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# A Chevalley formula in equivariant $K$-theory

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**Abstract**

The aim of this paper is to give a recursive formula to compute the product of a line bundle with the structure sheaf of a Schubert variety in the equivariant $K$-theory of a flag variety.


*Keywords:* Equivariant $K$-theory; Flag varieties

**Contents**



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## 1. Introduction

Let $G$ be a complex semi-simple connected group of rank $r$, $B \subset G$ a Borel subgroup of $G$, and $H \subset B$ a maximal torus of $B$. We denote by $R[H]$ the ring of representations of $H$ and $X = G/B$ the flag variety of $G$. The $H$-equivariant $K$-theory $K(H, X)$ of $X$ has an $R[H]$-basis $[\mathcal{O}_{\overline{X}_w}]^H$ indexed by $W = N_G(H)/H$ the Weyl group of $G$, where $[\mathcal{O}_{\overline{X}_w}]^H$ is the class of the structure sheaf of the Schubert variety $\overline{X}_w$. The Schubert variety $\overline{X}_w \subset X$ is the closure of the $B$-orbit of $w \in W$. Let $\mathfrak{h}$ be the Lie algebra of $H$, we denote by $\rho_i \in \mathfrak{h}^*$, $1 \leqslant i \leqslant r$, the fundamental weights, and by $\mathfrak{h}^*_{\mathbb{Z}} = \bigoplus_{1 \leqslant i \leqslant r} \mathbb{Z}\rho_i \subset \mathfrak{h}^*$ the weight lattice which is identified canonically with $X(H)$, the group of characters of $H$. Then for all $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$, we denote by $e^\lambda \in X(H)$ the corresponding character, and by $\mathcal{L}^X_\lambda$ the canonical line bundle over $X$. The torus $H$ acts on $\mathcal{L}^X_\lambda$, and it defines a class $[\mathcal{L}^X_\lambda]^H$ in $K(H, X)$. In fact, $K(H, X)$ is generated as an $R[H]$-algebra by these line bundles, and then $K(H, X)$ is canonically isomorphic to the $R[H]$-algebra $(R[H] \otimes_{\mathbb{Z}} R[H])/\mathcal{I}$ where $\mathcal{I} = \langle f \otimes 1 - 1 \otimes f \mid f \in R[H]^W \rangle$. More precisely, $K(H, X)$ is generated by the line bundles $\mathcal{L}_{\rho_1}, \dots, \mathcal{L}_{\rho_r}, \mathcal{L}_{-\rho_1}, \dots, \mathcal{L}_{-\rho_r}$, but we do not know the relations in terms of polynomials in these generators. If we want to understand the link between the basis $\{[\mathcal{O}_{\overline{X}_w}]^H\}_{w \in W}$ and this presentation, it is interesting to find a "Giambelli formula" which expresses $[\mathcal{O}_{\overline{X}_w}]^H$ in terms of $\{[\mathcal{L}^X_\lambda]^H\}_{\lambda \in \mathfrak{h}^*_{\mathbb{Z}}}$ and a "Chevalley formula," i.e. to find the coefficients $q^\lambda_{w,v} \in R[H]$ satisfying:

$$\left[\mathcal{L}^X_\lambda\right]^H [\mathcal{O}_{\overline{X}_w}]^H = \sum_{v \in W} q^\lambda_{w,v} [\mathcal{O}_{\overline{X}_v}]^H.$$

Such a formula has been known for a long time in cohomology (see [2] in ordinary cohomology and [8] in equivariant cohomology). In [13] Pittie and Ram give a Chevalley formula in ordinary $K$-theory for a dominant weight $\lambda$ by using L–S paths. Littelmann and Seshadri generalize this formula to $H$-equivariant $K$-theory in [11]. Such a formula was first given in the case $G = SL(n, \mathbb{C})$ by Fulton and Lascoux in [7] by using "tableaux" of shape $\lambda$. In [10] Lenart and Postnikov give a formula which works for all weights (even for non dominant weights). The aim of this paper is to find a new algorithm to compute these coefficients $q^\lambda_{w,v}$. Our formula is valid for all weights. The idea of this algorithm can be found in [1] in the setting of complex cobordism.

Let us explain our main result (Theorem 4). Let $\{\alpha_i\}_{1 \leqslant i \leqslant r} \subset \mathfrak{h}^*_{\mathbb{Z}}$ be a system of simple roots. We denote by $\{s_i\}_{1 \leqslant i \leqslant r} \subset W$ the corresponding simple reflections.

For all simple roots $\alpha$, we define two $\mathbb{Z}$-linear maps $T^0_\alpha$ and $T^1_\alpha$ from $R[H]$ to $R[H]$ by

$$T^1_\alpha\left(e^\lambda\right) = e^{s_\alpha \lambda},$$

$$T^0_\alpha\big(e^\lambda\big) = \begin{cases} 0 & \text{if } \lambda(\alpha^\vee) = 0, \\ e^\lambda + e^{\lambda-\alpha} + \cdots + e^{\lambda-(\lambda(\alpha^\vee)-1)\alpha} & \text{if } \lambda(\alpha^\vee) > 0, \\ -e^{\lambda+\alpha} - \cdots - e^{\lambda-\lambda(\alpha^\vee)\alpha} & \text{if } \lambda(\alpha^\vee) < 0, \end{cases}$$

for all characters $e^\lambda \in X(H)$, where $\alpha^\vee \in \mathfrak{h}$ is the coroot of $\alpha$.

