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Integrality over fixed rings of automorphisms in a Lie nilpotent setting

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ABSTRACT

Let R be a Lie nilpotent algebra of index $k \geq 1$ over a field K of characteristic zero. If G is an n-element subgroup $G \subseteq \mathrm{Aut}_K(R)$ of K-automorphisms, then we prove that R is right integral over $\mathrm{Fix}(G)$ of degree n^k. In the presence of a primitive n-th root of unity $e \in K$, for a K-automorphism $\delta \in \mathrm{Aut}_K(R)$ with $\delta^n = \mathrm{id}_R$, we prove that the skew polynomial algebra $R[w, \delta]$ is right integral of degree n^k over $\mathrm{Fix}(\delta)[w^n]$.

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1. Introduction

The notions of algebraicity and integrality play an important role in algebra and algebraic number theory. The aim of our paper is to present two integrality results, which are non-commutative generalizations of well known facts, the first one is closely

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related to classical Galois theory and the second one is related to the integrality of the commutative polynomial algebra $C[w]$ over $C[w^n]$.

A natural replacement of commutativity is the following: a ring R is called Lie nilpotent of index k, if the left normed commutator product (with $[x, y] = xy - yx$)

$$[[\dots[[x_1, x_2], x_3], \dots, x_k], x_{k+1}] = 0$$

is a polynomial identity on R.

In section (5) we prove the following two integrality theorems.

Theorem (A). *Let R be a Lie nilpotent algebra of index $k \geq 1$ over a field K of characteristic zero. If G is an n-element subgroup $G \subseteq \mathrm{Aut}_K(R)$ of K-automorphisms, then R is right integral over $\mathrm{Fix}(G)$ of degree n^k. In other words, for any $r \in R$ we have*

$$c_0 + rc_1 + \cdots + r^{n^k-1}c_{n^k-1} + r^{n^k} = 0$$

for some $c_i \in \mathrm{Fix}(G)$, $0 \leq i \leq n^k - 1$.

Remarks to (A). Let $K \subseteq L$ be a field extension and $G \subseteq \mathrm{Aut}_K(L)$ be a finite subgroup of the K-automorphisms of L. Then $a \in L$ is a root of the monic polynomial $f(w) = \Pi_{\gamma\in G}(w - \gamma(a))$ of degree $n = |G|$ and the coefficients of $f(w) \in L[w]$ are in the fixed intermediate field $K \subseteq \mathrm{Fix}(G) \subseteq L$ of G. Thus L is integral over $\mathrm{Fix}(G)$ of degree n. We note that the replacement of L by a commutative K-algebra R (in the above argument) immediately yields Theorem (A) for $k = 1$.

If $\mathrm{Fix}(G) = K$ and $a \in L$ is also a root of a K-irreducible polynomial $q(w) \in K[w]$, then $q(w)$ is a divisor of $f(w)$. For a finite normal field extension $K \subseteq L$, this observation shows that the Galois group $G = \mathrm{Aut}_K(L)$ acts transitively on the roots of $q(w)$.

Theorem (B). *Let R be a Lie nilpotent algebra of index $k \geq 1$ over a field K of characteristic zero. If $e \in K$ is a primitive n-th root of unity and $\delta \in \mathrm{Aut}_K(R)$ is a K-automorphism with $\delta^n = \mathrm{id}_R$, then the skew polynomial algebra $R[w, \delta]$ is right integral of degree n^k over the K-subalgebra $\mathrm{Fix}(\delta)[w^n]$ of $R[w, \delta]$. In other words, for any $f(w) \in R[w, \delta]$ we have*

$$g_0(w^n) + f(w)g_1(w^n) + \cdots + f^{n^k-1}(w)g_{n^k-1}(w^n) + f^{n^k}(w) = 0$$

for some $g_i(w^n) \in \mathrm{Fix}(\delta)[w^n]$, $0 \leq i \leq n^k - 1$ (notice, that w^n is central in $R[w, \delta]$).

Remarks to (B). For a commutative K-algebra C, the choice $R = C$ and $\delta = \mathrm{id}_C$ in Theorem (B) immediately yields the well-known fact, that the polynomial algebra $C[w]$ is integral of degree n over the C-subalgebra $C[w^n]$ generated by the power w^n. A simple illustration of the case $n = 2$ is the following: any $f(w) \in C[w]$ can be uniquely written as $f(w) = p(w^2) + wq(w^2)$ and $f^2(w) - 2p(w^2)f(w) + (p^2(w^2) - w^2q^2(w^2)) = 0$ proves the integrality of $C[w]$ over $C[w^2]$ of degree 2.

The proofs of Theorem (A) and (B) are heavily based on the use of certain matrix algebras over the given Lie nilpotent algebra R. The so called Lie nilpotent Cayley–Hamilton theorem from [3] is an indispensable ingredient of our development. In order to provide a self-contained treatment, we present the necessary prerequisites in sections (2), (3) and (4).

Throughout the paper an algebra R means a not necessarily commutative unitary algebra over a field K of characteristic zero, all K-subalgebras contain the identity and all K-endomorphisms preserve the identity. The group of units in R is denoted by $\mathrm{U}(R)$ and the centre of R is denoted by $\mathrm{Z}(R)$. The notation for the full $n \times n$ matrix algebra over R is $\mathrm{M}_n(R)$ and $\mathrm{GL}_n(R) = \mathrm{U}(\mathrm{M}_n(R))$. The matrix $E_{i,j} \in \mathrm{M}_n(R)$ has 1 in the (i, j) position and zeros in all other positions. The fixed K-subalgebra of a K-endomorphism $\delta : R \longrightarrow R$ is $\mathrm{Fix}(\delta) = \{r \in R \mid \delta(r) = r\}$.

