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Another class of simple graded Lie conformal algebras that cannot be embedded into general Lie conformal algebras ☆

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A B S T R A C T

In a previous paper by the authors, we obtain the first example of a finitely freely generated simple $\mathbb{Z}$-graded Lie conformal algebra of linear growth that cannot be embedded into any general Lie conformal algebra. In this paper, we obtain, as a byproduct, another class of such Lie conformal algebras by classifying $\mathbb{Z}$-graded simple Lie conformal algebras $\mathcal{G} = \oplus_{i=-1}^{\infty}\mathcal{G}_i$ satisfying the following,

(1) $\mathcal{G}_0 \cong$ Vir, the Virasoro conformal algebra;
(2) Each $\mathcal{G}_i$ for $i \geq -1$ is a Vir-module of rank one.

These algebras include some Lie conformal algebras of Block type.

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1. Introduction

Conformal algebras, first introduced in [15], appear naturally in the context of formal distribution Lie algebras and play important roles in quantum field theory and conformal field theory (e.g., [4,7,17]). They also turn out to be effective tools in the study of infinite-dimensional Lie or associative algebras satisfying the locality property, and their representations [16].

In recent years, the structure theory, representation theory and cohomology theory of Lie conformal algebras have been extensively studied (e.g., [3,5–14,19,22]). In particular, simple finite Lie conformal algebras were classified in [8], which turn out to be isomorphic either to the Virasoro conformal algebra or the current Lie conformal algebra $\mathrm{Cur}\,\mathfrak{g}$ associated to a simple finite-dimensional Lie algebra $\mathfrak{g}$. Finite irreducible conformal modules over the Virasoro conformal algebra were determined in [6] and of their extensions in [7]. The cohomology theory of conformal algebras with coefficients in an arbitrary module has been developed in [3,9]. However, the theory of simple infinite Lie conformal algebras is far from being well developed, it is more complicated than the theory of Lie or associative algebras (e.g., [13,14]).

In order to better understand the theory of simple infinite Lie conformal algebras, it is very natural to first study some important examples. It is well-known that the general Lie conformal algebra gc_N (which is a simple infinite Lie conformal algebra) plays the same important role in the theory of Lie conformal algebras as the general Lie algebra gl_N does in the theory of Lie algebras. Thus the study of Lie conformal algebras related to the general Lie conformal algebra gc_N has drawn lots of attention in literature (e.g., [1,2,12,18–21]). In particular, in [20], we study filtered Lie conformal algebras whose associated graded algebras are isomorphic to that of the general Lie conformal algebra gc_1, and as a byproduct we obtain the first example of a finitely freely generated simple $\mathbb{Z}$-graded Lie conformal algebra of linear growth that cannot be embedded into a general Lie conformal algebra gc_N for any N, namely, the Lie conformal algebra $\mathrm{gr}\,gc_1$ (the associated graded conformal algebra of gc_1, which is also called a Lie conformal algebra of Block type), see [20, Theorem 1.1]. Motivated by the facts that a simple Lie conformal algebra of rank one is isomorphic to the Virasoro conformal algebra Vir and that a finite simple Vir-module is of rank one [6–8], in this paper, we study $\mathbb{Z}$-graded Lie conformal algebras $\mathcal{G} = \oplus_{i=-1}^{\infty}\mathcal{G}_i$ satisfying the following reasonable conditions

(C1) $\mathcal{G}_0 \cong$ Vir, the Virasoro conformal algebra;
(C2) Each $\mathcal{G}_i$ for $i \geq -1$ is a Vir-module of rank one;
(C3) $\mathcal{G}$ is simple.

To state the main result, we first give the following definitions.

Definition 1.1. Let $\alpha \in \mathbb{C}$, $s = 1, 2$. Denote by $B(s, \alpha)$ the Lie conformal algebra with $\mathbb{C}[\partial]$-basis $\{G_i \,|\, i \in \mathbb{Z}_{\geq -1}\}$ and the following λ-brackets,

$$
\begin{aligned}
B(1,\alpha):\ & [G_{-1}{}_\lambda G_{-1}]=0,\ \ [G_{-1}{}_\lambda G_0]=(\alpha-\partial)G_{-1},\ \ [G_{-1}{}_\lambda G_j]=(j+1)G_{j-1},\ \ j\geq 1,\\
& [G_i{}_\lambda G_j]=\Big((j-i)\alpha+(i+j+2)\lambda+(i+1)\partial\Big)G_{i+j},\ \ i,j\in\mathbb{Z}_+,\\
B(2,\alpha):\ & [G_i{}_\lambda G_j]=\Big((j-i)\alpha+(i+j+2)\lambda+(i+1)\partial\Big)G_{i+j},\ \ i,j\in\mathbb{Z}_{\geq -1}.
\end{aligned}
\tag{1.1}
$$

Definition 1.2. Let $\alpha\in\mathbb{C}$. Denote by $\mathcal{B}(\alpha)=\oplus_{i=-1}^{\infty}\mathcal{B}_i$ the $\mathbb{Z}$-graded simple Lie conformal algebra with the λ-brackets $[\mathfrak{b}_0{}_\lambda\mathfrak{b}_i]=\big(i\alpha+(i+2)\lambda+\partial\big)\mathfrak{b}_i$ for $i>0$, which satisfies

(i) $\mathcal{B}_0\cong$ Vir, the Virasoro conformal algebra,

(ii) $\mathcal{B}_{-1}$ is a Vir-module of rank one,

(iii) Each $\mathcal{B}_i$ for $i>0$ is a Vir-module of finite rank,

where $\mathfrak{b}_0$ is the $\mathbb{C}[\partial]$-generator of $\mathcal{B}_0$ and $\mathfrak{b}_i$ is any one of $\mathbb{C}[\partial]$-generators of $\mathcal{B}_i$ for $i>0$.

The main result of the present paper is the following.

Theorem 1.3.

(1) *The Lie conformal algebra $B(s,\alpha)$ is simple for any $\alpha\in\mathbb{C}$ and $s=1,2$.*
(2) *For $\alpha_1,\alpha_2\in\mathbb{C}$, $s_1,s_2\in\{1,2\}$, $B(s_1,\alpha_1)\cong B(s_2,\alpha_2)$ if and only if $(s_1,\alpha_1)=(s_2,\alpha_2)$.*
(3) *Let $\mathcal{G}=\oplus_{i=-1}^{\infty}\mathcal{G}_i$ be a simple Lie conformal algebra satisfying conditions* (C1) *and* (C2)*. Then $\mathcal{G}\cong B(s,\alpha)$ for some $\alpha\in\mathbb{C}$ and $s=1,2$.*
(4) *For any $\alpha\in\mathbb{C}$, the Lie conformal algebra $\mathcal{B}(\alpha)$ does not have a nontrivial representation on any finite $\mathbb{C}[\partial]$-module. In particular, $\mathcal{B}(\alpha)$ is a finitely freely generated simple Lie conformal algebra of linear growth that cannot be embedded into gc_N for any N.*

Therefore, Theorem 1.3(4) provides another class $\mathcal{B}(\alpha)$ of finitely freely generated simple $\mathbb{Z}$-graded Lie conformal algebras of linear growth that cannot be embedded into a general Lie conformal algebra gc_N for any N.

The paper is organized as follows. In section 2, we briefly recall some definitions and preliminary results. In section 3, we first study the structure of the Lie conformal algebra $B(s,\alpha)$, then we give the proof of Theorem 1.3(4). In order to classify $\mathbb{Z}$-graded simple Lie conformal algebras $\mathcal{G}$, some technical lemmas were given in section 4. Then in section 5, we use these technical lemmas to determine all simple Lie conformal algebras satisfying conditions (C1), (C2), and complete the proof of Theorem 1.3.

Throughout the paper, we denote by $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{Z}_{\geq -1}$ the sets of complex numbers, nonzero complex numbers, integers, nonnegative integers and integers greater than -2 respectively.

2. Definitions and preliminary results

In this section, we summarize some basic definitions and results concerning Lie conformal algebras. More details can be found in [3,6,15].

