Journal of Algebra 485 (2017) 118–141

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Decomposition of Garside groups and self-similar L-algebras

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A R T I C L E   I N F O

Article history:
Received 7 January 2017
Available online 5 May 2017
Communicated by Patrick Dehornoy

Dedicated to B.V.M.

MSC:
primary 20F60, 20M30, 20F36, 08A05, 05E18
secondary 17D99

Keywords:
Right ℓ-group
Crossed product
L-algebra
Self-similar
Garside group
Artin group
Quantum Yang–Baxter equation

A B S T R A C T

Picantin's iterated crossed product representation of Garside monoids is extended and reproved for a wide class of not necessarily noetherian partially ordered groups with a right invariant lattice structure. It is shown that the tree-like structure of such an iterated crossed product is equivalent to a partial cycle set, closely related to a class of set-theoretic solutions of the quantum Yang–Baxter equation. The decomposition of finite square-free solutions is related to the crossed product representation of the corresponding structure group.

Introduction

The theory of Garside groups and Garside monoids arose from the discovery of a fundamental element Δ_n, the least common left and right multiple of the atoms, in the positive braid monoid of the n-strand braid group $\mathscr{B}_n$. Garside's observation [33] was

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http://dx.doi.org/10.1016/j.jalgebra.2017.04.023

soon generalized to Artin–Tits groups of spherical type [14,27] and eventually led to the class of *Garside groups* [26,21], comprising all spherical Artin groups, all torus knot groups ([26], Example 4), all one-relator groups with non-trivial centre [42], all braid groups $\pi_1(W\backslash V^{\mathrm{reg}})$ of well-generated finite complex reflection groups W [7], and the structure groups of unitary non-degenerate set-theoretic solutions of the quantum Yang–Baxter equation [17,23,25]. Every Garside monoid has two lattice structures according to left and right divisibility, and a *Garside element* Δ for which the sets of left and right divisors coincide.

Shortly before a first general definition of Garside groups had been released [26], a new development was initiated by Birman, Ko, and Lee's discovery of a *dual* Garside structure [10] on Artin's braid groups. Again, the new concept was soon extended to all spherical Artin groups [5]. For Artin groups of Euclidean type, the classical monoid of positive elements no longer gives rise to a lattice, while the dual monoid still yields a Garside structure for groups of type $\widetilde{A}_n$ or $\widetilde{C}_n$ [28,29]. Precisely, such a group is *quasi-Garside* [25] in the sense that the set of atoms is not assumed to be finite. Beyond Euclidean groups, a dual quasi-Garside structure for free groups was obtained by Bessis [6]. Using a careful analysis of factorizations of Euclidean isometries into reflections [13], McCammond [39] recently proved that $\widetilde{A}_n$, $\widetilde{C}_n$, and $\widetilde{G}_2$ (see [48]) are the only Euclidean Artin groups which admit a dual quasi-Garside structure. On the other hand, McCammond and Sulway [40] embed the dual Artin monoid of *any* Euclidean type into a quasi-Garside monoid, which implies, for instance, that irreducible Euclidean Artin groups are torsion-free and have a trivial centre.

The framework of Garside structure is large, but many groups of a somewhat similar type are not covered by that scheme. For example, braid groups with infinitely many strands no longer admit a Garside element. Another failure occurs even for very small groups like the fundamental group $\mathbb{Z} \rtimes \mathbb{Z}$ of the Klein bottle, where the noetherian property neither holds for the left- nor the right-hand ordering which both are linear. As observed in [47], however, these and many other non-noetherian groups are *right ℓ-groups*, that is, groups with a lattice order, so that the right multiplications $a \mapsto ab$ are lattice automorphisms.

One of the advantages to study Garside-like groups as right ℓ-groups comes from their intimate connection with (two-sided) ℓ-groups, providing a big source of inspiration and a wealth of proof techniques from ℓ-groups to be adapted to that wider scope. For the theory of ℓ-groups, the reader is referred to [4,8,19,37].

For example, the proof that braid groups are torsion-free caused a lot of trouble [31, 32,20,44] until Dehornoy eliminated the noetherian hypothesis to get a much simpler proof [22]. In the wider context of right ℓ-groups, the proof has now become trivial ([47], Proposition 3). Secondly, Picantin's decomposition of the quasi-centre of a Garside group [41] easily extends to right ℓ-groups with a noetherian quasi-centre [47], where the decomposition follows immediately from an old theorem of Birkhoff [9] on noetherian ℓ-groups. In the same paper [41], Picantin obtains an iterated crossed product representation for every Garside group. The factors of the crossed product act on each other, subject to compatibility conditions, which leads to some complication. “Iterated” means that after a first maximal crossed product decomposition, the quasi-centre of the factors need not be cyclic, so that each factor may further decompose into a crossed product, until the whole process stops at groups with cyclic quasi-centre. This leads to a tree-like structure of any Garside group.

In the present paper, we extend Picantin's main result [41] to the context of right ℓ-groups and beyond, using a conceptual description of crossed products which reduces all the technicalities to a minimum. To make this precise, we start with the familiar case of Garside groups, and gradually extend the scope. Let us note first that there are two extreme types of Garside-like groups: groups with cyclic quasi-centre which we call *quasi-cyclic*, and the structure groups G_X of non-degenerate unitary set-theoretic solutions $R\colon X^2 \to X^2$ of the quantum Yang–Baxter equation. For finite X, Chouraqui [17] observed that the groups G_X are Garside groups. In [45] it was shown that the solutions X are equivalent to a binary operation $\cdot$ on X with bijective left multiplications satisfying the equation

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z).$$

Such a structure $(X; \cdot)$ is called a *cycle set*. If $x \cdot x = x$ holds for all $x \in X$, the cycle set is said to be *square-free* [45].

For a Garside group G, our main result implies that G is an iterated crossed product of quasi-cyclic groups, and that the tree structure of this iterated crossed product is determined by the set X of atoms as a *partial cycle set*, that is, a partition $X = \bigsqcup_{i \in I} X_i$ and a product $x \cdot y$ on X which is defined whenever x and y belong to different subsets X_i. Furthermore, it is assumed that the above equation holds, whenever defined, and that the left multiplication by $x \in X_i$ is a permutation on $X \smallsetminus X_i$. So the X_i are singletons if and only if X is equivalent to a square-free cycle set, and in this case, G is the structure group of the solution given by X. Otherwise, the X_i are the atom sets of quasi-cyclic Garside groups. A general Garside group can thus be regarded as an intermediate structure between quasi-cyclic groups and structure groups of certain solutions of the quantum Yang–Baxter equation.

It should be recalled that the concept of square-free cycle set arose from a conjecture of Gateva-Ivanova [34] on n-generated semigroups of *skew-polynomial type* [35]. The corresponding semigroup algebras are Artin–Schelter regular [2,3] of global dimension n, and any such semigroup gives rise to a solution of the quantum Yang–Baxter equation with a marvellous tree structure. In the language of cycle sets, this means that the associated finite (square-free) cycle set X admits a decomposition $X = \bigsqcup_{i \in I} X_i$ into subsets X_i which are invariant under left multiplication, and that $|I| > 1$ whenever $|X| > 1$. Since every X_i is again a square-free cycle set, the iterated decompositions make X into a tree. The conjecture that every finite square-free cycle set comes from such an Artin–Schelter regular algebra was verified in [45]. Via *partial* cycle sets, the tree structure reappears in the iterated crossed product decomposition of a Garside group.

Beyond Garside groups, we prove that the quasi-centre of any right ℓ-group G is a (two-sided) ℓ-group (Proposition 5), removing the archimedean hypothesis in a similar result of [47]. The *quasi-centre* $N(G)$ consists of the elements $u \in G$ which are *normal* in the sense that the positive cone (with respect to the current fixed order) is invariant under conjugation with u. In particular, a Garside element is the same as a *strong order unit*, a normal element $u \geqslant 1$ such that the interval $[1, u]$ generates the group. If $N(G)$ is noetherian, this immediately gives a decomposition of $N(G)$ into a cardinal sum of cyclic groups. Based on this decomposition of $N(G)$, we obtain a crossed product representation for right ℓ-groups G with a noetherian quasi-centre such that G is the smallest convex subgroup containing $N(G)$ (Theorem 2).