2. The algebra of skew centralizing matrices

Using the natural (element-wise) extension $\delta_n : \mathrm{M}_n(R) \longrightarrow \mathrm{M}_n(R)$ of a K-endomorphism $\delta : R \longrightarrow R$ and a matrix $W \in \mathrm{M}_n(R)$, one can define the subalgebra of skew (δ, W)-centralizing matrices in $\mathrm{M}_n(R)$ as

$$\mathrm{M}_n(R, \delta, W) = \{A \in \mathrm{M}_n(R) \mid \delta_n(A)W = WA\}.$$

For the sake of brevity, in the rest of the paper we omit the qualifier “skew”.

If $W \in \mathrm{M}_n(\mathrm{Fix}(\delta))$, then $\mathrm{M}_n(R, \delta, W)$ is closed with respect to the action of δ_n.

Example (1). The natural $\mathbb{Z}_2$-grading $E = E_0 \oplus E_1$ defines an automorphism $\varepsilon(g_0 + g_1) = g_0 - g_1$ of the Grassmann algebra

$$E = K \langle v_1, \ldots, v_i, \ldots \mid v_i v_j + v_j v_i = 0 \text{ for all } 1 \le i \le j \rangle$$

generated by the infinite sequence of anticommutative indeterminates $(v_i)_{i \ge 1}$ with $v_i^2 = 0$.

Using the invertible diagonal matrix

$$Q_d = E_{1,1} + \cdots + E_{d,d} - E_{d+1,d+1} - \cdots - E_{n,n}$$

in $\mathrm{M}_n(K)$, we obtain the classical supermatrix algebra $\mathrm{M}_{n,d}(E)$ as

$$\mathrm{M}_{n,d}(E) = \mathrm{M}_n(E, \varepsilon, Q_d) = \{A \in \mathrm{M}_n(E) \mid \varepsilon_n(A)Q_d = Q_d A\}.$$

The shape of a matrix $A \in \mathrm{M}_{n,d}(E)$ is

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix},$$

where the square blocks $A_{1,1}$ and $A_{2,2}$ are of sizes $d \times d$ and $(n-d) \times (n-d)$ and the rectangular blocks $A_{1,2}$ and $A_{2,1}$ are of sizes $d \times (n-d)$ and $(n-d) \times d$. The entries of $A_{1,1}$ and $A_{2,2}$ are in the even part E_0 of E, while the entries of $A_{1,2}$ and $A_{2,1}$ are in the odd part E_1 of E.

We note that E is Lie nilpotent of index 2, the K-algebras $\mathrm{M}_n(E)$ and $\mathrm{M}_{n,d}(E)$ play an important role in Kemer's classification of T-prime T-ideals (see [2]). $\square$

Example (2). A $\mathbb{Z}_n$-grading of R is an n-tuple $(R_0, R_1, \dots, R_{n-1})$, where each R_i is a K-subspace of R such that

$$R = R_0 \oplus R_1 \oplus \dots \oplus R_{n-1}$$

is a direct sum and $R_i R_j \subseteq R_{i+j}$ for all $i, j \in \{0, 1, \dots, n-1\}$, where $i + j$ is taken in $\{0, 1, \dots, n-1\}$ modulo n. Using such a $\mathbb{Z}_n$-grading of R, the K-subalgebra $\mathrm{M}_n^g(R)$ of $\mathrm{M}_n(R)$ is defined as follows:

$$\mathrm{M}_n^g(R) = \{A = [a_{i,j}] \in \mathrm{M}_n(R) \mid a_{i,j} \in R_{j-i} \text{ for all } 1 \le i, j \le n\} =$$
$$\begin{bmatrix} R_0 & R_1 & \cdots & R_{n-2} & R_{n-1} \\ R_{n-1} & R_0 & R_1 & \ddots & R_{n-2} \\ \vdots & R_{n-1} & \ddots & \ddots & \vdots \\ R_2 & \ddots & \ddots & R_0 & R_1 \\ R_1 & R_2 & \cdots & R_{n-1} & R_0 \end{bmatrix},$$

where $j - i$ is taken in $\{0, 1, \dots, n-1\}$ modulo n.

If $e \in K$ is a primitive n-th root of unity, then

$$\widehat{e}(r_0 + r_1 + \dots + r_{n-1}) = r_0 + er_1 + \dots + e^{n-1} r_{n-1}$$

defines a K-automorphism $\widehat{e} : R \longrightarrow R$ of the $\mathbb{Z}_n$-graded K-algebra $R = R_0 \oplus R_1 \oplus \dots \oplus R_{n-1}$ and

$$D_e = e^{-1} E_{1,1} + e^{-2} E_{2,2} + \dots + e^{-n} E_{n,n}$$

is a diagonal matrix in $\mathrm{GL}_n(K)$. Now $\mathrm{M}_n^g(R)$ can be obtained as the algebra of $(\widehat{e}, D_e)$-centralizing matrices:

$$\mathrm{M}_n^g(R) = \mathrm{M}_n(R, \widehat{e}, D_e) = \{A \in \mathrm{M}_n(R) \mid \widehat{e}_n(A) D_e = D_e A\}. \quad \square$$

One of the remarkable properties of (δ, W)-centralizing matrices is the following.