Definition 2.1. A *Lie conformal algebra* is a $\mathbb{C}[\partial]$-module A with a λ-bracket $[\cdot_\lambda \cdot]$ which defines a $\mathbb{C}$-bilinear map $A \times A \to A[\lambda]$, where $A[\lambda] = \mathbb{C}[\lambda] \otimes A$ is the space of polynomials of λ with coefficients in A, such that for $x, y, z \in A$,

$$[\partial x_\lambda y] = -\lambda [x_\lambda y], \qquad [x_\lambda \partial y] = (\partial + \lambda)[x_\lambda y] \quad \text{(conformal sesquilinearity)}, \tag{2.1}$$

$$[x_\lambda y] = -[y_{-\lambda-\partial} x] \quad \text{(skew-symmetry)}, \tag{2.2}$$

$$[x_\lambda [y_\mu z]] = [[x_\lambda y]_{\lambda+\mu} z] + [y_\mu [x_\lambda z]] \quad \text{(Jacobi identity)}. \tag{2.3}$$

A subset $S \subset A$ is called a *generating set* if S generates A as a $\mathbb{C}[\partial]$-module. If there exists a finite generating set, then A is called *finite*. Otherwise, it is called *infinite*.

For a given Lie conformal algebra A, from [15], we know that there is an important Lie algebra associated to it. For each $j \in \mathbb{Z}_+$, regarding $[a_\lambda b] \in \mathbb{C}[\lambda] \otimes A$ as a formal polynomial in λ, we can define the *j th product* $a_{(j)} b$ by the coefficient of λ^j in $[a_\lambda b]$, i.e. $a_{(j)} b$ for all $a, b \in A$ as follows:

$$[a_\lambda b] = \sum_{j \in \mathbb{Z}_+} (a_{(j)} b) \frac{\lambda^j}{j!}. \tag{2.4}$$

Now we can give the definition of this Lie algebra.

Definition 2.2. An *annihilation algebra* of a Lie conformal algebra A is a Lie algebra with $\mathbb{C}$-basis $\{a_{(n)} \mid a \in A,\ n \in \mathbb{Z}_+\}$ and relations

$$[a_{(m)}, b_{(n)}] = \sum_{j \in \mathbb{Z}_+} \binom{m}{j} (a_{(j)} b)_{(m+n-j)}, \qquad \partial(a_{(n)}) = -n a_{(n-1)}.$$

The *Virasoro conformal algebra* Vir is the simplest nontrivial Lie conformal algebra. It is a free $\mathbb{C}[\partial]$-module of rank one with generator L and can be defined by

$$\mathrm{Vir} = \mathbb{C}[\partial]L: \qquad [L_\lambda L] = (\partial + 2\lambda)L. \tag{2.5}$$

It is known that any simple Lie conformal algebra of free rank one over $\mathbb{C}[\partial]$ is isomorphic to Vir [8].

The *general Lie conformal algebra* gc_N can be defined as the infinite rank $\mathbb{C}[\partial]$-module $\mathbb{C}[\partial, x] \otimes gl_N$, with the λ-bracket

$$[f(\partial,x)A\,{}_\lambda\, g(\partial,x)B] = f(-\lambda,x+\partial+\lambda)g(\partial+\lambda,x)AB - f(-\lambda,x)g(\partial+\lambda,x-\lambda)BA, \quad (2.6)$$

for $f(\partial,x), g(\partial,x) \in \mathbb{C}[\partial,x]$, $A, B \in gl_N$, where gl_N is the space of $N \times N$ matrices, and we have identified $f(\partial,x)\otimes A$ with $f(\partial,x)A$. If we set $J^n_A = x^n A$, then

$$[J^m_A\,{}_\lambda\, J^n_B] = \sum_{s=0}^{m}\binom{m}{s}(\lambda+\partial)^s J^{m+n-s}_{AB} - \sum_{s=0}^{n}\binom{n}{s}(-\lambda)^s J^{m+n-s}_{BA},$$

for $m,n \in \mathbb{Z}_+$, $A,B \in gl_N$, where $\binom{m}{s} = m(m-1)\cdots(m-s+1)/s!$ if $s \geq 0$ and $\binom{m}{s} = 0$ otherwise, is the binomial coefficient.

Definition 2.3. A *module over a Lie conformal algebra* A is a $\mathbb{C}[\partial]$-module M with a λ-action $\cdot_\lambda\cdot : A \times M \to M[[\lambda]]$, where $M[[\lambda]]$ is the set of formal power series of λ with coefficients in M, such that for $x, y \in A,\ v \in M$,

$$x_\lambda(y_\mu v) - y_\mu(x_\lambda v) = [x_\lambda y]_{\lambda+\mu} v, \quad (2.7)$$

$$(\partial x)_\lambda v = -\lambda x_\lambda v, \quad x_\lambda(\partial v) = (\partial+\lambda)x_\lambda v. \quad (2.8)$$

An A-module M is called *conformal* if $x_\lambda v \in M[\lambda]$ for $x \in A,\ v \in M$ and *finite* if M is finitely generated over $\mathbb{C}[\partial]$.

According to [6], we know that all free nontrivial conformal Vir-modules of rank one over $\mathbb{C}[\partial]$ are $M_{\Delta,\alpha}$ for $\Delta, \alpha \in \mathbb{C}$, where

$$M_{\Delta,\alpha} = \mathbb{C}[\partial]v : \qquad L_\lambda v = (\alpha+\partial+\Delta\lambda)v. \quad (2.9)$$

The module $M_{\Delta,\alpha}$ is irreducible if and only if $\Delta \neq 0$.

3. Graded Lie conformal algebras $B(s,\alpha)$

It is straightforward to verify that (1.1) indeed defines Lie conformal algebras $B(1,\alpha)$ and $B(2,\alpha)$. Furthermore, the annihilation algebra of $B(2,\alpha)$ is the Lie algebra $\mathcal{A} = \mathrm{span}_{\mathbb{C}}\{G_{i,m} \mid i,m \in \mathbb{Z}_{\geq -1}\}$ with Lie brackets

$$[G_{i,m}, G_{j,n}] = (j-i)\alpha G_{i+j,m+n+1} + \big((j+1)(m+1) - (n+1)(i+1)\big)G_{i+j,m+n}.$$

When $\alpha = 0$, this Lie algebra has close relation to the Block-type Lie algebras studied in [12,19].

Now we study the structure of the Lie conformal algebra $B(s,\alpha)$ for $\alpha \in \mathbb{C}$ and $s = 1, 2$. First we need some definitions. For any $x \in B(s,\alpha)$, we define the operator $(\mathrm{ad}\,x)_\lambda : B(s,\alpha) \to B(s,\alpha)[\lambda]$ such that $(\mathrm{ad}\,x)_\lambda(y) = [x_\lambda y]$ for any $y \in B(s,\alpha)$. An element $x \in B(s,\alpha)$ is *locally nilpotent* if for any $y \in B(s,\alpha)$, there exists $1 \leq n \in \mathbb{Z}_+$ such that $(\mathrm{ad}\,x)^n_\lambda(y) = 0$. We have

Lemma 3.1. *The set of locally nilpotent elements of $B(s,\alpha)$ is equal to $\mathbb{C}[\partial]G_{-1}$.*

Proof. Denote by $\mathcal{N}$ the set of locally nilpotent elements of $B(s,\alpha)$. First by (1.1) and conformal sesquilinearity, we have the following for $a_0(\partial), c_j(\partial) \in \mathbb{C}[\partial]$,

$$[a_0(\partial)G_{-1\,\lambda} c_j(\partial)G_j] = \begin{cases} (\alpha-\partial)a_0(-\lambda)c_j(\partial+\lambda)G_{-1} & \text{if } \ s=1,\ j=0, \\ (j+1)a_0(-\lambda)c_j(\partial+\lambda)G_{j-1} & \text{if } \ s=1,\ 1\le j\in\mathbb{Z}, \\ (j+1)(\alpha+\lambda)a_0(-\lambda)c_j(\partial+\lambda)G_{j-1} & \text{if } \ s=2,\ 0\le j\in\mathbb{Z}. \end{cases} \tag{3.1}$$

