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Dade's Inductive Conjecture for the Ree Groups of Type G_2 in the Defining Characteristic

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We verify the inductive form of Dade's conjecture for the finite simple groups ${}^2G_2(3^{2m+1})$, where m is a positive integer, for the prime $p = 3$. Together with work by J. An (1994, *Indian J. Math.* **36**, 7–27) this completes the verification of the conjecture for this series of groups.

Key Words: Alperin's conjecture; Dade's conjecture; character theory; finite groups of Lie type; modular representation theory.

1. INTRODUCTION

In [7, 8, 9] Dade presents a series of conjectures, the strongest of which (the inductive conjecture) he claims admits a reduction to finite simple groups. The conjectures originate from the reformulation of Alperin's weight conjecture (see [1]) given by Knörr and Robinson in [11], and consequently a verification of any of Dade's conjectures implies that Alperin's weight conjecture holds. It should be noted, however, that the verification of Dade's conjecture for a given group does not imply that Alperin's conjecture holds for that group. However, Alperin's conjecture is known to hold for finite groups with split BN-pairs in the defining characteristic and so does indeed hold in this case (see [4]).

Let G be a finite group and p a prime. We say that a p-subgroup Q of G is *radical* if $Q = O_p(N_G(Q))$, where $O_p(H)$ is the unique maximal normal p-subgroup of H. We say that a chain $\sigma: Q_0 < \cdots < Q_n$ of

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p-subgroups of G (where inclusions are strict) has length $|\sigma| = n$, and write $G_\sigma = N_G(Q_0) \cap \cdots \cap N_G(Q_n)$ for its stabilizer under the conjugation action of G. Write σ_i for the p-chain $Q_0 < \cdots < Q_i$ and $V_\sigma = Q_0$. We say that the *p-chain* σ is *radical* if $Q_i = O_p(N_G(\sigma_i))$ for each i, i.e., if Q_0 is a radical p-subgroup of G and Q_i is a radical p-subgroup of $N_G(\sigma_{i-1})$ for each $i \neq 0$. Write $\mathcal{R} = \mathcal{R}(G)$ for the set of radical p-chains of G and write $\mathcal{R}(G \mid Q) = \{\sigma \in \mathcal{R}(G) : V_\sigma = Q\}$. Write $\mathcal{R}(G \mid Q)/G$ for a set of orbit representatives under the action of G. If $N \lhd G$ and $\mu \in \mathrm{Irr(N)}$ (the set of irreducible characters of N), then denote by $\mathrm{Irr(G}, \mu)$ the set of irreducible characters of G covering μ.

Let R be a local principal ideal domain whose residue field $k = R/J(R)$ has characteristic p and whose field of fractions K has characteristic zero and is a splitting field for G. For example, we could take (K, R, k) to be a complete p-modular system in which R contains a primitive $|G|$th root of unity.

As mentioned in [9], it suffices in verifying Dade's inductive conjecture for the groups ${}^2G_2(3^{2m+1})$ to test them only for the invariant conjecture, since $\mathrm{Out}({}^2\mathrm{G}_2(3^{2\mathrm{m}+1}))$ is cyclic and ${}^2G_2(3^{2m+1})$ has trivial Schur multiplier (see [6]). We state here only the invariant conjecture, and refer the reader to [9] for a statement of the inductive conjecture.

Conjecture 1 (*Dade's invariant*). Let B be a p-block of positive defect of a finite group G satisfying $O_p(G) = 1$, and suppose that $G \lhd E$. Write $\mathrm{Irr_d}(\mathrm{G}_\sigma, \mathrm{B})$ for the set of irreducible characters χ of G_σ lying in p-blocks which are Brauer correspondents of B and whose degrees satisfy $\chi(1)_p p^d = |G|_p$ (we say that χ has defect d). If $G \lhd H \leq E$ and $\sigma \in \mathcal{R}(G)$, then denote by $\mathrm{Irr_d}(\mathrm{G}_\sigma, \mathrm{B}, \mathrm{H})$ the set of those characters in $\mathrm{Irr_d}(\mathrm{G}_\sigma, \mathrm{B})$ with inertial subgroup $N_H(\sigma)$ in $N_E(\sigma)$. Write $k_d(G_\sigma, B, H) = |\mathrm{Irr_d}(\mathrm{G}_\sigma, \mathrm{B}, \mathrm{H})|$. Then

$$\sum_{\sigma \in \mathcal{R}(G|1)/G} (-1)^{|\sigma|} k_d(G_\sigma, B, H) = 0.$$

