 1)\mathcal{S}^{13}(x_1+x_2)\mathcal{S}^{23}(x_1).\end{aligned}$$

From (3.12), (4.5) and (3.11), we obtain

$$\begin{aligned}\mathcal{S}^{12}(x)\mathcal{S}^{23}(z)\mathcal{S}^{13}(x+z) &= \mathcal{S}^{12}(x)\mathcal{S}^{13}(x+z)\mathcal{S}^{23}(z)\\ &= \mathcal{S}^{23}(z)\mathcal{S}^{13}(x+z)\mathcal{S}^{12}(x)\\ &= \mathcal{S}^{23}(z)\mathcal{S}^{12}(x)\mathcal{S}^{13}(x+z).\end{aligned}$$

By the invertibility of $\mathcal{S}(x)$, we have

$$\mathcal{S}^{12}(x)\mathcal{S}^{23}(z) = \mathcal{S}^{23}(z)\mathcal{S}^{12}(x).$$

This concludes the proof. □

4.2. Twisted tensor product nonlocal vertex algebras

Let U and V be two nonlocal vertex algebras. We have an (ordinary) tensor product nonlocal vertex algebra $U\otimes V$, where the vacuum vector is $\mathbf{1}\otimes\mathbf{1}$ and the vertex operator map is given by

$$Y(u\otimes v,x)(u'\otimes v') = Y(u,x)u'\otimes Y(v,x)v' \quad \text{for } u,u'\in U,\ v,v'\in V.$$

That is,

$$Y_{U\otimes V}(x) = (Y_U(x)\otimes Y_V(x))\,\sigma^{23},$$

where σ^{23} is the linear operator on $(U\otimes V)^{\otimes 2}$, defined by

$$\sigma^{23}(u\otimes v\otimes u'\otimes v') = u\otimes u'\otimes v\otimes v'$$

for $u,u'\in U$, $v,v'\in V$.

We now recall the notion of a twisting operator, which was formulated and studied in [14].

Definition 4.7. Let U and V be nonlocal vertex algebras. A *twisting operator* for the ordered pair (U, V) is a linear map

$$R(x): \ V \otimes U \to U \otimes V \otimes \mathbb{C}((x)),$$

satisfying the following conditions:

$$R(x)(v \otimes \mathbf{1}) = \mathbf{1} \otimes v \quad \text{for } v \in V, \tag{4.6}$$

$$R(x)(\mathbf{1} \otimes u) = u \otimes \mathbf{1} \quad \text{for } u \in U, \tag{4.7}$$

$$R(x_1)(1 \otimes Y(x_2)) = (Y(x_2) \otimes 1)R^{23}(x_1)R^{12}(x_1 + x_2), \tag{4.8}$$

$$R(x_1)(Y(x_2) \otimes 1) = (1 \otimes Y(x_2))R^{12}(x_1 - x_2)R^{23}(x_1). \tag{4.9}$$

Remark 4.8. It can be readily seen that the flip operator σ on $V \otimes V$ for a nonlocal vertex algebra V is a twisting operator and we in particular have

$$\sigma(1 \otimes Y(x_2)) = (Y(x_2) \otimes 1)\sigma^{23}\sigma^{12}, \tag{4.10}$$

$$\sigma(Y(x_2) \otimes 1) = (1 \otimes Y(x_2))\sigma^{12}\sigma^{23}. \tag{4.11}$$

We will use these two equalities frequently in the following.

The following gives the twisted tensor products of nonlocal vertex algebras, which was proved in [14].

Theorem 4.9. *Let U, V be nonlocal vertex algebras and let $R(x)$ be a twisting operator of the ordered pair (U, V). Set*

$$Y_R(x) = (Y(x) \otimes Y(x))R^{23}(-x). \tag{4.12}$$

Then $(U \otimes V, Y_R(x), \mathbf{1} \otimes \mathbf{1})$ carries the structure of a nonlocal vertex algebra, which contains U and V canonically as nonlocal vertex subalgebras.

We get the relation between a twisting operator and a twistor in the following two lemmas. Firstly, we can construct a twistor from a twisting operator:

Lemma 4.10. *Let $U \otimes_R V$ be a twisted tensor product nonlocal vertex algebra, then the operator*

$$T(x) = \sigma^{23}R^{23}(-x): \ (U \otimes V) \otimes (U \otimes V) \to (U \otimes V) \otimes (U \otimes V) \otimes \mathbb{C}((x))$$

is a twistor for $U \otimes V$ and $(U \otimes V)^T = U \otimes_R V$.

Proof. For $u \in U$, $v \in V$, using (4.6), we get

$$T(x)(u \otimes v \otimes \mathbf{1} \otimes \mathbf{1}) = \sigma^{23} R^{23}(-x)(u \otimes v \otimes \mathbf{1} \otimes \mathbf{1}) = u \otimes v \otimes \mathbf{1} \otimes \mathbf{1},$$

and by (4.7), we have

$$T(x)(\mathbf{1} \otimes \mathbf{1} \otimes u \otimes v) = \sigma^{23} R^{23}(-x)(\mathbf{1} \otimes \mathbf{1} \otimes u \otimes v) = \mathbf{1} \otimes \mathbf{1} \otimes u \otimes v.$$

From (4.8) and (4.11), we have

$$\begin{aligned}
&T(x_1)(1_{U\otimes V} \otimes Y_{U\otimes V}(x_2)) \\
&= \sigma^{23} R^{23}(-x_1)(1 \otimes 1 \otimes Y(x_2) \otimes Y(x_2))\sigma^{45} \\
&= \sigma^{23}(1 \otimes Y(x_2) \otimes 1 \otimes Y(x_2)) R^{34}(-x_1) R^{23}(-x_1 + x_2)\sigma^{45} \\
&= (1 \otimes 1 \otimes Y(x_2) \otimes Y(x_2))\sigma^{23}\sigma^{34} R^{34}(-x_1) R^{23}(-x_1 + x_2)\sigma^{45},
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
&(1_{U\otimes V} \otimes Y_{U\otimes V}(x_2)) T^{13}(x_1) T^{12}(x_1 - x_2) \\
&= (1 \otimes 1 \otimes Y(x_2) \otimes Y(x_2))\sigma^{45}\sigma^{25} R^{25}(-x_1)\sigma^{23} R^{23}(-x_1 + x_2).
\end{aligned}$$

Then we have

$$T(x_1)(1_{U\otimes V} \otimes Y_{U\otimes V}(x_2)) = (1_{U\otimes V} \otimes Y_{U\otimes V}(x_2)) T^{13}(x_1) T^{12}(x_1 - x_2).$$

By (4.9) and (4.10), we get

$$\begin{aligned}
&T(x_1)(Y_{U\otimes V}(x_2) \otimes 1_{U\otimes V}) \\
&= \sigma^{23} R^{23}(-x_1)(Y(x_2) \otimes Y(x_2) \otimes 1 \otimes 1)\sigma^{23} \\
&= \sigma^{23}(Y(x_2) \otimes 1 \otimes Y(x) \otimes 1) R^{34}(-x_1 - x_2) R^{45}(-x_1)\sigma^{23} \\
&= (Y(x_2) \otimes Y(x_2) \otimes 1 \otimes 1)\sigma^{45}\sigma^{34} R^{34}(-x_1 - x_2) R^{45}(-x_1)\sigma^{23},
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
&(Y_{U\otimes V}(x_2) \otimes 1_{U\otimes V}) T^{13}(x_1 + x_2) T^{23}(x_1) \\
&= (Y(x_2) \otimes Y(x_2) \otimes 1 \otimes 1)\sigma^{23}\sigma^{25} R^{25}(-x_1 - x_2)\sigma^{45} R^{45}(-x_1).
\end{aligned}$$

Then we obtain

$$T(x_1)(Y_{U\otimes V}(x_2) \otimes 1_{U\otimes V}) = (Y_{U\otimes V}(x_2) \otimes 1_{U\otimes V}) T^{13}(x_1 + x_2) T^{23}(x_1).$$

It is easy to see that

$$\begin{aligned}T^{12}(x_1)T^{23}(x_2) &= \sigma^{23}R^{23}(-x_1)\sigma^{45}R^{45}(-x_2)\\ &= \sigma^{45}R^{45}(-x_2)\sigma^{23}R^{23}(-x_1)\\ &= T^{23}(x_2)T^{12}(x_1).\end{aligned}$$

Furthermore, we have

$$\begin{aligned}Y_T(x) &= Y_{U\otimes V}(x)T(x)\\ &= Y_{U\otimes V}(x)\sigma^{23}R^{23}(-x)\\ &= (Y(x)\otimes Y(x))\sigma^{23}\sigma^{23}R^{23}(-x)\\ &= (Y(x)\otimes Y(x))R^{23}(-x).\end{aligned}$$

That is, $(U\otimes V)^T = U\otimes_R V$. □

Conversely, we have:

Lemma 4.11. *Let $R(x) : V\otimes U \to U\otimes V\otimes \mathbb{C}((x))$ be a linear map and let $T(x)$ be a twistor for $U\otimes V$ defined by $T(x) = \sigma^{23}R^{23}(-x)$. Then $R(x)$ is a twisting operator for the ordered pair (U, V) and $U\otimes_R V = (U\otimes V)^T$.*

Proof. For $u\in U$, $v\in V$,

$$u\otimes v\otimes \mathbf{1}\otimes \mathbf{1} = T(x)(u\otimes v\otimes \mathbf{1}\otimes \mathbf{1}) = \sigma^{23}R^{23}(-x)(u\otimes v\otimes \mathbf{1}\otimes \mathbf{1}).$$

Then we get $R^{23}(-x)(u\otimes v\otimes \mathbf{1}\otimes \mathbf{1}) = u\otimes \mathbf{1}\otimes v\otimes \mathbf{1}$. That is,

$$R(x)(v\otimes \mathbf{1}) = \mathbf{1}\otimes v.$$

Similarly,

$$\mathbf{1}\otimes \mathbf{1}\otimes u\otimes v = T(x)(\mathbf{1}\otimes \mathbf{1}\otimes u\otimes v) = \sigma^{23}R^{23}(-x)(\mathbf{1}\otimes \mathbf{1}\otimes u\otimes v).$$

Then we have $R^{23}(-x)(\mathbf{1}\otimes \mathbf{1}\otimes u\otimes v) = \mathbf{1}\otimes u\otimes \mathbf{1}\otimes v$. That is,

$$R(x)(\mathbf{1}\otimes u) = u\otimes \mathbf{1}.$$

And

$$T(x_1)(1_{U\otimes V}\otimes Y_{U\otimes V}(x_2)) = \sigma^{23}R^{23}(-x_1)(1\otimes 1\otimes Y(x_2)\otimes Y(x_2))\sigma^{45},$$

and on the other hand, we have

$$\begin{aligned}&(1_{U\otimes V}\otimes Y_{U\otimes V}(x_2))T^{13}(x_1)T^{12}(x_1-x_2)\\ &=(1\otimes 1\otimes Y(x_2)\otimes Y(x_2))\sigma^{45}\sigma^{25}R^{25}(-x_1)\sigma^{23}R^{23}(-x_1+x_2).\end{aligned}$$

Then applying to $\mathbf{1}\otimes v\otimes u\otimes \mathbf{1}\otimes u'\otimes \mathbf{1}$, for $u,u'\in U$, $v\in V$, we obtain

$$R(x_1)(1\otimes Y(x_2))=(Y(x_2)\otimes 1)R^{23}(x_1)R^{12}(x_1+x_2).$$

Similarly, we have

$$T(x_1)(Y_{U\otimes V}(x_2)\otimes 1_{U\otimes V})=\sigma^{23}R^{23}(-x_1)(Y(x_2)\otimes Y(x_2)\otimes 1\otimes 1)\sigma^{23},$$

and on the other hand, we get

$$\begin{aligned}&(Y_{U\otimes V}(x_2)\otimes 1_{U\otimes V})T^{13}(x_1+x_2)T^{23}(x_1)\\ &=(Y(x_2)\otimes Y(x_2)\otimes 1\otimes 1)\sigma^{23}\sigma^{25}R^{25}(-x_1-x_2)\sigma^{45}R^{45}(-x_1).\end{aligned}$$

Then applying to $\mathbf{1}\otimes v\otimes \mathbf{1}\otimes v'\otimes u\otimes \mathbf{1}$, for $u\in U$, $v,v'\in V$, we have

$$R(x_1)(Y(x_2)\otimes 1)=(1\otimes Y(x_2))R^{12}(x_1-x_2)R^{23}(x_1).$$

Finally, we have

$$\begin{aligned}Y_R(x)&=(Y(x)\otimes Y(x))R^{23}(-x)\\ &=(Y(x)\otimes Y(x))\sigma^{23}\sigma^{23}R^{23}(-x)\\ &=Y_{U\otimes V}(x)\sigma^{23}R^{23}(-x)\\ &=Y_{U\otimes V}(x)T(x).\end{aligned}$$

Then we obtain $U\otimes_R V=(U\otimes V)^T$. $\square$

We will say that the twistor $T(x)$ is afforded by the twisting operator $R(x)$.

Remark 4.12. In [12], a notion of nonlocal vertex bialgebra and a notion of nonlocal vertex module-algebra for a nonlocal vertex bialgebra were formulated, and a smash product construction of nonlocal vertex algebras was established. Given a nonlocal vertex bialgebra H and for a nonlocal vertex H-module algebra U, we had a smash product $U\sharp H$. In [14], the smash product construction was slightly generalized, and it was shown that the smash product $U\sharp V$ is a twisted tensor product with respect to a canonical twisting operator. Thus from Lemma 4.10 and Lemma 4.11, we know that the smash products in [14] also can be related with twistors.

Furthermore, we have:

Proposition 4.13. *Let U and V be two nonlocal vertex algebras and let $R(x), S(x) : V \otimes U \to U \otimes V \otimes \mathbb{C}((x))$ be two twisting operators satisfying:*

$$R^{12}(x_1)S^{23}(x_2)\sigma^{12} = \sigma^{23}S^{12}(x_2)R^{23}(x_1), \tag{4.13}$$

$$\sigma^{12}R^{23}(x_1)S^{12}(x_2) = S^{23}(x_2)R^{12}(x_1)\sigma^{23}. \tag{4.14}$$

Denote by $T_R(x)$ (resp. $T_S(x)$) the twistor for $U \otimes V$ afforded by $R(x)$ (resp. $S(x)$). Define $(RS)(x)$ (resp. $(SR)(x)$) : $V \otimes U \to U \otimes V \otimes \mathbb{C}((x))$ by

$$(RS)(x) = R(x)\sigma S(x) \quad (\textit{resp. } (SR)(x) = S(x)\sigma R(x)).$$

Then:

(1) $T_S(x)$ (resp. $T_R(x)$) is a twistor for $U \otimes_R V$ (resp. $U \otimes_S V$).
(2) $T_R(x)T_S(x)$ (resp. $T_S(x)T_R(x)$) is a twistor for $U \otimes V$ afforded by the twisting operator $(RS)(x)$ (resp. $(SR)(x)$).
(3) $(U \otimes_R V)^{T_S} = U \otimes_{RS} V$ (resp. $(U \otimes_S V)^{T_R} = U \otimes_{SR} V$).