From this and using conformal sesquilinearity, we immediately obtain that $\mathbb{C}[\partial]G_{-1}\subset\mathcal{N}$. Now let $x=\sum_{i=-1}^{\infty} b_i(\partial)G_i\in\mathcal{N}$, suppose $max\{i\,|\,b_i(\partial)\neq 0\}=i_0$. If $i_0\geq 0$, by (1.1), we have $[G_{i_0\,\lambda}G_j] = \big((j-i_0)\alpha+(i_0+j+2)\lambda+(i_0+1)\partial\big)G_{i_0+j}$ for $j\in\mathbb{Z}_+$. Then applying $(\mathrm{ad}\,x)_\lambda^n$ to G_j, we can obtain that the coefficient of G_{ni_0+j} in the expression of $(\mathrm{ad}\,x)_\lambda^n(G_j)$ is nonzero for any $1\le n\in\mathbb{Z}_+$ and $j\in\mathbb{Z}_+$. This is a contradiction with $x\in\mathcal{N}$. Therefore we must have $i_0=-1$, then the lemma follows. $\square$

Proof of Theorem 1.3(1) and (2). (1) Let J be a nonzero ideal of $B(s,\alpha)$ for some $\alpha\in\mathbb{C}$ and $s=1,2$. Then there exists at least one nonzero element $x=\sum_{j=-1}^{m} b_j(\partial)G_j\in J$ for some $b_j(\partial)\in\mathbb{C}[\partial]$, where $m\in\mathbb{Z}_+$ such that $b_m(\partial)\neq 0$. We claim that $a_0(\partial)G_{-1}\in J$ for some nonzero $a_0(\partial)\in\mathbb{C}[\partial]$. If $m=-1$, we immediately have the claim. Otherwise, we can apply the operator $(\mathrm{ad}\,G_{-1})_\lambda^{m+1}$ to x, we have the following for $b_m(\partial)\in\mathbb{C}[\partial]$,

$$J\ni(\mathrm{ad}\,G_{-1})_\lambda^{m+1}(x) = \begin{cases} (m+1)!(\alpha-\partial)b_m(\partial+\lambda)G_{-1} & \text{if } \ s=1, \\ (m+1)!(\alpha+\lambda)^m b_m(\partial+\lambda)G_{-1} & \text{if } \ s=2. \end{cases} \tag{3.2}$$

Then we inductively deduce from (3.1) that all $G_j\in J$ for $j\in\mathbb{Z}_{\geq -1}$, i.e., $J=B(s,\alpha)$. Therefore, $B(s,\alpha)$ is simple.

(2) It is obvious that the sufficient condition holds. We only need to prove the necessary condition. For $\alpha_1,\alpha_2\in\mathbb{C}$, $s_1,s_2\in\{1,2\}$, we suppose $B(s_1,\alpha_1)\cong B(s_2,\alpha_2)$, $\{G_i\,|\,i\in\mathbb{Z}_{\geq-1}\}$ and $\{G_i'\,|\,i\in\mathbb{Z}_{\geq-1}\}$ are the $\mathbb{C}[\partial]$-bases of $B(s_1,\alpha_1)$ and $B(s_2,\alpha_2)$ respectively. By (1.1), we can immediately conclude that $s_1=s_2$.

First suppose $s_1=1$. Let $\varphi: B(1,\alpha_1)\longrightarrow B(1,\alpha_2)$ be an isomorphism. By Lemma 3.1, we can assume $\varphi(G_{-1})=a(\partial)G_{-1}'$ and $\varphi(G_0)=\sum_{i=-1}^{\infty} b_i(\partial)G_i'$ for some $a(\partial)$, $b_i(\partial)\in\mathbb{C}[\partial]$ with $i\in\mathbb{Z}_{\geq-1}$. Applying the isomorphism φ to the both sides of $[G_{-1\,\lambda}G_0]=(\alpha_1-\partial)G_{-1}$, comparing the coefficients of G_{i-1}' for $1\le i\in\mathbb{Z}$ and G_{-1}' respectively, we can deduce that $b_i(\partial)=0$ for $1\le i\in\mathbb{Z}$ and

$$(\alpha_2-\partial)a(-\lambda)b_0(\lambda+\partial)=(\alpha_1-\partial)a(\partial). \tag{3.3}$$

By (1.1), we have $[G_{0\,\lambda}G_0]=(2\lambda+\partial)G_0$. Applying the isomorphism φ to this equation, then comparing the coefficients of G_0', we can obtain that

$$b_0(-\lambda)b_0(\lambda+\partial)=b_0(\partial). \tag{3.4}$$

Comparing the degrees of λ in (3.4) and using the fact that φ is an isomorphism, we have $b_0(\partial)=1$. Then by (3.3), we can conclude that $\alpha_1=\alpha_2$.

Now suppose $s_1=2$. Similarly, if $B(2,\alpha_1)\cong B(2,\alpha_2)$, we can also obtain that $\alpha_1=\alpha_2$. Therefore, if $\varphi: B(s,\alpha_1)\longrightarrow B(s,\alpha_2)$ is an isomorphism, then $(s_1,\alpha_1)=(s_2,\alpha_2)$. $\square$

In order to prove Theorem 1.3 (4), we need some preparations. Assume V is a finitely freely $\mathbb{C}[\partial]$-generated nontrivial $\mathcal{B}(\alpha)$-module. Regarding V as a module over Vir, by [6, Theorem 3.2(1)], we can choose a composition series,

$$V=V_N\supset V_{N-1}\supset\cdots\supset V_1\supset V_0=0,$$

such that for each $i=1,2,...,N$, the composition factor $\overline{V}_i=V_i/V_{i-1}$ is either a rank one free module M_{Δ_i,β_i} with $\Delta_i\neq 0$, or else a 1-dimensional trivial module $\mathbb{C}_{\beta_i}$ with trivial λ-action and with ∂ acting as the scalar β_i. Denote by $\bar{v}_i$ a $\mathbb{C}[\partial]$-generator of $\overline{V}_i$ and $v_i\in V_i$ the preimage of $\bar{v}_i$. Then $\{v_i\,|\,1\le i\le N\}$ is a $\mathbb{C}[\partial]$-generating set of V, such that the λ-action of $\mathfrak{b}_0$ on v_i is a $\mathbb{C}[\lambda,\partial]$-combination of $v_1,...,v_i$.

Lemma 3.2. *For all $i\gg 0$, the λ-action of $\mathfrak{b}_i$ on v_1 is trivial, namely, ${\mathfrak{b}_i}_\lambda v_1=0$.*

Proof. Assume $i\gg 0$ is fixed and suppose ${\mathfrak{b}_i}_\lambda v_1\neq 0$, and let $k_i\geq 1$ be the largest integer such that ${\mathfrak{b}_i}_\lambda v_1\not\subset V_{k_i-1}[\lambda]$. We consider the following possibilities.

Case 1: $V_1=M_{\Delta_1,\beta_1}$, $\overline{V}_{k_i}=M_{\Delta_{k_i},\beta_{k_i}}$.