In the absence of a noetherian property, we have to deal with right ℓ-groups in general. By [47], Theorem 1, a right ℓ-group is determined by its negative cone, which is a self-similar left and right hoop. Recall that a *left hoop* [46] is a monoid H with a binary operation $\to$ such that the following are satisfied for all $a, b, c \in H$:

$$a \to a = 1$$

$$ab \to c = a \to (b \to c)$$

$$(a \to b)a = (b \to a)b.$$

The multiplication of H and the operation $\to$ determine each other. So a left hoop can be regarded as a special type of monoid. If H is right cancellative, it is called *self-similar*. (For equational descriptions of self-similarity, see Proposition 1.) Every left hoop is a $\wedge$-semilattice with respect to the partial order

$$a \leqslant b \iff a \to b = 1 \tag{0}$$

and satisfies the left Ore condition. Therefore, a self-similar left hoop H admits a group of left fractions $G(H)$ which, in a sense, can be viewed as a “quarter of an ℓ-group”, that is to say, “one half” of a right ℓ-group. Indeed, the negative cone of a right ℓ-group G has a second operation $\rightsquigarrow$, satisfying

$$b \leqslant a \rightsquigarrow c \iff ab \leqslant c \iff a \leqslant b \to c,$$

which makes the negative cone G^- into a self-similar *right* hoop. Accordingly, there is a second partial order, similar to (0). We will show, however, that Picantin’s concept of crossed product extends to a framework of one-sided (self-similar) hoops (Theorem 1).

With respect to the binary operation $\to$, every left hoop H is an *L-algebra* [46], that is, the relation (0) is a partial order with greatest element 1, and the equations $1 \to a = a$ and

$$(a \to b) \to (a \to c) = (b \to a) \to (b \to c)$$

hold for all $a, b, c \in H$. Note the equivalence of this equation with the above mentioned one for a cycle set (see the remark before Theorem 3). The "L" and the arrow notation for L-algebras come from their origin in algebraic logic [46]. Since 1 stands for the greatest element, the *logical unit*, the negative cone is the preferred one. Conversely, every L-algebra X embeds into a self-similar left hoop $S(X)$ such that X generates $S(X)$ as a monoid. Up to isomorphism, the *self-similar closure* $S(X)$ of X is unique.

Now the concept of L-algebra allows a succinct description of the crossed product of self-similar left hoops H_i, $i \in I$, as follows. Assume that the H_i act on each other by order automorphisms. We say that the actions are *compatible* if the *wedge* $\bigwedge H_i$, the disjoint union $\bigsqcup_{i \in I} H_i$ with the unit elements $1 \in H_i$ identified, is an L-algebra. As the main equation defining an L-algebra depends on three variables, this compatibility does not involve more than three factors, as Picantin [41] observed in the context of Garside monoids. The *crossed product* $\boxtimes_{i \in I} H_i$ of the H_i is just the self-similar closure $S(\bigwedge_{i \in I} M_i)$ (Definition 5). By this definition, the infinite distributivity of a crossed product, and the unique ordered factorization of its elements (Theorem 1) follow immediately. Moreover, the lattice structure of a crossed product is the cardinal sum of its factors (Corollary 1), which explains the distributivity of the lattice ordering for structure groups associated to the quantum Yang–Baxter equation. An intrinsic description of crossed products in terms of the submonoids H_i (Corollary 3) is proved in a few lines. In essence, this shows that the pattern behind Picantin's amazing result is a decomposition theorem for L-algebras, extending the decomposition [45] of square-free cycle sets (Theorem 3).

1. Preliminaries

A group G with a partial order $\leqslant$ is said to be *right partially ordered* if

$$a \leqslant b \implies ac \leqslant bc \tag{1}$$

holds for $a, b, c \in G$. In terms of the *positive cone* $G^+ := \{a \in G \mid a \geqslant 1\}$, the partial order can be expressed by $a \leqslant b \iff ba^{-1} \in G^+$, and then G is right partially ordered if and only if $P := G^+$ is a submonoid with trivial unit group $P^\times = P \cap P^{-1}$. Every right partial order $\leqslant$ corresponds to a *left partial order*

$$a \preccurlyeq b \;:\Longleftrightarrow\; b^{-1} \leqslant a^{-1} \iff a^{-1}b \in G^+ \iff b^{-1}a \in G^-, \tag{2}$$

where $G^- := \{a \in G \mid a \leqslant 1\}$ denotes the *negative cone* of G. Thus, a right partially ordered group G is a lattice with respect to $\leqslant$ if and only if G is a lattice with respect to $\preccurlyeq$, and then we call G a *right ℓ-group* [47]. A *lattice-ordered* group is a right ℓ-group where both partial orders $\leqslant$ and $\preccurlyeq$ coincide. For the theory of lattice-ordered groups (*ℓ-groups* for short), we refer to [8,19].

The negative cone of a right ℓ-group G can be described as follows. Recall that a *left hoop* [46] is a monoid M with a binary operation $\to$ satisfying

$$a \to a = 1 \tag{3}$$

$$ab \to c = a \to (b \to c) \tag{4}$$

$$(a \to b)a = (b \to a)b \tag{5}$$

for $a, b, c \in M$. By [46], Proposition 3, the unit element 1 of a left hoop M is a *logical unit*, that is,

$$a \to a = a \to 1 = 1, \qquad 1 \to a = a \tag{6}$$

holds for every $a \in M$. Note that an element 1 satisfying (6) is unique. Moreover, every left hoop M is a $\wedge$-semilattice with

$$a \wedge b = (a \to b)a = (b \to a)b. \tag{7}$$

In particular, this provides M with a partial order

$$a \leqslant b \;:\Longleftrightarrow\; a \to b = 1 \;\Longleftrightarrow\; \exists x \in M\colon a = xb. \tag{8}$$

Therefore, Eq. (4) yields the adjointness relation $ab \leqslant c \Longleftrightarrow a \leqslant b \to c$, which shows that the multiplication and the operation $\to$ determine each other. Furthermore, the adjointness relation easily implies that

$$a \to (b \wedge c) = (a \to b) \wedge (a \to c)$$

holds for all $a, b, c \in M$ (see [46], Proposition 4).

Recall that a monoid M is said to be *right cancellative* if the map $a \mapsto ab$ is injective for all $b \in M$. By [46], Proposition 5, we have

Proposition 1. *For a left hoop M, the following are equivalent.*

(a) *M is right cancellative.*
(b) *The map $b \mapsto (a \to b)$ is surjective for all $a \in M$.*
(c) *$a \to ba = b$ holds for all $a, b \in M$.*
(d) *$a \to bc = ((c \to a) \to b)(a \to c)$ holds for $a, b, c \in M$.*

A left hoop M is called *self-similar* [46] if the equivalent conditions of Proposition 1 are satisfied. Any left hoop M gives rise to a *right hoop* $(M^{\mathrm{op}}; \rightsquigarrow)$ (with multiplication reversed). The equations (3)–(5) then turn into

$$a \rightsquigarrow a = 1 \tag{9}$$

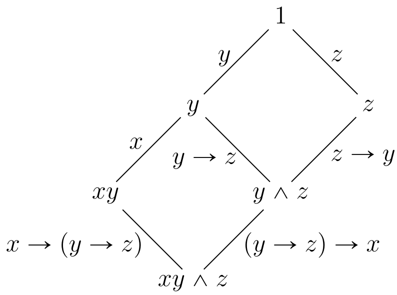
$$ab \rightsquigarrow c = b \rightsquigarrow (a \rightsquigarrow c) \tag{10}$$

$$a(a \rightsquigarrow b) = b(b \rightsquigarrow a). \tag{11}$$

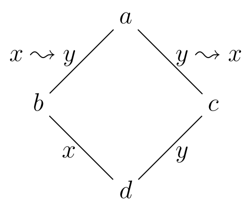
Accordingly, we call a right hoop *self-similar* if it is left cancellative.

By [47], Theorem 1, the negative cone of a right ℓ-group can be characterized as a self-similar left and right hoop, and every self-similar left and right hoop arises in this way. For brevity, self-similar left and right hoops are called *right ℓ-cones*. To avoid confusion, we warn the reader that a “left hoop” is not a “hoop”. In fact, *hoops* [12,11] are even more special than left and right hoops: they are commutative, which means that the two operations $\to$ and $\rightsquigarrow$ coincide.

The equations (3)–(5) and (d) of Proposition 1 can be visualized by the following Hasse diagram (cf. [21], Fig. 1.2):



For every covering relation $a < b$ there is a unique atom x with $a = xb$. The upper square of the diagram shows how $y \to z$ is obtained for a pair of distinct atoms y, z. For a right ℓ-cone, the operation $\rightsquigarrow$ is obtained similarly by reversing the partial order:



This explains the relation $x(x \rightsquigarrow y) = y(y \rightsquigarrow x)$ as well as (18) in Section 3.