An has verified the inductive conjecture for the groups ${}^2G_2(3^{2m+1})$ for all primes except $p = 3$ (see [2]), and so the results presented here complete the verification of the conjecture for these groups.

Henceforth let $G = {}^2G_2(3^{2m+1})$. We make the trivial observation that G has a split BN-pair of rank one and so the only radical 3-subgroups are the trivial group 1 and the Sylow 3-subgroups of G. Let $P \in \mathrm{Syl}_3(G)$ be the unipotent radical, so that $\mathcal{R}(G \mid 1)/G$ may be taken to be $\{1, 1 < P\}$ and the conjecture reduces to checking that $k_d(G, B, H) = k_d(N_G(P), B, H)$ for each choice of B, H, and d.

In Section 2 we investigate the irreducible characters of $N_G(P)$, the characters of G having been investigated fully in [15]. In Section 3 we examine the action of the automorphisms of G on the irreducible characters of G and $N_G(P)$, and so verify the conjecture in this case. Blau and Michler have verified a simpler form of the conjecture for these groups, and given the action of outer automorphisms on the 3-regular classes (see [3]). Our results may be viewed as an extension of theirs.

2. IRREDUCIBLE CHARACTERS OF $N_G(P)$

We realize G as a twisted Chevalley group (see, for example, [5, 13]). Write $q = 3^{2m+1}$, where $m > 0$.

LEMMA 2. *$|P| = q^3$ and we may label its elements as $x(t,u,v)$, where $t,u,v \in \mathbb{F}_q$, with multiplication given by*

$$x(t_1,u_1,v_1)x(t_2,u_2,v_2) = x(t_1+t_2, u_1+u_2-t_1t_2^{3f}, v_1+v_2-t_2u_1+t_1t_2^{3f+1}-t_1^2t_2^{3f}),$$

where $f = 3^m$.

Conjugation within P is given by

$$x(t_1,u_1,v_1)x(t_2,u_2,v_2)x(t_1,u_1,v_1)^{-1} = x(t_2, u_2 - t_1t_2^{3f} + t_2t_1^{3f}, v_2 - t_2u_1 + t_1u_2 + t_1t_2^{3f+1} + t_1^2t_2^{3f} + t_1^{3f}t_2^2).$$

$Z(P) = \{x(0,0,v) \mid v \in \mathbb{F}_q\}$ and $|Z(P)| = q$. The subgroup

$$P_1 = \{x(0,u,v) \in P \mid u,v \in \mathbb{F}_q\}$$

of order q^2 is normal in P and $P_1 = P' = \Phi(P)$, so P/P_1 is elementary abelian.

Proof. The description of the elements and their multiplication follows from [5]. From this it is easily seen that $x(t,u,v)^{-1} = x(-t, -u-t^{3f+1}, -v-tu+t^{3f+2})$ for each $t,u,v \in \mathbb{F}_q$, and that the conjugation action is as given. The rest follows easily. □

We may calculate the conjugacy class lengths explicitly:

LEMMA 3. *P has q conjugacy classes of length 1, $q-1$ of length q, and $3(q-1)$ of length $\frac{1}{3}q^2$.*

Proof. Note that

$$x(t_1,u_1,v_1)x(0,u_2,0)x(t_1,u_1,v_1)^{-1} = x(0,u_2,t_1u_2),$$

so that each $x(0,u_2,0)$ with $u_2 \in \mathbb{F}_q^\times$ lies in a distinct conjugacy class of length q contained in P_1. It is clear that we have q conjugacy classes of length one, thus accounting for every element of P_1.