We denote by $\underline{W}$ the monoid generated by the elements $\{\underline{s}_i\}_{1\leqslant i\leqslant r}$ with the relations $\underline{s}_i^2 = \underline{s}_i$ and the braid relations of $W$. We denote by $T : W \to \underline{W}$ the canonical bijection (of sets) between $W$ and $\underline{W}$.

Then Theorem 4 can be formulated as follows.

**Theorem.** *Let $w = s_{i_1} \cdots s_{i_N}$ be a reduced decomposition of $w \in W$ as a product of simple reflections. For all $\epsilon = (\epsilon_1, \ldots, \epsilon_N) \in \{0, 1\}^N$, we define an element $\underline{v}(\epsilon)$ of $\underline{W}$ by*

$$\underline{v}(\epsilon) = \prod_{\substack{1\leqslant j\leqslant N \\ \epsilon_j=1}} \underline{s}_{i_j} \in \underline{W}.$$

*Then, for all weights $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$, and all $v \in W$, the coefficient $q^\lambda_{w,v} \in R[H]$ is given by the formula*

$$q^\lambda_{w,v} = \sum_{\substack{\epsilon\in\{0,1\}^N \\ \underline{v}(\epsilon)=T(v)}} T^{\epsilon_1}_{\alpha_{i_1}} \cdots T^{\epsilon_N}_{\alpha_{i_N}}\big(e^\lambda\big).$$

To apply this formula, we need to find a reduced decomposition of $w \in W$, and all solutions in $\{0, 1\}^N$ for the equation $\underline{v}(\epsilon) = T(v)$.

Let us describe our strategy. We follow the same method as in [14] to find restrictions to fixed points in equivariant cohomology and $K$-theory. First we describe an $R[H]$-basis of the $H$-equivariant $K$-theory of a Bott–Samelson variety $\Gamma$ and we decompose the class of a line bundle $[\mathcal{L}^\Gamma_\lambda]^H$ in this basis. To find this formula, we use the structure of iterated fibrations with fiber $\mathbb{C}P^1$ of Bott–Samelson varieties. Then we use the standard map $g : \Gamma \to X$ to deduce a Chevalley formula in $K(H, X)$. In [5,6] Haibao Duan also used Bott–Samelson varieties to find formulas in Schubert calculus and we used this idea in [15,16] to find similar formulas in the equivariant setting. In these two papers we study Bott towers i.e. all varieties which have a structure of iterated fibrations with fiber $\mathbb{C}P^1$.

The paper is organized as follows.

In Section 2, we recall basic definitions on semi-simple groups and their flag varieties.

In Section 3, we recall the definition of the Bott–Samelson variety associated to a sequence of simple roots and we define a cell decomposition of this variety. For more details on this section, see [9,15].

In Section 4, we recall the definition of the $H$-equivariant $K$-theory of an algebraic $H$-variety and we introduce the notion of restriction to fixed points which will be the main tool of our proofs.

In Section 5, we construct an $R[H]$-basis of the $H$-equivariant $K$-theory of a Bott–Samelson variety $\Gamma$ and for all $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$, we decompose the line bundle $\mathcal{L}^\Gamma_\lambda$ in this basis (Theorem 2).

In Section 6, if $g : \Gamma \to X$ is the standard map from a Bott–Samelson variety $\Gamma$ to the flag variety $X$, we describe the morphism $g_*$ induced in $K$-theory (Theorem 3) and we deduce from this result the main theorem of this paper (Theorem 4) which gives a Chevalley formula in equivariant $K$-theory.

In Section 7, we restrict our calculations to ordinary $K$-theory (Theorem 5).

## 2. Preliminaries and notation

### 2.1. Root system

Let $G$ be a connected and simply connected complex semi-simple group of rank $r$. We denote by $e$ the neutral element of $G$. Let $B \subset G$ be a Borel subgroup of $G$ and $H \subset B$ the Cartan subgroup of $B$. We denote by $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ the Lie algebras of $H$, $B$ and $G$.

We choose a system of simple roots $\pi = \{\alpha_i\}_{1\leqslant i\leqslant r} \subset \mathfrak{h}^*$ and simple coroots $\pi^\vee = \{\alpha_i^\vee\}_{1\leqslant i\leqslant r} \subset \mathfrak{h}$, such that

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta_+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}),$$

where for $\lambda \in \mathfrak{h}^*$, $\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \text{ such that } [h,x] = \lambda(h)x,\ \forall h \in \mathfrak{h}\}$, and where we define $\Delta_+$ by $\Delta_+ = \{\alpha \in \sum_{i=1}^r \mathbb{N}\alpha_i \text{ such that } \alpha \neq 0 \text{ and } \mathfrak{g}_\alpha \neq 0\}$. We set $\Delta = \Delta_+ \cup \Delta_-$ where $\Delta_- = -\Delta_+$. We call $\Delta_+$ (respectively $\Delta_-$) the set of positive roots (respectively negative).