2.1. Proposition. *If $W \in \mathrm{GL}_n(R) \cap \mathrm{M}_n(\mathrm{Z}(R))$ is an invertible matrix with central elements, then the trace of a matrix $A \in \mathrm{M}_n(R, \delta, W)$ is in the fixed ring of δ, i.e. $\mathrm{tr}(A) \in \mathrm{Fix}(\delta)$.*

Proof. Since the entries of W are central and $W^{-1}W = I_n$, a straightforward computation gives that $\mathrm{tr}(WAW^{-1}) = \mathrm{tr}(A)$ for all $A \in \mathrm{M}_n(R)$. Thus for a matrix $A \in \mathrm{M}_n(R, \delta, W)$ we have

$$\delta(\mathrm{tr}(A)) = \mathrm{tr}(\delta_n(A)) = \mathrm{tr}(WAW^{-1}) = \mathrm{tr}(A),$$

whence $\mathrm{tr}(A) \in \mathrm{Fix}(\delta)$ follows. $\square$

Any K-endomorphism $\delta : R \longrightarrow R$ can be naturally extended to a K-endomorphism $\delta_z : R[z] \longrightarrow R[z]$ of the polynomial ring $R[z]$: for $r_0, r_1 \ldots, r_t \in R$ take

$$\delta_z(r_0 + r_1 z + \cdots + r_t z^t) = \delta(r_0) + \delta(r_1)z + \cdots + \delta(r_t)z^t.$$

The elements of the skew polynomial ring $R[w, \delta]$ in the skew indeterminate w are left polynomials of the form $f(w) = r_0 + r_1 w + \cdots + r_t w^t$ with $r_0, r_1, \ldots, r_t \in R$. Besides the obvious addition, we have the following multiplication rule in $R[w, \delta]$: $wr = \delta(r)w$. If $\delta^n = \mathrm{id}_R$ (such a δ is an automorphism), then w^n is a central element of $R[w, \delta]$.

Let $G = \{\sigma_1, \sigma_2, \ldots, \sigma_n\} \subseteq \mathrm{Aut}_K(R)$ be an n-element subgroup of the group of all K-automorphisms of R (with $\sigma_n = \mathrm{id}_R$). The fixed K-subalgebra of G is

$$\mathrm{Fix}(G) = \{r \in R \mid \sigma(r) = r \text{ for all } \sigma \in G\} = \textstyle\bigcap_{\sigma \in G} \mathrm{Fix}(\sigma).$$

Any element $\tau \in G$ defines a permutation $\pi \in \mathrm{S}_n$ by

$$\tau \circ \sigma_1 = \sigma_{\pi(1)}, \ldots, \tau \circ \sigma_n = \sigma_{\pi(n)}$$

and the corresponding $n \times n$ permutation matrix is

$$P_\tau = E_{1,\pi(1)} + \cdots + E_{n,\pi(n)}.$$

2.2. Theorem. *Let $G = \{\sigma_1, \sigma_2, \ldots, \sigma_n\} \subseteq \mathrm{Aut}_K(R)$ be an n-element subgroup of K-automorphisms. If $r \in R$, then*

$$\Gamma_G(r) = \sum_{i=1}^{n} \sigma_i(r) E_{i,i}$$

defines a natural diagonal embedding $\Gamma_G : R \longrightarrow \mathrm{M}_n(R)$ of rings. For $r \in R$ and $c \in \mathrm{Fix}(G)$ we have $\Gamma_G(cr) = c\Gamma_G(r)$, $\Gamma_G(rc) = \Gamma_G(r)c$ and $\Gamma_G(c) = cI_n$.

If $\tau \in G$, then $\Gamma_G : R \longrightarrow \mathrm{M}_n(R, \tau, P_\tau)$ is an embedding of K-algebras, where $P_\tau \in \mathrm{GL}_n(K)$ is the permutation matrix defined by τ.

Proof. Since $\Gamma_G(r)$ is diagonal, the verification of the fact that Γ_G is an injective ring homomorphism is straightforward.

If $r \in R$ and $c \in \operatorname{Fix}(G)$, then $\Gamma_G(cr) = c\Gamma_G(r)$, $\Gamma_G(rc) = \Gamma_G(r)c$ and $\Gamma_G(c) = cI_n$ follow from the fact that $\sigma(cr) = c\sigma(r)$, $\sigma(rc) = \sigma(r)c$ and $\sigma(c) = c$ for all $\sigma \in G$.

The calculations

$$\tau_n(\Gamma_G(r)) = \sum_{i=1}^{n} \tau(\sigma_i(r))E_{i,i} = \sum_{i=1}^{n} \sigma_{\pi(i)}(r)E_{i,i}$$

and

$$\tau_n(\Gamma_G(r))P_\tau = \left(\Sigma_{i=1}^{n}\sigma_{\pi(i)}(r)E_{i,i}\right)\left(\Sigma_{j=1}^{n}E_{j,\pi(j)}\right) = \Sigma_{i=1}^{n}\sigma_{\pi(i)}(r)E_{i,i}E_{i,\pi(i)} =$$
$$\Sigma_{i=1}^{n}\sigma_{\pi(i)}(r)E_{i,\pi(i)} = \Sigma_{i=1}^{n}E_{i,\pi(i)}\sigma_{\pi(i)}(r)E_{\pi(i),\pi(i)} =$$
$$\left(\Sigma_{i=1}^{n}E_{i,\pi(i)}\right)\left(\Sigma_{k=1}^{n}\sigma_k(r)E_{k,k}\right) = P_\tau\Gamma_G(r)$$

show that $\Gamma_G(r) \in \mathrm{M}_n(R, \tau, P_\tau)$. $\square$

2.3. Theorem. *If $\delta : R \longrightarrow R$ is a K-endomorphism and $W \in \mathrm{M}_n(R)$, then any homomorphism $\varphi : R \longrightarrow \mathrm{M}_n(R, \delta, W)$ of K-algebras with $\varphi \circ \delta = \delta_n \circ \varphi$ has a unique K-homomorphic extension*

$$\varphi^{(\delta)} : R[w, \delta] \longrightarrow \mathrm{M}_n(R[z])$$

such that $\varphi^{(\delta)}(r) = \varphi(r)$ and $\varphi^{(\delta)}(w) = Wz$. For $\varphi^{(\delta)}$ we have

$$\varphi^{(\delta)}(r_0 + r_1w + \cdots + r_tw^t) = \varphi(r_0) + \varphi(r_1)Wz + \cdots + \varphi(r_t)W^tz^t.$$