We can write

$${\mathfrak{b}_i}_\lambda v_1\equiv p_i(\lambda,\partial)v_{k_i}\pmod{V_{k_i-1}[\lambda]}\quad\text{for some}\quad p_i(\lambda,\partial)\in\mathbb{C}[\lambda,\partial]. \tag{3.5}$$

Applying the operator ${\mathfrak{b}_0}_\mu$ to (3.5), we obtain

$$(\beta_{k_i}+\partial+\Delta_{k_i}\mu)p_i(\lambda,\mu+\partial)=\big(i\alpha+(1+i)\mu-\lambda\big)p_i(\lambda+\mu,\partial)+(\beta_1+\lambda+\partial+\Delta_1\mu)p_i(\lambda,\partial). \tag{3.6}$$

Letting $\partial=0$, we have

$$p_i(\lambda,\mu)=\frac{1}{\beta_{k_i}+\Delta_{k_i}\mu}\Big(\big(i\alpha+(1+i)\mu-\lambda\big)p_i(\lambda+\mu,0)+(\beta_1+\lambda+\Delta_1\mu)p_i(\lambda,0)\Big). \tag{3.7}$$

Using this in (3.6) with $\lambda=i\alpha+(1+i)\mu$ and $\partial=-\beta_{k_i}-\Delta_{k_i}\mu$, we can deduce that

$$\begin{aligned}&(i+1)\big((\Delta_{k_i}+1)\mu+\beta_{k_i}\big)p_i\big((i+1-\Delta_{k_i})\mu-\beta_{k_i}+i\alpha,0\big)\\&\quad=\big((i+1-\Delta_1\Delta_{k_i})\mu+\beta_1-\beta_{k_i}\Delta_1\big)p_i\big((i+1)\mu+i\alpha,0\big).\end{aligned}$$

Suppose $p_i(\lambda, 0)$ has degree m_i. Comparing the coefficients of μ^{m_i+1} in the above equation, we obtain (note that the following equation does not depend on the coefficients of $p_i(\lambda, \mu)$)

$$(i+1)(\Delta_{k_i}+1)(i+1-\Delta_{k_i})^{m_i} = (i+1-\Delta_1\Delta_{k_i})(i+1)^{m_i}. \tag{3.8}$$

When i is sufficiently large, one can easily see that (3.8) cannot hold if $m_i > 1$ (note that $\Delta_1, \Delta_{k_i} \neq 0$, and there are only finitely many choices for what Δ_{k_i} can be, since $1 \leq k_i \leq N$). Thus $m_i \leq 1$ if $i \gg 0$. Then from (3.7), we obtain that $p_i(\lambda, \mu)$ is a polynomial of degree ≤ 1. Thus suppose $p_i(\lambda, \mu) = a_{i,0} + a_{i,1}\lambda + a_{i,2}\mu$. Then by comparing the coefficients of $\lambda\mu$, $\mu\partial$ and μ respectively in (3.6), we immediately have $p_i(\lambda, \mu) = 0$.

Case 2: $V_1 = \mathbb{C}_{\beta_1}$, $\overline{V}_{k_i} = M_{\Delta_{k_i}, \beta_{k_i}}$.

In this case, we can still assume (3.5). Applying the operator $\mathfrak{b}_{0\mu}$ to (3.5), we obtain

$$p_i(\lambda, \mu+\partial)(\beta_{k_i} + \partial + \Delta_{k_i}\mu) = (i\alpha + (1+i)\mu - \lambda)p_i(\lambda+\mu, \partial). \tag{3.9}$$

Taking $\mu = \partial = 0$, we get $p_i(\lambda, 0) = 0$. Then letting $\partial = 0$, we obtain $p_i(\lambda, \mu) = 0$.

Case 3: $V_1 = M_{\Delta_1, \beta_1}$, $\overline{V}_{k_i} = \mathbb{C}_{\beta_{k_i}}$.

In this case, since ∂ acts on $\bar{v}_{k_i}$ as the scalar β_{k_i}, i.e., $\partial v_{k_i} \equiv \beta_{k_i} v_{k_i} \pmod{V_{k_i-1}[\lambda]}$, we can write

$$\mathfrak{b}_{i\,\lambda} v_1 \equiv p_i(\lambda) v_{k_i} \pmod{V_{k_i-1}[\lambda]} \quad \text{for some} \quad p_i(\lambda) \in \mathbb{C}[\lambda]. \tag{3.10}$$

Applying the operator $\mathfrak{b}_{0\mu}$ to (3.10), we obtain

$$0 = \big(i\alpha + (1+i)\mu - \lambda\big)p_i(\lambda+\mu) + (\beta_1 + \lambda + \partial + \Delta_1\mu)p_i(\lambda). \tag{3.11}$$

By comparing the coefficients of ∂, we immediately get $p_i(\lambda) = 0$.

Case 4: $V_1 = \mathbb{C}_{\beta_1}$, $\overline{V}_{k_i} = \mathbb{C}_{\beta_{k_i}}$.

As above, we can still assume (3.10). Applying the operator $\mathfrak{b}_{0\mu}$ to (3.10), in this case we obtain

$$0 = \big(i\alpha + (1+i)\mu - \lambda\big)p_i(\lambda+\mu). \tag{3.12}$$

It is obvious that $p_i(\lambda) = 0$. $\square$

Finally we can give the proof of Theorem 1.3(4). By induction on $j \leq N$, we obtain $\mathfrak{b}_{i\,\lambda} v_j = 0$, i.e., the λ-action of $\mathfrak{b}_i$ is trivial. From this, we immediately obtain that the λ-action of $\mathcal{B}(\alpha)$ on V is trivial since $\mathcal{B}(\alpha)$ is a simple Lie conformal algebra. Then Theorem 1.3(4) follows.

4. Some technical lemmas

Let $\mathcal{G} = \oplus_{i=-1}^{\infty}\mathcal{G}_i$ be a simple Lie conformal algebra satisfying conditions (C1) and (C2). The main problem to be addressed in this paper is to classify these $\mathbb{Z}$-graded Lie conformal algebras. In order to solve the main problem, we need some preparations. Since $\mathcal{G} = \oplus_{i=-1}^{\infty}\mathcal{G}_i$ satisfying conditions (C1) and (C2), denote by $\mathfrak{g}_i$ a $\mathbb{C}[\partial]$-generator of $\mathcal{G}_i$, then $\{\mathfrak{g}_i \,|\, i \in \mathbb{Z}_{\geq -1}\}$ is a $\mathbb{C}[\partial]$-generating set of $\mathcal{G}$. By (2.9), we can suppose

$$[\mathfrak{g}_{-1\lambda}\mathfrak{g}_{-1}] = 0, \tag{4.1}$$

$$[\mathfrak{g}_{0\lambda}\mathfrak{g}_j] = (\alpha_j + \partial + \Delta_j\lambda)\mathfrak{g}_j, \tag{4.2}$$

$$[\mathfrak{g}_{i\lambda}\mathfrak{g}_j] = g_{i,j}(\lambda,\partial)\mathfrak{g}_{i+j}, \tag{4.3}$$

where $\alpha_j,\ \Delta_j \in \mathbb{C}$ for $j \in \mathbb{Z}_{\geq -1}$ and $g_{i,j}(\lambda,\partial) \in \mathbb{C}[\lambda,\partial]$ for $i, j \in \mathbb{Z}_{\geq -1}$ are polynomials of λ and ∂. From (4.3), we can see that $\mathfrak{g}_{i+j}$ can be generated by $\mathfrak{g}_i$ and $\mathfrak{g}_j$ for $i, j \in \mathbb{Z}_{\geq -1}$. It is very natural to firstly consider the cases $g_{i,j}(\lambda,\partial)$ for $i = 0$, $j \in \mathbb{Z}_{\geq -1}$ and $i = 1$, $j = -1$. This is also the aim of this section. Based on this, in the next section we will determine all $g_{i,j}(\lambda,\partial)$ for $i, j \in \mathbb{Z}_{\geq -1}$ in the proof of Theorem 1.3(3). Since $\mathcal{G}$ is simple, we also know that

$$[\mathfrak{g}_{-1\lambda}\mathfrak{g}_j] \neq 0 \quad \text{for} \quad 1 \leq j \in \mathbb{Z}. \tag{4.4}$$

Lemma 4.1. *In (4.2), we have $\Delta_0 = 2$ and $\alpha_j = j\alpha_1$ for $j \in \mathbb{Z}_{\geq -1}$; in particular, we get $\alpha_0 = 0$. Thus for $j \in \mathbb{Z}_{\geq -1}$ we can suppose*

$$g_{0,0}(\lambda,\partial) = \partial + 2\lambda, \tag{4.5}$$

$$g_{0,j}(\lambda,\partial) = j\alpha_1 + \partial + \Delta_j\lambda. \tag{4.6}$$