We associate to $(\mathfrak{g},\mathfrak{h})$ the Weyl group $W \subset \mathrm{Aut}(\mathfrak{h}^*)$ generated by the simple reflections $\{s_i\}_{1\leqslant i\leqslant r}$ defined by

$$\forall \lambda \in \mathfrak{h}^*, \quad s_i(\lambda) = \lambda - \lambda\big(\alpha_i^\vee\big)\alpha_i.$$

By dualizing, we get an action of $W$ on $\mathfrak{h}$.

If we denote by $S$ the set of simple reflections, the couple $(W,S)$ is a Coxeter system. Thus we have a notion of Bruhat order denoted by $u \leqslant v$ and a notion of length denoted by $l(w) \in \mathbb{N}$. We denote by 1 the neutral element of $W$.

We have $\Delta = W\pi$, and for $\beta = w\alpha_i \in \Delta^+$, we set $s_\beta = ws_iw^{-1} \in W$ (which does not depend on the couple $(w,\alpha_i)$ satisfying $\beta = w\alpha_i$), and $\beta^\vee = w\alpha_i^\vee \in \mathfrak{h}$.

We define the fundamental weights $\rho_i \in \mathfrak{h}^*$ $(1 \leqslant i \leqslant r)$ by

$$\rho_i\big(\alpha_j^\vee\big) = \delta_{i,j}, \quad \text{for all } 1 \leqslant i, j \leqslant r,$$

and the weight lattice $\mathfrak{h}^*_{\mathbb{Z}}$ by

$$\mathfrak{h}^*_{\mathbb{Z}} = \bigoplus_{1\leqslant i\leqslant r} \mathbb{Z}\rho_i \subset \mathfrak{h}^*.$$

### 2.2. Flag varieties

Let $N_G(H)$ be the normalizer of $H$ in $G$, the quotient group $N_G(H)/H$ can be identified to $W$. We set $X = G/B$. It is a flag variety. The group $G$ acts on $X$ by multiplication on the left. This action yields an action of $B$ and $H$ on $X$. The set of fixed points of this action of $H$ on $X$ can be identified to $W$. For $w \in W$, we define $C(w) = B \cup BwB$ and for all simple roots $\alpha$, we define the subgroup $P_\alpha$ of $G$ by $P_\alpha = C(s_\alpha)$. We have the Bruhat decomposition of $G = \bigsqcup_{w\in W} BwB$

and if we define $X_w = BwB/B$, then $X = \coprod_{w\in W} X_w$. For all $w \in W$, the Schubert cell $X_w$ is isomorphic to $\mathbb{C}^{l(w)}$. Thus we get an $H$-equivariant cell decomposition of $X$ where all cells have even real dimension.

For all $w \in W$, the Schubert variety $\overline{X}_w$ is the closure of the cell $X_w$. It is an irreducible $H$-equivariant subvariety of $X$ of complex dimension $l(w)$. In general Schubert varieties are not smooth. For all $w \in W$, we have the decomposition

$$\overline{X}_w = \coprod_{w' \leqslant w} X_{w'}.$$

*2.3. The monoid $\underline{W}$*

We define the monoid $\underline{W}$ as the monoid generated by the elements $\{\underline{s}_i\}_{1\leqslant i\leqslant r}$ with the relations $\underline{s}_i^2 = \underline{s}_i$ and the braid relations of $W$:

$$\begin{cases} \underline{s}_i^2 = \underline{s}_i, \\ \underbrace{\underline{s}_i\,\underline{s}_j\cdots}_{m_{i,j}\text{ terms}} = \underbrace{\underline{s}_j\,\underline{s}_i\cdots}_{m_{i,j}\text{ terms}} \quad \text{if } m_{i,j} < \infty, \end{cases}$$

where $m_{i,j}$ is the order of $s_i s_j$ in $W$.

We denote by $T : W \to \underline{W}$ the bijection defined by $T(w) = \underline{s}_{i_1}\cdots\underline{s}_{i_l}$ if $w = s_{i_1}\cdots s_{i_l}$ is a reduced decomposition of $w$ (i.e. $l = l(w)$).

## 3. Bott–Samelson varieties

Let $N \geqslant 1$ be a positive integer. We use the notation of Section 2.