Proof. (1) We just prove $T_S(x)$ is a twistor for $U \otimes_R V$. For $u \in U$, $v \in V$, using (4.7), we have

$$T_S(x)(\mathbf{1} \otimes \mathbf{1} \otimes u \otimes v) = \sigma^{23}S^{23}(-x)(\mathbf{1} \otimes \mathbf{1} \otimes u \otimes v) = \mathbf{1} \otimes \mathbf{1} \otimes u \otimes v,$$

and similarly using (4.6), we get

$$T_S(x)(u \otimes v \otimes \mathbf{1} \otimes \mathbf{1}) = \sigma^{23}S^{23}(-x)(u \otimes v \otimes \mathbf{1} \otimes \mathbf{1}) = u \otimes v \otimes \mathbf{1} \otimes \mathbf{1}.$$

From (4.8) and (4.11), we obtain

$$\begin{aligned}
&T_S(x_1)(1_{U\otimes V} \otimes Y_{U\otimes_R V}(x_2)) \\
&= \sigma^{23}S^{23}(-x_1)(1 \otimes 1 \otimes Y(x_2) \otimes Y(x_2))R^{45}(-x_2) \\
&= \sigma^{23}(1 \otimes Y(x_2) \otimes 1 \otimes Y(x_2))S^{34}(-x_1)S^{23}(-x_1 + x_2)R^{45}(-x_2) \\
&= (1 \otimes 1 \otimes Y(x_2) \otimes Y(x_2))\sigma^{23}\sigma^{34}S^{34}(-x_1)S^{23}(-x_1 + x_2)R^{45}(-x_2) \\
&= (1 \otimes 1 \otimes Y(x_2) \otimes Y(x_2))\sigma^{23}\sigma^{34}S^{34}(-x_1)R^{45}(-x_2)S^{23}(-x_1 + x_2),
\end{aligned}$$

and on the other hand, we have

$$\begin{aligned}
&(1_{U\otimes V} \otimes Y_{U\otimes_R V}(x_2))T_S^{13}(x_1)T_S^{12}(x_1 - x_2) \\
&= (1 \otimes 1 \otimes Y(x_2) \otimes Y(x_2))R^{45}(-x_2)\sigma^{25}S^{25}(-x_1)\sigma^{23}S^{23}(-x_1 + x_2),
\end{aligned}$$

and by (4.14), we get that

$$T_S(x_1)(1_{U\otimes V}\otimes Y_{U\otimes_R V}(x_2)) = (1_{U\otimes V}\otimes Y_{U\otimes_R V}(x_2))T_S^{13}(x_1)T_S^{12}(x_1-x_2).$$

Similarly, by (4.9) and (4.10), we have

$$\begin{aligned}
&T_S(x_1)(Y_{U\otimes_R V}(x_2)\otimes 1_{U\otimes V}),\\
&=\sigma^{23}S^{23}(-x_1)(Y(x_2)\otimes Y(x_2)\otimes 1\otimes 1)R^{23}(-x_2)\\
&=\sigma^{23}(Y(x_2)\otimes 1\otimes Y(x_2)\otimes 1)S^{34}(-x_1-x_2)S^{45}(-x_1)R^{23}(-x_2)\\
&=(Y(x_2)\otimes Y(x_2)\otimes 1\otimes 1)\sigma^{45}\sigma^{34}S^{34}(-x_1-x_2)S^{45}(-x_1)R^{23}(-x_2)\\
&=(Y(x_2)\otimes Y(x_2)\otimes 1\otimes 1)\sigma^{45}\sigma^{34}S^{34}(-x_1-x_2)R^{23}(-x_2)S^{45}(-x_1),
\end{aligned}$$

and on the other hand, we have

$$\begin{aligned}
&(Y_{U\otimes_R V}(x_2)\otimes 1_{U\otimes V})T_S^{13}(x_1+x_2)T_S^{23}(x_1)\\
&=(Y(x_2)\otimes Y(x_2)\otimes 1\otimes 1)R^{23}(-x_2)\sigma^{25}S^{25}(-x_1-x_2)\sigma^{45}S^{45}(-x_1),
\end{aligned}$$

and by (4.13), we obtain that

$$T_S(x_1)(Y_{U\otimes_R V}(x_2)\otimes 1_{U\otimes V}) = (Y_{U\otimes_R V}(x_2)\otimes 1_{U\otimes V})T_S^{13}(x_1+x_2)T_S^{23}(x_1).$$

And obviously,

$$\begin{aligned}
T_S^{12}(x_1)T_S^{23}(x_2) &= \sigma^{23}S^{23}(-x_1)\sigma^{45}S^{45}(-x_2)\\
&=\sigma^{45}S^{45}(-x_2)\sigma^{23}S^{23}(-x_1)\\
&=T_S^{23}(x_2)T_S^{12}(x_1).
\end{aligned}$$

(2) We need to prove $(RS)(x)$ is a twisting operator for ordered pair (U,V). For $v\in V$, using (4.6), we have

$$(RS)(x)(v\otimes\mathbf{1}) = R(x)\sigma S(x)(v\otimes\mathbf{1}) = R(x)\sigma(\mathbf{1}\otimes v) = R(x)(v\otimes\mathbf{1}) = \mathbf{1}\otimes v,$$

similarly, for $u\in U$, using (4.7), we get

$$(RS)(x)(\mathbf{1}\otimes u) = R(x)\sigma S(x)(\mathbf{1}\otimes u) = R(x)\sigma(u\otimes\mathbf{1}) = R(x)(\mathbf{1}\otimes u) = u\otimes\mathbf{1}.$$

By (4.8) and (4.11), we have

$$\begin{aligned}
(RS)(x_1)(1\otimes Y(x_2)) &= R(x_1)\sigma S(x_1)(1\otimes Y(x_2))\\
&=R(x_1)\sigma(Y(x_2)\otimes 1)S^{23}(x_1)S^{12}(x_1+x_2)\\
&=R(x_1)(1\otimes Y(x_2))\sigma^{12}\sigma^{23}S^{23}(x_1)S^{12}(x_1+x_2)\\
&=(Y(x_2)\otimes 1)R^{23}(x_1)R^{12}(x_1+x_2)\sigma^{12}\sigma^{23}S^{23}(x_1)S^{12}(x_1+x_2),
\end{aligned}$$

and on the other hand, by (4.14), we obtain

$$\begin{aligned}
&(Y(x_2)\otimes 1)(RS)^{23}(x_1)(RS)^{12}(x_1+x_2)\\
&=(Y(x_2)\otimes 1)R^{23}(x_1)\sigma^{23}S^{23}R^{12}(x_1+x_2)\sigma^{12}S^{12}(x_1+x_2)\\
&=(Y(x_2)\otimes 1)R^{23}(x_1)\sigma^{23}\sigma^{12}R^{23}(x_1+x_2)S^{12}(x_1)\sigma^{23}\sigma^{12}S^{12}(x_1+x_2),
\end{aligned}$$

from a direct calculation, we have

$$(RS)(x_1)(1\otimes Y(x_2))=(Y(x_2)\otimes 1)(RS)^{23}(x_1)(RS)^{12}(x_1+x_2).$$

Similarly, using (4.9) and (4.10), we get

$$\begin{aligned}
(RS)(x_1)(Y(x_2)\otimes 1)&=R(x_1)\sigma S(x_1)(Y(x_2)\otimes 1)\\
&=R(x_1)\sigma(1\otimes Y(x_2))S^{12}(x_1-x_2)S^{23}(x_1)\\
&=R(x_1)(Y(x_2)\otimes 1)\sigma^{23}\sigma^{12}S^{12}(x_1-x_2)S^{23}(x_1)\\
&=(1\otimes Y(x_2))R^{12}(x_1-x_2)R^{23}(x_1)\sigma^{23}\sigma^{12}S^{12}(x_1-x_2)S^{23}(x_1),
\end{aligned}$$

and on the other hand, by (4.13), we have

$$\begin{aligned}
&(1\otimes Y(x_2))(RS)^{12}(x_1-x_2)(RS)^{23}(x_1)\\
&=(1\otimes Y(x_2))R^{12}(x_1-x_2)\sigma^{12}S^{12}(x_1-x_2)R^{23}(x_1)\sigma^{23}S^{23}(x_1)\\
&=(1\otimes Y(x_2))R^{12}(x_1-x_2)\sigma^{12}\sigma^{23}R^{12}(x_1)S^{23}(x_1-x_2)\sigma^{12}\sigma^{23}S^{23}(x_1),
\end{aligned}$$

from a direct calculation, we obtain

$$(RS)(x_1)(Y(x_2)\otimes 1)=(1\otimes Y(x_2))(RS)^{12}(x_1-x_2)(RS)^{23}(x_1).$$

Furthermore, we get

$$T_R(x)T_S(x)=\sigma^{23}R^{23}(-x)\sigma^{23}S^{23}(-x)=\sigma^{23}(RS)^{23}(-x).$$

That is, $T_R(x)T_S(x)$ is a twistor for $U\otimes V$ afforded by the twisting operator $(RS)(x)$.

(3) It is easy to see that

$$\begin{aligned}
Y_{U\otimes_R V}(x)T_S(x)&=(Y(x)\otimes Y(x))R^{23}(-x)\sigma^{23}S^{23}(-x)\\
&=(Y(x)\otimes Y(x))(RS)^{23}(-x)\\
&=Y_{U\otimes_{RS}V}(x).
\end{aligned}$$

This concludes the proof. □

4.3. Iterated twisted tensor product nonlocal vertex algebras

First of all, we review some results of the construction of iterated twisted tensor products of nonlocal vertex algebras from [17].

Let U, V and W be nonlocal vertex algebras, let

$$\begin{aligned} R_1(x): &\quad V \otimes U \to U \otimes V \otimes \mathbb{C}((x)), \\ R_2(x): &\quad W \otimes V \to V \otimes W \otimes \mathbb{C}((x)), \\ R_3(x): &\quad W \otimes U \to U \otimes W \otimes \mathbb{C}((x)) \end{aligned}$$

be twisting operators for the ordered pairs (U, V), (V, W) and (U, W), respectively. Define operators

$$\begin{aligned} T_1(x) = R_2^{23}(x)R_3^{12}(x): &\quad W \otimes (U \otimes_{R_1} V) \to (U \otimes_{R_1} V) \otimes W \otimes \mathbb{C}((x)), \\ T_2(x) = R_1^{12}(x)R_3^{23}(x): &\quad (V \otimes_{R_2} W) \otimes U \to U \otimes (V \otimes_{R_2} W) \otimes \mathbb{C}((x)). \end{aligned}$$

The following was proved in [17]:

Theorem 4.14. *Let U, V and W be nonlocal vertex algebras, let $R_1(x)$, $R_2(x)$ and $R_3(x)$ be twisting operators for the ordered pairs (U, V), (V, W) and (U, W), respectively. Then the following conditions are equivalent:*

(1) $T_1(x) = R_2^{23}(x)R_3^{12}(x)$ is a twisting operator.
(2) $T_2(x) = R_1^{12}(x)R_3^{23}(x)$ is a twisting operator.
(3) The twisting operators $R_1(x)$, $R_2(x)$ and $R_3(x)$ satisfy the following compatibility condition (called the hexagon equation):

$$R_2^{23}(x_1 - x_2)R_3^{12}(x_1)R_1^{23}(x_2) = R_1^{12}(x_2)R_3^{23}(x_1)R_2^{12}(x_1 - x_2). \tag{4.15}$$

Furthermore, if all the three conditions are satisfied, then the nonlocal vertex algebras $U \otimes_{T_2} (V \otimes_{R_2} W)$ and $(U \otimes_{R_1} V) \otimes_{T_1} W$ are equal. In this case, we will denote this nonlocal vertex algebra by $U \otimes_{R_1} V \otimes_{R_2} W$, which contains U, V and W canonically as nonlocal vertex subalgebras.

In particular, if we take one of the compatible twisting operators $R_1(x)$, $R_2(x)$, $R_3(x)$ as a usual flip, respectively, then we have

$$R_1^{12}(x_1)\sigma^{23}R_2^{12}(x_2) = R_2^{23}(x_2)\sigma^{12}R_1^{23}(x_1), \tag{4.16}$$

$$R_1^{12}(x_1)R_3^{23}(x_2)\sigma^{12} = \sigma^{23}R_3^{12}(x_2)R_1^{23}(x_1), \tag{4.17}$$

$$\sigma^{12}R_3^{23}(x_1)R_2^{12}(x_2) = R_2^{23}(x_2)R_3^{12}(x_1)\sigma^{23}. \tag{4.18}$$

That is, $(R_1(x), R_2(x), \sigma)$, $(R_1(x), \sigma, R_3(x))$ and $(\sigma, R_2(x), R_3(x))$ are compatible, respectively. These three equalities are very crucial in the proof of the following results.