Suppose now that $x(t_2, u_2, v_2) \in P$ with $t_2 \neq 0$. If $x(t_1, u_1, v_1) \in C_P(x(t_2, u_2, v_2))$, then $t_2 t_1^{3f} = t_1 t_2^{3f}$. Let $\phi: \mathbb{F}_q \to \mathbb{F}_q$ be the additive endomorphism sending λ to $t_2\lambda^{3f} - \lambda t_2^{3f}$. If $0 \neq \lambda \in \ker(\phi)$, then $\lambda^{3f} = \lambda t_2^{3f-1}$. Without loss of generality, assume $t_2^{3f-1} = 1$, so $\lambda^{3f} = \lambda$. Hence

$$\lambda^{3^{m+1}-1} = 1 = \lambda^{q-1} = \lambda^{3^m(3^{m+1}-1)+3^m-1}$$

and

$$1 = \lambda^{q-1} = \lambda^{3(3^m-1)(3^m+1)+2} = \lambda^2,$$

so $|\ker(\phi)| = 3$. This gives $|C_P(x(t_2, u_2, v_2))| = 3q$ since we can choose u_1 to give $t_2 u_1$ any value in $\mathbb{F}_q$. Hence we have found $3(q-1)$ conjugacy classes of length $\frac{1}{3}q^2$. This accounts for all the elements of P. □

We use the action of P on $\mathrm{Irr}(\mathrm{P}_1)$ to calculate the nature and degrees of the irreducible characters of P.

LEMMA 4. *P has q linear characters, $q-1$ irreducible characters of degree q, and $3(q-1)$ of degree 3^m.*

Proof. Considering $\mathbb{F}_q$ as an $n = (2m+1)$-dimensional vector space over $\mathbb{F}_3$, write $\zeta \in \mathbb{F}_q$ as $\zeta = (\zeta_1, \ldots, \zeta_n)$. Write the elements of $\mathrm{Irr}(\mathrm{P}_1)$ as $\chi_{x,y}$, $x, y \in \mathbb{F}_q$, where, for $x(0, u, v) \in P_1$,

$$\chi_{x,y}(x(0,u,v)) = \prod_{k=1}^{n} e^{2u_k x_k \pi i/3} \prod_{j=1}^{n} e^{2v_j y_j \pi i/3}.$$

Recall that

$$\begin{aligned} &x(t_1, u_1, v_1)x(0, u_2, 0)x(t_1, u_1, v_1)^{-1} \\ &\quad = x(0, u_2, t_1 u_2) \qquad \forall\, t_1, u_1, v_1, u_2 \in \mathbb{F}_q. \end{aligned}$$

From this we see that for any $x \in \mathbb{F}_q$, we have $I_P(\chi_{x,0}) = P$ and that for any $x \in \mathbb{F}_q$, $y \in \mathbb{F}_q^\times$, we have $I_P(\chi_{x,y}) = P_1$. This gives us q orbits of length one and $q-1$ orbits of length q. Representatives of the long orbits are $\chi_{0,y}$, $y \in \mathbb{F}_q^\times$.

By Clifford theory representatives of the long orbits induce to give distinct irreducible characters of P of degree q. Also $\chi_{0,0} = 1_{P_1}$ extends to P in q distinct ways to give q linear characters of P. Since P_1 is the derived subgroup of P, every linear character of P is an extension of $\chi_{0,0}$.

Now consider the characters $\chi_{x,0}$, where $x \in \mathbb{F}_q^\times$. We show that all irreducible characters of P covering such characters of P_1 have the same degree.