*3.1. Definition*

Let $\mu_1, \ldots, \mu_N$ be a sequence of $N$ simple roots (repetitions may occur). We define

$$\Gamma(\mu_1, \ldots, \mu_N) = P_{\mu_1} \times_B P_{\mu_2} \times_B \cdots \times_B P_{\mu_N}/B,$$

as the space of orbits of $B^N$ acting on $P_{\mu_1} \times P_{\mu_2} \times \cdots \times P_{\mu_N}$ by

$$(g_1, g_2, \ldots, g_N)(b_1, b_2, \ldots, b_N) = \left(g_1 b_1, b_1^{-1} g_2 b_2, \ldots, b_{N-1}^{-1} g_N b_N\right), \quad b_i \in B,\ g_i \in P_{\mu_i}.$$

It is an irreducible complex projective variety of dimension $N$. We denote by $[g_1, g_2, \ldots, g_N]$ the class of $(g_1, g_2, \ldots, g_N)$ in $\Gamma(\mu_1, \ldots, \mu_N)$ and by $g_{\mu_i} \in P_{\mu_i}$ a representative of the reflection of $N_{P_{\mu_i}}(H)/H \simeq \mathbb{Z}/2\mathbb{Z}$.

We define a left action of $B$ on $\Gamma(\mu_1, \ldots, \mu_N)$ by

$$b[g_1, g_2, \ldots, g_N] = [bg_1, g_2, \ldots, g_N], \quad b \in B,\ g_i \in P_{\mu_i}.$$

By restricting this action to $H$, we get an action of $H$ on $\Gamma(\mu_1, \ldots, \mu_N)$.

In the following two sections we denote $\Gamma(\mu_1, \ldots, \mu_N)$ by $\Gamma$.

*3.2. Cell decomposition*

For $\epsilon \in \{0,1\}^N$, we denote by $\Gamma_\epsilon \subset \Gamma$ the set of classes $[g_1, g_2, \dots, g_N]$ satisfying for all integers $1 \leqslant i \leqslant N$

$$g_i \in B \quad \text{if } \epsilon_i = 0, \qquad g_i \notin B \quad \text{if } \epsilon_i = 1.$$

For $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_N) \in \{0,1\}^N$, we denote by $l(\epsilon)$ the cardinal of $\{1 \leqslant i \leqslant N,\ \epsilon_i = 1\}$. It is called the length of $\epsilon$. We define a partial order on $\{0,1\}^N$ by

$$\epsilon \leqslant \epsilon' \quad \Leftrightarrow \quad (\forall 1 \leqslant i \leqslant N,\ \epsilon_i = 1 \Rightarrow \epsilon'_i = 1).$$

The following proposition is obvious.

**Proposition 1.**

(i) *For all $\epsilon \in \{0,1\}^N$, $\Gamma_\epsilon$ is a complex affine space of dimension $l(\epsilon)$ which is invariant under the action of $B$, and this action induces a linear action of the torus $H$ on $\Gamma_\epsilon$.*
(ii) *For all $\epsilon \in \{0,1\}^N$, $\overline{\Gamma}_\epsilon = \coprod_{\epsilon' \leqslant \epsilon} \Gamma_{\epsilon'}$.*
(iii) $\Gamma = \coprod_{\epsilon \in \{0,1\}^N} \Gamma_\epsilon$.
(iv) *For all $\epsilon \in \{0,1\}^N$, $\overline{\Gamma}_\epsilon$ can be identified with the Bott–Samelson variety $\Gamma\ (\mu_i, \epsilon_i = 1)$ and is an irreducible smooth subvariety of $\Gamma$.*

For $\epsilon \in \{0,1\}^N$ and $1 \leqslant i \leqslant N$, we define

$$v_i(\epsilon) = \prod_{\substack{1 \leqslant j \leqslant i \\ \epsilon_j = 1}} s_{\mu_j} \in W,$$

where, by convention, $\prod_\emptyset = 1$. We set $v(\epsilon) = v_N(\epsilon) \in W$.
Moreover, we define the root $\alpha_i(\epsilon) \in \Delta$ by

$$\alpha_i(\epsilon) = v_i(\epsilon)\mu_i.$$

Let $\Gamma^H$ be the set of fixed points of the action of $H$ on $\Gamma$, we can identify $\Gamma^H$ with $\{0,1\}^N$ thanks to the following lemma.

**Lemma 1.**

(i) $\Gamma^H \simeq \prod_{1 \leqslant i \leqslant N} N_{P_{\mu_i}}(H)/H \simeq \prod_{1 \leqslant i \leqslant N} \{e, g_{\mu_i}\} \simeq \{0,1\}^N$, *where we identify $e$ with* 0 *and $g_{\mu_i}$ with* 1.
(ii) *For all $\epsilon \in \{0,1\}^N$, $\Gamma_\epsilon$ is the $B$-orbit of $\epsilon \in \Gamma^H$.*
(iii) *For $(\epsilon, \epsilon') \in (\{0,1\}^N)^2$*

$$\epsilon \in \overline{\Gamma}_{\epsilon'} \quad \Leftrightarrow \quad \epsilon \leqslant \epsilon',$$

*and if we denote by $T^\epsilon_{\epsilon'}$ the tangent space to $\overline{\Gamma}_{\epsilon'}$ at $\epsilon$, then the weights of the representation of $H$ in $T^\epsilon_{\epsilon'}$ are $\{-\alpha_i(\epsilon)\}_{i,\ \epsilon'_i = 1}$.*