For a right ℓ-cone M, Eqs. (5) and (11) imply the left and right Ore condition. As M is left and right cancellative, it follows that M has a (left and right) group of fractions $G(M)$. We call a right ℓ-cone M and its group of fractions *noetherian* [47] if ascending sequences $a_0 \leqslant a_1 \leqslant a_2 \leqslant \cdots$ or $a_0 \preccurlyeq a_1 \preccurlyeq a_2 \preccurlyeq \cdots$ become *stationary*, that is, $a_n = a_{n+1}$ for some $n \in \mathbb{N}$. Note that M is noetherian if and only if $G(M)$ satisfies the ascending and descending chain condition for bounded sequences with respect to $\leqslant$, or equivalently, with respect to $\preccurlyeq$. Clearly, a noetherian right ℓ-cone M is *atomic* in the sense that every element of M can be represented as a product of maximal elements $x < 1$, also called *atoms*. The set of atoms of M will be denoted by $X(M)$. If M is noetherian, it may still happen that a bounded interval has finite chains of arbitrary length. Therefore, we call M *bounded atomic* [47] if every element of M can be written as a product of atoms such that the number of factors in any such product is bounded. Every bounded atomic right ℓ-cone is noetherian, but not vice versa.

An element u of a right ℓ-group G is called *normal* if $uG^+u^{-1} = G^+$. The set of normal elements is a partially ordered subgroup $N(G)$, the *quasi-centre* [47] of G. If, in addition, every $a \in G^+$ satisfies $a \leqslant u^n$ for some $n \in \mathbb{N}$, we call u a *strong order unit* [47]. A *quasi-Garside group* [25] is then equivalent to a right ℓ-group with a strong order unit and a bounded atomic positive cone. If the number of atoms is finite, such a group is called *Garside* [21,25].

For the next section, we recall the concept of *L-algebra* [46], that is, a set X with a binary operation $\to$ such that (0) is a partial order with greatest element 1, and the equations $1 \to x = x$ and

$$(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$$

hold for all $x, y, z \in X$. A *morphism* $f\colon X \to Y$ of L-algebras is a map which satisfies $f(x \to y) = f(x) \to f(y)$ and $f(1) = 1$. For an inclusion morphism $f\colon X \hookrightarrow Y$, we call X a *L-subalgebra* of Y.

Every left hoop is an L-algebra. By [46], Section 2, the multiplication of a left hoop is uniquely determined by the underlying L-algebra. In other words, left hoops form a special class of L-algebras. An L-algebra X is called *self-similar* [46] if for each $x \in X$, the map $y \mapsto (x \to y)$ is a bijection from $\{y \in X \mid y \leqslant x\}$ onto X. By [46], Theorem 1 and Proposition 5, an L-algebra is self-similar if and only if it is a self-similar left hoop. Every L-algebra X has a *self-similar closure* $S(X)$, a unique self-similar left hoop generated by X as a monoid [46]. In other words, an L-algebra X has a unique partial multiplication which is everywhere defined if and only if X is self-similar. Typical examples of L-algebras are intervals $[a, 1]$ in an abelian ℓ-group G where a^{-1} is a strong order unit of G. Here the self-similar closure coincides with the negative cone G^- of G. See also Examples 1 and 2 in the following section.

2. Crossed products of self-similar left hoops

For any set M, we denote the group of permutations on M by $\mathfrak{S}(M)$. Thus $\mathfrak{S}(M)$ is a subgroup of the monoid M^M of maps $M \to M$. If M is partially ordered, we write $\mathfrak{S}^+(M)$ for the group of order automorphisms of M.

If M is a self-similar left hoop, the natural action of $\mathfrak{S}^+(M)$ on M gives rise to a *retro-action* of M on $\mathfrak{S}^+(M)$, given by

$$(a \to \pi)(b) := \pi(a) \to \pi(ba) \tag{12}$$

for $a, b \in M$ and $\pi \in \mathfrak{S}^+(M)$. Namely, we have

Proposition 2. *Let M be a self-similar left hoop. The retro-action is a well-defined monoid homomorphism $M \to \mathfrak{S}^+(M)^{\mathfrak{S}^+(M)}$ which satisfies the 1-cocycle condition*

$$a \to \pi\varrho = (\varrho(a) \to \pi)(a \to \varrho)$$

for all $a \in M$ *and* $\pi, \varrho \in \mathfrak{S}^+(M)$*.*

Proof. We show first that the inverse of the map (12) is given by

$$(a \to \pi)^{-1}(c) := a \to \pi^{-1}(c\pi(a)).$$

Indeed, $a \to \pi^{-1}\big((\pi(a) \to \pi(ba))\pi(a)\big) = a \to \pi^{-1}(\pi(a) \wedge \pi(ba)) = a \to \pi^{-1}(\pi(ba)) = a \to ba = b$ by Proposition 1, and $\pi(a) \to \pi\big((a \to \pi)^{-1}(c)a\big) = \pi(a) \to c\pi(a) = c$. Thus $a \to \pi$ is bijective. Eqs. (4) and (7) imply that $b \leqslant c \Longrightarrow a \to b \leqslant a \to c$ holds in M. Hence $a \to \pi \in \mathfrak{S}^+(M)$. For $a, b, c \in M$ and $\pi \in \mathfrak{S}^+(M)$, we have $1 \to \pi = \pi$ and

$$\begin{aligned}(ab \to \pi)(c) &= \pi(ab) \to \pi(cab) = (\pi(ab) \to \pi(b)) \to (\pi(ab) \to \pi(cab)) \\ &= (\pi(b) \to \pi(ab)) \to (\pi(b) \to \pi(cab)) = (b \to \pi)(a) \to (b \to \pi)(ca) \\ &= (a \to (b \to \pi))(c).\end{aligned}$$

Thus $a \mapsto (a \to \pi)$ is a monoid homomorphism.

Finally, $(a \to \pi\varrho)(b) = \pi\varrho(a) \to \pi\varrho(ba) = \pi\varrho(a) \to \pi\big((\varrho(a) \to \varrho(ba))\varrho(a)\big) = \pi\varrho(a) \to \pi\big((a \to \varrho)(b)\varrho(a)\big) = (\varrho(a) \to \pi)(a \to \varrho)(b)$ for all $b \in M$. $\square$

Note that $\mathfrak{S}^+(M)$ is just a monoid, not a left hoop. For left hoops, we consider a stronger type of action.

Definition 1. Let H and M be left hoops. An *action* of H on M is given by a monoid homomorphism $\sigma\colon H \to \mathfrak{S}^+(M)$. For $h \in H$ and $a \in M$, we write $h \to a := \sigma(h)(a)$.

For mutual actions of self-similar left hoops, the retro-action has to be taken into account. Proposition 2 leads to the following

Definition 2. Let M_1 and M_2 be self-similar left hoops. We say that two mutual actions $\sigma_{j,i}\colon M_i \to \mathfrak{S}^+(M_j)$ with $\{i, j\} = \{1, 2\}$ are *compatible* if the 1-cocycle conditions

$$a_i \to a_j b_j = \big((b_j \to a_i) \to a_j\big)(a_i \to b_j)$$

are satisfied for $a_1, b_1 \in M_1$ and $a_2, b_2 \in M_2$.

It is easily checked that two mutual actions $\sigma_{j,i}\colon M_i \to \mathfrak{S}^+(M_j)$ are compatible if and only if the diagrams

$$\begin{tikzcd} M_i \arrow[r, "\sigma_{j,i}"] \arrow[d, "\sigma_{i,j}(a_j)"] & \mathfrak{S}^+(M_j) \arrow[d, "\varrho_j(a_j)"] \\ M_i \arrow[r, "\sigma_{j,i}"] & \mathfrak{S}^+(M_j) \end{tikzcd}$$

commute for all $a_j \in M_j$ and $\{i, j\} = \{1, 2\}$, where $\varrho_j \colon M_j \to \mathfrak{S}^+(M_j)^{\mathfrak{S}^+(M_j)}$ denotes the retro-action. There is a simpler and more elegant description of compatibility.

Proposition 3. *Let H and M be self-similar left hoops. Two actions $H \to \mathfrak{S}^+(M)$ and $M \to \mathfrak{S}^+(H)$ are compatible if and only if the equation*

$$(h \to a) \to (h \to u) = (a \to h) \to (a \to u) \tag{13}$$

holds for $a \in M$, $h \in H$, and $u \in H \cup M$.

Proof. For $u \in M$, we have

$$\big((h \to a) \to (h \to u)\big)(h \to a) = (h \to a) \wedge (h \to u) = h \to (a \wedge u) = h \to (a \to u)a.$$

Since M is right cancellative, this shows that Eq. (13) with $u \in M$ is equivalent to

$$h \to (a \to u)a = \big((a \to h) \to (a \to u)\big)(h \to a).$$

By Proposition 1(b), every element of $v \in M$ is of the form $v = a \to u$ for some $u \in M$. Therefore, Eq. (13) with $u \in M$ can be written as

$$h \to va = \big((a \to h) \to v\big)(h \to a).$$

Since Eq. (13) is symmetric in a and h, this completes the proof. □

Definition 3. For a family of L-algebras X_i, $i \in I$, we define the *wedge* $\bigwedge_{i \in I} X_i$ to be the partially ordered set obtained from the disjoint union of the X_i by identifying the unit elements 1 of the X_i. So each X_i can be regarded as a subset of $\bigwedge_{i \in I} X_i$. For $I = \{1, \ldots, n\}$, we also write $X_1 \wedge \cdots \wedge X_n$ instead of $\bigwedge_{i \in I} X_i$.