(1) *If $f(w) \in R[w, \delta]$ and $\varphi(c) = cI_n$ for some $c \in R$, then $\varphi^{(\delta)}(cf(w)) = c\varphi^{(\delta)}(f(w))$.*
(2) *If φ is injective and $W \in \mathrm{GL}_n(R)$ is invertible, then $\varphi^{(\delta)}$ is also injective.*
(3) *If $W \in \mathrm{M}_n(\operatorname{Fix}(\delta))$, then $\varphi^{(\delta)}$ is an $R[w, \delta] \longrightarrow \mathrm{M}_n(R[z], \delta_z, W)$ homomorphism of K-algebras.*

Proof. The K-linearity of $\varphi^{(\delta)}$ is clear. In order to see the multiplicative property of $\varphi^{(\delta)}$, it is enough to prove that

$$\varphi^{(\delta)}((rw^i)(sw^j)) = \varphi^{(\delta)}(rw^i)\varphi^{(\delta)}(sw^j)$$

for all $r, s \in R$ and $0 \le i, j$.

Using $\varphi \circ \delta = \delta_n \circ \varphi$ and $\varphi(s) \in \mathrm{M}_n(R, \delta, W)$ we obtain that $\varphi(\delta(s))W = \delta_n(\varphi(s))W = W\varphi(s)$. We proceed by induction and assume that $\varphi(\delta^i(s))W^i = W^i\varphi(s)$ holds for some $i \ge 1$. Now the substitution of $\delta(s)$ into the place of s gives

$$\varphi(\delta^{i+1}(s))W^{i+1} = \varphi(\delta^i(\delta(s)))W^iW = W^i\varphi(\delta(s))W = W^iW\varphi(s) = W^{i+1}\varphi(s).$$

Thus we have

$$\varphi^{(\delta)}((rw^i)(sw^j)) = \varphi^{(\delta)}(r\delta^i(s)w^{i+j}) = \varphi(r\delta^i(s))W^{i+j}z^{i+j} =$$
$$\varphi(r)\varphi(\delta^i(s))W^iW^jz^{i+j} = \varphi(r)W^i\varphi(s)W^jz^{i+j} =$$
$$\varphi(r)W^iz^i\varphi(s)W^jz^j = \varphi^{(\delta)}(rw^i)\varphi^{(\delta)}(sw^j).$$

(1) It follows from $\varphi(cr_i) = c\varphi(r_i)$.
(2) It follows from the fact, that $\varphi(r_i)W^i = 0$ implies $r_i = 0$.
(3) The application of $\delta_n \circ \varphi = \varphi \circ \delta$ and $\delta_n(W) = W$ gives

$$(\delta_z)_n(\varphi^{(\delta)}(w))W = (\delta_z)_n(\varphi(r)Wz)W = \delta_n(\varphi(r)W)zW =$$
$$\delta_n(\varphi(r))\delta_n(W)zW = \varphi(\delta(r))WWz = W\varphi(r)Wz = W\varphi^{(\delta)}(w),$$

whence $\varphi^{(\delta)}(w) \in \mathrm{M}_n(R[z], \delta_z, W)$ follows. Since $\varphi^{(\delta)}(r_i) = \varphi(r_i) \in \mathrm{M}_n(R[z], \delta_z, W)$, we obtain that $\varphi^{(\delta)}(f(w)) \in \mathrm{M}_n(R[z], \delta_z, W)$ for all $f(w) \in R[w, \delta]$. □

2.4. Corollary. *For an automorphism $\delta \in \mathrm{Aut}_K(R)$ with $\delta^n = \mathrm{id}_R$, the number of elements of the cyclic subgroup $\langle\delta\rangle = \{\delta^i \mid 1 \le i \le n\}$ of $\mathrm{Aut}_K(R)$ is a divisor of n (and equality not necessarily holds). A natural diagonal embedding $\Delta : R \longrightarrow \mathrm{M}_n(R)$ of K-algebras can be defined by $\Delta(r) = \Sigma_{i=1}^n \delta^i(r)E_{i,i}$, where $r \in R$. Using $H = E_{1,2} + E_{2,3} + \cdots + E_{n-1,n} + E_{n,1}$, a straightforward calculation shows that $\delta_n(\Delta(r))H = H\Delta(r)$. It follows that Δ is an $R \longrightarrow \mathrm{M}_n(R, \delta, H)$ map. Since $\Delta \circ \delta = \delta_n \circ \Delta$ and $H \in \mathrm{GL}_n(K) \cap \mathrm{M}_n(\mathrm{Fix}(\delta))$, there is a unique homomorphic extension $\Delta^{(\delta)} : R[w, \delta] \longrightarrow \mathrm{M}_n(R[z])$ of Δ such that $\Delta^{(\delta)}(r) = \Delta(r)$, $\Delta^{(\delta)}(w) = Hz$ and*

$$\Delta^{(\delta)}(r_0 + r_1w + \cdots + r_tw^t) = \Delta(r_0) + \Delta(r_1)Hz + \cdots + \Delta(r_t)H^tz^t.$$

This map is a

$$\Delta^{(\delta)} : R[w, \delta] \longrightarrow \mathrm{M}_n(R[z], \delta_z, H)$$

embedding of K-algebras. If $c \in \mathrm{Fix}(\delta)$, then $\Delta^{(\delta)}(cf(w)) = c\Delta^{(\delta)}(f(w))$ for all $f(w) \in R[w, \delta]$. Now $cH = Hc$ ensures that $\Delta^{(\delta)}(f(w)c) = \Delta^{(\delta)}(f(w))c$.