*3.3. Fibrations of Bott–Samelson varieties*

For all $2 \leqslant k \leqslant N$, let denote by $\pi_k : \Gamma(\mu_1, \ldots, \mu_k) \to \Gamma(\mu_1, \ldots, \mu_{k-1})$ the projection defined by

$$\pi_k\big([g_1, \ldots, g_k]\big) = [g_1, \ldots, g_{k-1}].$$

If we denote by $\pi_1 : \Gamma(\mu_1) \to \{\text{point}\}$ the trivial projection, we get the following diagram:

$$\begin{array}{c} \Gamma(\mu_1, \ldots, \mu_N) \\ \downarrow{\scriptstyle \pi_N} \\ \Gamma(\mu_1, \ldots, \mu_{N-1}) \\ \downarrow{\scriptstyle \pi_{N-1}} \\ \vdots \\ \downarrow{\scriptstyle \pi_3} \\ \Gamma(\mu_1, \mu_2) \\ \downarrow{\scriptstyle \pi_2} \\ \Gamma(\mu_1) \simeq \mathbb{C}P^1 \\ \downarrow{\scriptstyle \pi_1} \\ \{\text{point}\} \end{array}$$

where each projection $\pi_k$ is a fibration with fiber $\mathbb{C}P^1$ (see [15] for more details).

*3.4. Line bundles*

We denote by $X(H)$ the group of characters of $H$. For all integral weights $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$, we denote by $e^\lambda : H \to S^1$ the corresponding character. This way we get an isomorphism between the additive group $\mathfrak{h}^*_{\mathbb{Z}}$ and $X(H)$.

Since $H \simeq B/U$, where $U$ is the unipotent radical of $B$, we can extend to $B$ all characters $e^\lambda \in X(H)$ (in fact $X(H) \simeq X(B)$). Then for all $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$, we denote by $\mathcal{L}^\Gamma_\lambda$ the $B$-equivariant line bundle over $\Gamma$ defined as the space of orbits of $B^N$ acting on $P_{\mu_1} \times P_{\mu_2} \times \cdots \times P_{\mu_N} \times \mathbb{C}$ by

$$\begin{aligned} &(g_1, g_2, \ldots, g_N, v)(b_1, b_2, \ldots, b_N) = \big(g_1 b_1, b_1^{-1} g_2 b_2, \ldots, b_{N-1}^{-1} g_N b_N, e^\lambda\big(b_N^{-1}\big)v\big), \\ &\quad b_i \in B,\ g_i \in P_{\mu_i},\ v \in \mathbb{C}. \end{aligned}$$

## 4. Equivariant *K*-theory

Let $Z$ be a complex algebraic $H$-variety, we denote by $Z^H \subset Z$ the set of fixed points of the action of $H$ on $Z$.

We denote by $K^0(H, Z)$ the Grothendieck group of $H$-equivariant complex vector bundles of finite rank over $Z$. The tensor product of vector bundles defines a product on $K^0(H, Z)$. Since $K^0(H, \text{point}) \simeq R[H]$, where $R[H] = \mathbb{Z}[X(H)]$ is the representation ring of the torus $H$, we get a $R[H]$-algebra structure on $K^0(H, Z)$.

For all $H$-equivariant algebraic maps $g : Z_1 \to Z_2$ we denote by $g^* : K^0(H, Z_2) \to K^0(H, Z_1)$ the morphism of $R[H]$-algebras defined by pulling back vector bundles. In particular, the inclusion $Z^H \subset Z$ gives a morphism $i_H^* : K^0(H, Z) \to K^0(H, Z^H)$ called restriction to fixed points. If the set of fixed points $Z^H$ is finite, $K^0(H, Z^H)$ can be identified with $F(Z^H; R[H])$ the $R[H]$-algebra of all maps $f : Z^H \mapsto R[H]$ and we get a morphism $i_H^* : K^0(H, Z) \to F(Z^H; R[H])$. Moreover, if $K^0(H, Z)$ is a free $R[H]$-module, then the morphism $i_H^*$ is injective. This is an easy consequence of the localization theorem (see [3, Section 5.10]).

If we assume that $Z$ is a complex projective smooth $H$-variety, then $K^0(H, Z)$ is isomorphic to $K_0(H, Z)$ the Grothendieck group of $H$-equivariant coherent sheaves on $Z$ (see [3, Chapter 5]). In this case, we identify these two groups and we denote them by $K(H, Z)$.

For all proper $H$-equivariant morphisms $g : Z_1 \to Z_2$, we denote by $g_* : K_0(H, Z_1) \to K_0(H, Z_2)$ the direct image morphism. For all $H$-equivariant subvarieties $Z' \subset Z$, we denote by $[\mathcal{O}_{Z'}]^H \in K_0(H, Z)$ the class of $i_*(\mathcal{O}_{Z'})$ where $\mathcal{O}_{Z'}$ is the structure sheaf of $Z'$ and $i$ is the inclusion of $Z'$ in $Z$.