Consider the nonlocal vertex algebra $K = U \otimes V \otimes W$ and the operator $T(x) : K \otimes K \to K \otimes K \otimes \mathbb{C}((x))$:

$$T(x) = \sigma^{34}\sigma^{45}\sigma^{23} R_1^{23}(-x) R_2^{45}(-x) R_3^{34}(-x). \tag{4.19}$$

As one of main results of this paper we have:

Theorem 4.15. *With notation as above, $T(x)$ is a twistor for K if and only if $(R_1(x), R_2(x), \sigma)$, $(R_1(x), \sigma, R_3(x))$ and $(\sigma, R_2(x), R_3(x))$ are compatible triples. Moreover, in this case it follows that $R_1(x)$, $R_2(x)$ and $R_3(x)$ are compatible twisting operators and $K^T = U \otimes_{R_1} V \otimes_{R_2} W$.*

Proof. For the "if" part, for $u \in U$, $v \in V$, $w \in W$, we need to prove

$$T(x)(\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes u \otimes v \otimes w) = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes u \otimes v \otimes w, \tag{4.20}$$

$$T(x)(u \otimes v \otimes w \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}) = u \otimes v \otimes w \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}, \tag{4.21}$$

$$\begin{aligned} &T(x_1)(1_{U\otimes V\otimes W} \otimes Y_{U\otimes V\otimes W}(x_2)) \\ &\quad = (1_{U\otimes V\otimes W} \otimes Y_{U\otimes V\otimes W}(x_2)) T^{13}(x_1) T^{12}(x_1 - x_2), \end{aligned} \tag{4.22}$$

$$\begin{aligned} &T(x_1)(Y_{U\otimes V\otimes W}(x_2) \otimes 1_{U\otimes V\otimes W}) \\ &\quad = (Y_{U\otimes V\otimes W}(x_2) \otimes 1_{U\otimes V\otimes W}) T^{13}(x_1 + x_2) T^{23}(x_1), \end{aligned} \tag{4.23}$$

$$T^{12}(x_1) T^{23}(x_2) = T^{23}(x_2) T^{12}(x_1). \tag{4.24}$$

At first, for $u \in U$, $v \in V$, $w \in W$, we can easily have

$$\begin{aligned} &T(x)(\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes u \otimes v \otimes w) \\ &= \sigma^{34}\sigma^{45}\sigma^{23} R_1^{23}(-x) R_2^{45}(-x) R_3^{34}(-x)(\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes u \otimes v \otimes w) \\ &= \sigma^{34}\sigma^{45}\sigma^{23} R_1^{23}(-x) R_2^{45}(-x)(\mathbf{1} \otimes \mathbf{1} \otimes u \otimes \mathbf{1} \otimes v \otimes w) \\ &= \sigma^{34}\sigma^{45}\sigma^{23}(\mathbf{1} \otimes u \otimes \mathbf{1} \otimes v \otimes \mathbf{1} \otimes w) \\ &= \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes u \otimes v \otimes w, \end{aligned}$$

and

$$\begin{aligned} &T(x)(u \otimes v \otimes w \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}) \\ &= \sigma^{34}\sigma^{45}\sigma^{23} R_1^{23}(-x) R_2^{45}(-x) R_3^{34}(-x)(u \otimes v \otimes w \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}) \\ &= \sigma^{34}\sigma^{45}\sigma^{23} R_1^{23}(-x) R_2^{45}(-x)(u \otimes v \otimes \mathbf{1} \otimes w \otimes \mathbf{1} \otimes \mathbf{1}) \\ &= \sigma^{34}\sigma^{45}\sigma^{23}(u \otimes \mathbf{1} \otimes v \otimes \mathbf{1} \otimes w \otimes \mathbf{1}) \\ &= u \otimes v \otimes w \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}. \end{aligned}$$

Concerning (4.22), from (4.8) and (4.11), we have

$$
\begin{aligned}
&T(x_1)(1_{U\otimes V\otimes W}\otimes Y_{U\otimes V\otimes W}(x_2))\\
&=\sigma^{34}\sigma^{45}\sigma^{23}R_1^{23}(-x_1)R_2^{45}(-x_1)R_3^{34}(-x_1)(1\otimes 1\otimes 1\otimes Y(x_2)\otimes Y(x_2)\otimes Y(x_2))\\
&\quad\cdot\sigma^{78}\sigma^{56}\sigma^{67}\\
&=\sigma^{34}\sigma^{45}\sigma^{23}R_1^{23}(-x_1)R_2^{45}(-x_1)(1\otimes 1\otimes Y(x_2)\otimes 1\otimes Y(x_2)\otimes Y(x_2))\\
&\quad\cdot R_3^{45}(-x_1)R_3^{34}(-x_1+x_2)\sigma^{78}\sigma^{56}\sigma^{67}\\
&=\sigma^{34}\sigma^{45}\sigma^{23}(1\otimes Y(x_2)\otimes 1\otimes Y(x_2)\otimes 1\otimes Y(x_2))R_1^{34}(-x_1)R_1^{23}(-x_1+x_2)\\
&\quad\cdot R_2^{67}(-x_1)R_2^{56}(-x_1+x_2)R_3^{45}(-x_1)R_3^{34}(-x_1+x_2)\sigma^{78}\sigma^{56}\sigma^{67}\\
&=\sigma^{34}(1\otimes 1\otimes Y(x_2)\otimes 1\otimes Y(x_2)\otimes Y(x_2))\sigma^{56}\sigma^{67}\sigma^{23}\sigma^{34}R_1^{34}(-x_1)R_1^{23}(-x_1+x_2)\\
&\quad\cdot R_2^{67}(-x_1)R_2^{56}(-x_1+x_2)R_3^{45}(-x_1)R_3^{34}(-x_1+x_2)\sigma^{78}\sigma^{56}\sigma^{67}\\
&=(1\otimes 1\otimes 1\otimes Y(x_2)\otimes Y(x_2)\otimes Y(x_2))\sigma^{34}\sigma^{45}\sigma^{56}\sigma^{67}\sigma^{23}\sigma^{34}R_1^{34}(-x_1)R_1^{23}(-x_1+x_2)\\
&\quad\cdot R_2^{67}(-x_1)R_2^{56}(-x_1+x_2)R_3^{45}(-x_1)R_3^{34}(-x_1+x_2)\sigma^{78}\sigma^{56}\sigma^{67},
\end{aligned}
$$

using (4.18), we get $R_2^{56}(-x_1+x_2)R_3^{45}(-x_1)=\sigma^{45}R_3^{56}(-x_1)R_2^{45}(-x_1+x_2)\sigma^{56}$, then

$$
\begin{aligned}
&T(x_1)(1_{U\otimes V\otimes W}\otimes Y_{U\otimes V\otimes W}(x_2))\\
&=(1\otimes 1\otimes 1\otimes Y(x_2)\otimes Y(x_2)\otimes Y(x_2))\sigma^{34}\sigma^{45}\sigma^{56}\sigma^{67}\sigma^{23}\sigma^{34}R_1^{34}(-x_1)R_1^{23}(-x_1+x_2)\\
&\quad\cdot R_2^{67}(-x_1)R_2^{56}(-x_1+x_2)R_3^{45}(-x_1)R_3^{34}(-x_1+x_2)\sigma^{78}\sigma^{56}\sigma^{67}\\
&=(1\otimes 1\otimes 1\otimes Y(x_2)\otimes Y(x_2)\otimes Y(x_2))\sigma^{34}\sigma^{45}\sigma^{56}\sigma^{67}\sigma^{23}\sigma^{34}R_1^{34}(-x_1)R_1^{23}(-x_1+x_2)\\
&\quad\cdot R_2^{67}(-x_1)\sigma^{45}R_3^{56}(-x_1)R_2^{45}(-x_1+x_2)\sigma^{56}R_3^{34}(-x_1+x_2)\sigma^{78}\sigma^{56}\sigma^{67}\\
&=(1\otimes 1\otimes 1\otimes Y(x_2)\otimes Y(x_2)\otimes Y(x_2))\sigma^{34}\sigma^{45}\sigma^{56}\sigma^{67}\sigma^{23}\sigma^{34}R_1^{34}(-x_1)\\
&\quad\cdot R_2^{67}(-x_1)\sigma^{45}R_3^{56}(-x_1)\sigma^{78}\sigma^{67}R_1^{23}(-x_1+x_2)R_2^{45}(-x_1+x_2)R_3^{34}(-x_1+x_2),
\end{aligned}
$$

and on the other hand, we have

$$
\begin{aligned}
&(1_{U\otimes V\otimes W}\otimes Y_{U\otimes V\otimes W}(x_2))T^{13}(x_1)T^{12}(x_1-x_2)\\
&=(1\otimes 1\otimes 1\otimes Y(x_2)\otimes Y(x_2)\otimes Y(x_2))\sigma^{78}\sigma^{56}\sigma^{67}\\
&\quad\cdot\sigma^{37}\sigma^{78}\sigma^{23}R_1^{23}(-x_1)R_2^{78}(-x_1)R_3^{37}(-x_1)\\
&\quad\cdot\sigma^{34}\sigma^{45}\sigma^{23}R_1^{23}(-x_1+x_2)R_2^{45}(-x_1+x_2)R_3^{34}(-x_1+x_2),
\end{aligned}
$$

and by direct calculation we know that

$$
\begin{aligned}
&\sigma^{34}\sigma^{45}\sigma^{56}\sigma^{67}\sigma^{23}\sigma^{34}R_1^{34}(-x_1)R_2^{67}(-x_1)\sigma^{45}R_3^{56}(-x_1)\sigma^{78}\sigma^{67}\\
&\cdot R_1^{23}(-x_1+x_2)R_2^{45}(-x_1+x_2)R_3^{34}(-x_1+x_2)
\end{aligned}
$$

$$
\begin{aligned}
&= \sigma^{78}\sigma^{56}\sigma^{67}\sigma^{37}\sigma^{78}\sigma^{23}R_1^{23}(-x_1)R_2^{78}(-x_1)R_3^{37}(-x_1)\sigma^{34}\sigma^{45}\sigma^{23} \\
&\quad \cdot R_1^{23}(-x_1+x_2)R_2^{45}(-x_1+x_2)R_3^{34}(-x_1+x_2).
\end{aligned}
$$

Then we have (4.22).

For (4.23), using (4.9) and (4.10), we have

$$
\begin{aligned}
&T(x_1)(Y_{U\otimes V\otimes W}(x_2)\otimes 1_{U\otimes V\otimes W}) \\
&= \sigma^{34}\sigma^{45}\sigma^{23}R_1^{23}(-x_1)R_2^{45}(-x_1)R_3^{34}(-x_1)(Y(x_2)\otimes Y(x_2)\otimes Y(x_2)\otimes 1\otimes 1\otimes 1) \\
&\quad \cdot \sigma^{45}\sigma^{23}\sigma^{34} \\
&= \sigma^{34}\sigma^{45}\sigma^{23}R_1^{23}(-x_1)R_2^{45}(-x_1)(Y(x_2)\otimes Y(x_2)\otimes 1\otimes Y(x_2)\otimes 1\otimes 1) \\
&\quad \cdot R_3^{56}(-x_1-x_2)R_3^{67}(-x_1)\sigma^{45}\sigma^{23}\sigma^{34} \\
&= \sigma^{34}\sigma^{45}\sigma^{23}(Y(x_2)\otimes 1\otimes Y(x_2)\otimes 1\otimes Y(x_2)\otimes 1)R_1^{34}(-x_1-x_2)R_1^{45}(-x_1) \\
&\quad \cdot R_2^{67}(-x_1-x_2)R_2^{78}(-x_1)R_3^{56}(-x_1-x_2)R_3^{67}(-x_1)\sigma^{45}\sigma^{23}\sigma^{34} \\
&= \sigma^{34}(Y(x_2)\otimes Y(x_2)\otimes 1\otimes Y(x_2)\otimes 1\otimes 1)\sigma^{78}\sigma^{67}\sigma^{45}\sigma^{34}R_1^{34}(-x_1-x_2)R_1^{45}(-x_1) \\
&\quad \cdot R_2^{67}(-x_1-x_2)R_2^{78}(-x_1)R_3^{56}(-x_1-x_2)R_3^{67}(-x_1)\sigma^{45}\sigma^{23}\sigma^{34} \\
&= (Y(x_2)\otimes Y(x_2)\otimes Y(x_2)\otimes 1\otimes 1\otimes 1)\sigma^{67}\sigma^{56}\sigma^{78}\sigma^{67}\sigma^{45}\sigma^{34}R_1^{34}(-x_1-x_2)R_1^{45}(-x_1) \\
&\quad \cdot R_2^{67}(-x_1-x_2)R_2^{78}(-x_1)R_3^{56}(-x_1-x_2)R_3^{67}(-x_1)\sigma^{45}\sigma^{23}\sigma^{34},
\end{aligned}
$$

by (4.17), we get $R_1^{45}(-x_1)R_3^{56}(-x_1-x_2) = \sigma^{56}R_3^{45}(-x_1-x_2)R_1^{56}(-x_1)\sigma^{45}$, then

$$
\begin{aligned}
&T(x_1)(Y_{U\otimes V\otimes W}(x_2)\otimes 1_{U\otimes V\otimes W}) \\
&= (Y(x_2)\otimes Y(x_2)\otimes Y(x_2)\otimes 1\otimes 1\otimes 1)\sigma^{67}\sigma^{56}\sigma^{78}\sigma^{67}\sigma^{45}\sigma^{34}R_1^{34}(-x_1-x_2)R_1^{45}(-x_1) \\
&\quad \cdot R_2^{67}(-x_1-x_2)R_2^{78}(-x_1)R_3^{56}(-x_1-x_2)R_3^{67}(-x_1)\sigma^{45}\sigma^{23}\sigma^{34} \\
&= (Y(x_2)\otimes Y(x_2)\otimes Y(x_2)\otimes 1\otimes 1\otimes 1)\sigma^{67}\sigma^{56}\sigma^{78}\sigma^{67}\sigma^{45}\sigma^{34} \\
&\quad \cdot R_1^{34}(-x_1-x_2)R_2^{67}(-x_1-x_2)R_1^{45}(-x_1)R_3^{56}(-x_1-x_2)R_2^{78}(-x_1)R_3^{67}(-x_1)\sigma^{45}\sigma^{23}\sigma^{34} \\
&= (Y(x_2)\otimes Y(x_2)\otimes Y(x_2)\otimes 1\otimes 1\otimes 1)\sigma^{67}\sigma^{56}\sigma^{78}\sigma^{67}\sigma^{45}\sigma^{34} \\
&\quad \cdot R_1^{34}(-x_1-x_2)R_2^{67}(-x_1-x_2)\sigma^{56}R_3^{45}(-x_1-x_2)R_1^{56}(-x_1)\sigma^{45}R_2^{78}(-x_1)R_3^{67}(-x_1) \\
&\quad \cdot \sigma^{45}\sigma^{23}\sigma^{34},
\end{aligned}
$$

and on the other hand,

$$
\begin{aligned}
&(Y_{U\otimes V\otimes W}(x_2)\otimes 1_{U\otimes V\otimes W})T^{13}(x_1+x_2)T^{23}(x_1) \\
&= (Y(x_2)\otimes Y(x_2)\otimes Y(x_2)\otimes 1\otimes 1\otimes 1)\sigma^{45}\sigma^{23}\sigma^{34} \\
&\quad \cdot \sigma^{37}\sigma^{78}\sigma^{23}R_1^{23}(-x_1-x_2)R_2^{78}(-x_1-x_2)R_3^{37}(-x_1-x_2) \\
&\quad \cdot \sigma^{67}\sigma^{78}\sigma^{56}R_1^{56}(-x_1)R_2^{78}(-x_1)R_3^{67}(-x_1),
\end{aligned}
$$

by the direct calculation we have

$$
\begin{aligned}
&\sigma^{67}\sigma^{56}\sigma^{78}\sigma^{67}\sigma^{45}\sigma^{34}R_1^{34}(-x_1-x_2)R_2^{67}(-x_1-x_2)\sigma^{56}R_3^{45}(-x_1-x_2)R_1^{56}(-x_1)\\
&\quad\cdot\sigma^{45}R_2^{78}(-x_1)R_3^{67}(-x_1)\sigma^{45}\sigma^{23}\sigma^{34}\\
&=\sigma^{45}\sigma^{23}\sigma^{34}\sigma^{37}\sigma^{78}\sigma^{23}R_1^{23}(-x_1-x_2)R_2^{78}(-x_1-x_2)R_3^{37}(-x_1-x_2)\\
&\quad\cdot\sigma^{67}\sigma^{78}\sigma^{56}R_1^{56}(-x_1)R_2^{78}(-x_1)R_3^{67}(-x_1).
\end{aligned}
$$

Then we have (4.23).