By Definition 3, Proposition 3 yields

Corollary. *Let H and M be self-similar left hoops. Two actions $H \to \mathfrak{S}^+(M)$ and $M \to \mathfrak{S}^+(H)$ are compatible if and only if they make $H \wedge M$ into an L-algebra.*

Accordingly, the concept of compatible actions naturally extends as follows.

Definition 4. Let M_i, $i \in I$, be a family of self-similar left hoops. We say that a system of actions $\sigma_{j,i} \colon M_i \to \mathfrak{S}^+(M_j)$ for $i \neq j$ is *compatible* if it makes $\bigwedge_{i \in I} M_i$ into an L-algebra.

Recall that every L-algebra X admits a *self-similar closure* [46], that is, a self-similar left hoop $S(X)$ with X as an L-subalgebra such that X generates $S(M)$ as a monoid. By [46], Theorem 3, $S(X)$ is unique up to isomorphism.

Example 1. The simplest example is given by the L-algebra $X_a := \{1, a\}$ with $a < 1$. The self-similar closure $S(X_a)$ is the negative cone of the ℓ-group $G(S(X_a)) \cong \mathbb{Z}$. It consists of the powers $1 > a > a^2 > a^3 > \cdots$ with $a^m \to a^n = a^{n-m}$ for $m \leqslant n$.

Further examples can be found in [46,47]. Here is a new one.[1]

Example 2. Let V be a finite dimensional vector space over a field K, and let $(\,,\,)\colon V \times V \to K$ be an *anisotropic* bilinear form, that is $(x, x) \neq 0$ unless $x = 0$. For $x, y \in V$, we define

$$x \to y := y - \frac{(y, x)}{(x, x)} x$$

if $x \neq 0$, and $0 \to y := y$. It is easily checked that the equation

$$(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$$

holds for all $x, y, z \in V$. Note that for $x \neq 0$, the one-dimensional subspace $K(x \to y)$ is determined by the subspaces Kx and Ky. Furthermore, $x \to y = 0$ if and only if $Kx = Ky$. Hence $\to$ induces a partial binary operation on the projective space $\mathbb{P}(V)$ which is defined for all pairs of distinct points in $\mathbb{P}(V)$.

Recall that an L-algebra is said to be *discrete* [47] if every element $x < 1$ is maximal. We adjoin 0 to the projective space $\mathbb{P}(V)$ such that 0 becomes a logical unit in $\widetilde{X}(V) := \mathbb{P}(V) \sqcup \{0\}$. With the Gram determinant

$$\Delta := (x, x)(y, y) - (x, y)(y, x),$$

we have

$$(x \to y, x \to y) = \frac{\Delta}{(x, x)}.$$

Hence $x \to y = 0$ if and only if $\Delta = 0$. For $x, y \neq 0$, this implies that $Kx = Ky$ if and only if $K(x \to y) = K(y \to x)$. Thus $\widetilde{X}(V)$ is a discrete L-algebra.

Using [47], Theorem 4, it can be shown that the self-similar closure of $\widetilde{X}(V)$ is the negative cone of a quasi-Garside group. If $\{x_0, \ldots, x_n\}$ is a basis of V, with corresponding points $P_i := Kx_i$, then $P_0 \wedge \cdots \wedge P_n$ is a Garside element of $\widetilde{X}(V)$, and the elements of the interval $[P_0 \wedge \cdots \wedge P_n, 0]$ in $S(\widetilde{X}(V))$ can be identified with the subspaces of $\mathbb{P}(V)$

[1] I am indebted to Carsten Dietzel for the idea behind this example.

with reverse inclusion as partial order. So the empty space stands for the logical unit 0, while $\mathbb{P}(V)$ corresponds to the Garside element $P_0 \wedge \cdots \wedge P_n$.

For finite fields K, there are no anisotropic forms of ranks $\geqslant 3$. On the other hand, notably for $K = \mathbb{R}$, there are many non-symmetric bilinear forms with non-isomorphic L-algebras which determine the same quadratic form $x \mapsto (x, x)$. In particular, the bilinear form $(\, , \,)$ is symmetric if and only if the associated quasi-Garside group is lattice-ordered.

Definition 5. Let M_i, $i \in I$, be a family of self-similar left hoops with a compatible system of actions $\sigma_{j,i}\colon M_i \to \mathfrak{S}^+(M_j)$. We define the *crossed product* $\boxtimes^{\sigma}_{i\in I} M_i$, or simply $\boxtimes_{i\in I} M_i$, of the M_i to be the self-similar closure $S(\bigwedge_{i\in I} M_i)$. If $I = \{1, \ldots, n\}$, we also write $M_1 \boxtimes \cdots \boxtimes M_n$.

So there are natural embeddings $M_i \hookrightarrow \boxtimes_{i\in I} M_i$. Since M_i acts bijectively on the M_j with $j \neq i$, the implication

$$a_i \leqslant a_j \implies a_j = 1 \tag{14}$$

holds for $a_i \in M_i$ and $a_j \in M_j$ with $i \neq j$.

Theorem 1. *Let $M = \boxtimes_{i\in I} M_i$ be a crossed product of self-similar left hoops.*

(a) *Every $a \in M$ can be uniquely written in the form $a = a_1 \wedge \cdots \wedge a_n$ with $n \in \mathbb{N}$ and $a_j \in M_{i_j} \smallsetminus \{1\}$ for distinct $i_1, \ldots, i_n \in I$. (The case $a = 1$ is covered by $n = 0$.)*
(b) *If I is totally ordered, any $a \in M$ can be uniquely written in the form $a = a_1 \cdots a_n$ with $n \in \mathbb{N}$ and $a_j \in M_{i_j} \smallsetminus \{1\}$ for distinct $i_1 < \ldots < i_n$ in I.*

Proof. a) For $b \in M_i$ and $c \in M_j$ with $i \neq j$, the equation $c \to x = b$ has a unique solution $x \in M_i$. Let us write $x = b^c$. Thus $c \to b^c = b$ (which is also $(c \to b)^c$). In particular, Eq. (7) yields

$$bc = b^c \wedge c. \tag{15}$$

So the existence of a representation $a = a_1 \wedge \cdots \wedge a_n$ follows. To prove uniqueness, let $J \subset I$ be finite and $a_1, \ldots, a_n \in \bigcup_{j\in J} M_j$. For $i \in I$ and $b \in M_i$, we verify the implication

$$a_1 \wedge \cdots \wedge a_n \leqslant b < 1 \implies i \in J. \tag{16}$$

For $|J| = 1$, this follows by (14). Proceeding by induction on $|J|$, we can assume that $a_1, \ldots, a_{n-1} \in \bigcup_{k\in J\smallsetminus\{j\}} M_k$ for some $j \in J$. With $c := a_1 \wedge \cdots \wedge a_{n-1}$, this gives $(a_n \to c)a_n \leqslant b$. So $(a_n \to a_1) \wedge \cdots \wedge (a_n \to a_{n-1}) = a_n \to c \leqslant a_n \to b$. Since $a_n \to a_\ell \in \bigcup_{k\in J\smallsetminus\{j\}} M_k$ for $\ell < n$, and $a_n \to b \in M_i$, the inductive hypothesis gives $i \in J \smallsetminus \{j\}$ or $a_n \to b = 1$.

Now let $a_1 \wedge \cdots \wedge a_n = b_1 \wedge \cdots \wedge b_m$ be two representations as in (a). If $n = 0$ or $m = 0$, the meets on both sides are empty, and there is nothing to prove. Thus, we assume that $m, n > 0$. So there is a unique $j \in \{1, \ldots, n\}$ with $a_j, b_m \in M_i$ for some $i \in I$. After rearrangement of the a_i, we can assume that $j = n$. As above, this implies that $(a_n \to a_1) \wedge \cdots \wedge (a_n \to a_{n-1}) \leqslant a_n \to b_m$. Hence (16) gives $a_n \leqslant b_m$, and by symmetry, $a_n \geqslant b_m$. By induction, this shows that the representation in (a) is unique.

b) Assume that $b \in M_i$ and $c \in M_j$ with $i \neq j$. Define ${}^b c := b^c \to c$. Then Eq. (5) can then be rewritten as a relation $bc = ({}^b c)(b^c)$ with ${}^b c \in M_j$ and $b^c \in M_i$. Hence, by Eq. (15), the product representation in (b) is equivalent to the $\wedge$-representation in (a). $\square$

Remark. Crossed products of groups and monoids have been considered by several authors [50,16,49,43,41,15,36,18]. Owing to the founders, they are also called *Zappa–Szép products*. Gebhardt and Tawn [36] define an internal Zappa–Szép product of a pair of submonoids G, H of a monoid K by the property that every element of K has a unique representation $gh = h'g'$ with $g, g' \in G$ and $h, h' \in H$. They show that Picantin's crossed products [41] are of that type. On the other hand, they give an example of an internal and external Zappa–Szép product in the sense of Brin [15] which is not a crossed product in Picantin's sense [41]. They prove that both concepts coincide for Garside monoids.