## 5. *K*-theory of Bott–Samelson varieties

We use the notation of Section 3. Let $N$ be a positive integer, and let $\mu_1, \mu_2, \ldots, \mu_N$ be a sequence of $N$ simple roots. Let $\Gamma$ be the Bott–Samelson variety $\Gamma = \Gamma(\mu_1, \mu_2, \ldots, \mu_N)$. For $1 \leqslant k \leqslant N$, we denote $\Gamma(\mu_1, \ldots, \mu_k)$ by $\Gamma^k$. By convention, $\Gamma^0 = \{\text{point}\}$. For $1 \leqslant k \leqslant N$, let $\pi_k$ be the projection $\Gamma^k \to \Gamma^{k-1}$ defined in Section 3.3.

We denote by $\Gamma^k = \coprod_{\epsilon \in \{0,1\}^k} \Gamma_\epsilon^k$ the cell decomposition defined in Section 3.2. For all $\epsilon \in \{0, 1\}^k$, let denote by $\overline{\Gamma}_\epsilon^k$ the closure of $\Gamma_\epsilon^k$ in $\Gamma^k$.

### *5.1. A basis of the K-theory of Bott–Samelson varieties*

For all integers $1 \leqslant k \leqslant N$, $\Gamma^k$ is a complex projective smooth $H$-variety, and then we denote by $K(H, \Gamma^k)$ its $H$-equivariant $K$-theory.

For all $\epsilon \in \{0, 1\}^k$, we set $\mathcal{O}_{k,\epsilon}^H = [\mathcal{O}_{\overline{\Gamma}_\epsilon^k}]^H \in K(H, \Gamma^k)$, and for $1 \leqslant i \leqslant k$, $\mathcal{O}_{k,i}^H = \mathcal{O}_{k,[^i_k]}^H$, where for $1 \leqslant j \leqslant k$, $[^i_k]_j = 1 - \delta_{i,j}$.

Since $\Gamma^k = \coprod_{\epsilon \in \{0,1\}^k} \Gamma_\epsilon^k$ is a cell decomposition of $\Gamma^k$, the family $\{\mathcal{O}_{k,\epsilon}^H\}_{\epsilon \in \{0,1\}^k}$ is a $R[H]$-basis of the module $K(H, \Gamma^k)$. Moreover, $(\Gamma^k)^H$ is finite and isomorphic to $\{0, 1\}^k$. Thus we have the following proposition.

**Proposition 2.**

(i) *The $H$-equivariant $K$-theory of $(\Gamma^k)^H$ can be identified with $F(\{0, 1\}^k; R[H])$.*
(ii) $K(H, \Gamma^k) = \bigoplus_{\epsilon \in \{0,1\}^k} R[H]\mathcal{O}_{k,\epsilon}^H$.
(iii) *The restriction to fixed points $i_H^* : K(H, \Gamma^k) \to F(\{0, 1\}^k; R[H])$ is injective.*

For all integers $1 \leqslant k \leqslant N$ and all $\lambda \in \mathfrak{h}_\mathbb{Z}^*$, we denote by $\mathcal{L}_\lambda^k$ the line bundle over $\Gamma^k$ defined in Section 3.4 and by $[\mathcal{L}_\lambda^k]^H$ its class in $K(H, \Gamma^k)$. For $k = 0$, $[\mathcal{L}_\lambda^0]^H = e^\lambda \in R[H] \simeq K(H, \text{point})$.

We will decompose $[\mathcal{L}_\lambda^k]^H$ in the $R[H]$-basis $\{\mathcal{O}_{k,\epsilon}^H\}_{\epsilon\in\{0,1\}^k}$. For this we need restrictions to fixed points.

*5.2. Restrictions to fixed points*

**Proposition 3.** *For all integer $1 \leqslant k \leqslant N$, $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$ and $(\epsilon, \epsilon') \in (\{0,1\}^k)^2$,*

$$i_H^*\big(\big[\mathcal{L}_\lambda^k\big]^H\big)(\epsilon) = e^{v(\epsilon)\lambda},$$

$$i_H^*\big(\mathcal{O}_{k,\epsilon'}^H\big)(\epsilon) = \begin{cases} \prod_{1\leqslant i\leqslant k,\ \epsilon'_i=0}(1-e^{-\alpha_i(\epsilon)}) & \textit{if } \epsilon \leqslant \epsilon', \\ 0 & \textit{otherwise}. \end{cases}$$

**Proof.** The first relation is obvious by definition of $\mathcal{L}_\lambda^k$.

Let us prove the second relation.