If $|\langle\delta\rangle| = n$, then the choice $G = \{\sigma_1 = \delta, \sigma_2 = \delta^2, \ldots, \sigma_n = \delta^n\}$ in Theorem 2.2 gives that $\Delta = \Gamma_G$ and $H = P_\delta$.

3. The Lie nilpotent Cayley–Hamilton theorem

A Lie nilpotent analogue of classical determinant theory was developed in [3], further details can be found in [1,5]. Here we present the basic definitions and results about the sequences of right determinants and right characteristic polynomials, including the so-called Lie nilpotent right Cayley–Hamilton identities.

For an $n \times n$ matrix $A = [a_{i,j}]$ over an arbitrary (possibly non-commutative) ring or algebra R with 1, the element

$$\mathrm{sdet}(A) = \sum_{\tau,\pi \in \mathrm{S}_n} \mathrm{sgn}(\pi) a_{\tau(1),\pi(\tau(1))} \cdots a_{\tau(t),\pi(\tau(t))} \cdots a_{\tau(n),\pi(\tau(n))}$$

of R is called the symmetric determinant of A. The symmetric adjoint $A^* = [a^*_{r,s}]$ of $A = [a_{i,j}]$ is defined as the following natural symmetrization of the classical adjoint:

$$a^*_{r,s} = \sum_{\tau,\pi} \mathrm{sgn}(\pi) a_{\tau(1),\pi(\tau(1))} \cdots a_{\tau(s-1),\pi(\tau(s-1))} a_{\tau(s+1),\pi(\tau(s+1))} \cdots a_{\tau(n),\pi(\tau(n))}$$

where the sum is taken over all $\tau, \pi \in \mathrm{S}_n$ with $\tau(s) = s$ and $\pi(s) = r$. We note that the (r, s) entry of A^* is exactly the signed symmetric determinant $(-1)^{r+s}\mathrm{sdet}(A_{s,r})$ of the $(n-1) \times (n-1)$ minor $A_{s,r}$ of A arising from the deletion of the s-th row and the r-th column of A. If R is commutative, then $\mathrm{sdet}(A) = n!\det(A)$ and $A^* = (n-1)!\mathrm{adj}(A)$, where $\det(A)$ and $\mathrm{adj}(A)$ denote the ordinary determinant and adjoint of A.

The next result of Domokos plays a fundamental role in the proof of Theorem 4.1, on which the rest of section (4) is based.

3.1. Theorem *([1]). Let R be an algebra over a field K of characteristic zero. If $A \in \mathrm{M}_n(R)$ and $T \in \mathrm{GL}_n(K)$, then $(T^{-1}AT)^* = T^{-1}A^*T$.*

The right adjoint sequence $(P_k)_{k\geq 1}$ of a matrix $A \in \mathrm{M}_n(R)$ is defined by the recursion: $P_1 = A^*$ and $P_{k+1} = (AP_1 \cdots P_k)^*$ for $k \geq 1$. It is easy to see that for any $i \geq 1$, the right adjoint sequence of the matrix $AP_1 \cdots P_i$ is $(P_k)_{k\geq i+1}$. The k-th right adjoint of A is defined as

$$\mathrm{radj}_{(k)}(A) = nP_1 \cdots P_k.$$

The k-th right determinant of A is the trace of $AP_1 \cdots P_k$:

$$\mathrm{rdet}_{(k)}(A) = \mathrm{tr}(AP_1 \cdots P_k).$$

We note that

$$\mathrm{rdet}_{(1)}(A) = \mathrm{tr}(AA^*) = \mathrm{sdet}(A) = \mathrm{tr}(A^*A)$$

and the following theorem shows that $\mathrm{radj}_{(k)}(A)$ and $\mathrm{rdet}_{(k)}(A)$ can play a role similar to that played by the ordinary adjoint and determinant in the commutative case.

3.2. Theorem *([3],[5]). If $\frac{1}{n} \in R$ and the ring R is Lie nilpotent of index k, then for a matrix $A \in \mathrm{M}_n(R)$ we have*

$$A\mathrm{radj}_{(k)}(A) = nAP_1 \cdots P_k = \mathrm{rdet}_{(k)}(A)I_n.$$

The above Theorem 3.2 is not used explicitly, however it helps our understanding and serves as a starting point in the proof of Theorem 3.4.

Let $R[x]$ denote the ring of polynomials of the single commuting indeterminate x, with coefficients in R. The k-th right characteristic polynomial of A is the k-th right determinant of the $n \times n$ matrix $xI_n - A$ in $\mathrm{M}_n(R[x])$:

$$p_{A,k}(x) = \mathrm{rdet}_{(k)}(xI_n - A).$$

3.3. Proposition *([5]). The k-th right characteristic polynomial $p_{A,k}(x) \in R[x]$ of $A \in \mathrm{M}_n(R)$ is of the form*

$$p_{A,k}(x) = \lambda_0^{(k)} + \lambda_1^{(k)} x + \cdots + \lambda_{n^k-1}^{(k)} x^{n^k-1} + \lambda_{n^k}^{(k)} x^{n^k},$$

where $\lambda_0^{(k)}, \lambda_1^{(k)}, \ldots, \lambda_{n^k-1}^{(k)}, \lambda_{n^k}^{(k)} \in R$ and $\lambda_{n^k}^{(k)} = n\left\{(n-1)!\right\}^{1+n+n^2+\cdots+n^{k-1}}$.