Definition 6. For a family of partially ordered sets M_i with a distinguished element 1, we define the *cardinal sum* $\biguplus_{i \in I} M_i$ to be the partially ordered set of all $(a_i) \in \prod_{i \in I} M_i$ with $a_i = 1$ for almost all $i \in I$ (cf. [19]).

For example, if the M_i are left hoops, $\biguplus_{i \in I} M_i$ is a $\wedge$-semilattice. As an immediate consequence of Theorem 1, we have

Corollary 1. *As a $\wedge$-semilattice, a crossed product $\boxtimes_{i \in I} M_i$ of self-similar left hoops is isomorphic to the cardinal sum $\biguplus_{i \in I} M_i$.*

Secondly, crossed products are associative in the following sense.

Corollary 2. *Let $\boxtimes_{i \in I} M_i$ be a crossed product of self-similar left hoops, and let $I = \bigsqcup_{j \in J} I_j$ be a partition. Then $\boxtimes_{i \in I} M_i \cong \boxtimes_{j \in J} \big(\boxtimes_{i \in I_j} M_i\big)$.*

Proof. We only have to verify that for distinct $j, k \in J$, any element $a \in \boxtimes_{i \in I_j} M_i$ acts bijectively on $\boxtimes_{i \in I_k} M_i$. Since $b \mapsto (a \to b)$ respects meets, it is enough to verify that a acts bijectively on each M_ℓ with $\ell \notin I_j$. By Eq. (4), this reduces to the case $a \in M_i$ with $i \in I_j$. $\square$

Note that the converse of Corollary 2 is false: a crossed product $\boxtimes_{j \in J} \big(\boxtimes_{i \in I_j} M_i\big)$ need not give rise to a crossed product $\boxtimes_{i \in I} M_i$. For example, $M_1 \boxtimes (M_2 \boxtimes M_3)$ fails to be a crossed product $M_1 \boxtimes M_2 \boxtimes M_3$ whenever M_1 permutes M_2 and M_3, as is typical for Garside groups arising from solutions of the quantum Yang–Baxter equation ([17,23]; [47], Theorem 2).

As a third consequence, we get a simple characterization of crossed products. Recall that a subset Δ of a partially ordered set Ω is *convex* if $a \leqslant b \leqslant c$ with $a, c \in \Delta$ and $b \in \Omega$ implies that $b \in \Delta$.

Corollary 3. *Let M be a self-similar left hoop with self-similar L-subalgebras M_i, $i \in I$. Then $M = \boxtimes_{i \in I} M_i$ if and only if the following are satisfied.*

(a) *Every element of M is of the form $a_1 \wedge \cdots \wedge a_n$ with $a_i \in \bigcup_{i \in I} M_i$.*
(b) *The M_i are convex with $M_i \cap M_j = \{1\}$ for distinct $i, j \in I$.*
(c) *If $a \in M_i$ and $b \in M_j$ with $i \neq j$, then $a \to b \in M_j$.*

Proof. The necessity of (a)–(c) is obvious. As to the convexity of M_i, assume that $a \geqslant b \in M_i$ with $a \in M_j$. If $i \neq j$, then $b \to a = b \to 1$, which gives $a = 1 \in M_i$.

Conversely, let (a)–(c) be satisfied. Then M is generated by $\bigwedge_{i \in I} M_i = \bigcup_{i \in I} M_i$. For distinct $i, j \in I$ and $a \in M_i$, we have to verify that the map $\sigma_a \colon M_j \to M_j$ with $\sigma_a(b) := a \to b$ is an order automorphism. Thus, assume that $a \to b \leqslant a \to b'$ holds for some $b, b' \in M_j$. Then $1 = (a \to b) \to (a \to b') = (b \to a) \to (b \to b')$. Hence $b \to a \leqslant b \to b'$. By (b) and (c), this implies that $b \to b' \in M_i \cap M_j$. Thus (b) yields $b \to b' = 1$, that is, $b \leqslant b'$. In particular, this shows that σ_a is injective.

To verify the surjectivity of σ_a, let $c \in M_j$ be given. By (a), we have $ca = b_1 \wedge \cdots \wedge b_n$ for some $b_k \in \bigcup_{\ell \in I} M_\ell$. Hence $c \leqslant a \to (b_1 \wedge \cdots \wedge b_n) \leqslant (a \to b_i)$, and (b) implies that $a \to b_i \in M_j$ for all $i \in \{1, \ldots, n\}$. So, if $b_i \in M_k$ for $k \neq j$, then $a \to b_i \in M_j \cap M_k = \{1\}$, which yields $ca = a \wedge b$ for some $b \in M_j$. Hence $ca = (a \to b)a$, and thus $c = a \to b$. □

Example 3. Let M, H be self-similar left hoops. By (8), the monoid automorphisms of M form a subgroup $\mathrm{Aut}(M)$ of $\mathfrak{S}^+(M)$. Hence every monoid homomorphism $\sigma \colon H \to \mathrm{Aut}(M)$ defines an action of H on M, and by Definition 2, this action is compatible with the trivial action of M on H. Conversely, if an action $\sigma \colon H \to \mathfrak{S}^+(M)$ is compatible with the trivial action of M on H, the image of σ is in $\mathrm{Aut}(M)$. The crossed product $M \boxtimes H$ then coincides with the semi-direct product $M \rtimes H$. Indeed, every element of $M \boxtimes H$ is of the form $x \wedge a = (x \to a)x = ax$ with $a \in H$ and $x \in M$. So $ax = (a \to x)a$, which shows that the action $\sigma \colon H \to \mathrm{Aut}(M)$ is induced by conjugation.

3. The quasi-centre of a right ℓ-group

In this section, we show that the quasi-centre $N(G) = \{a \in G \mid aG^+a^{-1} = G^+\}$ of a right ℓ-group G with positive cone G^+ is lattice-ordered and, under a weak noetherian hypothesis, gives rise to a crossed product decomposition of G.

Proposition 4. *Let G be a right ℓ-group. For $a \in G$, the following are equivalent.*

(a) *$a \in N(G)$.*

(b) *The map $b \mapsto ab$ is an order isomorphism.*
(c) $\forall\, b \in G\colon a \leqslant b \iff a \preccurlyeq b$.
(d) $\forall\, b \in G\colon b \leqslant a \iff b \preccurlyeq a$.

Proof. (a) ⇒ (b): We have $b \leqslant c \overset{(1)}{\iff} cb^{-1} \in G^+ \overset{(a)}{\iff} acb^{-1}a^{-1} \in G^+ \overset{(1)}{\iff} ab \leqslant ac$.

(b) ⇒ (c): $a \leqslant b \overset{(b)}{\iff} 1 \leqslant a^{-1}b \overset{(1)}{\iff} b^{-1} \leqslant a^{-1} \overset{(2)}{\iff} a \preccurlyeq b$.

(c) ⇒ (d): We have $b \leqslant a \overset{(1)}{\iff} a \leqslant ab^{-1}a \overset{(c)}{\iff} a \preccurlyeq ab^{-1}a \overset{(2)}{\iff} a^{-1}ba^{-1} \leqslant a^{-1} \overset{(1)}{\iff} a^{-1} \leqslant b^{-1} \overset{(2)}{\iff} b \preccurlyeq a$.

(d) ⇒ (a): $1 \leqslant b \overset{(1)}{\iff} a^{-1} \leqslant ba^{-1} \overset{(2)}{\iff} ab^{-1} \preccurlyeq a \overset{(d)}{\iff} ab^{-1} \leqslant a \overset{(1)}{\iff} 1 \leqslant aba^{-1}$. □

The following proposition (see Proposition 20 of [47]) underlines the importance of the quasi-centre for an arbitrary right ℓ-group. For example, it implies that the quasi-centre is always a distributive sublattice.