For (4.24), from (4.19), we have

$$
\begin{aligned}
&T^{12}(x_1)T^{23}(x_2)\\
&=\sigma^{34}\sigma^{45}\sigma^{23}R_1^{23}(-x_1)R_2^{45}(-x_1)R_3^{34}(-x_1)\sigma^{67}\sigma^{78}\sigma^{56}R_1^{56}(-x_2)R_2^{78}(-x_2)R_3^{67}(-x_2)\\
&=\sigma^{67}\sigma^{78}\sigma^{34}\sigma^{45}\sigma^{23}R_1^{23}(-x_1)R_2^{45}(-x_1)\sigma^{56}R_1^{56}(-x_2)R_2^{78}(-x_2)R_3^{67}(-x_2)R_3^{34}(-x_1),
\end{aligned}
$$

from (4.16), we have $R_2^{45}(-x_1)\sigma^{56}R_1^{56}(-x_2)=\sigma^{45}\sigma^{56}R_1^{56}(-x_2)\sigma^{45}R_2^{45}(-x_1)$, then

$$
\begin{aligned}
&T^{12}(x_1)T^{23}(x_2)\\
&=\sigma^{67}\sigma^{78}\sigma^{34}\sigma^{45}\sigma^{23}R_1^{23}(-x_1)R_2^{45}(-x_1)\sigma^{56}R_1^{56}(-x_2)R_2^{78}(-x_2)R_3^{67}(-x_2)R_3^{34}(-x_1)\\
&=\sigma^{67}\sigma^{78}\sigma^{34}\sigma^{45}\sigma^{23}R_1^{23}(-x_1)\sigma^{45}\sigma^{56}R_1^{56}(-x_2)\sigma^{45}R_2^{45}(-x_1)R_2^{78}(-x_2)R_3^{67}(-x_2)R_3^{34}(-x_1)\\
&=\sigma^{67}\sigma^{78}\sigma^{56}R_1^{56}(-x_2)R_2^{78}(-x_2)R_3^{67}(-x_2)\sigma^{34}\sigma^{45}\sigma^{23}R_1^{23}(-x_1)R_2^{45}(-x_1)R_3^{34}(-x_1)\\
&=T^{23}(x_2)T^{12}(x_1).
\end{aligned}
$$

Next we prove the "only if" part. For (4.16), for $u''\in U$, $v'\in V$, $w\in W$, we take $\mathbf{1}\otimes\mathbf{1}\otimes w$, $\mathbf{1}\otimes v'\otimes\mathbf{1}$ and $u''\otimes\mathbf{1}\otimes\mathbf{1}$, then

$$
\begin{aligned}
&T^{12}(x_1)T^{23}(x_2)(\mathbf{1}\otimes\mathbf{1}\otimes w\otimes\mathbf{1}\otimes v'\otimes\mathbf{1}\otimes u''\otimes\mathbf{1}\otimes\mathbf{1})\\
&=\sigma^{34}\sigma^{45}\sigma^{23}R_1^{23}(-x_1)R_2^{45}(-x_1)R_3^{34}(-x_1)\sigma^{67}\sigma^{78}\sigma^{56}R_1^{56}(-x_2)R_2^{78}(-x_2)R_3^{67}(-x_2)\\
&\quad\cdot(\mathbf{1}\otimes\mathbf{1}\otimes w\otimes\mathbf{1}\otimes v'\otimes\mathbf{1}\otimes u''\otimes\mathbf{1}\otimes\mathbf{1})\\
&=\sigma^{34}\sigma^{45}\sigma^{23}R_1^{23}(-x_1)R_2^{45}(-x_1)R_3^{34}(-x_1)\sigma^{67}\sigma^{56}R_1^{56}(-x_2)\\
&\quad\cdot(\mathbf{1}\otimes\mathbf{1}\otimes w\otimes\mathbf{1}\otimes v'\otimes u''\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1})\\
&=\sigma^{67}\sigma^{34}\sigma^{45}R_2^{45}(-x_1)\sigma^{56}R_1^{56}(-x_2)(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\otimes w\otimes v'\otimes u''\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}),
\end{aligned}
$$

and on other hand, we also have

$$
\begin{aligned}
&T^{23}(x_2)T^{12}(x_1)(\mathbf{1}\otimes\mathbf{1}\otimes w\otimes\mathbf{1}\otimes v'\otimes\mathbf{1}\otimes u''\otimes\mathbf{1}\otimes\mathbf{1})\\
&=\sigma^{67}\sigma^{78}\sigma^{56}R_1^{56}(-x_2)R_2^{78}(-x_2)R_3^{67}(-x_2)\sigma^{34}\sigma^{45}\sigma^{23}R_1^{23}(-x_1)R_2^{45}(-x_1)R_3^{34}(-x_1)\\
&\quad\cdot(\mathbf{1}\otimes\mathbf{1}\otimes w\otimes\mathbf{1}\otimes v'\otimes\mathbf{1}\otimes u''\otimes\mathbf{1}\otimes\mathbf{1})
\end{aligned}
$$

$$
\begin{aligned}
&= \sigma^{67}\sigma^{78}\sigma^{56}R_1^{56}(-x_2)R_2^{78}(-x_2)R_3^{67}(-x_2)\sigma^{34}\sigma^{45}R_2^{45}(-x_1) \\
&\quad \cdot (\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\otimes w\otimes v'\otimes\mathbf{1}\otimes u''\otimes\mathbf{1}\otimes\mathbf{1}) \\
&= \sigma^{67}\sigma^{56}R_1^{56}(-x_2)\sigma^{34}\sigma^{45}R_2^{45}(-x_1)(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\otimes w\otimes v'\otimes u''\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}),
\end{aligned}
$$

from (3.5), we know that (4.16) holds.

Concerning (4.17), we also take $\mathbf{1}\otimes\mathbf{1}\otimes w$, $\mathbf{1}\otimes v'\otimes\mathbf{1}$ and $u''\otimes\mathbf{1}\otimes\mathbf{1}$, using (4.9) and (4.10), we have

$$
\begin{aligned}
&T(x_1)(Y_{U\otimes V\otimes W}(x_2)\otimes 1_{U\otimes V\otimes W})(\mathbf{1}\otimes\mathbf{1}\otimes w\otimes\mathbf{1}\otimes v'\otimes\mathbf{1}\otimes u''\otimes\mathbf{1}\otimes\mathbf{1}) \\
&= \sigma^{34}\sigma^{45}\sigma^{23}R_1^{23}(-x_1)R_2^{45}(-x_1)R_3^{34}(-x_1)(Y(x_2)\otimes Y(x_2)\otimes Y(x_2)\otimes 1\otimes 1\otimes 1) \\
&\quad \cdot \sigma^{45}\sigma^{23}\sigma^{34}(\mathbf{1}\otimes\mathbf{1}\otimes w\otimes\mathbf{1}\otimes v'\otimes\mathbf{1}\otimes u''\otimes\mathbf{1}\otimes\mathbf{1}) \\
&= (Y(x_2)\otimes Y(x_2)\otimes Y(x_2)\otimes 1\otimes 1\otimes 1)\sigma^{67}\sigma^{56}\sigma^{78}\sigma^{67}\sigma^{45}\sigma^{34}R_1^{34}(-x_1-x_2)R_1^{45}(-x_1) \\
&\quad \cdot R_2^{67}(-x_1-x_2)R_2^{78}(-x_1)R_3^{56}(-x_1-x_2)R_3^{67}(-x_1)\sigma^{45}\sigma^{23}\sigma^{34} \\
&\quad \cdot (\mathbf{1}\otimes\mathbf{1}\otimes w\otimes\mathbf{1}\otimes v'\otimes\mathbf{1}\otimes u''\otimes\mathbf{1}\otimes\mathbf{1}) \\
&= (Y(x_2)\otimes Y(x_2)\otimes Y(x_2)\otimes 1\otimes 1\otimes 1)\sigma^{67}\sigma^{56}\sigma^{78}\sigma^{67}\sigma^{45}\sigma^{34}R_1^{34}(-x_1-x_2)R_1^{45}(-x_1) \\
&\quad \cdot R_2^{67}(-x_1-x_2)R_3^{56}(-x_1-x_2)(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\otimes v'\otimes w\otimes u''\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}),
\end{aligned}
$$

and on the other hand,

$$
\begin{aligned}
&(Y_{U\otimes V\otimes W}(x_2)\otimes 1_{U\otimes V\otimes W})T^{13}(x_1+x_2)T^{23}(x_1) \\
&\quad \cdot (\mathbf{1}\otimes\mathbf{1}\otimes w\otimes\mathbf{1}\otimes v'\otimes\mathbf{1}\otimes u''\otimes\mathbf{1}\otimes\mathbf{1}) \\
&= (Y(x_2)\otimes Y(x_2)\otimes Y(x_2)\otimes 1\otimes 1\otimes 1)\sigma^{45}\sigma^{23}\sigma^{34} \\
&\quad \cdot \sigma^{37}\sigma^{78}\sigma^{23}R_1^{23}(-x_1-x_2)R_2^{78}(-x_1-x_2)R_3^{37}(-x_1-x_2) \\
&\quad \cdot \sigma^{67}\sigma^{78}\sigma^{56}R_1^{56}(-x_1)R_2^{78}(-x_1)R_3^{67}(-x_1)(\mathbf{1}\otimes\mathbf{1}\otimes w\otimes\mathbf{1}\otimes v'\otimes\mathbf{1}\otimes u''\otimes\mathbf{1}\otimes\mathbf{1}) \\
&= (Y(x_2)\otimes Y(x_2)\otimes Y(x_2)\otimes 1\otimes 1\otimes 1)\sigma^{45}\sigma^{23}\sigma^{34} \\
&\quad \cdot \sigma^{37}\sigma^{78}\sigma^{23}R_1^{23}(-x_1-x_2)R_2^{78}(-x_1-x_2)R_3^{37}(-x_1-x_2) \\
&\quad \cdot \sigma^{67}\sigma^{56}R_1^{56}(-x_1)(\mathbf{1}\otimes\mathbf{1}\otimes w\otimes\mathbf{1}\otimes v'\otimes u''\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}),
\end{aligned}
$$

then from (3.4) we can easily have (4.17).

For (4.18), we still take $\mathbf{1}\otimes\mathbf{1}\otimes w$, $\mathbf{1}\otimes v'\otimes\mathbf{1}$ and $u''\otimes\mathbf{1}\otimes\mathbf{1}$, from (4.8) and (4.11), we have

$$
\begin{aligned}
&T(x_1)(1_{U\otimes V\otimes W}\otimes Y_{U\otimes V\otimes W}(x_2))(\mathbf{1}\otimes\mathbf{1}\otimes w\otimes\mathbf{1}\otimes v'\otimes\mathbf{1}\otimes u''\otimes\mathbf{1}\otimes\mathbf{1}) \\
&= \sigma^{34}\sigma^{45}\sigma^{23}R_1^{23}(-x_1)R_2^{45}(-x_1)R_3^{34}(-x_1)(1\otimes 1\otimes 1\otimes Y(x_2)\otimes Y(x_2)\otimes Y(x_2)) \\
&\quad \cdot \sigma^{78}\sigma^{56}\sigma^{67}(\mathbf{1}\otimes\mathbf{1}\otimes w\otimes\mathbf{1}\otimes v'\otimes\mathbf{1}\otimes u''\otimes\mathbf{1}\otimes\mathbf{1}) \\
&= (1\otimes 1\otimes 1\otimes Y(x_2)\otimes Y(x_2)\otimes Y(x_2))\sigma^{34}\sigma^{45}\sigma^{56}\sigma^{67}\sigma^{23}\sigma^{34}R_1^{34}(-x_1)R_1^{23}(-x_1+x_2)
\end{aligned}
$$

$$
\begin{aligned}
&\cdot R_2^{67}(-x_1)R_2^{56}(-x_1+x_2)R_3^{45}(-x_1)R_3^{34}(-x_1+x_2)\sigma^{78}\sigma^{56}\sigma^{67}\\
&\cdot (\mathbf{1}\otimes\mathbf{1}\otimes w\otimes\mathbf{1}\otimes v'\otimes\mathbf{1}\otimes u''\otimes\mathbf{1}\otimes\mathbf{1})\\
=&\ (1\otimes 1\otimes 1\otimes Y(x_2)\otimes Y(x_2)\otimes Y(x_2))\sigma^{34}\sigma^{45}\sigma^{56}\sigma^{67}\sigma^{23}\sigma^{34}R_1^{34}(-x_1)\\
&\cdot R_2^{67}(-x_1)R_2^{56}(-x_1+x_2)R_3^{45}(-x_1)(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\otimes w\otimes u''\otimes v'\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}),
\end{aligned}
$$

and on the other hand, we have

$$
\begin{aligned}
&(1_{U\otimes V\otimes W}\otimes Y_{U\otimes V\otimes W}(x_2))T^{13}(x_1)T^{12}(x_1-x_2)\\
&\cdot (\mathbf{1}\otimes\mathbf{1}\otimes w\otimes\mathbf{1}\otimes v'\otimes\mathbf{1}\otimes u''\otimes\mathbf{1}\otimes\mathbf{1})\\
=&\ (1\otimes 1\otimes 1\otimes Y(x_2)\otimes Y(x_2)\otimes Y(x_2))\sigma^{78}\sigma^{56}\sigma^{67}\\
&\cdot \sigma^{37}\sigma^{78}\sigma^{23}R_1^{23}(-x_1)R_2^{78}(-x_1)R_3^{37}(-x_1)\sigma^{34}\sigma^{45}\sigma^{23}\\
&\cdot R_1^{23}(-x_1+x_2)R_2^{45}(-x_1+x_2)R_3^{34}(-x_1+x_2)(\mathbf{1}\otimes\mathbf{1}\otimes w\otimes\mathbf{1}\otimes v'\otimes\mathbf{1}\otimes u''\otimes\mathbf{1}\otimes\mathbf{1})\\
=&\ (1\otimes 1\otimes 1\otimes Y(x_2)\otimes Y(x_2)\otimes Y(x_2))\sigma^{78}\sigma^{56}\sigma^{67}\\
&\cdot \sigma^{37}\sigma^{78}\sigma^{23}R_1^{23}(-x_1)R_2^{78}(-x_1)R_3^{37}(-x_1)\sigma^{34}\sigma^{45}\\
&\cdot R_2^{45}(-x_1+x_2)(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\otimes w\otimes v'\otimes\mathbf{1}\otimes u''\otimes\mathbf{1}\otimes\mathbf{1}),
\end{aligned}
$$

then from (3.3) we directly get (4.18).