Fix $\mu = \chi_{x,0}$, $x \in \mathbb{F}_q^\times$. Since $I_P(\chi_{x,0}) = P$, there is a degree preserving 1–1 correspondence $\mathrm{Irr}(\mathrm{P}, \mu) \leftrightarrow \mathrm{Irr}((\widehat{\mathrm{P}/\mathrm{P}_1}), \widehat{\mu})$, where $(\widehat{P/P_1})$ is a central extension of P/P_1 with kernel A (i.e., $A \leq Z((\widehat{P/P_1})) \cap (\widehat{P/P_1})'$ and

$(\widehat{P/P_1})/A \cong P/P_1)$ and $\widehat{\mu} \in \mathrm{Irr(A)}$ (see, for example, [10, Chap. 11]). We may take A to be a cyclic p-group (see [12]). However, the Schur multiplier of an elementary abelian group is itself elementary abelian, and so $|A| = 3$. Note that $\widehat{\mu}$ is non-trivial since we may not extend μ to P (as all the linear characters of P cover $\chi_{0,0}$). This demonstrates that the degrees of the elements of $\mathrm{Irr(P}, \chi_{\mathrm{x},0})$ are independent of the choice of $x \in \mathbb{F}_q^\times$. Since P possesses $5q-4$ conjugacy classes we then have $|\mathrm{Irr(P}, \chi_{\mathrm{x},0})| = 3$, and so all irreducible characters of P lying over $\chi_{x,0}$ must have the same degree, which must be 3^m, as required. $\square$

We examine the action of $N_G(P)$ on the irreducible characters of P to calculate the irreducible character degrees of $N_G(P)$. We note that by using Ward's character table for G (see [15]) and the fact that the Knörr–Robinson reformulation of Alperin's weight conjecture is known to hold for finite groups with split BN-pairs in the defining characteristic (see [11, 5.3]), we may already observe that $N_G(P)$ possesses $q+7$ conjugacy classes.

LEMMA 5. *$N_G(P)$ has irreducible characters of degrees $1, q-1, 3^m(q-1)/2, 3^m(q-1), q(q-1)$ with multiplicity $q-1, 1, 4, 2, 1$, respectively.*

Proof. By the results of [13] $N_G(P) = PW$, where $W \cong \mathbb{F}_q^\times$ acts transitively on the non-trivial elements of P/P_1 and of $Z(P)$. Hence W acts transitively on the $q-1$ non-trivial linear characters of P, and so by Clifford's theorem we obtain $q-1$ linear characters of $N_G(P)$ and one irreducible character of degree $q-1$ (where the linear characters are the irreducible characters of $N_G(P)/P$ regarded as characters for $N_G(P)$, and the irreducible character of degree $q-1$ is $\lambda^{N_G(P)}$, where λ is a non-trivial linear character of P). The irreducible characters of degree q of P are also permuted transitively by W, giving one irreducible character of degree $q(q-1)$. Since $k(N_G(P)) = q+7$ we are left with 6. The remaining irreducible characters must cover the irreducible characters of P of degree 3^m.

By Ree [13], W possesses an unique involution, h_0, and $C_P(h_0) = \{x(0, u, 0) \mid u \in \mathbb{F}_q\}$. Hence $W/ < h_0 >$ acts on $P_1/Z(P)$ with two orbits of length $(q-1)/2$ and one of length one (noting that by [13] W stabilizes P_1). It follows from Brauer's theorem that W acts on the irreducible characters $\chi_{x,0}$, where $x \in \mathbb{F}_q^\times$, with two orbits, which must then both be of length $(q-1)/2$. Recall that each of these $\chi_{x,0}$ is covered by three irreducible characters of P. It follows that the irreducible characters of $N_G(P)$ of height m must consist of two of degree $3^m(q-1)$ and four of degree $3^m(q-1)/2$. $\square$

3. DADE'S CONJECTURE

Blau and Michler give a description of the action of an arbitrary outer automorphism on the semisimple classes of G:

LEMMA 6 (Blau and Michler [3]). Out(G) *is cyclic and consists of automorphisms of the form* τ^r, *where* τ *is the Frobenius automorphism* $x \to x^3$. *We have* $|\mathrm{Out(G)}| = 2\mathrm{m} + 1$, *and* τ^r *stabilizes* P.