Proposition 5. *The quasi-centre $N(G)$ of a right ℓ-group G is an ℓ-subgroup of G.*

Proof. Since $N(G)$ is a subgroup, it suffices to show that $N(G)$ is a sublattice. Thus, let $g, h \in N(G)$ be given. For any $a \in G^+$, we have $ga \geqslant g$ and $ha \geqslant h$ by Proposition 4(b). Hence $(g \wedge h)a = ga \wedge ha \geqslant g \wedge h$, and thus $(g \wedge h)G^+(g \wedge h)^{-1} \subset G^+$.

Now Proposition 4(b) implies that $1 \leqslant 1 \vee gh^{-1} = g(g^{-1} \vee h^{-1})$, which yields $(g^{-1} \vee h^{-1})^{-1} \leqslant g$. By symmetry, we obtain $(g^{-1} \vee h^{-1})^{-1} \leqslant g \wedge h$. Furthermore, $(g^{-1} \vee h^{-1})(g \wedge h) = g^{-1}(g \wedge h) \vee h^{-1}(g \wedge h) = (1 \wedge g^{-1}h) \vee (h^{-1}g \wedge 1) \leqslant 1$. Hence $(g \wedge h)^{-1}(g^{-1} \vee h^{-1})^{-1} \geqslant 1$. Conjugation with $g \wedge h$ yields $(g^{-1} \vee h^{-1})^{-1}(g \wedge h)^{-1} \geqslant 1$. Thus $g \wedge h \leqslant (g^{-1} \vee h^{-1})^{-1} \leqslant g \wedge h$, which proves that

$$g \wedge h = (g^{-1} \vee h^{-1})^{-1}.$$

So it is enough to verify that $g \wedge h \in N(G)$. Assume that $(g \wedge h)a(g \wedge h)^{-1} \in G^+$ for some $a \in G$. Then $g \wedge h \leqslant (g \wedge h)a \leqslant ga$, which gives $g^{-1}(g \wedge h) \leqslant a$. Similarly, $h^{-1}(g \wedge h) \leqslant a$. So we obtain $1 = (g^{-1} \vee h^{-1})(g \wedge h) = g^{-1}(g \wedge h) \vee h^{-1}(g \wedge h) \leqslant a$, which proves that $g \wedge h$ is normal. □

As usual, an *interval* $[a, b]$ of a partially ordered set Ω with $a \leqslant b$ consists of the elements $c \in \Omega$ with $a \leqslant c \leqslant b$.

Corollary. *Let G be a right ℓ-group, and let $a, b \in N(G)$ with $a \vee b = 1$.*

(a) *Every element $c \in [ab, 1]$ is of the form $c = (c \wedge a) \vee (c \wedge b) = (c \vee a) \wedge (c \vee b)$.*
(b) *The maps $c \mapsto c \vee b$ and $c \mapsto c \wedge a$ give a lattice isomorphism $[ab, a] \cong [b, 1]$.*

Proof. Since a and b are normal, we have $ac \leqslant a$ and $bc \leqslant b$. Hence $c = (a \vee b)c = ac \vee bc \leqslant (c \wedge a) \vee (c \wedge b) \leqslant c$. By Proposition 4, $a(c \vee b) = ac \vee ab \leqslant c$. Hence $c \vee b \leqslant a^{-1}c$, and similarly, $c \vee a \leqslant b^{-1}c$. Thus $c \leqslant (c \vee a) \wedge (c \vee b) \leqslant b^{-1}c \wedge a^{-1}c = (b^{-1} \wedge a^{-1})c = c$.

b) This follows by the special case $c \in [ab, a]$, respectively $c \in [b, 1]$. □

Definition 7. We call a right ℓ-group G *quasi-noetherian* if $N(G)$ is noetherian. The elements of $P(G) := X(N(G)^-)$ will be called *primes* of G. We say that G has *enough normal elements* if any $a \in G^-$ admits a normal element $g \leqslant a$. If, in addition, $N(G)$ is cyclic, we call G *quasi-cyclic*.

If G has enough normal elements, Proposition 4 implies that every $g \in G$ belongs to some interval $[a, b]$ with $a, b \in N(G)$. By definition, a quasi-cyclic right ℓ-group G has a unique smallest strong order unit. However, G itself need not be noetherian, and even if G is noetherian, it need not be a Garside group.

Example 4. Let X be a set with a permutation $\pi \in \mathfrak{S}(X)$. Define $x \to y := \pi(y)$ if x, y belong to different π-orbits, and $x \to y := x$ otherwise, if $x \neq y$. Adjoining a logical unit 1, this makes $\widetilde{X} := X \sqcup \{1\}$ into a discrete L-algebra. The criterion [47], Proposition 19, implies that $S(\widetilde{X})$ is modular as a lattice. (We take the opportunity to correct an inaccuracy: in [47], the criterion states that the self-similar closure $S(Y)$ of a discrete L-algebra Y is modular if and only if $(x \to y) \to u = (y \to x) \to v$ with $x, y, u, v \in Y \smallsetminus \{1\}$ and $x \neq y$ implies that there exists $z \in X$ with $x \to z = u$ and $y \to z = v$. The correct statement should either allow $z \in S(Y)$ or restrict the assumption to $(x \to y) \to u = (y \to x) \to v < 1$.) Precisely, $S(\widetilde{X})$ is generated by X, with relations $x^2 = y^2$ whenever x and y belong to the same π-orbit. Furthermore, $S(\widetilde{X})$ is the negative cone of a right ℓ-group.

Unless π is an involution, the lattice of G is not distributive, that is, G is not the structure group of a cycle set. For any π-orbit P containing $x \in X$, the meet $\bigwedge P$ exists and coincides with x^2 if $|P| > 1$, and these meets are the primes of G. They generate the quasi-centre, and G has enough normal elements. The subgroup generated by each π-orbit is quasi-cyclic. So there is no iteration in the crossed product decomposition of G, or in other words, the underlying tree of this crossed product is of height 1. Example 6 will show that trees of infinite height also arise.

Note that G is always noetherian, while G is quasi-Garside if and only if π has finitely many orbits.

Now we apply the results of Section 2 to quasi-noetherian right ℓ-groups G. For any $a \in N(G) \cap G^-$, consider the right ℓ-subgroup

$$G_a := \bigcup_{n \in \mathbb{N}} [a^n, a^{-n}].$$

Theorem 2. *Let G be a quasi-noetherian right ℓ-group with enough normal elements. Then* $G^- = \boxtimes_{p\in P(G)} G_p^-$.

Proof. By Proposition 4, each G_p is a right ℓ-subgroup of G. Birkhoff's theorem [9] implies that $N(G)$ is a cardinal sum of the G_p with $p \in P(G)$. To show that G^- is a crossed product of the G_p^-, we apply Corollary 3 of Theorem 1. Condition (a) follows by the corollary of Proposition 5, while (b) is trivial. To verify (c), assume that $a \in [p^n, 1]$ and $b \in [q^n, 1]$ with distinct primes $p, q \in P(G)$. Since q^n is normal, $q^n a \leqslant q^n \leqslant b$. Whence $q^n \leqslant a \to b \leqslant 1$. $\square$

Corollary 1. *As a lattice, every quasi-noetherian right ℓ-group G with enough normal elements is a cardinal sum* $G = \boxplus_{p\in P(G)} G_p$.

Proof. As $G = N(G)G^-$, this follows immediately by Corollary 1 of Theorem 1. $\square$

Corollary 2. *For a quasi-noetherian right ℓ-group G with distinct primes $p, q \in P(G)$ and $a \in G_p^-$, the map $\sigma_a : G_q^- \to G_q^-$ with $\sigma_a(b) := a \to b$ is an order automorphism.*

Proof. Since G_{pq} has enough normal elements, Theorem 2 applies. $\square$

If G is noetherian with enough normal elements, any atom $x \in X(G^-)$ majorizes a unique prime $p \in P(G)$, and the map $x \mapsto p$ is a surjection $\pi : X(G^-) \twoheadrightarrow P(G)$. The fibers of π give a partition

$$X(G^-) = \bigsqcup_{p\in P(G)} X(G_p^-). \tag{17}$$

Corollary 2 implies that $x \to y \in X(G_q^-)$ holds for $x \in X(G_p^-)$ and $y \in X(G_q^-)$ whenever $p \neq q$. The operation $x \rightsquigarrow y$ gives rise to similar permutations of the $X(G_q^-)$, and both operations $x \to y$ and $x \rightsquigarrow y$ are related by a duality which was first observed in the context of non-degenerate cycle sets ([45], Definition 1):

$$(x \to y) \rightsquigarrow (y \to x) = x = (x \rightsquigarrow y) \to (y \rightsquigarrow x). \tag{18}$$

By [47], Proposition 4, Eq. (18) even holds for any pair of distinct atoms $x, y \in X(G^-)$. However, $x \to y$ or $x \rightsquigarrow y$ need no longer be atoms if $\pi(x) = \pi(y)$.