The degree and the leading coefficient of the k-th right characteristic polynomial in Proposition 3.3 will provide the degree of right integrality in Theorems (A) and (B).

3.4. Theorem *([3],[5]). If $\frac{1}{n} \in R$ and the ring R is Lie nilpotent of index k, then a right Cayley–Hamilton identity*

$$(A)p_{A,k} = I_n \lambda_0^{(k)} + A\lambda_1^{(k)} + \cdots + A^{n^k-1}\lambda_{n^k-1}^{(k)} + A^{n^k}\lambda_{n^k}^{(k)} = 0$$

with right scalar coefficients holds for $A \in \mathrm{M}_n(R)$. We also have $(A)u = 0$, where $u(x) = p_{A,k}(x)h(x)$ and $h(x) \in R[x]$ is arbitrary.

4. The right characteristic polynomials of a skew centralizing matrix

4.1. Theorem. *Let R be an algebra over a field K of characteristic zero, $\delta : R \longrightarrow R$ be a K-endomorphism and $W \in \mathrm{GL}_n(K)$ be an invertible matrix. If $A \in \mathrm{M}_n(R, \delta, W)$ is a (δ, W)-centralizing matrix, then $A^* \in \mathrm{M}_n(R, \delta, W)$. In other words, the matrix algebra $\mathrm{M}_n(R, \delta, W)$ is closed with respect to taking the symmetric adjoint.*

Proof. The definition of A^* and the element-wise action of δ_n ensure that $\delta_n(A^*) = \delta_n(A)^*$ for all $A \in \mathrm{M}_n(R)$. Since $W \in \mathrm{GL}_n(K)$, Theorem 3.1 gives that $(WAW^{-1})^* = WA^*W^{-1}$ for all $A \in \mathrm{M}_n(R)$. Thus for a matrix $A \in \mathrm{M}_n(R, \delta, W)$ we have

$$\delta_n(A^*) = \delta_n(A)^* = (WAW^{-1})^* = WA^*W^{-1},$$

whence $A^* \in \mathrm{M}_n(R, \delta, W)$ follows. $\square$

Remark. If $A \in \mathrm{M}_{n,d}(E) = \mathrm{M}_n(E, \varepsilon, Q_d)$ as in Example (1), then the application of Theorem 4.1 gives that $A^* \in \mathrm{M}_{n,d}(E) = \mathrm{M}_n(E, \varepsilon, Q_d)$. This is one of the main results in [4].

4.2. Theorem. *Let R be an algebra over a field K of characteristic zero, $\delta : R \longrightarrow R$ be a K-endomorphism and $W \in \mathrm{GL}_n(K)$ be an invertible matrix. If $A \in \mathrm{M}_n(R, \delta, W)$ is a (δ, W)-centralizing matrix and $k \geq 1$ is an integer, then for the k-th right determinant we have $\mathrm{rdet}_{(k)}(A) \in \mathrm{Fix}(\delta)$.*

Proof. The repeated application of Theorem 4.1 gives that the recursion $P_1 = A^*$ and $P_{k+1} = (AP_1 \cdots P_k)^*$ starting from a (δ, W)-centralizing matrix $A \in \mathrm{M}_n(R, \delta, W)$ defines a sequence $(P_k)_{k\geq 1}$ in $\mathrm{M}_n(R, \delta, W)$. Since $\mathrm{rdet}_{(k)}(A) = \mathrm{tr}(AP_1 \cdots P_k)$ is the trace of $AP_1 \cdots P_k \in \mathrm{M}_n(R, \delta, W)$ and the trace of a (δ, W)-centralizing matrix is in $\mathrm{Fix}(\delta)$ by Proposition 2.1, the proof is complete. $\square$

4.3. Corollary. *Let R be an algebra over a field K of characteristic zero, $\delta : R \longrightarrow R$ be a K-endomorphism and $W \in \mathrm{GL}_n(K)$ be an invertible matrix. If $A \in \mathrm{M}_n(R, \delta, W)$ is a (δ, W)-centralizing matrix and $k \geq 1$ is an integer, then for the k-th right characteristic polynomial we have $p_{A,k}(x) \in \mathrm{Fix}(\delta)[x]$. In other words, the coefficients of the k-th right $p_{A,k}(x) = \mathrm{rdet}_{(k)}(xI_n - A)$ characteristic polynomial are in $\mathrm{Fix}(\delta)$.*

Proof. We use the natural extension $\delta_x : R[x] \longrightarrow R[x]$ of δ. Since

$$xI_n - A \in \mathrm{M}_n(R[x], \delta_x, W),$$

Theorem 4.2 gives that $p_{A,k}(x) = \mathrm{rdet}_{(k)}(xI_n - A)$ is in $\mathrm{Fix}(\delta_x) = \mathrm{Fix}(\delta)[x]$. $\square$

Remark. If $A \in \mathrm{M}_{n,d}(E) = \mathrm{M}_n(E, \varepsilon, Q_d)$ as in Example (1), then the application of Corollary 4.3 gives that $p_{A,2}(x) \in \mathrm{Fix}(\varepsilon)[x] = E_0[x]$. This fact was exploited in [4].