Proof. From $\mathcal{G}_0 \cong \text{Vir}$, by (2.5), we can conclude that $\Delta_0 = 2$, $\alpha_0 = 0$ in (4.2), thus (4.5) holds. Now applying the operator $\mathfrak{g}_{0_\mu}$ to (4.3), using the Jacobi identity $[\mathfrak{g}_{0\mu}[\mathfrak{g}_{i\lambda}\mathfrak{g}_j]] = [[\mathfrak{g}_{0\mu}\mathfrak{g}_i]_{\lambda+\mu}\mathfrak{g}_j] + [\mathfrak{g}_{i\lambda}[\mathfrak{g}_{0\mu}\mathfrak{g}_j]]$ and comparing the coefficients of $\mathfrak{g}_{i+j}$ for $i, j \in \mathbb{Z}_{\geq -1}$, we obtain

$$\begin{aligned}(\alpha_{j+i} + \partial + \Delta_{j+i}\mu)g_{i,j}(\lambda,\mu+\partial) - (\alpha_j + \partial + \lambda + \Delta_j\mu)g_{i,j}(\lambda,\partial)\\ = \big(\alpha_i - \lambda + (\Delta_i - 1)\mu\big)g_{i,j}(\lambda+\mu,\partial).\end{aligned} \tag{4.7}$$

Now taking $\mu = 0$ in (4.7), one can immediately get that $(\alpha_{j+i} - \alpha_j - \alpha_i)g_{i,j}(\lambda,\partial) = 0$ for all $i, j \in \mathbb{Z}_{\geq -1}$. Then the lemma follows. □

In order to determine all the polynomials $g_{i,j}(\lambda,\partial)$, we would first like to deal with the case with $i = 1$ and $j = -1$. There is no need to compute $g_{-1,1}(\lambda,\partial)$ as it can be determined from $g_{1,-1}(\lambda,\partial)$ by skew-symmetry. Comparing the coefficients of $\mathfrak{g}_{j+1}$ on both sides of the Jacobi identity $[\mathfrak{g}_{0\lambda}[\mathfrak{g}_{1\mu}\mathfrak{g}_j]] = [[\mathfrak{g}_{0\lambda}\mathfrak{g}_1]_{\lambda+\mu}\mathfrak{g}_j] + [\mathfrak{g}_{1\mu}[\mathfrak{g}_{0\lambda}\mathfrak{g}_j]]$, by (4.3) and Lemma 4.1, we have

$$\begin{aligned}\big((j+1)\alpha_1+\partial+\Delta_{j+1}\lambda\big)g_{1,j}(\mu,\lambda+\partial)-(j\alpha_1+\partial+\mu+\Delta_j\lambda)g_{1,j}(\mu,\partial)\\ =\big(\alpha_1-\mu+(\Delta_1-1)\lambda\big)g_{1,j}(\lambda+\mu,\partial).\end{aligned}\tag{4.8}$$

Taking $\partial = 0$ and $j = -1$, by Lemma 4.1, we obtain

$$\begin{aligned}g_{1,-1}(\mu,\lambda)=\frac{1}{2\lambda}\Big(\big(\alpha_1-\mu+(\Delta_1-1)\lambda\big)g_{1,-1}(\lambda+\mu,0)\\ -(\alpha_1-\mu-\Delta_{-1}\lambda)g_{1,-1}(\mu,0)\Big).\end{aligned}\tag{4.9}$$

Similarly, applying the operator $\mathfrak{g}_{-1_\lambda}$ to $[\mathfrak{g}_{1\mu}\mathfrak{g}_j]$, using the Jacobi identity and comparing the coefficients of $\mathfrak{g}_j$, by (4.3) and Lemma 4.1, we obtain

$$\begin{aligned}g_{-1,j+1}(\lambda,\partial)g_{1,j}(\mu,\lambda+\partial)-g_{-1,j}(\lambda,\partial+\mu)g_{1,j-1}(\mu,\partial)\\ =\big(j\alpha_1+\partial+\Delta_j(\lambda+\mu)\big)g_{-1,1}(\lambda,-\lambda-\mu).\end{aligned}\tag{4.10}$$

Setting $j = -1$ and replacing ∂, λ by $-\lambda$, $-\mu-\partial$ in (4.10) respectively, by (2.2), (4.1), (4.3) and Lemma 4.1, we can deduce that

$$(\alpha_1-\mu-\partial)g_{1,-1}(\mu,0)=\big(\alpha_1-\mu+(\Delta_{-1}-1)\partial\big)g_{1,-1}(\mu,\partial).\tag{4.11}$$

Using (4.9) in the above formula, we get

$$\begin{aligned}\big(\alpha_1-\mu+(\Delta_1-1)\partial\big)\big(\alpha_1-\mu+(\Delta_{-1}-1)\partial\big)g_{1,-1}(\mu+\partial,0)\\ =\Big(2\partial(\alpha_1-\mu-\partial)+(\alpha_1-\mu-\Delta_{-1}\partial)\big(\alpha_1-\mu+(\Delta_{-1}-1)\partial\big)\Big)g_{1,-1}(\mu,0).\end{aligned}\tag{4.12}$$

Letting $\mu = 0$ implies the following,

$$\begin{aligned}\big(\alpha_1+(\Delta_1-1)\partial\big)\big(\alpha_1+(\Delta_{-1}-1)\partial\big)g_{1,-1}(\partial,0)\\ =\Big(2\partial(\alpha_1-\partial)+(\alpha_1-\Delta_{-1}\partial)\big(\alpha_1+(\Delta_{-1}-1)\partial\big)\Big)g_{1,-1}(0,0).\end{aligned}\tag{4.13}$$

Taking $\partial = -\mu$ in (4.12), and replacing μ by ∂, we obtain

$$\begin{aligned}\Big(-2\partial\alpha_1+(\alpha_1-\Delta_{-1}\partial)\big(\alpha_1+(\Delta_{-1}-1)\partial\big)\Big)g_{1,-1}(\partial,0)\\ =(\alpha_1-\Delta_1\partial)(\alpha_1-\Delta_{-1}\partial)g_{1,-1}(0,0).\end{aligned}\tag{4.14}$$

Since $g_{1,-1}(\lambda,\partial)$ is a polynomial of λ and ∂, we can write $g_{1,-1}(\partial,0)=\sum_{i=0}^{m}a_{1,-1}^{i}\partial^{i}$ for some $a_{1,-1}^{i}\in\mathbb{C}$ with $0\le i\le m$. We need to consider whether or not $\alpha_1\neq 0$. First assume $\alpha_1 \neq 0$. If $a^0_{1,-1} = 0$, then comparing the degrees of ∂ on both sides of (4.14), we immediately have $g_{1,-1}(\partial, 0) = 0$, which implies that $g_{1,-1}(\lambda, \partial) = 0$ by (4.9), a contradiction with (4.4). If $a^0_{1,-1} \neq 0$, we have the following.

Lemma 4.2. *Assume $\alpha_1 \neq 0$ and $a^0_{1,-1} \neq 0$. Then $\Delta_{-1} = 0,\ \Delta_1 = 3$ or $\Delta_{-1} = 1,\ \Delta_1 = 3$. Furthermore,*

$$g_{1,-1}(\lambda,\partial) = \begin{cases} a^0_{1,-1} & \text{if} \quad \Delta_{-1} = 0,\ \Delta_1 = 3, \\ \dfrac{a^0_{1,-1}}{\alpha_1}(\alpha_1 - \lambda - \partial) & \text{if} \quad \Delta_{-1} = 1,\ \Delta_1 = 3. \end{cases} \tag{4.15}$$

Proof. Since $\alpha_1 \neq 0$ and $g_{1,-1}(0,0) = a^0_{1,-1} \neq 0$, comparing the degrees of ∂ on both sides of (4.14), we get $a^i_{1,-1} = 0$ for $2 \leq i \leq m$. And we need to consider the following possibilities.

Case 1: $\Delta_{-1} = 0$.