If $\epsilon \nleqslant \epsilon'$, the fixed point $\epsilon \notin \overline{\Gamma}_{\epsilon'}^k$, and then by the localization theorem,

$$i_H^*\big(\mathcal{O}_{k,\epsilon'}^H\big)(\epsilon) = 0.$$

If $\epsilon \leqslant \epsilon'$, $\epsilon \in \overline{\Gamma}_{\epsilon'}^k$. Since $\overline{\Gamma}_{\epsilon'}^k$ and $\Gamma^k$ are smooth, we can use the self-intersection formula (see [3, Proposition 5.4.10]) to get $i_H^*(\mathcal{O}_{k,\epsilon'}^H)(\epsilon) = i_H^*(\lambda(T^*_{\overline{\Gamma}_{\epsilon'}^k}\Gamma^k))(\epsilon)$, where $T^*_{\overline{\Gamma}_{\epsilon'}^k}\Gamma^k$ is the normal bundle of $\overline{\Gamma}_{\epsilon'}^k$ in $\Gamma^k$ and for all vector bundles $V$, $\lambda(V) = \sum_{0\leqslant i\leqslant \dim(V)}(-1)^i\Lambda^i(V)$.

Then Lemma 1 and the relation $\lambda(V_1 \oplus V_2) = \lambda(V_1) \otimes \lambda(V_2)$ give us

$$i_H^*\big(\mathcal{O}_{k,\epsilon'}^H\big)(\epsilon) = \prod_{1\leqslant i\leqslant k,\ \epsilon'_i=0}\big(1-e^{-\alpha_i(\epsilon)}\big). \qquad \square$$

Since $i_H^*$ is injective, we deduce the following formula.

**Corollary 1.** *For all integers $1 \leqslant k \leqslant N$ and all $\epsilon \in \{0,1\}^k$,*

$$\mathcal{O}_{k,\epsilon}^H = \prod_{1\leqslant i\leqslant k,\ \epsilon_i=0} \mathcal{O}_{k,i}^H. \tag{1}$$

**Remark 1.** This corollary is also a consequence of the fact that

$$\overline{\Gamma}_\epsilon^k = \bigcap_{\substack{1\leqslant i\leqslant k\\ \epsilon'_i=0}} \overline{\Gamma}_{[\substack{i\\k}]}^k$$

is a transversal intersection.

*5.3. Decomposition of line bundles*

**Theorem 1.** *For all integers $1 \leqslant k \leqslant N$, and all $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$,*

$$\big[\mathcal{L}_\lambda^k\big]^H = \pi_k^*\big(\big[\mathcal{L}_{s_{\mu_k}\lambda}^{k-1}\big]^H\big) + \mathcal{O}_{k,k}^H\pi_k^*\big(\big[\mathcal{L}_{\lambda,\mu_k}^{k-1}\big]^H\big),$$

*where*

$$\big[\mathcal{L}^{k-1}_{\lambda,\mu_k}\big]^H = \begin{cases} 0 & \textit{if } \lambda(\mu_k^\vee) = 0, \\ [\mathcal{L}^{k-1}_{\lambda}]^H + [\mathcal{L}^{k-1}_{\lambda-\mu_k}]^H + \cdots + [\mathcal{L}^{k-1}_{\lambda-(\lambda(\mu_k^\vee)-1)\mu_k}]^H & \textit{if } \lambda(\mu_k^\vee) > 0, \\ -[\mathcal{L}^{k-1}_{\lambda+\mu_k}]^H - \cdots - [\mathcal{L}^{k-1}_{\lambda-\lambda(\mu_k^\vee)\mu_k}]^H & \textit{if } \lambda(\mu_k^\vee) < 0. \end{cases}$$

**Proof.** To prove this theorem, we check these relations after restriction to fixed points.

Let $\epsilon$ be an element of $\{0,1\}^k$. We denote by $\bar{\epsilon}$ the element of $\{0,1\}^{k-1}$ corresponding to the fixed point $\pi_k(\epsilon)$ in $\Gamma^{k-1}$ (by convention $\{0,1\}^0 = \{0\}$, and $v(0) = 1 \in W$).

If we use Proposition 3 for $\Gamma^k$ and $\Gamma^{k-1}$, we find:

$$i_H^*\big(\big[\mathcal{L}^k_\lambda\big]^H\big)(\epsilon) = e^{v(\epsilon)\lambda},$$

$$\begin{aligned} &i_H^*\big(\pi_k^*\big(\big[\mathcal{L}^{k-1}_{s_{\mu_k}\lambda}\big]^H\big) + \mathcal{O}^H_{k,k}\pi_k^*\big(\big[\mathcal{L}^{k-1}_{\lambda,\mu_k}\big]^H\big)\big)(\epsilon) \\ &\quad = e^{v(\bar{\epsilon})s_{\mu_k}\lambda} + \delta_{\epsilon_k,0}\big(1 - e^{-v(\epsilon)\mu_k}\big)\big(i_H^*\big(\big[\mathcal{L}^{k-1}_{\lambda,\mu_k}\big]^H\big)\big)(\bar{\epsilon}). \end{aligned}$$

If $\epsilon_k = 1$, we have to check $e^{v(\epsilon)\lambda} = e^{v(\bar{\epsilon})s_{\mu_k}\lambda}$. This is obvious since $v(\epsilon) = v(\bar{\epsilon})s_{\mu_k}$.