4.4. Corollary. *Let $R = R_0 \oplus R_1 \oplus \cdots \oplus R_{n-1}$ be a $\mathbb{Z}_n$-graded algebra over a field K of characteristic zero and $e \in K$ be a primitive n-th root of unity. If $k \geq 1$ is an integer and $A \in \mathrm{M}_n^g(R)$ is a "graded" $n \times n$ matrix with respect to the given $\mathbb{Z}_n$-grading, then for the k-th right determinant and k-th right characteristic polynomial of A we have*

$$\mathrm{rdet}_{(k)}(A) \in R_0 \ \textit{and}\ p_{A,k}(x) = \mathrm{rdet}_{(k)}(xI_n - A) \in R_0[x].$$

Proof. Since $\mathrm{M}_n^g(R) = \mathrm{M}_n(R, \widehat{e}, D_e)$ (see Example (2)) and $\mathrm{Fix}(\widehat{e}) = R_0$, Theorem 4.2 and Corollary 4.3 can be applied. $\square$

Remark. In the absence of a primitive n-th root of unity, the direct proof (not using Example (2), Theorem 4.2 and Corollary 4.3) of the implications

$$A \in \mathrm{M}_n^g(R) \Longrightarrow A^* \in \mathrm{M}_n^g(R),\ \mathrm{rdet}_{(k)}(A) \in R_0,\ p_{A,k}(x) = \mathrm{rdet}_{(k)}(xI_n - A) \in R_0[x]$$

is rather technical.

The combination of Theorem 3.4 and Corollary 4.3 gives the following.

4.5. Theorem. *Let R be an algebra over a field K of characteristic zero, $\delta : R \longrightarrow R$ be an endomorphism and $W \in \mathrm{GL}_n(K)$ be an invertible matrix. If R is Lie nilpotent of index k and $A \in \mathrm{M}_n(R, \delta, W)$, then a right Cayley–Hamilton identity*

$$(A)p_{A,k} = I_n\lambda_0^{(k)} + A\lambda_1^{(k)} + \cdots + A^{n^k-1}\lambda_{n^k-1}^{(k)} + A^{n^k}\lambda_{n^k}^{(k)} = 0$$

holds, where the coefficients $\lambda_i^{(k)}$, $0 \le i \le n^k$ of $p_{A,k}(x) = \mathrm{rdet}_{(k)}(xI_n - A)$ are in $\mathrm{Fix}(\delta)$. Since $\lambda_{n^k}^{(k)} = n\{(n-1)!\}^{1+n+n^2+\cdots+n^{k-1}}$ is invertible (in K), the above identity provides the right integrality of $\mathrm{M}_n(R, \delta, W)$ over $\mathrm{Fix}(\delta)$ of degree n^k.

5. The proofs of Theorem (A) and (B)

Proof of Theorem (A). Take an element $\tau \in G$ and consider the embedding $\Gamma_G : R \longrightarrow \mathrm{M}_n(R, \tau, P_\tau)$ in Theorem 2.2. For $r \in R$ Theorem 4.5 ensures that

$$I_n\lambda_0^{(k)} + \Gamma_G(r)\lambda_1^{(k)} + \cdots + (\Gamma_G(r))^{n^k-1}\lambda_{n^k-1}^{(k)} + (\Gamma_G(r))^{n^k}\lambda_{n^k}^{(k)} = 0$$

holds for the (τ, P_τ)-centralizing matrix $\Gamma_G(r) \in \mathrm{M}_n(R, \tau, P_\tau)$, where

$$p_{\Gamma_G(r),k}(x) = \mathrm{rdet}_{(k)}(xI_n - \Gamma_G(r)) = \lambda_0^{(k)} + \lambda_1^{(k)}x + \cdots + \lambda_{n^k-1}^{(k)}x^{n^k-1} + \lambda_{n^k}^{(k)}x^{n^k},$$

$\lambda_{n^k}^{(k)} = q \ge 1$ is an integer and $\lambda_i^{(k)} \in \mathrm{Fix}(\tau)$ for each $0 \le i \le n^k$. Thus we have $c_i = \frac{1}{q}\lambda_i^{(k)} \in \mathrm{Fix}(G)$ for each index $0 \le i \le n^k - 1$. It follows that

$$\begin{gathered}\Gamma_G(c_0 + rc_1 + \cdots + r^{n^k-1}c_{n^k-1} + r^{n^k}) = \\ I_nc_0 + \Gamma_G(r)c_1 + \cdots + (\Gamma_G(r))^{n^k-1}c_{n^k-1} + (\Gamma_G(r))^{n^k} = \\ \left(I_n\lambda_0^{(k)} + \Gamma_G(r)\lambda_1^{(k)} + \cdots + (\Gamma_G(r))^{n^k-1}\lambda_{n^k-1}^{(k)} + (\Gamma_G(r))^{n^k}\lambda_{n^k}^{(k)}\right) \cdot \frac{1}{q} = 0\end{gathered}$$

and $\ker(\Gamma_G) = \{0\}$ gives the desired right integrality. $\square$

Proof of Theorem (B). Consider the embeddings

$$\Delta : R \longrightarrow \mathrm{M}_n(R, \delta, H) \text{ and } \Delta^{(\delta)} : R[w, \delta] \longrightarrow \mathrm{M}_n(R[z], \delta_z, H)$$

in Corollary 2.4, where $\Delta(r) = \Sigma_{i=1}^n \delta^i(r)E_{i,i}$, $H = E_{1,2} + E_{2,3} + \cdots + E_{n-1,n} + E_{n,1}$ and

$$\Delta^{(\delta)}(r_0 + r_1w + \cdots + r_tw^t) = \Delta(r_0) + \Delta(r_1)Hz + \cdots + \Delta(r_t)H^tz^t.$$