Finally, from (4.18), (4.16) and (4.17), we have

$$
\begin{aligned}
R_2^{23}(x_1-x_2)R_3^{12}(x_1)R_1^{23}(x_2) &= \sigma^{12}R_3^{23}(x_1)R_2^{12}(x_1-x_2)\sigma^{23}R_1^{23}(x_2)\\
&= \sigma^{12}R_3^{23}(x_1)\sigma^{13}R_1^{13}(x_2)R_2^{12}(x_1-x_2)\\
&= R_1^{12}(x_2)R_3^{23}(x_1)R_2^{12}(x_1-x_2).
\end{aligned}
$$

That is, $R_1(x)$, $R_2(x)$ and $R_3(x)$ are compatible. It is easy to see that

$$
\begin{aligned}
&Y_T(x) = Y_{U\otimes V\otimes W}(x)T(x)\\
&= (Y(x)\otimes Y(x)\otimes Y(x))\sigma^{45}\sigma^{23}\sigma^{34}\sigma^{34}\sigma^{45}\sigma^{23}R_1^{23}(-x)R_2^{45}(-x)R_3^{34}(-x)\\
&= (Y(x)\otimes Y(x)\otimes Y(x))R_1^{23}(-x)R_2^{45}(-x)R_3^{34}(-x).
\end{aligned}
$$

That is, $K^T = U\otimes_{R_1} V\otimes_{R_2} W$. $\square$

4.4. N-factor iterated twisted tensor product nonlocal vertex algebras

In this subsection, we begin by recalling the construction of a twisted tensor product of any number of factors.

The following is lifted from [17]:

Lemma 4.16. *Let $V_1, V_2, \ldots, V_n$ be nonlocal vertex algebras, let $R_{ij}(x) : V_j \otimes V_i \to V_i \otimes V_j \otimes \mathbb{C}((x))$ be twisting operators for $1 \le i < j \le n$, such that for any $i < j < k$ the twisting operators $R_{ij}(x)$, $R_{jk}(x)$ and $R_{ik}(x)$ satisfying the hexagon equation, let $T^i_{n-1,n}(x) = R^{12}_{i,n-1}(x)R^{23}_{in}(x)$ be twisting operators defined by*

$$T^i_{n-1,n}(x) : \quad (V_{n-1} \otimes_{R_{n-1,n}} V_n) \otimes V_i \to V_i \otimes (V_{n-1} \otimes_{R_{n-1,n}} V_n) \otimes \mathbb{C}((x))$$

for $i = 1, \ldots, n-2$. Then for $1 < i < j \le n-2$, the twisting operators $R_{ij}(x)$, $T^j_{n-1,n}(x)$ and $T^i_{n-1,n}(x)$ satisfy the hexagon equation.

The following was proved in [17]:

Theorem 4.17 *(Coherence Theorem). Let $V_1, V_2, \ldots, V_n$ be nonlocal vertex algebras, let $R_{ij}(x) : V_j \otimes V_i \to V_i \otimes V_j \otimes \mathbb{C}((x))$ be twisting operators for $1 \le i < j \le n$, such that for any $i < j < k$ the twisting operators $R_{ij}(x)$, $R_{jk}(x)$ and $R_{ik}(x)$ satisfy the hexagon equation, and let $T^i_{j-1,j}(x)$ be twisting operators defined by*

$$\begin{aligned} T^i_{j-1,j}(x) &= R^{12}_{i,j-1}(x)R^{23}_{ij}(x) : \\ &\quad (V_{j-1} \otimes_{R_{j-1,j}} V_j) \otimes V_i \to V_i \otimes (V_{j-1} \otimes_{R_{j-1,j}} V_j) \otimes \mathbb{C}((x)) \quad \text{for } i \le j-2, \\ T^i_{j-1,j}(x) &= R^{23}_{ji}(x)R^{12}_{j-1,i}(x) : \\ &\quad V_i \otimes (V_{j-1} \otimes_{R_{j-1,j}} V_j) \to (V_{j-1} \otimes_{R_{j-1,j}} V_j) \otimes V_i \otimes \mathbb{C}((x)) \quad \text{for } i \ge j+1. \end{aligned}$$

Then for $i, k \notin \{j-1, j\}$ the twisting operators $R_{ik}(x)$, $T^k_{j-1,j}(x)$ and $T^i_{j-1,j}(x)$ satisfy the hexagon equation. Furthermore, for any $1 \le i \le n$ the (inductively defined) twisted tensor product nonlocal vertex algebras

$$V_1 \otimes_{R_{12}} \cdots \otimes_{R_{i-3,i-2}} V_{i-2} \otimes_{T^{i-2}_{i-1,i}} (V_{i-1} \otimes_{R_{i-1,i}} V_i) \otimes_{T^{i+1}_{i-1,i}} V_{i+1} \otimes_{R_{i+1,i+2}} \cdots \otimes_{R_{n-1,n}} V_n$$

are all equal.

Next we construct twistors from n-factor iterated twisted tensor products:

Theorem 4.18. *Let $V_1, V_2, \ldots, V_n$ be nonlocal vertex algebras, let $R_{ij}(x) : V_j \otimes V_i \to V_i \otimes V_j \otimes \mathbb{C}((x))$ be twisting operators for $1 \le i < j \le n$, and let $K = V_1 \otimes V_2 \otimes \cdots \otimes V_n$. Then the following two conditions are equivalent:*

(1) The operator $T(x) : K \otimes K \to K \otimes K \otimes \mathbb{C}((x))$ defined by

$$\begin{aligned} &T(x) \\ &= \sigma^{n,n+1} \left(\sigma^{n-1,n}\sigma^{n+1,n+2}\right) \\ &\quad \cdots \left(\sigma^{n-k,n-k+1}\sigma^{n-k+2,n-k+3} \cdots \sigma^{n+k-2,n+k-1}\sigma^{n+k,n+k+1}\right) \end{aligned}$$

$$\cdots \left(\sigma^{23}\sigma^{45}\cdots\sigma^{2n-2,2n-1}\right)\left(R^{23}_{12}(-x)R^{45}_{23}(-x)\cdots R^{2n-2,2n-1}_{n-1,n}(-x)\right)$$
$$\cdots\left(R^{n-k,n-k+1}_{1,n-k}(-x)R^{n-k+2,n-k+3}_{2,n-k+1}(-x)\cdots R^{n+k-2,n+k-1}_{k,n-1}(-x)R^{n+k,n+k+1}_{k+1,n}(-x)\right)$$
$$\cdots\left(R^{n-1,n}_{1,n-1}(-x)R^{n+1,n+2}_{2n}(-x)\right)R^{n,n+1}_{1n}(-x)$$

is a twistor.

(2) *For any triple $i<j<k\in\{1,\dots,n\}$, $\{R_{ij}(x),R_{jk}(x),\sigma\}$, $\{R_{ij}(x),\sigma,R_{ik}(x)\}$ and $\{\sigma,R_{jk}(x),R_{ik}(x)\}$ are sets of compatible twisting operators.*

Moreover, if the conditions are satisfied, then the twisting operators $\{R_{ij}(x)\}_{1\le i<j\le n}$ are compatible, and we have $K^T=V_1\otimes_{R_{12}}\otimes V_2\otimes_{R_{23}}\cdots\otimes_{R_{n-1,n}}V_n$.

Proof. The proof is by induction on the number of factors $n\ge 3$. For $n=3$, the result is Theorem 4.15. Now, assuming the result holds for $n-1$ factors with their corresponding twisting operators, and given $V_1,V_2,\dots,V_n$ nonlocal vertex algebras, satisfying the hypothesis of the theorem, we consider the algebras $U_1=V_1,\dots,U_{n-2}=V_{n-2},U_{n-1}=V_{n-1}\otimes_{R_{n-1,n}}V_n$, with the twisting operators defined as in the Coherence Theorem.

For simplicity of notation, in the following we only prove the case of $n=4$.

Let V_i be nonlocal vertex algebras for $i=1,2,3,4$, let

$$\begin{aligned}
&R_{12}(x): \quad V_2\otimes V_1\to V_1\otimes V_2\otimes\mathbb{C}((x)), &&R_{23}(x): \quad V_3\otimes V_2\to V_2\otimes V_3\otimes\mathbb{C}((x)),\\
&R_{13}(x): \quad V_3\otimes V_1\to V_1\otimes V_3\otimes\mathbb{C}((x)), &&R_{14}(x): \quad V_4\otimes V_1\to V_1\otimes V_4\otimes\mathbb{C}((x)),\\
&R_{24}(x): \quad V_4\otimes V_2\to V_2\otimes V_4\otimes\mathbb{C}((x)), &&R_{34}(x): \quad V_4\otimes V_3\to V_3\otimes V_4\otimes\mathbb{C}((x))
\end{aligned}$$

be twisting operators for the ordered pairs (V_1,V_2), (V_2,V_3), (V_1,V_3), (V_1,V_4), (V_2,V_4) and (V_3,V_4), respectively.

If we take one of the compatible twisting operators $R_{12}(x)$, $R_{23}(x)$, $R_{13}(x)$ as a usual flip, respectively, then we have

$$R^{12}_{12}(x_1)\sigma^{23}R^{12}_{23}(x_2)=R^{23}_{23}(x_2)\sigma^{12}R^{23}_{12}(x_1), \tag{4.25}$$
$$R^{12}_{12}(x_1)R^{23}_{13}(x_2)\sigma^{12}=\sigma^{23}R^{12}_{13}(x_2)R^{23}_{12}(x_1), \tag{4.26}$$
$$\sigma^{12}R^{23}_{13}(x_1)R^{12}_{23}(x_2)=R^{23}_{23}(x_2)R^{12}_{13}(x_1)\sigma^{23}. \tag{4.27}$$

That is, $(R_{12}(x),R_{23}(x),\sigma)$, $(R_{12}(x),\sigma,R_{13}(x))$ and $(\sigma,R_{23}(x),R_{13}(x))$ are compatible, respectively.

If we take one of the compatible twisting operators $R_{12}(x)$, $R_{24}(x)$, $R_{14}(x)$ as a usual flip, respectively, then we have

$$R^{12}_{12}(x_1)\sigma^{23}R^{12}_{24}(x_2)=R^{23}_{24}(x_2)\sigma^{12}R^{23}_{12}(x_1), \tag{4.28}$$
$$R^{12}_{12}(x_1)R^{23}_{14}(x_2)\sigma^{12}=\sigma^{23}R^{12}_{14}(x_2)R^{23}_{12}(x_1), \tag{4.29}$$
$$\sigma^{12}R^{23}_{14}(x_1)R^{12}_{24}(x_2)=R^{23}_{24}(x_2)R^{12}_{14}(x_1)\sigma^{23}. \tag{4.30}$$

That is, $(R_{12}(x), R_{24}(x), \sigma)$, $(R_{12}(x), \sigma, R_{14}(x))$ and $(\sigma, R_{24}(x), R_{14}(x))$ are compatible, respectively.

If we take one of the compatible twisting operators $R_{13}(x)$, $R_{34}(x)$, $R_{14}(x)$ as a usual flip, respectively, then we have

$$R_{13}^{12}(x_1)\sigma^{23}R_{34}^{12}(x_2) = R_{34}^{23}(x_2)\sigma^{12}R_{13}^{23}(x_1), \tag{4.31}$$

$$R_{13}^{12}(x_1)R_{14}^{23}(x_2)\sigma^{12} = \sigma^{23}R_{14}^{12}(x_2)R_{13}^{23}(x_1), \tag{4.32}$$

$$\sigma^{12}R_{14}^{23}(x_1)R_{34}^{12}(x_2) = R_{34}^{23}(x_2)R_{14}^{12}(x_1)\sigma^{23}. \tag{4.33}$$

That is, $(R_{13}(x), R_{34}(x), \sigma)$, $(R_{13}(x), \sigma, R_{14}(x))$ and $(\sigma, R_{34}(x), R_{14}(x))$ are compatible, respectively.

If we take one of the compatible twisting operators $R_{23}(x)$, $R_{34}(x)$, $R_{24}(x)$ as a usual flip, respectively, then we have

$$R_{23}^{12}(x_1)\sigma^{23}R_{34}^{12}(x_2) = R_{34}^{23}(x_2)\sigma^{12}R_{23}^{23}(x_1), \tag{4.34}$$

$$R_{23}^{12}(x_1)R_{24}^{23}(x_2)\sigma^{12} = \sigma^{23}R_{24}^{12}(x_2)R_{23}^{23}(x_1), \tag{4.35}$$

$$\sigma^{12}R_{24}^{23}(x_1)R_{34}^{12}(x_2) = R_{34}^{23}(x_2)R_{24}^{12}(x_1)\sigma^{23}. \tag{4.36}$$

That is, $(R_{23}(x), R_{34}(x), \sigma)$, $(R_{23}(x), \sigma, R_{24}(x))$ and $(\sigma, R_{34}(x), R_{24}(x))$ are compatible, respectively.

We need to prove these conditions (4.25)–(4.36) together are equivalent to

$$\begin{aligned} T(x) = \sigma^{45}\left(\sigma^{34}\sigma^{56}\right)\left(\sigma^{23}\sigma^{45}\sigma^{67}\right)\left(R_{12}^{23}(-x)R_{23}^{45}(-x)R_{34}^{67}(-x)\right) \\ \cdot \left(R_{13}^{34}(-x)R_{24}^{56}(-x)\right)R_{14}^{45}(-x) \end{aligned}$$

is a twistor.