Example 5. The *pure braid group* $\mathscr{P}_n$ on n strands is the kernel of the natural epimorphism $\mathscr{B}_n \twoheadrightarrow \mathfrak{S}_n$ from the braid group $\mathscr{B}_n$ onto the symmetric group. Artin's combing of braids [1] gives rise to a split short exact sequence $F_n \hookrightarrow \mathscr{P}_{n+1} \overset{p}{\twoheadrightarrow} \mathscr{P}_n$, where F_n denotes the free group with n generators and p is the projection which forgets the $(n+1)$-th strand. Thus $\mathscr{P}_{n+1} \cong F_n \rtimes \mathscr{P}_n$, and by induction, $\mathscr{P}_{n+1} \cong F_n \rtimes \cdots \rtimes F_2 \rtimes F_1$. The action of $\mathscr{P}_n$ on F_n is induced by Artin's action of $\mathscr{B}_n$ on F_n (see [38], Section 3). If we endow

F_n with Bessis' dual quasi-Garside structure [6], the negative cone F_n^- is stable under the action of $\mathscr{B}_n$, and the Garside element of F_n^- is fixed under this action. Therefore, Example 3 yields a crossed product decomposition

$$\mathscr{P}_n^- \cong F_{n-1}^- \boxtimes \cdots \boxtimes F_2^- \boxtimes F_1^-$$

which makes $\mathscr{P}_n$ into a quasi-Garside group. We remark that the semidirect product $\mathscr{P}_n \cong F_{n-1} \rtimes \cdots \rtimes F_2 \rtimes F_1$ has also been used to make $\mathscr{P}_n$ into a totally ordered group, using the Magnus expansion of free groups [38]. With a slight modification of the argument, this implies that the fundamental group of the complement of a fibre-type hyperplane arrangement admits a (two-sided) total ordering ([38], Theorem 19). The pure braid group arises as a special case, first considered by Fadell and Neuwirth [31], where the hyperplanes form the zero set of the discriminant of a polynomial.

The conjugation by normal elements yields another permutation of the atoms.

Proposition 6. *Let G be a right ℓ-group, and $a \in N(G)$. For any atom $x \in X(G^-)$,*

$$axa^{-1} = \begin{cases} a \to x & \text{for } a \nleqslant x \\ (x \to a) \to a & \text{for } a \leqslant x \end{cases} \qquad a^{-1}xa = \begin{cases} a \rightsquigarrow x & \text{for } a \nleqslant x \\ (x \rightsquigarrow a) \rightsquigarrow a & \text{for } a \leqslant x \end{cases}$$

Proof. Since a is normal, there are atoms $y, z \in X(G^-)$ with $axa^{-1} = y$ and $a^{-1}xa = z$. Hence $ax = ya$, and thus $y = a \to ax = \big((x \to a) \to a\big)(a \to x)$. This gives the first dichotomy. The second one follows by $z = a \rightsquigarrow xa = (a \rightsquigarrow x)\big((x \rightsquigarrow a) \rightsquigarrow a\big)$, using Proposition 4. □

In general, the G_p of Theorem 2 need not be quasi-cyclic. If every convex subgroup of G is quasi-normal, each G_p may further decompose into a crossed product, which yields a tree-like crossed product representation of G^-:

$$G^- = \mathop{\boxtimes}_{p \in P(G)} \mathop{\boxtimes}_{q \in P(G_p)} \cdots (G_{pq\cdots})^-.$$

Remark. If G is noetherian, the partition (17) shows that the stacked crossed product representation goes parallel to an iterated partition of the atoms. Starting with the partition (17), each $X(G_p^-)$ admits a similar partition, and so on. Ultimately, this leads to a partition

$$X(G^-) = \bigsqcup_{i \in I} X(G_i^-) \tag{19}$$

where each G_i is a quasi-cyclic subgroup of G.

In particular, $x \to y$ is a partial operation on $X(G^-)$ which is defined everywhere unless x and y belong to one of the sets $X(G_i^-)$ of the refined partition (19). Apart from this exception, the structure of $X(G^-)$ is that of a cycle set.

Recall that a *cycle set* [45] is a set X with a binary operation $\cdot$ such that the left multiplication $y \mapsto x \cdot y$ is bijective and the equation

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z) \tag{20}$$

holds for all $x, y, z \in X$. If the square map $x \mapsto x \cdot x$ is bijective, too, the cycle set X is called non-degenerate. A cycle set X is said to be *square-free* [45] if the square map $x \mapsto x \cdot x$ is the identity. (So X is non-degenerate in this case.)

Definition 8. Let X be a set with a partition $X = \bigsqcup_{i \in I} X_i$ and a partial binary operation $\cdot$ such that $x \cdot y$ is defined for $x \in X_i$ and $y \in X_j$ with $i \neq j$. We call X a *partial cycle set* if for any $x \in X_i$, the left multiplication $y \mapsto x \cdot y$ is a permutation of $\bigsqcup_{j \in I \smallsetminus \{i\}} X_j$, and Eq. (20) holds whenever the products on both sides are defined.

Thus, in the noetherian case, Corollary 2 of Theorem 2 states that (19) is a partial cycle set with $x \cdot y := x \to y$. Since $x \cdot x$ is never defined, the concept of non-degeneracy does not immediately apply to a partial cycle set. However, by [45], Proposition 2, a cycle set $(X; \to)$ is non-degenerate if and only if it admits a *dual*, that is, a cycle set structure $(X; \rightsquigarrow)$ such that both operations are related by Eq. (18). In this sense, the partial cycle set (19) is non-degenerate. On the other hand, the definition of a partial cycle set $X = \bigsqcup_{i \in I} X_i$ already implies that for $x \in X_i$ and $y \notin X_i$, the product $x \cdot y$ never belongs to X_i. Thus, in a negative sense, partial cycle sets are even "square-free".

Every non-degenerate cycle set $(X, \cdot)$ admits a unique extension to a cycle set on the free abelian group $\mathbb{Z}^{(X)}$, so that the equations

$$\begin{aligned} a \cdot (b + c) &= (a \cdot b) + (a \cdot c) \\ (a + b) \cdot c &= (a \cdot b) \cdot (a \cdot c) \end{aligned} \tag{21}$$

hold for the extended operation on $\mathbb{Z}^{(X)}$. On the same set $\mathbb{Z}^{(X)}$, there is another group operation $\circ$, henceforth written as juxtaposition, given by the equation

$$a + b = (a \cdot b) \circ a = (a \cdot b)a, \tag{22}$$

which makes $\mathbb{Z}^{(X)}$ into a group G_X, the *structure group* [30,45] of X. Chouraqui [17] observed that G_X is a Garside group if X is finite. In general, G_X is a noetherian right ℓ-group with negative cone $\mathbb{N}^{(X)}$.

Remark. Note the similarity between Eq. (22) and the hoop equation (7): For distinct atoms a, b, the sum $a + b$ is equal to the meet $a \wedge b$. Accordingly, the operation $\cdot$ in

X corresponds to the operation $\to$ of a left hoop. Indeed, Eq. (24) of [47] states that $x \cdot y = x \to y$ holds for distinct atoms x, y. For $x = y$, however, both structures fall apart: $x \to x = 1$ in a left hoop, while for a cycle set X, the square map $D(x) := x \cdot x$ is a bijection if and only if X is non-degenerate. For details, see [47], Section 4.

Eq. (4) also has a striking analogue in G_X, namely, the equation

$$ab \cdot c = a \cdot (b \cdot c). \tag{23}$$

So the structure group G_X of a non-degenerate cycle set X acts on $X = X(G_X^-)$, which leads to a partition $X = \bigsqcup_{i \in I} X_i$ into orbits under G_X.

Theorem 3. *Let X be a square-free cycle set. The structure group G_X has enough normal elements if and only if its orbits on X are finite. If G_X has finite orbits, then $Y \mapsto \bigwedge Y$ gives a bijection between the orbits $Y \subset X$ and the primes of G_X.*

Proof. Since G_X is generated by atoms, Eq. (23) shows that a subset $Y \subset X$ is invariant under G_X if and only if the equivalence $y \in Y \Leftrightarrow x \cdot y \in Y$ holds for all $x \neq y$ in X. By the above remark, this statement remains true if $x \cdot y$ is replaced by $x \to y$. Now assume that G_X has enough normal elements, and let $y \in X$ be an atom. Then there is a normal element $a \leqslant y$. For any $x \in X$, this implies that $ax \leqslant a$. Hence $a \leqslant x \to a \leqslant x \to y$. By [47], Proposition 6, G_X is distributive as a lattice. So the set of atoms $y \geqslant a$ is finite, which shows that the orbits of G_X are finite.