When r is an integer dividing $2m + 1$, τ^r *fixes* 3^r *3-regular conjugacy classes of* G.

We give a combinatorial lemma included in [3]:

LEMMA 7 [3]. *Let* A *be a finite group acting on two sets* S_1 *and* S_2 *as a permutation group. Let* $H \leq A$ *and write* $f_i(H) = |\{s \in S_i \mid s^a = s \ \forall\ a \in H\}|$ *and* $m_i(H) = |\{s \in S_i \mid C_A(s) = H\}|$.

If $f_1(H) = f_2(H)$ *for all* $H \leq A$ *then* $m_1(H) = m_2(H)$ *for all* $H \leq A$.

We come to the main result of the paper:

THEOREM 8. *Dade's inductive conjecture holds for G for the prime* $p = 3$.

Proof. As remarked earlier it suffices to verify Conjecture 1 for G. By [15] G possesses only one 3-block of defect zero, namely that containing the Steinberg character, and $C_G(P) \subset P$. Hence there is only one 3-block of maximal defect (by a well-known result of block theory). But defect groups are radical p-subgroups and the Sylow 3-subgroups are the only non-trivial radical 3-subgroups of G, so the principal 3-block B is the unique 3-block of non-zero defect. Suppose that $G \lhd E$ and choose $t \mid 2m + 1$ such that τ^t generates E/G, where τ is as in Lemma 6. Let $G \lhd H \leq E$, and choose $r|2m + 1$ such that τ^r generates H/G.

From the character table for G (see [15, p. 87]), we have $k_{6m+3}(G, B) = q$, $k_{5m+3}(G, B) = 6$ and $k_{4m+2}(G, B) = 1$, this accounting for all the irreducible characters of G in B.

Consider first the action of τ^r on Irr(G, B). Examination of the character table given in [15] gives us that τ^r fixes every 3-singular conjugacy class of G; hence by Lemma 6 τ^r fixes $3^r + 8$ conjugacy classes of G. So τ^r fixes $3^r + 7$ irreducible characters of B, by Brauer's theorem. Further examination of the character table and using the fact that τ^r has odd order reveals that τ^r fixes every irreducible character of B of positive height, and so 3^r irreducible characters of height zero, i.e., $f_{\mathrm{Irr}_{6m+3}(G,B)}(\langle \tau^r \rangle) = 3^r$, $f_{\mathrm{Irr}_{5m+3}(G,B)}(\langle \tau^r \rangle) = 6$, and $f_{\mathrm{Irr}_{4m+2}(G,B)}(\langle \tau^r \rangle) = 1$ in the notation of Lemma 7.

Now consider the action of τ^r on $\mathrm{Irr(N_G(P))}$. The 3-regular conjugacy classes of $N_G(P) = PW$ are represented by the elements of W, and so τ^r fixes $3^r - 1$ 3-regular conjugacy classes of $N_G(P)$. Hence τ^r fixes $3^r - 1$ irreducible Brauer characters of $N_G(P)$, and so $3^r - 1$ linear characters and the irreducible character of degree $q-1$. It remains to examine the action on the characters of positive height. Clearly the unique irreducible character of degree $q(q-1)$ is fixed by τ^r, as are the two of degree $3^m(q-1)$ since τ^r has odd order. Finally it is clear from the construction of the remaining characters that τ^r cannot act on the four irreducible characters of degree $3^m(q-1)/2$ with an orbit of length three, and so τ^r fixes these also. Hence $f_{\mathrm{Irr}_{6m+3}(\mathrm{N_G(P),B_0})}(\langle\tau^r\rangle) = 3^r$, $f_{\mathrm{Irr}_{5m+3}(\mathrm{N_G(P),B_0})}(\langle\tau^r\rangle) = 6$, and $f_{\mathrm{Irr}_{4m+2}(\mathrm{N_G(P),B_0})}(\langle\tau^r\rangle) = 1$.

The result then follows from Lemma 7. □

Remark. The above results also yield a verification of Robinson's conjecture (see [14]) for G.

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