If $\Delta_{-1} = 0$, by (4.14), we must have $\Delta_1 = 3$ and $g_{1,-1}(\partial, 0) = a^0_{1,-1}$. Therefore, by (4.9) we have $g_{1,-1}(\lambda, \partial) = a^0_{1,-1}$, i.e., the first case of (4.15) holds.

Case 2: $\Delta_{-1} = 1$.

In this case, (4.14) shows that $\Delta_1 = 3$ and $a^1_{1,-1} = -\frac{a^0_{1,-1}}{\alpha_1}$. Using this in (4.9), we get the second case of (4.15).

Case 3: $\Delta_{-1} \neq 0$ and $\Delta_{-1} \neq 1$.

If $\Delta_{-1} \neq 0$ and $\Delta_{-1} \neq 1$, comparing the degrees of ∂ on both sides of (4.14), we know that $g_{1,-1}(\partial, 0) = g_{1,-1}(0,0) \neq 0$. Then comparing the coefficients of ∂^i for $i = 1, 2$ on both sides of (4.14) respectively, we have $\Delta_{-1} + \Delta_1 = 3$ and $\Delta_{-1} + \Delta_1 = 1$, a contradiction. Hence the lemma follows. □

Now we deal with the case $\alpha_1 = 0$. If $a^0_{1,-1} \neq 0$, taking $\alpha_1 = 0$ in (4.13) and (4.14) and comparing the coefficients of ∂^2 on both sides respectively, we can deduce that $\Delta_{-1} = 0$, $\Delta_1 = 3$ and $g_{1,-1}(\partial, 0) = a^0_{1,-1}$. Then by (4.9) we immediately obtain the following.

Lemma 4.3. *If $\alpha_1 = 0$ and $a^0_{1,-1} \neq 0$, then $\Delta_{-1} = 0$, $\Delta_1 = 3$ and $g_{1,-1}(\lambda, \partial) = a^0_{1,-1}$.*

Now we can consider the most complicated case that $\alpha_1 = 0$ and $a^0_{1,-1} = 0$. In this case (4.13) and (4.14) turn into

$$(\Delta_1 - 1)(\Delta_{-1} - 1)g_{1,-1}(\partial, 0) = 0, \tag{4.16}$$

$$\Delta_{-1}(1 - \Delta_{-1})g_{1,-1}(\partial, 0) = 0. \tag{4.17}$$

Lemma 4.4. *If $\alpha_1 = 0$ and $a^0_{1,-1} = 0$, then we can deduce that $\Delta_{-1} = 1,\ \Delta_1 = 3$ and*

$$g_{1,-1}(\lambda, \partial) = a^1_{1,-1}(\lambda + \partial) \qquad \text{for} \quad a^1_{1,-1} \in \mathbb{C}^*, \tag{4.18}$$

Proof. By (4.4) and (4.17), our discussion will be divided into the following two cases.
Case 1: $\Delta_{-1} = 0$.

From (4.16), it follows that $(\Delta_1 - 1)g_{1,-1}(\partial, 0) = 0$. If $\Delta_1 \neq 1$, then it is obvious that $g_{1,-1}(\partial, 0) = 0$. This together with (4.9) shows that $g_{1,-1}(\lambda, \partial) = 0$, i.e., a contradiction with (4.4). Now we suppose $\Delta_1 = 1$. Taking $\lambda = -\mu \neq 0$ in (4.9), using the fact that $a^0_{1,-1} = 0$, we obtain

$$g_{1,-1}(\mu, -\mu) = -\frac{1}{2}\, g_{1,-1}(\mu, 0). \tag{4.19}$$

In addition, (4.11) shows that

$$g_{1,-1}(\mu, \partial) = g_{1,-1}(\mu, 0) \quad \text{for } \ \partial \neq -\mu. \tag{4.20}$$

Letting $\alpha_1 = 0$, $\Delta_{-1} = 0$ and $\Delta_1 = 1$ in (4.12), we have $\mu g_{1,-1}(\mu + \partial, 0) = (\mu - 2\partial)g_{1,-1}(\mu, 0)$ for $\mu + \partial \neq 0$. Then setting $\mu = 1$ and $\partial = -\mu + 1$ in this formula respectively, we get

$$g_{1,-1}(\mu, 0) = (3 - 2\mu)g_{1,-1}(1, 0) \quad \text{for } \ \mu \neq 0, \tag{4.21}$$

$$\mu g_{1,-1}(1, 0) = (3\mu - 2)g_{1,-1}(\mu, 0). \tag{4.22}$$

Inserting (4.21) into (4.22) gives that $(\mu - 1)^2 g_{1,-1}(1, 0) = 0$ for $\mu \neq 0$. It leads to $g_{1,-1}(1, 0) = 0$. Then (4.19), (4.20) together with (4.21) show that $g_{1,-1}(\lambda, \partial) = 0$, i.e., we also get a contradiction with (4.4).
Case 2: $\Delta_{-1} = 1$.

Taking $\alpha_1 = 0$ and $\Delta_{-1} = 1$ in (4.12), we have

$$\mu\big((\Delta_1 - 1)\partial - \mu\big)g_{1,-1}(\mu + \partial, 0) = (\mu + \partial)(2\partial - \mu)g_{1,-1}(\mu, 0).$$

Letting $\partial = -\mu + 1$ in the above formula, then replacing μ by ∂, we can obtain

$$(2 - 3\partial)g_{1,-1}(\partial, 0) = \partial(\Delta_1 - 1 - \Delta_1\partial)g_{1,-1}(1, 0). \tag{4.23}$$

Recall that $g_{1,-1}(\partial, 0) = \sum_{i=1}^{m} a^i_{1,-1}\partial^i$. Comparing the coefficients of ∂^i for $1 \leq i \leq m+1$ on both sides of (4.23), we deduce that $a^i_{1,-1} = 0$ for $2 \leq i \leq m$, $a^1_{1,-1} = g_{1,-1}(1, 0)$ and $(\Delta_1 - 3)a^1_{1,-1} = 0$. Thus we have that $g_{1,-1}(\partial, 0) = a^1_{1,-1}\partial$ and $(\Delta_1 - 3)a^1_{1,-1} = 0$. If $a^1_{1,-1} = 0$, it follows that $g_{1,-1}(\partial, 0) = 0$, then (4.9) leads to $g_{1,-1}(\lambda, \partial) = 0$. Therefore, by (4.4) and (4.9), we have $g_{1,-1}(\lambda, \partial) = a^1_{1,-1}(\lambda + \partial) \neq 0$. And this lemma holds. □

5. Classification of graded Lie conformal algebras

In this section, we determine all $g_{i,j}(\lambda, \partial)$ for $i, j \in \mathbb{Z}_{\geq -1}$, so that we can classify $\mathbb{Z}$-graded simple Lie conformal algebras $\mathcal{G} = \oplus_{i=-1}^{\infty}\mathcal{G}_i$.

Proof of Theorem 1.3(3). Applying $\mathfrak{g}_{-1_\lambda}$ to $[\mathfrak{g}_{0\mu}\mathfrak{g}_j]$, by (2.3), we have the Jacobi identity $[\mathfrak{g}_{-1\lambda}[\mathfrak{g}_{0\mu}\mathfrak{g}_j]] = [[\mathfrak{g}_{-1\lambda}\mathfrak{g}_0]_{\lambda+\mu}\mathfrak{g}_j] + [\mathfrak{g}_{0\mu}[\mathfrak{g}_{-1\lambda}\mathfrak{g}_j]]$. Then comparing the coefficients of $\mathfrak{g}_{j-1}$, by (4.3) and Lemma 4.1, we obtain

$$\begin{aligned}(j\alpha_1+\lambda+\partial+\Delta_j\mu)g_{-1,j}(\lambda,\partial) &- \big((j-1)\alpha_1+\partial+\Delta_{j-1}\mu\big)g_{-1,j}(\lambda,\partial+\mu)\\ &= \big(\alpha_1+\lambda-(\Delta_{-1}-1)\mu\big)g_{-1,j}(\lambda+\mu,\partial).\end{aligned} \tag{5.1}$$

Taking $j=1$ in (4.10), we have

$$\begin{aligned}g_{-1,2}(\lambda,\partial)g_{1,1}(\mu,\lambda+\partial) &- g_{-1,1}(\lambda,\partial+\mu)g_{1,0}(\mu,\partial)\\ &= \big(\alpha_1+\partial+\Delta_1(\lambda+\mu)\big)g_{-1,1}(\lambda,-\lambda-\mu).\end{aligned} \tag{5.2}$$

By skew-symmetry, we have $g_{-1,1}(\lambda,\partial) = -g_{1,-1}(-\lambda-\partial,\partial)$, so for convenience, we suppose $a^0_{-1,1} = -a^0_{1,-1}$. By Lemma 4.2–4.4, we need to consider the following three cases.