For the "only if" part, we shall use the hypothesis of the theorem for $V_1 \otimes_{R_{12}} V_2 \otimes_{T_{34}^2} (V_3 \otimes V_4)$. And we have the following twisting operators:

$$\begin{aligned} &R_{12}(x) : V_2 \otimes V_1 \to V_1 \otimes V_2 \otimes \mathbb{C}((x)), \\ &T_{34}^2(x) = R_{23}^{12}(x)R_{24}^{23}(x) : (V_3 \otimes V_4) \otimes V_2 \to V_2 \otimes (V_3 \otimes V_4) \otimes \mathbb{C}((x)), \\ &T_{34}^1(x) = R_{13}^{12}(x)R_{14}^{23}(x) : (V_3 \otimes V_4) \otimes V_1 \to V_1 \otimes (V_3 \otimes V_4) \otimes \mathbb{C}((x)). \end{aligned}$$

If we take one of the twisting operators $R_{12}(x)$, $T_{34}^2(x)$, $T_{34}^1(x)$ as a usual flip, respectively, then we need to prove

$$\begin{aligned} &R_{12}^{12}(x_1)(\sigma^{23}\sigma^{34})(T_{34}^2)^{12}(x_2) = (T_{34}^2)^{23}(x_2)(\sigma^{12}\sigma^{23})R_{12}^{34}(x_1), \\ &R_{12}^{12}(x_1)(T_{34}^1)^{23}(x_2)(\sigma^{12}\sigma^{23}) = (\sigma^{23}\sigma^{34})(T_{34}^1)^{12}(x_2)R_{12}^{34}(x_1), \\ &\sigma^{12}(T_{34}^1)^{23}(x_1)(T_{34}^2)^{12}(x_2) = (T_{34}^2)^{23}(x_2)(T_{34}^1)^{12}(x_1)\sigma^{23}. \end{aligned}$$

That is, $(R_{12}(x), T^2_{34}(x) = R^{12}_{23}(x)R^{23}_{24}(x), \sigma^{12}\sigma^{23})$, $(R_{12}(x), \sigma^{12}\sigma^{23}, T^1_{34}(x) = R^{12}_{13}(x) \times R^{23}_{14}(x))$ and $(\sigma, T^2_{34}(x) = R^{12}_{23}(x)R^{23}_{24}(x), T^1_{34}(x) = R^{12}_{13}(x)R^{23}_{14}(x))$ are compatible, respectively. These are also equivalent to:

$$R^{12}_{12}(x_1)(\sigma^{23}\sigma^{34})(R^{12}_{23}(x_2)R^{23}_{24}(x_2)) = (R^{23}_{23}(x_2)R^{34}_{24}(x_2))(\sigma^{12}\sigma^{23})R^{34}_{12}(x_1), \tag{4.37}$$

$$R^{12}_{12}(x_1)(R^{23}_{13}(x_2)R^{34}_{14}(x_2))(\sigma^{12}\sigma^{23}) = (\sigma^{23}\sigma^{34})(R^{12}_{13}(x_2)R^{23}_{14}(x_2))R^{34}_{12}(x_1), \tag{4.38}$$

$$\begin{aligned}&\sigma^{12}(R^{23}_{13}(x_1)R^{34}_{14}(x_1))(R^{12}_{23}(x_2)R^{23}_{24}(x_2)) \\ &\quad = (R^{23}_{23}(x_2)R^{34}_{24}(x_2))(R^{12}_{13}(x_1)R^{23}_{14}(x_1))\sigma^{23}.\end{aligned} \tag{4.39}$$

For (4.37), from (4.25) and (4.28), we have

$$\begin{aligned}R^{12}_{12}(x_1)(\sigma^{23}\sigma^{34})(R^{12}_{23}(x_2)R^{23}_{24}(x_2)) &= R^{12}_{12}(x_1)\sigma^{23}R^{12}_{23}(x_2)\sigma^{34}R^{23}_{24}(x_2) \\ &= R^{23}_{23}(x_2)\sigma^{12}R^{23}_{12}(x_1)\sigma^{34}R^{23}_{24}(x_2) \\ &= R^{23}_{23}(x_2)\sigma^{12}R^{34}_{24}(x_2)\sigma^{23}R^{34}_{12}(x_1) \\ &= (R^{23}_{23}(x_2)R^{34}_{24}(x_2))(\sigma^{12}\sigma^{23})R^{34}_{12}(x_1).\end{aligned}$$

Concerning (4.38), by (4.26) and (4.29), we get

$$\begin{aligned}R^{12}_{12}(x_1)(R^{23}_{13}(x_2)R^{34}_{14}(x_2))(\sigma^{12}\sigma^{23}) &= \sigma^{23}R^{12}_{13}(x_2)R^{23}_{12}(x_1)\sigma^{12}R^{34}_{14}(x_2)\sigma^{12}\sigma^{23} \\ &= \sigma^{23}R^{12}_{13}(x_2)R^{23}_{12}(x_1)R^{34}_{14}(x_2)\sigma^{23} \\ &= \sigma^{23}R^{12}_{13}(x_2)\sigma^{34}R^{23}_{14}(x_2)R^{34}_{12}(x_1).\end{aligned}$$

For (4.39), using (4.27) and (4.30), we obtain

$$\begin{aligned}\sigma^{12}(R^{23}_{13}(x_1)R^{34}_{14}(x_1))(R^{12}_{23}(x_2)R^{23}_{24}(x_2)) &= \sigma^{12}R^{23}_{13}(x_1)R^{12}_{23}(x_2)R^{34}_{14}(x_1)R^{23}_{24}(x_2) \\ &= R^{23}_{23}(x_2)R^{12}_{13}(x_1)\sigma^{23}R^{34}_{14}(x_1)R^{23}_{24}(x_2) \\ &= R^{23}_{23}(x_2)R^{12}_{13}(x_1)R^{34}_{24}(x_2)R^{23}_{14}(x_1)\sigma^{34} \\ &= (R^{23}_{23}(x_2)R^{34}_{24}(x_2))(R^{12}_{13}(x_1)R^{23}_{14}(x_1))\sigma^{34}.\end{aligned}$$

With the induction hypothesis, these conditions (4.37)–(4.39) hold if and only if

$$\begin{aligned}T_2(x) &= (\sigma^{45}\sigma^{34})(\sigma^{56}\sigma^{45})\sigma^{23}R^{23}_{12}(-x)(T^2_{34})^{45}(-x)(T^1_{34})^{34}(-x) \\ &= (\sigma^{45}\sigma^{34})(\sigma^{56}\sigma^{45})\sigma^{23}R^{23}_{12}(-x)(R^{45}_{23}(-x)R^{56}_{24}(-x))(R^{34}_{13}(-x)R^{45}_{14}(-x))\end{aligned}$$

is a twistor for $V_1 \otimes_{R_{12}} V_2 \otimes_{T^2_{34}} (V_3 \otimes V_4)$.

We also have another twistor $T_1(x)$ for $V_1 \otimes V_2 \otimes (V_3 \otimes_{R_{34}} V_4)$:

$$\begin{aligned}T_1(x) &= \sigma^{45}(\sigma^{34}\sigma^{56})(\sigma^{23}\sigma^{45}\sigma^{67})(\sigma^{23}\sigma^{45}R^{67}_{34}(-x))(\sigma^{34}\sigma^{56}\sigma^{45}) \\ &= \sigma^{45}\sigma^{56}\sigma^{67}R^{67}_{34}(-x)\sigma^{56}\sigma^{45}.\end{aligned}$$

By (4.31) and (4.34), we have

$$
\begin{aligned}
T_1^{12}(x_1)T_2^{23}(x_2) &= \sigma^{45}\sigma^{56}\sigma^{67}R_{34}^{67}(-x_1)\sigma^{56}\sigma^{45} \\
&\quad \cdot \sigma^{89}\sigma^{78}\sigma^{9,10}\sigma^{89}\sigma^{67}R_{12}^{67}(-x_2)R_{23}^{89}(-x_2)R_{24}^{9,10}(-x_2)R_{13}^{78}(-x_2)R_{14}^{89}(-x_2) \\
&= \sigma^{89}\sigma^{78}\sigma^{9,10}\sigma^{89}\sigma^{67}R_{12}^{67}(-x_2)R_{23}^{89}(-x_2)R_{24}^{9,10}(-x_2)R_{13}^{78}(-x_2)R_{14}^{89}(-x_2) \\
&\quad \cdot \sigma^{45}\sigma^{56}\sigma^{67}R_{34}^{67}(-x_1)\sigma^{56}\sigma^{45} \\
&= T_2^{23}(x_2)T_1^{12}(x_1).
\end{aligned}
$$

Using (4.33) and (4.36), we get

$$
\begin{aligned}
T_1^{12}(x_1)T_2^{13}(x_2) &= \sigma^{45}\sigma^{56}\sigma^{67}R_{34}^{67}(-x_1)\sigma^{56}\sigma^{45} \\
&\quad \cdot \sigma^{49}\sigma^{34}\sigma^{9,10}\sigma^{49}\sigma^{23}R_{12}^{23}(-x_2)R_{23}^{49}(-x_2)R_{24}^{9,10}(-x_2)R_{13}^{34}(-x_2)R_{14}^{49}(-x_2) \\
&= \sigma^{49}\sigma^{34}\sigma^{9,10}\sigma^{49}\sigma^{23}R_{12}^{23}(-x_2)R_{23}^{49}(-x_2)R_{24}^{9,10}(-x_2)R_{13}^{34}(-x_2)R_{14}^{49}(-x_2) \\
&\quad \cdot \sigma^{45}\sigma^{56}\sigma^{67}R_{34}^{67}(-x_1)\sigma^{56}\sigma^{45} \\
&= T_2^{13}(x_2)T_1^{12}(x_1).
\end{aligned}
$$

Furthermore, we can easily have

$$
\begin{aligned}
T_1^{23}(x_1)T_2^{12}(x_2) &= \sigma^{89}\sigma^{9,10}\sigma^{10,11}R_{34}^{10,11}(-x_1)\sigma^{9,10}\sigma^{89} \\
&\quad \cdot \sigma^{45}\sigma^{34}\sigma^{56}\sigma^{45}\sigma^{23}R_{12}^{23}(-x_2)R_{23}^{45}(-x_2)R_{24}^{56}(-x_2)R_{13}^{34}(-x_2)R_{14}^{45}(-x_2) \\
&= \sigma^{45}\sigma^{34}\sigma^{56}\sigma^{45}\sigma^{23}R_{12}^{23}(-x_2)R_{23}^{45}(-x_2)R_{24}^{56}(-x_2)R_{13}^{34}(-x_2)R_{14}^{45}(-x_2) \\
&\quad \cdot \sigma^{89}\sigma^{9,10}\sigma^{10,11}R_{34}^{10,11}(-x_1)\sigma^{9,10}\sigma^{89} \\
&= T_2^{12}(x_2)T_1^{23}(x_1),
\end{aligned}
$$

and

$$
\begin{aligned}
T_1^{23}(x_1)T_2^{13}(x_2) &= \sigma^{89}\sigma^{9,10}\sigma^{10,11}R_{34}^{10,11}(-x_1)\sigma^{9,10}\sigma^{89} \\
&\quad \cdot \sigma^{49}\sigma^{34}\sigma^{9,10}\sigma^{49}\sigma^{23}R_{12}^{23}(-x_2)R_{23}^{49}(-x_2)R_{24}^{9,10}(-x_2)R_{13}^{34}(-x_2)R_{14}^{49}(-x_2) \\
&= \sigma^{49}\sigma^{34}\sigma^{9,10}\sigma^{49}\sigma^{23}R_{12}^{23}(-x_2)R_{23}^{49}(-x_2)R_{24}^{9,10}(-x_2)R_{13}^{34}(-x_2)R_{14}^{49}(-x_2) \\
&\quad \cdot \sigma^{89}\sigma^{9,10}\sigma^{10,11}R_{34}^{10,11}(-x_1)\sigma^{9,10}\sigma^{89} \\
&= T_2^{13}(x_2)T_1^{23}(x_1).
\end{aligned}
$$

From the second result in Proposition 3.5, we know that $T(x) = T_1(x)T_2(x)$ is a twistor for $V_1 \otimes V_2 \otimes V_3 \otimes V_4$.

To see the "if" part, given a twistor $T(x)$ for $V_1 \otimes_{R_{12}} V_2 \otimes_{T^2_{34}} (V_3 \otimes V_4)$, we have:

$$\begin{aligned}
&T(x_1)(1_{V_1\otimes V_2\otimes V_3\otimes V_4} \otimes Y_{V_1\otimes V_2\otimes V_3\otimes V_4}(x_2)) \\
&\qquad = (1_{V_1\otimes V_2\otimes V_3\otimes V_4} \otimes Y_{V_1\otimes V_2\otimes V_3\otimes V_4}(x_2))T^{13}(x_1)T^{12}(x_1 - x_2), \qquad (4.40)\\
&T(x_1)(Y_{V_1\otimes V_2\otimes V_3\otimes V_4}(x_2) \otimes 1_{V_1\otimes V_2\otimes V_3\otimes V_4}) \\
&\qquad = (Y_{V_1\otimes V_2\otimes V_3\otimes V_4}(x_2) \otimes 1_{V_1\otimes V_2\otimes V_3\otimes V_4})T^{13}(x_1 + x_2)T^{23}(x_1), \qquad (4.41)\\
&T^{12}(x_1)T^{23}(x_2) = T^{23}(x_2)T^{12}(x_1). \qquad (4.42)
\end{aligned}$$

From (4.40), for $v_1'' \in V_1$, $v_2', v_2'' \in V_2$, $v_3, v_3' \in V_3$, $v_4 \in V_4$, taking $\mathbf{1}\otimes\mathbf{1}\otimes v_3\otimes\mathbf{1}\otimes\mathbf{1}\otimes v_2'\otimes\mathbf{1}\otimes\mathbf{1}\otimes v_1''\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}$, $\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\otimes v_4\otimes\mathbf{1}\otimes v_2'\otimes\mathbf{1}\otimes\mathbf{1}\otimes v_1''\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}$, $\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\otimes v_4\otimes\mathbf{1}\otimes\mathbf{1}\otimes v_3'\otimes\mathbf{1}\otimes v_1''\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}$ and $\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\otimes v_4\otimes\mathbf{1}\otimes\mathbf{1}\otimes v_3'\otimes\mathbf{1}\otimes\mathbf{1}\otimes v_2''\otimes\mathbf{1}\otimes\mathbf{1}$, respectively, we can have (4.27), (4.30), (4.33) and (4.36), correspondingly. Similarly, using (4.41), taking the same elements, we get (4.26), (4.29), (4.32) and (4.35), correspondingly. Finally, by (4.42), taking the above elements, we get (4.25), (4.28), (4.31) and (4.34), respectively.

For $1 \le i < j < k \le 4$, from (4.25)–(4.36), we have

$$\begin{aligned}
R^{23}_{jk}(x_1 - x_2)R^{12}_{ik}(x_1)R^{23}_{ij}(x_2) &= \sigma^{12}R^{23}_{ik}(x_1)R^{12}_{jk}(x_1 - x_2)\sigma^{23}R^{23}_{ij}(x_2)\\
&= \sigma^{12}R^{23}_{ik}(x_1)\sigma^{13}R^{13}_{ij}(x_2)R^{12}_{jk}(x_1 - x_2)\\
&= R^{12}_{ij}(x_2)R^{23}_{ik}(x_1)R^{12}_{jk}(x_1 - x_2).
\end{aligned}$$

That is, $R_{ij}(x)$, $R_{jk}(x)$ and $R_{ik}(x)$ are compatible. It is easy to see that

$$\begin{aligned}
Y_T(x) &= Y_{V_1\otimes V_2\otimes V_3\otimes V_4}(x)T(x)\\
&= (Y(x)\otimes Y(x)\otimes Y(x)\otimes Y(x))\sigma^{23}\sigma^{45}\sigma^{67}\sigma^{34}\sigma^{56}\sigma^{45}\sigma^{45}\sigma^{56}\sigma^{34}\sigma^{67}\sigma^{45}\sigma^{23}\\
&\quad\cdot R^{23}_{12}(-x)R^{45}_{23}(-x)R^{67}_{34}(-x)R^{34}_{13}(-x)R^{56}_{24}(-x)R^{45}_{14}(-x)\\
&= (Y(x)\otimes Y(x)\otimes Y(x)\otimes Y(x))R^{23}_{12}(-x)R^{45}_{23}(-x)R^{67}_{34}(-x)\\
&\quad\cdot R^{34}_{13}(-x)R^{56}_{24}(-x)R^{45}_{14}(-x).
\end{aligned}$$

That is, $K^T = V_1 \otimes_{R_{12}} V_2 \otimes_{R_{23}} V_3 \otimes_{R_{34}} V_4$. This concludes the proof. □

4.5. L-R-twisted tensor product nonlocal vertex algebras

Now we come to L-R-twisted tensor product nonlocal vertex algebras. Firstly, we recall the notion of an L-twisting operator, which was formulated and studied in [18].