Since $R[z]$ is also Lie nilpotent of index k, Theorem 4.5 gives that

$$\begin{gathered}I_n\lambda_0^{(k)}(z) + \Delta^{(\delta)}(f(w))\lambda_1^{(k)}(z) + \cdots \\ \cdots + (\Delta^{(\delta)}(f(w))^{n^k-1}\lambda_{n^k-1}^{(k)}(z) + (\Delta^{(\delta)}(f(w))^{n^k}\lambda_{n^k}^{(k)}(z) = 0,\end{gathered}$$

where the coefficients of the k-th right characteristic polynomial

$$\mathrm{rdet}_{(k)}(xI_n - \Delta^{(\delta)}(f(w))) = \lambda_0^{(k)}(z) + \lambda_1^{(k)}(z)x + \cdots + \lambda_{n^k-1}^{(k)}(z)x^{n^k-1} + \lambda_{n^k}^{(k)}(z)x^{n^k}$$

of the (δ_z, H)-centralizing matrix $\Delta^{(\delta)}(f(w)) \in \mathrm{M}_n(R[z], \delta_z, H)$ are in $\mathrm{Fix}(\delta_z) = \mathrm{Fix}(\delta)[z]$ and $\lambda_{n^k}^{(k)}(z) = q \geq 1$ is an integer.

According to Example (2), the natural $\mathbb{Z}_n$-grading

$$R[z] = R[z^n] \oplus zR[z^n] \oplus \cdots \oplus z^{n-1}R[z^n]$$

of the polynomial algebra $R[z]$ defines $\mathrm{M}_n^g(R[z])$ as

$$\mathrm{M}_n^g(R[z]) = \{A = [a_{i,j}(z)] \in \mathrm{M}_n(R[z]) \mid a_{i,j}(z) \in z^{j-i}R[z^n] \text{ for all } 1 \leq i, j \leq n\} =$$

$$\begin{bmatrix} R[z^n] & zR[z^n] & \cdots & z^{n-2}R[z^n] & z^{n-1}R[z^n] \\ z^{n-1}R[z^n] & R[z^n] & zR[z^n] & \ddots & z^{n-2}R[z^n] \\ \vdots & z^{n-1}R[z^n] & \ddots & \ddots & \vdots \\ z^2R[z^n] & \ddots & \ddots & R[z^n] & zR[z^n] \\ zR[z^n] & z^2R[z^n] & \cdots & z^{n-1}R[z^n] & R[z^n] \end{bmatrix} = \mathrm{M}_n(R[z], \widehat{e}, D_e).$$

Clearly, $\Delta(r_i) \in \mathrm{M}_n^g(R[z])$ and $Hz \in \mathrm{M}_n^g(R[z])$ imply that $\Delta(r_i)H^iz^i \in \mathrm{M}_n^g(R[z])$ for all $i \geq 0$. Thus we have $\Delta^{(\delta)}(f(w)) \in \mathrm{M}_n^g(R[z])$ for all $f(w) \in R[w, \delta]$. The use of Corollary 4.4 gives that

$$\mathrm{rdet}_{(k)}(xI_n - \Delta^{(\delta)}(f(w))) \in (R[z^n])[x],$$

whence $\lambda_i^{(k)}(z) = \mu_i(z^n) \in \mathrm{Fix}(\delta)[z^n]$ follows for all $0 \leq i \leq n^k - 1$.

Since $H^n = I_n$, for any polynomial $\mu(z^n) \in \mathrm{Fix}(\delta)[z^n]$ there exists a (unique) polynomial $g(w^n) \in \mathrm{Fix}(\delta)[w^n]$ such that

$$\mu(z^n)I_n = c_0I_n + c_1z^nI_n + \cdots + c_dz^{nd}I_n =$$
$$\Delta(c_0) + \Delta(c_1)H^nz^n + \cdots + \Delta(c_d)H^{nd}z^{nd} =$$
$$\Delta^{(\delta)}(c_0 + c_1w^n + \cdots + c_dw^{nd}) = \Delta^{(\delta)}(g(w^n)),$$

where the coefficients $c_0, c_1, \ldots, c_d \in \mathrm{Fix}(\delta)$ of $\mu(z^n)$ and $g(w^n)$ coincide. For each $0 \leq i \leq n^k - 1$ take $g_i(w^n) \in \mathrm{Fix}(\delta)[w^n]$ such that

$$\Delta^{(\delta)}(g_i(w^n)) = \frac{1}{q}\mu_i(z^n)I_n = \frac{1}{q}\lambda_i^{(k)}(z)I_n.$$

Thus

$$\Delta^{(\delta)}\left(g_0(w^n)+f(w)g_1(w^n)+\cdots+f^{n^k-1}(w)g_{n^k-1}(w^n)+f^{n^k}(w)\right)=$$
$$\frac{1}{q}\left(I_n\lambda_0^{(k)}(z)+\Delta^{(\delta)}(f(w))\lambda_1^{(k)}(z)+\cdots\right.$$
$$\left.\cdots+(\Delta^{(\delta)}(f(w))^{n^k-1}\lambda_{n^k-1}^{(k)}(z)+(\Delta^{(\delta)}(f(w))^{n^k}\lambda_{n^k}^{(k)}(z)\right)=0$$

and $\ker(\Delta^{(\delta)})=\{0\}$ gives the desired integrality. □

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