Case 1: $\Delta_{-1}=0$, $\Delta_1=3$ and $g_{1,-1}(\lambda,\partial) = -a^0_{-1,1}\neq 0$.

Since we have skew-symmetry, by Lemma 4.1, we have $g_{-1,1}(\lambda,\partial) = a^0_{-1,1}$ and $g_{1,0}(\lambda,\partial) = -\alpha_1+3\lambda+2\partial$. Then (5.2) turns into

$$g_{-1,2}(\lambda,\partial)g_{1,1}(\mu,\lambda+\partial) = 3a^0_{-1,1}(\lambda+2\mu+\partial).$$

Note that both $g_{-1,2}(\lambda,\partial)$ and $g_{1,1}(\lambda,\partial)$ are polynomials of λ and ∂, so by comparing the coefficients on both sides of the above formula, we can deduce that

$$g_{-1,2}(\lambda,\partial) = a^0_{-1,2}, \tag{5.3}$$

$$g_{1,1}(\lambda,\partial) = \frac{3a^0_{-1,1}}{a^0_{-1,2}}(2\lambda+\partial), \tag{5.4}$$

where $a^0_{-1,2}\in\mathbb{C}^*$. Setting $j=2$ in (5.1) and noting that $g_{-1,2}(\lambda,\partial) = a^0_{-1,2}\neq 0$, we get

$$\Delta_2 = \Delta_1+1 = 4. \tag{5.5}$$

By (4.10), (5.1) and (5.3)–(5.5), we can inductively deduce that

$$\Delta_j = j+2, \tag{5.6}$$

$$g_{-1,j}(\lambda,\partial) = a^0_{-1,j}, \tag{5.7}$$

$$g_{1,j}(\lambda,\partial) = \frac{a^0_{-1,1}}{2a^0_{-1,j+1}}(j+2)\Big((j-1)\alpha_1+(j+3)\lambda+2\partial\Big), \tag{5.8}$$

where $1\le j\in\mathbb{Z}$ and $a^0_{-1,j}\in\mathbb{C}^*$.

Now we want to determine $g_{j,2}(\lambda,\partial)$ for $2\le j\in\mathbb{Z}$. Comparing the coefficients of $\mathfrak{g}_{j+2}$ on both sides of $[\mathfrak{g}_{j\lambda}[\mathfrak{g}_{1\mu}\mathfrak{g}_1]] = [[\mathfrak{g}_{j\lambda}\mathfrak{g}_1]_{\lambda+\mu}\mathfrak{g}_1] + [\mathfrak{g}_{1\mu}[\mathfrak{g}_{j\lambda}\mathfrak{g}_1]]$, we obtain

$$\begin{aligned}&g_{j,2}(\lambda,\partial)g_{1,1}(\mu,\lambda+\partial)-g_{j,1}(\lambda,\partial+\mu)g_{1,j+1}(\mu,\partial)\\&\quad=g_{j,1}(\lambda,-\lambda-\mu)g_{j+1,1}(\lambda+\mu,\partial).\end{aligned}$$

From this, using (5.8) and skew-symmetry, we obtain, for $1\le j\in\mathbb{Z}$,

$$g_{j,2}(\lambda,\partial)=\frac{-a^0_{-1,1}a^0_{-1,2}}{6a^0_{-1,j+1}a^0_{-1,j+2}}(j+2)(j+3)\Big((j-2)\alpha_1-(j+4)\lambda-(j+1)\partial\Big).\tag{5.9}$$

Finally, we can determine all the polynomials $g_{j,i}(\lambda,\partial)$ as follows. Noting from $[\mathfrak{g}_{j\lambda}[\mathfrak{g}_{1\mu}\mathfrak{g}_i]]=[[\mathfrak{g}_{j\lambda}\mathfrak{g}_1]_{\lambda+\mu}\mathfrak{g}_i]+[\mathfrak{g}_{1\mu}[\mathfrak{g}_{j\lambda}\mathfrak{g}_i]]$, we obtain

$$\begin{aligned}&g_{j,i+1}(\lambda,\partial)g_{1,i}(\mu,\lambda+\partial)-g_{j,i}(\lambda,\partial+\mu)g_{1,j+i}(\mu,\partial)\\&\quad=g_{j,1}(\lambda,-\lambda-\mu)g_{j+1,i}(\lambda+\mu,\partial).\end{aligned}\tag{5.10}$$

By (5.8)–(5.10), we can inductively deduce, for $1\le i\in\mathbb{Z}$, $1\le j\in\mathbb{Z}$,

$$\begin{aligned}g_{j,i}(\lambda,\partial)=&\frac{-a^0_{-1,1}a^0_{-1,2}\cdots a^0_{-1,i}}{a^0_{-1,j+1}a^0_{-1,j+2}\cdots a^0_{-1,j+i}}\times\frac{(j+2)(j+3)\cdots(j+i+1)}{(i+1)!}\\&\times\Big((j-i)\alpha_1-(j+i+2)\lambda-(j+1)\partial\Big).\end{aligned}\tag{5.11}$$

In order for the polynomials $g_{j,i}(\lambda,\partial)$ to have some suitable forms, for $1\le j\in\mathbb{Z}$, we replace $\mathfrak{g}_j$ by $\mathfrak{g}'_j=\frac{(j+1)!}{a^0_{-1,1}a^0_{-1,2}\cdots a^0_{-1,j}}\mathfrak{g}_j$, so that $g_{-1,j}(\lambda,\partial)$ and $g_{j,i}(\lambda,\partial)$ have the following forms,

$$\begin{aligned}&g_{-1,j}(\lambda,\partial)=j+1\ \text{ for }\ 1\le j\in\mathbb{Z},\\&g_{j,i}(\lambda,\partial)=(i-j)\alpha_1+(i+j+2)\lambda+(j+1)\partial\ \text{ for }\ 1\le i\in\mathbb{Z},1\le j\in\mathbb{Z}.\end{aligned}$$

Since in this case $\Delta_{-1}=0$, using (4.6) and skew-symmetry, we obtain $g_{-1,0}(\lambda,\partial)=\alpha_1-\partial$. By (4.6) and (5.6), it is not hard to check that the second equation of the above also holds for $i=0$ or $j=0$. Therefore, in this case we obtain $\mathcal{G}\cong B(1,\alpha)$ for some $\alpha\in\mathbb{C}$.

Case 2: $\Delta_{-1}=1$, $\Delta_1=3$, $\alpha_1=0$ and $g_{1,-1}(\lambda,\partial)=a^1_{-1,1}(\lambda+\partial)$, where $a^1_{-1,1}\in\mathbb{C}^*$.