Definition 4.19. Let U and V be nonlocal vertex algebras. An *L-twisting operator* for the ordered pair (U, V) is a linear map

$$Q(x):\ U\otimes V \to U\otimes V\otimes \mathbb{C}((x)),$$

satisfying the following conditions:

$$Q(x)(u \otimes \mathbf{1}) = u \otimes \mathbf{1} \quad \text{for } u \in U, \tag{4.43}$$

$$Q(x)(\mathbf{1} \otimes v) = \mathbf{1} \otimes v \quad \text{for } v \in V, \tag{4.44}$$

$$Q(x_1)(1 \otimes Y(x_2)) = (1 \otimes Y(x_2))Q^{13}(x_1)Q^{12}(x_1 - x_2), \tag{4.45}$$

$$Q(x_1)(Y(x_2) \otimes 1) = (Y(x_2) \otimes 1)Q^{13}(x_1 + x_2)Q^{23}(x_1). \tag{4.46}$$

Remark 4.20. Note that if we take $U = V$ in the Definition 4.19, we know that an L-twisting operator satisfies all the conditions in the definition of a twistor except (3.5).

Now we present the L-R-twisted tensor products from [18].

Theorem 4.21. *Let U, V be nonlocal vertex algebras, let $R(x)$ be a twisting operator and let $Q(x)$ be an L-twisting operator of the ordered pair (U, V), respectively, satisfying the following conditions:*

$$R^{12}(x_1)Q^{23}(x_2) = Q^{13}(x_2)R^{12}(x_1), \tag{4.47}$$

$$R^{23}(x_1)Q^{12}(x_2) = Q^{13}(x_2)R^{23}(x_1). \tag{4.48}$$

Set

$${}_QY_R(x) = (Y(x) \otimes Y(x))Q^{14}(x)R^{23}(-x). \tag{4.49}$$

Then $(U \otimes V, {}_QY_R(x), \mathbf{1} \otimes \mathbf{1})$ carries the structure of a nonlocal vertex algebra, which contains U and V canonically as nonlocal vertex subalgebras.

Next, we study the relations among a twisting operator, an L-twisting operator and a twistor in the following lemmas. Firstly, the following result enables us to get a twistor from an L-R-twisted tensor product.

Lemma 4.22. *Let $U\,{}_Q\!\otimes_R V$ be an L-R-twisted tensor product of nonlocal vertex algebras U and V. Then ${}_QT_R(x) = Q^{14}(x)\sigma^{23}R^{23}(-x)$ is a twistor for nonlocal vertex algebra $U \otimes V$.*

Proof. For $u \in U$, $v \in V$, by (4.7) and (4.44), we have

$$\begin{aligned}
{}_QT_R(x)(\mathbf{1} \otimes \mathbf{1} \otimes u \otimes v) &= Q^{14}(x)\sigma^{23}R^{23}(-x)(\mathbf{1} \otimes \mathbf{1} \otimes u \otimes v) \\
&= Q^{14}(x)\sigma^{23}(\mathbf{1} \otimes u \otimes \mathbf{1} \otimes v) \\
&= \mathbf{1} \otimes \mathbf{1} \otimes u \otimes v,
\end{aligned}$$

and using (4.6) and (4.43), we get

$$\begin{aligned} {}_QT_R(x)(u\otimes v\otimes \mathbf{1}\otimes \mathbf{1}) &= Q^{14}(x)\sigma^{23}R^{23}(-x)(u\otimes v\otimes \mathbf{1}\otimes \mathbf{1}) \\ &= Q^{14}(x)\sigma^{23}(u\otimes \mathbf{1}\otimes v\otimes \mathbf{1}) \\ &= u\otimes v\otimes \mathbf{1}\otimes \mathbf{1}. \end{aligned}$$

From (4.8), (4.11) and (4.45), we have

$$\begin{aligned} &{}_QT_R(x_1)(1_{U\otimes V}\otimes Y_{U\otimes V}(x_2)) \\ &= {}_QT_R(x_1)(1\otimes 1\otimes Y(x_2)\otimes Y(x_2))\sigma^{45} \\ &= Q^{14}(x_1)\sigma^{23}R^{23}(-x_1)(1\otimes 1\otimes Y(x_2)\otimes Y(x_2))\sigma^{45} \\ &= Q^{14}(x_1)\sigma^{23}(1\otimes Y(x_2)\otimes 1\otimes Y(x_2))R^{34}(-x_1)R^{23}(-x_1+x_2)\sigma^{45} \\ &= Q^{14}(x_1)(1\otimes 1\otimes Y(x_2)\otimes Y(x_2))\sigma^{23}\sigma^{34}R^{34}(-x_1)R^{23}(-x_1+x_2)\sigma^{45} \\ &= (1\otimes 1\otimes Y(x_2)\otimes Y(x_2))Q^{16}(x_1)Q^{15}(x_1-x_2)\sigma^{23}\sigma^{34}R^{34}(-x_1)R^{23}(-x_1+x_2)\sigma^{45} \\ &= (1\otimes 1\otimes Y(x_2)\otimes Y(x_2))Q^{16}(x_1)Q^{15}(x_1-x_2)\sigma^{23}\sigma^{34}R^{34}(-x_1)\sigma^{45}R^{23}(-x_1+x_2), \end{aligned}$$

and on the other hand,

$$\begin{aligned} &(1_{U\otimes V}\otimes Y_{U\otimes V}(x_2))\,{}_QT_R^{13}(x_1)\,{}_QT_R^{12}(x_1-x_2) \\ &= (1\otimes 1\otimes Y(x_2)\otimes Y(x_2))\sigma^{45}Q^{16}(x_1)\sigma^{25}R^{25}(-x_1)Q^{14}(x_1-x_2)\sigma^{23}R^{23}(-x_1+x_2) \\ &= (1\otimes 1\otimes Y(x_2)\otimes Y(x_2))Q^{16}(x_1)\sigma^{45}\sigma^{25}R^{25}(-x_1)Q^{14}(x_1-x_2)\sigma^{23}R^{23}(-x_1+x_2). \end{aligned}$$

Then we get

$${}_QT_R(x_1)(1_{U\otimes V}\otimes Y_{U\otimes V}(x_2)) = (1_{U\otimes V}\otimes Y_{U\otimes V}(x_2))\,{}_QT_R^{13}(x_1)\,{}_QT_R^{12}(x_1-x_2).$$

Similarly, by (4.9), (4.10) and (4.46), we have

$$\begin{aligned} &{}_QT_R(x_1)(Y_{U\otimes V}(x_2)\otimes 1_{U\otimes V}) \\ &= Q^{14}(x_1)\sigma^{23}R^{23}(-x_1)(Y(x_2)\otimes Y(x_2)\otimes 1\otimes 1)\sigma^{23} \\ &= Q^{14}(x_1)\sigma^{23}(Y(x_2)\otimes 1\otimes Y(x_2)\otimes 1)R^{34}(-x_1-x_2)R^{45}(-x_1)\sigma^{23} \\ &= Q^{14}(x_1)(Y(x_2)\otimes Y(x_2)\otimes 1\otimes 1)\sigma^{45}\sigma^{34}R^{34}(-x_1-x_2)R^{45}(-x_1)\sigma^{23} \\ &= (Y(x_2)\otimes Y(x_2)\otimes 1\otimes 1)Q^{16}(x_1+x_2)Q^{26}(x_1)\sigma^{45}\sigma^{34}R^{34}(-x_1-x_2)R^{45}(-x_1)\sigma^{23} \\ &= (Y(x_2)\otimes Y(x_2)\otimes 1\otimes 1)Q^{16}(x_1+x_2)Q^{26}(x_1)\sigma^{45}\sigma^{34}R^{34}(-x_1-x_2)\sigma^{23}R^{45}(-x_1), \end{aligned}$$

and on the other hand,

$$\begin{aligned} &(Y_{U\otimes V}(x_2)\otimes 1_{U\otimes V})\,{}_QT_R^{13}(x_1+x_2)\,{}_QT_R^{23}(x_1) \\ &= (Y(x_2)\otimes Y(x_2)\otimes 1\otimes 1)\sigma^{23}Q^{16}(x_1+x_2)\sigma^{25}R^{25}(-x_1-x_2)Q^{36}(x_1)\sigma^{45}R^{45}(-x_1) \\ &= (Y(x_2)\otimes Y(x_2)\otimes 1\otimes 1)Q^{16}(x_1+x_2)\sigma^{23}\sigma^{25}R^{25}(-x_1-x_2)Q^{36}(x_1)\sigma^{45}R^{45}(-x_1). \end{aligned}$$

Then we get

$$_QT_R(x_1)(Y_{U\otimes V}(x_2)\otimes 1_{U\otimes V}) = (Y_{U\otimes V}(x_2)\otimes 1_{U\otimes V})_QT_R^{13}(x_1+x_2)_QT_R^{23}(x_1).$$

Using (4.47), we get

$$\begin{aligned}_QT_R^{12}(x_1)_QT_R^{23}(x_2) &= Q^{14}(x_1)\sigma^{23}R^{23}(-x_1)Q^{36}(x_2)\sigma^{45}R^{45}(-x_2)\\ &= Q^{14}(x_1)\sigma^{23}Q^{26}(x_2)R^{23}(-x_1)\sigma^{45}R^{45}(-x_2)\\ &= \sigma^{23}Q^{14}(x_1)Q^{26}(x_2)\sigma^{45}R^{23}(-x_1)R^{45}(-x_2),\end{aligned}$$

and from (4.48), we have

$$\begin{aligned}_QT_R^{23}(x_2)_QT_R^{12}(x_1) &= Q^{36}(x_2)\sigma^{45}R^{45}(-x_2)Q^{14}(x_1)\sigma^{23}R^{23}(-x_1)\\ &= \sigma^{45}Q^{36}(x_2)Q^{15}(x_1)R^{45}(-x_2)\sigma^{23}R^{23}(-x_1)\\ &= \sigma^{45}Q^{36}(x_2)Q^{15}(x_1)\sigma^{23}R^{23}(-x_1)R^{45}(-x_2).\end{aligned}$$

By the direct calculation, we finally have:

$$_QT_R^{12}(x_1)_QT_R^{23}(x_2) = {}_QT_R^{23}(x_2)_QT_R^{12}(x_1).$$

This concludes the proof. □

Conversely, we have:

Lemma 4.23. *Let $R(x): V\otimes U \to U\otimes V\otimes \mathbb{C}((x))$ be a twisting operator, $Q(x): U\otimes V \to U\otimes V\otimes \mathbb{C}((x))$ be a linear map and let $_QT_R(x)$ be a twistor for $U\otimes V$ defined by $_QT_R(x) = Q^{14}(x)\sigma^{23}R^{23}(-x)$. Then $Q(x)$ is an L-twisting operator for the ordered pair (U, V) and $U\ {}_Q\!\otimes_R V = (U\otimes V)^{_QT_R}$.*

Proof. For $u\in U$, by (3.2), we have

$$\begin{aligned}u\otimes \mathbf{1}\otimes \mathbf{1}\otimes \mathbf{1} &= {}_QT_R(x)(u\otimes \mathbf{1}\otimes \mathbf{1}\otimes \mathbf{1})\\ &= Q^{14}(x)\sigma^{23}R^{23}(-x)(u\otimes \mathbf{1}\otimes \mathbf{1}\otimes \mathbf{1})\\ &= Q^{14}(x)(u\otimes \mathbf{1}\otimes \mathbf{1}\otimes \mathbf{1}),\end{aligned}$$

then $Q(x)(u\otimes \mathbf{1}) = u\otimes \mathbf{1}$. Similarly, for $v\in V$, using (3.1), we get

$$\begin{aligned}\mathbf{1}\otimes \mathbf{1}\otimes \mathbf{1}\otimes v &= {}_QT_R(x)(\mathbf{1}\otimes \mathbf{1}\otimes \mathbf{1}\otimes v)\\ &= Q^{14}(x)\sigma^{23}R^{23}(-x)(\mathbf{1}\otimes \mathbf{1}\otimes \mathbf{1}\otimes v)\\ &= Q^{14}(\mathbf{1}\otimes \mathbf{1}\otimes \mathbf{1}\otimes v),\end{aligned}$$

then $Q(x)(\mathbf{1}\otimes v) = \mathbf{1}\otimes v$.