Conversely, assume that G_X has finite orbits $Y \subset X$. Define $p := \bigwedge Y$. For any $x \in X$, we have $x \to p = \bigwedge_{y \in Y}(x \to y)$. For $x \notin Y$, this gives $x \to p = p$. If $x \in Y$, we obtain $x \to p = \bigwedge(Y \setminus \{x\})$. Thus, in any case, $p \leqslant x \to p$, which yields $px \leqslant p$. Hence $pG^- p^{-1} \subset G^-$. More precisely, the case $x \notin Y$ gives $px = (x \to p)x = (p \to x)p$, that is, $pxp^{-1} = p \to x$. Therefore, the conjugation $x \mapsto pxp^{-1}$ is bijective on $X \setminus Y$. So it acts bijectively on Y, too, which proves that p is normal. Thus G_X has enough normal elements. Now let $y \in Y$ be given. For $x \in X$, let $\sigma(x)\colon X \to X$ be the permutation $\sigma(x)(z) := x \cdot z$. Since Y is finite, we have $\sigma(x)^{-1}(y) = \sigma^i(y)$ for some $i \in \mathbb{N}$. Therefore, any element of Y is of the form $\sigma(x_1) \cdots \sigma(x_n)(y)$ with $x_1, \dots, x_n \in X$. For every $x \in X$, and any normal element $a \in G$ with $a \leqslant y$, we have $ax \leqslant a \leqslant y$, hence $a \leqslant x \to y$. Thus $a \leqslant p$, which proves that p is prime. By the first paragraph, every prime p arises in this way. Since G_X is distributive, each atom $y \geqslant p$ belongs to Y. Whence $Y \mapsto \bigwedge Y$ is bijective. □

Corollary. *Let X be a square-free cycle set. Assume that the structure group G_X has enough normal elements. Then G_X is quasi-cyclic if and only if $G_X \cong \mathbb{Z}$.*

Proof. By Theorem 3, G_X has finite orbits, which are again cycle sets. Thus, if G_X is quasi-cyclic, X must be finite, hence a singleton by [45], Theorem 1. □

Example 5 in [45] shows that there are infinite square-free cycle sets with a single orbit. Thus, if the structure group G_X has infinite orbits, the concept "quasi-cyclic" has to be replaced by "transitive" with respect to the action on X. The following example shows that the corresponding tree may be of infinite height.

Example 6. Let X be the polynomial ring $\mathbb{F}_2[t]$ over the prime field $\mathbb{F}_2$, and let $v\colon \mathbb{F}_2[t] \to \mathbb{Z} \sqcup \{\infty\}$ be its t-adic valuation. For $x, y \in X$, we define

$$x \cdot y := y + t^{v(x+y)+1}.$$

For $x = y$, this gives $v(x + y) = \infty$, that is, $x \cdot x = x$ if we agree that $t^\infty = 0$. Since $v(x + (x \cdot y)) = v(x + y)$, we have

$$x \cdot (x \cdot y) = y$$

for all $x, y \in X$. So the left multiplication $y \mapsto x \cdot y$ is bijective. To verify Eq. (20), we have to show that

$$z + t^{v(x+z)+1} + t^{v(y+z+t^{v(x+y)+1}+t^{v(x+z)+1})+1} = z + t^{v(y+z)+1} + t^{v(x+z+t^{v(x+y)+1}+t^{v(y+z)+1})+1}$$

holds for $x, y, z \in X$. Substituting $a := x + z$ and $b := y + z$, the equation becomes

$$t^{v(a)+1} + t^{v(b+t^{v(a+b)+1}+t^{v(a)+1})+1} = t^{v(b)+1} + t^{v(a+t^{v(a+b)+1}+t^{v(b)+1})+1}.$$

As this equation is symmetric in a and b, we can assume that $v(a) < v(b)$. Thus, we have to verify

$$v(a) = v(a + t^{v(a+b)+1} + t^{v(b)+1}), \qquad v(b) = v(b + t^{v(a+b)+1} + t^{v(a)+1}).$$

This follows since $v(a + b) = v(a)$. So X is a square-free cycle set. Furthermore, one can show that X is self-dual (18), that is, it satisfies

$$(x \cdot y) \cdot (y \cdot x) = x$$

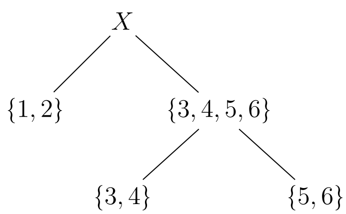
for all $x, y \in X$.

Now let $y = \sum_{i\in\mathbb{N}} a_i t^i \in X$ with $a_i \in \mathbb{F}_2$ be given. For $x := \sum_{i<m} a_i t^i + (a_m + 1)t^m$ and $m \in \mathbb{N}$, we have $v(x + y) = m$. Thus $x \cdot y = y + t^{m+1}$. Therefore, X splits in two orbits $X_0 := \{x \in X \mid v(x) = 0\}$ and $X_1 := \{x \in X \mid v(x) > 0\}$. Furthermore, the maps $x \mapsto tx$ and $x \mapsto 1 + tx$ are isomorphisms $X \xrightarrow{\sim} X_1$ and $X \xrightarrow{\sim} X_0$, respectively. Thus, each orbit X_i splits again into two orbits under G_{X_i}, and so on.

Theorem 3 and its corollary shed some light upon the decomposition of a cycle set into its orbits under the structure group. On the other hand, due to the partial cycle set structure (19) arising from an iterated crossed product representation, right ℓ-groups can be regarded as vast generalizations of the structure groups of cycle sets.

Note that the partial cycle set (19) completely describes the tree structure of G^- as an iterated crossed product. One is tempted to believe that a partial cycle set like (19) induces a cycle set structure on the index set I. However, the following example shows that the "blocks" X_i of a partial cycle set $X = \bigsqcup_{i \in I} X_i$ do not always admit an induced cycle set structure.

Example 7. Let $X = \{1, 2, 3, 4, 5, 6\}$ be the cycle set with left multiplications $\sigma_x(y) := x \cdot y$ given by $\sigma_1 := (12)(35)(46)$, $\sigma_2 := (12)(34)(56)$, and $\sigma_3 = \sigma_4 = \sigma_5 = \sigma_6 := (34)(56)$. The tree structure of X is given by



The ultimate partition $X = \{1, 2\} \sqcup \{3, 4\} \sqcup \{5, 6\}$ does not induce a cycle set. In fact, $\{1, 2\} \cdot \{3, 4\}$ is not well defined since $1 \cdot 3 = 5$, while $2 \cdot 3 = 4$.

Concluding remark. There are many ways to generate a right ℓ-group G by an L-algebra. In general, the negative cone G^- is the greatest L-algebra with this property. For quasi-cyclic G, every non-trivial interval $[u, 1]$ with $u \in N(G)$ is such an L-algebra. If G is quasi-noetherian with enough normal elements, Theorem 2 represents G as a crossed product of right ℓ-groups G_i. Thus, if each G_i is generated by an L-algebra X_i, Theorem 1 shows that the wedge of the X_i (Definition 3) gives a generating L-algebra of G. Now the reader may ask how L-algebras are related to *Garside families* and *Garside germs* [24]. For a Garside group G, the set of *primitive* elements [21] in G^- is defined to be the closure with respect to $\to$ of the set of atoms in G^-. The elements of the closure with respect to $\to$ and $\wedge$ are called *simple* [21]. Both sets are L-algebras and generate G^- as a monoid. More generally, if G is quasi-Garside with Garside element Δ, the Garside family of divisors of Δ is an L-algebra.

In contrast to L-algebras, Garside families are defined in the context of categories rather than monoids. For L-algebras, such an extension is possible, but it has not been carried out. Garside families formalize the subsets $\mathscr{S}$ of a category $\mathscr{C}$ with epic morphisms so that every $f \in \mathscr{C}$ factors through a "longest" morphism in $\mathscr{S}$ while this process, applied to the remaining factor of f, yields a factorization $f = s_1 \cdots s_n$ with $s_i \in \mathscr{S}$. If a Garside family is closed with respect to *right divisors* (see [24], Definition 4.7), the ambient category $\mathscr{C}$ is completely determined by $\mathscr{S}$ as a *germ*, that is, with respect to the partial multiplication in $\mathscr{S}$. Similarly, every L-algebra X embeds uniquely into a monoid

$S(X)$, the self-similar closure of X, and there is a canonical morphism $S(X) \to G(X)$ into a group, the *structure group* of X. Under favourable conditions, $S(X) \to G(X)$ is an embedding and $G(X)$ is a right ℓ-group. Theorem 1 deals with such a case, and its main point is that any crossed product representation of $G(X)$ (or its negative cone) is obtained on the level of L-algebras.

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