In this case, using skew-symmetry, we obtain that $g_{-1,1}(\lambda,\partial)=a^1_{-1,1}\lambda$ and $g_{1,0}(\lambda,\partial)=3\lambda+2\partial$. Then (5.2) turns into

$$g_{-1,2}(\lambda,\partial)g_{1,1}(\mu,\lambda+\partial)=3a^1_{-1,1}\lambda(\lambda+2\mu+\partial).$$

Since $g_{-1,2}(\lambda,\partial)$ and $g_{1,1}(\lambda,\partial)$ are polynomials of λ and ∂, we get from the above formula,

$$g_{-1,2}(\lambda,\partial)=a^1_{-1,2}\lambda,\tag{5.12}$$

$$g_{1,1}(\lambda,\partial)=\frac{3a^1_{-1,1}}{a^1_{-1,2}}(2\lambda+\partial),\tag{5.13}$$

for some $a^1_{-1,2} \in \mathbb{C}^*$. Taking $j = 2$, $\Delta_{-1} = 1$, $\Delta_1 = 3$ and $\alpha_1 = 0$ in (5.1), and noting that $a^1_{-1,2} \neq 0$, we get

$$\Delta_2 = 4. \tag{5.14}$$

Therefore, by (4.10), (5.1) and (5.12)–(5.14), we can inductively prove that

$$\begin{aligned} &\Delta_j = j+2, \\ &g_{-1,j}(\lambda,\partial) = a^1_{-1,j}\lambda, \\ &g_{1,j}(\lambda,\partial) = \frac{a^1_{-1,1}}{2a^1_{-1,j+1}}(j+2)\Big((j+3)\lambda + 2\partial\Big), \end{aligned}$$

where $1 \leq j \in \mathbb{Z}$ and $a^1_{-1,j} \in \mathbb{C}^*$. Similar to Case 1, the above three formulae together with (5.10) inductively show the following,

$$\begin{aligned} g_{j,i}(\lambda,\partial) = &\frac{a^1_{-1,1}a^1_{-1,2}\cdots a^1_{-1,i}}{a^1_{-1,j+1}a^1_{-1,j+2}\cdots a^1_{-1,j+i}} \times \frac{(j+2)(j+3)\cdots(j+i+1)}{(i+1)!} \\ &\times \Big((j+i+2)\lambda + (j+1)\partial\Big), \end{aligned}$$

for $1 \leq i \in \mathbb{Z}$ and $1 \leq j \in \mathbb{Z}$. Replace $\mathfrak{g}_j$ by $\mathfrak{g}'_j = \frac{(j+1)!}{a^1_{-1,1}a^1_{-1,2}\cdots a^1_{-1,j}}\mathfrak{g}_j$ for $1 \leq j \in \mathbb{Z}$, so that $g_{-1,j}(\lambda,\partial)$ and $g_{j,i}(\lambda,\partial)$ have the following forms,

$$\begin{aligned} &g_{-1,j}(\lambda,\partial) = (j+1)\lambda \quad \text{for} \quad 1 \leq j \in \mathbb{Z}, \\ &g_{j,i}(\lambda,\partial) = (i+j+2)\lambda + (j+1)\partial \quad \text{for} \quad 1 \leq i \in \mathbb{Z},\ 1 \leq j \in \mathbb{Z}. \end{aligned}$$

Using Lemma 4.1 and the fact that $\Delta_j = j + 2$ for $1 \leq j \in \mathbb{Z}$, we can immediately obtain that the above two formulae hold for all $i, j \in \mathbb{Z}_{\geq -1}$. Therefore, we have proved that $g_{j,i}(\lambda,\partial) = (i + j + 2)\lambda + (j + 1)\partial$ for all $i, j \in \mathbb{Z}_{\geq -1}$, which is equivalent to that $\mathcal{G} \cong B(2, 0)$.

Case 3: $\Delta_{-1} = 1, \Delta_1 = 3, \alpha_1 \neq 0$ and $g_{1,-1}(\lambda,\partial) = -\frac{a^0_{-1,1}}{\alpha_1}\big(\alpha_1-\lambda-\partial\big)$, where $a^0_{-1,1} \in \mathbb{C}^*$.

In this case, by skew-symmetry, we get $g_{-1,1}(\lambda,\partial) = a^0_{-1,1}\big(1 + \frac{1}{\alpha_1}\lambda\big)$ and $g_{1,0}(\lambda,\partial) = -\alpha_1 + 3\lambda + 2\partial$. Then (5.2) leads to

$$g_{-1,2}(\lambda,\partial)g_{1,1}(\mu,\lambda+\partial) = 3a^0_{-1,1}(\lambda + 2\mu + \partial)\Big(1 + \frac{1}{\alpha_1}\lambda\Big).$$

Setting $\lambda = 0$, $\partial = 0$ and $\lambda = \partial = 0$ in the above formula respectively, noting that $g_{-1,2}(\lambda,\partial)$ and $g_{1,1}(\lambda,\partial)$ are polynomials of λ and ∂, we obtain

$$g_{-1,2}(\lambda,\partial) = a^0_{-1,2}\Big(1 + \frac{1}{\alpha_1}\lambda\Big), \tag{5.15}$$

$$g_{1,1}(\lambda,\partial) = \frac{3a^0_{-1,1}}{a^0_{-1,2}}\big(2\lambda + \partial\big), \tag{5.16}$$

where $a^0_{-1,2} \in \mathbb{C}^*$. Taking $j = 2$, $\Delta_{-1} = 1$ and $\Delta_1 = 3$ in (5.1), noting that $a^0_{-1,2} \neq 0$, we get

$$\Delta_2 = 4. \tag{5.17}$$

By (4.10), (5.1) and (5.15)–(5.17), we can inductively deduce

$$\begin{aligned}
&\Delta_j = j + 2,\\
&g_{-1,j}(\lambda, \partial) = a^0_{-1,j}\Big(1 + \frac{1}{\alpha_1}\lambda\Big),\\
&g_{1,j}(\lambda, \partial) = \frac{a^0_{-1,1}}{2a^0_{-1,j+1}}(j+2)\Big((j-1)\alpha_1 + (j+3)\lambda + 2\partial\Big),
\end{aligned}$$

where $1 \le j \in \mathbb{Z}$ and $a^0_{-1,j} \in \mathbb{C}^*$. Similar to Case 1, from (5.10) and the above three formulae, it inductively follows that

$$\begin{aligned}
g_{j,i}(\lambda, \partial) = {} & \frac{a^0_{-1,1}a^0_{-1,2}\cdots a^0_{-1,i}}{a^0_{-1,j+1}a^0_{-1,j+2}\cdots a^0_{-1,j+i}} \times \frac{(j+2)(j+3)\cdots(j+i+1)}{(i+1)!}\\
& \times \Big((i-j)\alpha_1 + (j+i+2)\lambda + (j+1)\partial\Big),
\end{aligned}$$

for $1 \le i \in \mathbb{Z}$ and $1 \le j \in \mathbb{Z}$. Since in this case $\alpha_1 \neq 0$, by replacing $\mathfrak{g}_j$ by $\mathfrak{g}'_j = \frac{(j+1)!\alpha_1^j}{a^0_{-1,1}a^0_{-1,2}\cdots a^0_{-1,j}}\mathfrak{g}_j$ for $1 \le j \in \mathbb{Z}$, we obtain that $g_{-1,j}(\lambda, \partial)$ and $g_{j,i}(\lambda, \partial)$ have the following forms,

$$\begin{aligned}
&g_{-1,j}(\lambda, \partial) = (j+1)(\alpha_1 + \lambda) \ \text{ for } \ 1 \le j \in \mathbb{Z},\\
&g_{j,i}(\lambda, \partial) = (i-j)\alpha_1 + (i+j+2)\lambda + (j+1)\partial \ \text{ for } \ 1 \le i \in \mathbb{Z},\ 1 \le j \in \mathbb{Z}.
\end{aligned}$$

By Lemma 4.1 and noting that $\Delta_j = j + 2$ for $1 \le j \in \mathbb{Z}$, we can immediately conclude that the above two formulae hold for all $i, j \in \mathbb{Z}_{\ge -1}$. Hence, we obtain that $g_{j,i}(\lambda, \partial) = (i-j)\alpha_1 + (i+j+2)\lambda + (j+1)\partial$ for all $i, j \in \mathbb{Z}_{\ge -1}$ and $\alpha_1 \neq 0$. It follows that in this case, we have $\mathcal{G} \cong B(2, \alpha)$ for some $\alpha \in \mathbb{C}^*$.

Therefore, the above three cases together show that Theorem 1.3(3) holds. □

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