For $u \in U$, $v', v'' \in V$, by (3.3), we have

$$\begin{aligned}
&{}_QT_R(x_1)(1_{U\otimes V} \otimes Y_{U\otimes V}(x_2))(u \otimes \mathbf{1} \otimes \mathbf{1} \otimes v' \otimes \mathbf{1} \otimes v'') \\
&= Q^{14}(x)\sigma^{23}R^{23}(-x)(1 \otimes 1 \otimes Y(x_2) \otimes Y(x_2))\sigma^{45}(u \otimes \mathbf{1} \otimes \mathbf{1} \otimes v' \otimes \mathbf{1} \otimes v'') \\
&= Q^{14}(x)(1 \otimes 1 \otimes 1 \otimes Y(x))(u \otimes \mathbf{1} \otimes \mathbf{1} \otimes v' \otimes v''),
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
&(1_{U\otimes V} \otimes Y_{U\otimes V}(x_2))\,{}_QT_R^{13}(x_1)\,{}_QT_R^{12}(x_1 - x_2)(u \otimes \mathbf{1} \otimes \mathbf{1} \otimes v' \otimes \mathbf{1} \otimes v'') \\
&= (1 \otimes 1 \otimes Y(x_2) \otimes Y(x_2))\sigma^{45}Q^{16}(x_1)\sigma^{25}R^{25}(-x_1) \\
&\quad \cdot Q^{14}(x_1 - x_2)\sigma^{23}R^{23}(-x_1 + x_2)(u \otimes \mathbf{1} \otimes \mathbf{1} \otimes v' \otimes \mathbf{1} \otimes v'') \\
&= (1 \otimes 1 \otimes Y(x_2) \otimes Y(x_2))Q^{16}(x_1)Q^{15}(x_1 - x_2)(u \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes v' \otimes v'') \\
&= (1 \otimes 1 \otimes Y(x_2) \otimes Y(x_2))Q^{15}(x_1)Q^{14}(x_1 - x_2)(u \otimes \mathbf{1} \otimes \mathbf{1} \otimes v' \otimes v''),
\end{aligned}$$

then

$$Q(x_1)(1 \otimes Y(x_2)) = (1 \otimes Y(x_2))Q^{13}(x_1)Q^{12}(x_1 - x_2).$$

For $u, u' \in U$, $v'' \in V$, from (3.4), we have

$$\begin{aligned}
&{}_QT_R(x_1)(Y_{U\otimes V}(x_2) \otimes 1_{U\otimes V})(u \otimes \mathbf{1} \otimes u' \otimes \mathbf{1} \otimes \mathbf{1} \otimes v'') \\
&= Q^{14}(x_1)\sigma^{23}R^{23}(-x_1)(Y(x_2) \otimes Y(x_2) \otimes 1 \otimes 1)\sigma^{23}(u \otimes \mathbf{1} \otimes u' \otimes \mathbf{1} \otimes \mathbf{1} \otimes v'') \\
&= Q^{14}(x_1)(Y(x_2) \otimes 1 \otimes 1 \otimes 1)(u \otimes u' \otimes \mathbf{1} \otimes \mathbf{1} \otimes v''),
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
&(Y_{U\otimes V}(x_2) \otimes 1_{U\otimes V})\,{}_QT_R^{13}(x_1 + x_2)\,{}_QT_R^{23}(x_1)(u \otimes \mathbf{1} \otimes u' \otimes \mathbf{1} \otimes \mathbf{1} \otimes v'') \\
&= (Y(x_2) \otimes Y(x_2) \otimes 1 \otimes 1)\sigma^{23}Q^{16}(x_1 + x_2)\sigma^{25}R^{25}(-x_1 - x_2) \\
&\quad \cdot Q^{36}(x_1)\sigma^{45}R^{45}(-x_1)(u \otimes \mathbf{1} \otimes u' \otimes \mathbf{1} \otimes \mathbf{1} \otimes v'') \\
&= (Y(x_2) \otimes Y(x_2) \otimes 1 \otimes 1)Q^{16}(x_1 + x_2)Q^{26}(x_1)(u \otimes u' \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes v'') \\
&= (Y(x_2) \otimes 1 \otimes 1 \otimes 1)Q^{15}(x_1 + x_2)Q^{25}(x_1)(u \otimes u' \otimes \mathbf{1} \otimes \mathbf{1} \otimes v'').
\end{aligned}$$

Thus

$$Q(x_1)(Y(x_2) \otimes 1) = (Y(x_2) \otimes 1)Q^{13}(x_1 + x_2)Q^{23}(x_1).$$

This concludes the proof. □

Remark 4.24. For $u \in U$, $v \in V$, we just used $R(x)(v \otimes \mathbf{1}) = \mathbf{1} \otimes v$ and $R(x)(u \otimes \mathbf{1}) = \mathbf{1} \otimes u$ in the proof of Lemma 4.23.

Similarly, we also have:

Lemma 4.25. *Let $R(x) : V \otimes U \to U \otimes V \otimes \mathbb{C}((x))$ be a linear map, $Q(x) : U \otimes V \to U \otimes V \otimes \mathbb{C}((x))$ be an L-twisting operator and let ${}_QT_R(x)$ be a twistor for $U \otimes V$ defined by ${}_QT_R(x) = Q^{14}(x)\sigma^{23}R^{23}(-x)$. Then $R(x)$ is a twisting operator for the ordered pair (U, V) and $U\ {}_Q\otimes_R V = (U \otimes V)^{{}_QT_R}$.*

4.6. The noncommutative 2l-planes

In this subsection we present a concrete example. It is shown that the nonlocal vertex algebras $V_{\mathbf{Q}}$ associated with the noncommutative $2l$-planes defined by Connes and Dubois-Violette in [3] could be realized as nonlocal vertex algebras constructed from $V_{11} \otimes V_{22} \otimes \cdots \otimes V_{ll}$ by a suitable twistor $T(x)$. And this nonlocal vertex algebra is very similar to the quantum vertex algebra of Zamolodchikov–Faddev type studied in [7].

Definition 4.26. Let l be a positive integer and let $\mathbf{Q} = (q_{ij})_{i,j=1}^{l}$ be a complex matrix such that

$$q_{ii} = q_{ij}q_{ji} = 1 \quad \text{for } 1 \le i,\ j \le l. \tag{4.50}$$

Define $\mathcal{A}_{\mathbf{Q}}$ to be the associative algebra with identity (over $\mathbb{C}$) with generators

$$X_{i,n}, \quad Y_{i,n} \quad (i = 1, \dots, l,\ n \in \mathbb{Z}),$$

subject to relations

$$X_{i,m}X_{j,n} = q_{ij}X_{j,n}X_{i,m}, \quad Y_{i,m}Y_{j,n} = q_{ij}Y_{j,n}Y_{i,m}, \quad X_{i,m}Y_{j,n} = q_{ji}Y_{j,n}X_{i,m} \tag{4.51}$$

for $i, j = 1, \dots, l$, $m, n \in \mathbb{Z}$.

Let $\{e_1, \dots, e_l\}$ denote the standard $\mathbb{Z}$-basis of $\mathbb{Z}^l$. It is straightforward to see that $\mathcal{A}_{\mathbf{Q}}$ is a $\mathbb{Z}^l$-graded algebra with the grading defined by

$$\deg X_{i,m} = e_i, \quad \deg Y_{i,m} = -e_i \quad \text{for } 1 \le i \le l,\ m \in \mathbb{Z}. \tag{4.52}$$

Set

$$\mathcal{A}_{\mathbf{Q}}^{+} = \langle X_{i,m},\ Y_{j,n} \mid i, j = 1, \dots, l,\ m, n \ge 0\rangle,$$
$$\mathcal{A}_{\mathbf{Q}}^{-} = \langle X_{i,m},\ Y_{j,n} \mid i, j = 1, \dots, l,\ m, n < 0\rangle,$$

which are $\mathbb{Z}^l$-graded subalgebras of $\mathcal{A}_{\mathbf{Q}}$.

A vector w in an $\mathcal{A}_{\mathbf{Q}}$-module W is called a *vacuum vector* if $\mathcal{A}_{\mathbf{Q}}^{+}w = 0$, and an $\mathcal{A}_{\mathbf{Q}}$-module W equipped with a vacuum vector which generates W is called a *vacuum $\mathcal{A}_{\mathbf{Q}}$-module*.

Set

$$V_{\mathbf{Q}} = \mathcal{A}_{\mathbf{Q}}/(\mathcal{A}_{\mathbf{Q}}\mathcal{A}_{\mathbf{Q}}^{+}), \tag{4.53}$$

a left $\mathcal{A}_{\mathbf{Q}}$-module, and set

$$\mathbf{1} = 1 + (\mathcal{A}_{\mathbf{Q}}\mathcal{A}_{\mathbf{Q}}^{+}) \in V_{\mathbf{Q}}.$$

Clearly, $\mathbf{1}$ is a vacuum vector and $V_{\mathbf{Q}}$ equipped with $\mathbf{1}$ is a vacuum $\mathcal{A}_{\mathbf{Q}}$-module.

For $1 \leq i \leq l$, set

$$u^{(i)} = X_{i,-1}\mathbf{1}, \quad v^{(i)} = Y_{i,-1}\mathbf{1} \in V_{\mathbf{Q}} \tag{4.54}$$

and set

$$X_i(x) = \sum_{n\in\mathbb{Z}} X_{i,n}x^{-n-1}, \quad Y_i(x) = \sum_{n\in\mathbb{Z}} Y_{i,n}x^{-n-1} \in \mathcal{A}_{\mathbf{Q}}[[x, x^{-1}]]. \tag{4.55}$$

Now we endow $V_{\mathbf{Q}}$ with the structure of a weak quantum vertex algebra (cf. [7,17]).

Theorem 4.27. *Let $\boldsymbol{Q} = (q_{ij})_{1\leq i,j\leq l}$ be a complex matrix such that $q_{ii} = 1$ and $q_{ij}q_{ji} = 1$ for $1 \leq i, j \leq l$, let $\mathcal{A}_{\boldsymbol{Q}}$ be the associative algebra associated with $\boldsymbol{Q}$ and let $V_{\boldsymbol{Q}}$ be the universal vacuum $\mathcal{A}_{\boldsymbol{Q}}$-module. There exists a (unique) weak quantum vertex algebra structure on $V_{\boldsymbol{Q}}$ with $\mathbf{1}$ as the vacuum vector such that*

$$Y(u^{(i)}, x) = X_i(x), \quad Y(v^{(i)}, x) = Y_i(x) \quad \textit{for } i = 1, \ldots, l.$$

Let W be any $\mathcal{A}_{\boldsymbol{Q}}$-module satisfying the condition that for any $w \in W$, $X_{i,m}w = Y_{i,m}w = 0$ for $1 \leq i \leq l$ and for m sufficiently large. Then there exists a (unique) $V_{\boldsymbol{Q}}$-module structure on W with

$$Y_W(u^{(i)}, x) = X_i(x), \quad Y_W(v^{(i)}, x) = Y_i(x) \quad \textit{for } i = 1, \ldots, l.$$

Conversely, any $V_{\boldsymbol{Q}}$-module W is an $\mathcal{A}_{\boldsymbol{Q}}$-module with

$$X_i(x) = Y_W(u^{(i)}, x), \quad Y_i(x) = Y_W(v^{(i)}, x) \quad \textit{for } i = 1, \ldots, l.$$

Furthermore, similar to Proposition 3.8 in [7], we have (cf. [17]):

Proposition 4.28. *Let V be any nonlocal vertex algebra and let ψ be any map from $\{u^{(i)}, v^{(i)} \mid i = 1, \ldots, l\}$ to V such that*

$$Y(\psi(u^{(i)}), x_1)Y(\psi(u^{(j)}), x_2) = q_{ij}Y(\psi(u^{(j)}), x_2)Y(\psi(u^{(i)}), x_1),$$
$$Y(\psi(v^{(i)}), x_1)Y(\psi(v^{(j)}), x_2) = q_{ij}Y(\psi(v^{(j)}), x_2)Y(\psi(v^{(i)}), x_1),$$
$$Y(\psi(u^{(i)}), x_1)Y(\psi(v^{(j)}), x_2) = q_{ji}Y(\psi(v^{(j)}), x_2)Y(\psi(u^{(i)}), x_1)$$

for $1 \leq i, j \leq l$. Then there exists a unique nonlocal-vertex-algebra homomorphism from $V_{\mathbf{Q}}$ to V, extending ψ.

For each $1 \leq i \leq l$, $n \in \mathbb{Z}$, the algebra $\mathcal{A}_{q_{ii}}$ (associated with 1×1 matrix q_{ii}) generated by the elements $X_{i,n}$ and $Y_{i,n}$ is commutative and is isomorphic to $\mathbb{C}[X_{i,n}, Y_{i,n}]$, which is a polynomial algebra. Let V_{ii} be the universal vacuum module constructed from $\mathcal{A}_{q_{ii}}$. From Theorem 4.27, we know that V_{ii} is a nonlocal vertex subalgebra of $V_{\mathbf{Q}}$. In the following we show that $V_{\mathbf{Q}}$ can be realized as a deformation through a twistor of $V_{11} \otimes \cdots \otimes V_{ll}$. Moreover, Theorem 4.18 provides an explicit formula for the twistor $T(x)$ which recovers the iterated twisted tensor product.

We make the following vector space identification

$$\begin{aligned}
&V_{\mathbf{Q}} \to V_{11} \otimes \cdots \otimes V_{ll}, \\
&u^{(i)} \mapsto 1 \otimes \cdots \otimes u^{(i)} \otimes \cdots \otimes 1, \\
&v^{(i)} \mapsto 1 \otimes \cdots \otimes v^{(i)} \otimes \cdots \otimes 1,
\end{aligned}$$

where $u^{(i)}$ maps to the position $2i-1$ and $v^{(i)}$ maps to the position $2i$ for $u^{(i)}, v^{(i)} \in V_{\mathbf{Q}}$, $1 \leq i \leq l$.

It is easy to see that:

Proposition 4.29. *Let V_{ii}, a nonlocal vertex subalgebra of $V_{\mathbf{Q}}$, be the universal vacuum $\mathcal{A}_{q_{ii}}$-module for $1 \leq i \leq l$. Let $T(x)$ be a linear map defined on generator as*

$$T(x): \; u^{(i)} \otimes u^{(j)} = \begin{cases} u^{(i)} \otimes u^{(j)}, & \text{if} \quad i \leq j, \\ q_{ij} u^{(i)} \otimes u^{(j)}, & \text{if} \quad i > j, \end{cases} \tag{4.56}$$

$$T(x): \; u^{(i)} \otimes v^{(j)} = \begin{cases} u^{(i)} \otimes v^{(j)}, & \text{if} \quad i \leq j, \\ q_{ji} u^{(i)} \otimes v^{(j)}, & \text{if} \quad i > j, \end{cases} \tag{4.57}$$

$$T(x): \; v^{(i)} \otimes u^{(j)} = \begin{cases} v^{(i)} \otimes u^{(j)}, & \text{if} \quad i \leq j, \\ q_{ji} v^{(i)} \otimes u^{(j)}, & \text{if} \quad i > j, \end{cases} \tag{4.58}$$

$$T(x): \; v^{(i)} \otimes v^{(j)} = \begin{cases} v^{(i)} \otimes v^{(j)}, & \text{if} \quad i \leq j, \\ q_{ij} v^{(i)} \otimes v^{(j)}, & \text{if} \quad i > j \end{cases} \tag{4.59}$$

for $u^{(i)}, v^{(i)} \in V_{ii}$, $u^{(j)}, v^{(j)} \in V_{jj}$, $i, j \in \{1, \ldots, l\}$. Then $T(x)$ is an invertible twistor for nonlocal vertex algebra $V_{11} \otimes V_{22} \otimes \cdots \otimes V_{ll}$.

Furthermore, we have:

Proposition 4.30. *The nonlocal vertex algebra $V_{\mathbf{Q}}$ is isomorphic to the nonlocal vertex algebra $(V_{11} \otimes V_{22} \otimes \cdots \otimes V_{ll})^T$ constructed from $V_{11} \otimes V_{22} \otimes \cdots \otimes V_{ll}$ by the twistor $T(x)$.*

Acknowledgments

The author would like to thank the referee for careful reading and valuable suggestions to put this paper in better shape. The author was supported by Natural Science Foundation of China grant Nos. 11671247 and 11371238, and by a grant of "The First-class Discipline of Universities in Shanghai".

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