If $\epsilon_k = 0$, $v(\epsilon) = v(\bar{\epsilon})$ and thus we have to check

$$e^\lambda = \begin{cases} e^{s_{\mu_k}\lambda} & \text{if } \lambda(\mu_k^\vee) = 0, \\ e^{s_{\mu_k}\lambda} + (1 - e^{-\mu_k})[e^\lambda + e^{\lambda-\mu_k} + \cdots + e^{\lambda-(\lambda(\mu_k^\vee)-1)\mu_k}] & \text{if } \lambda(\mu_k^\vee) > 0, \\ e^{s_{\mu_k}\lambda} - (1 - e^{-\mu_k})[e^{\lambda+\mu_k} + \cdots + e^{\lambda-\lambda(\mu_k^\vee)\mu_k}] & \text{if } \lambda(\mu_k^\vee) < 0. \end{cases}$$

These relations hold since $s_{\mu_k}\lambda = \lambda - \lambda(\mu_k^\vee)\mu_k$. $\square$

**Definition 1.** Let $\beta$ be a simple root, we define two $\mathbb{Z}$-linear maps $T^0_\beta$ and $T^1_\beta$ from $R[H]$ to $R[H]$ by

$$T^1_\beta\big(e^\lambda\big) = e^{s_\beta\lambda},$$

$$T^0_\beta\big(e^\lambda\big) = \begin{cases} 0 & \text{if } \lambda(\beta^\vee) = 0, \\ e^\lambda + e^{\lambda-\beta} + \cdots + e^{\lambda-(\lambda(\beta^\vee)-1)\beta} & \text{if } \lambda(\beta^\vee) > 0, \\ -e^{\lambda+\beta} - \cdots - e^{\lambda-\lambda(\beta^\vee)\beta} & \text{if } \lambda(\beta^\vee) < 0, \end{cases}$$

for all characters $e^\lambda \in X(H)$.

**Remark 2.** The operators $T^0_\beta$ are called Demazure operators. Such operators were first defined by Demazure in [4].

**Theorem 2.** *For all integers $1 \leqslant k \leqslant N$ and all $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$,*

$$\left[\mathcal{L}^k_\lambda\right]^H = \sum_{\epsilon \in \{0,1\}^k} R^{\lambda,\epsilon}_{\mu_1,\dots,\mu_k} \mathcal{O}^H_{k,\epsilon},$$

*where $R^{\lambda,\epsilon}_{\mu_1,\dots,\mu_k} = T^{\epsilon_1}_{\mu_1} T^{\epsilon_2}_{\mu_2} \cdots T^{\epsilon_k}_{\mu_k}(e^\lambda)$.*

**Proof.** We prove this theorem by induction on $k$.

For $k = 1$, the theorem is a consequence of Theorem 1 in the case $k = 1$ and of the fact that $[\mathcal{L}^0_\lambda]^H = e^\lambda$ for all $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$.

We assume that the relation is proved for $k-1$ ($1 \leqslant k-1 \leqslant N-1$) and for all weights $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$. We assume, for example, that $\lambda(\mu_k^\vee) > 0$. Then by Theorem 1, we get

$$\left[\mathcal{L}^k_\lambda\right]^H = \sum_{\epsilon \in \{0,1\}^{k-1}} R^{s_{\mu_k}\lambda,\epsilon}_{\mu_1,\dots,\mu_{k-1}} \pi_k^*\left(\mathcal{O}^H_{k-1,\epsilon}\right) + \sum_{j=0}^{\lambda(\mu_k^\vee)-1} \sum_{\epsilon \in \{0,1\}^{k-1}} \mathcal{O}^H_{k,k} R^{\lambda - j\mu_k,\epsilon}_{\mu_1,\dots,\mu_{k-1}} \pi_k^*\left(\mathcal{O}^H_{k-1,\epsilon}\right).$$

Since $\pi_k$ is a smooth $H$-equivariant morphism between smooth $H$-varieties, for all $\epsilon \in \{0,1\}^{k-1}$,

$$\pi_k^*\left(\mathcal{O}^H_{k-1,\epsilon}\right) = [\mathcal{O}_{\pi_k^{-1}(\overline{\Gamma}^{k-1}_\epsilon)}]^H = \mathcal{O}^H_{k,\tilde{\epsilon}^1},$$

where $\tilde{\epsilon}^1 = (\epsilon_1, \epsilon_2, \dots, \epsilon_{k-1}, 1) \in \{0,1\}^k$. Moreover, by Corollary 1,

$$\mathcal{O}^H_{k,k} \pi_k^*\left(\mathcal{O}^H_{k-1,\epsilon}\right) = \mathcal{O}^H_{k,k} \mathcal{O}^H_{k,\tilde{\epsilon}^1} = \mathcal{O}^H_{k,\tilde{\epsilon}^0},$$

where $\tilde{\epsilon}^0 = (\epsilon_1, \epsilon_2, \dots, \epsilon_{k-1}, 0) \in \{0,1\}^k$.

Then we get

$$\left[\mathcal{L}^k_\lambda\right]^H = \sum_{\substack{\epsilon \in \{0,1\}^k \\ \epsilon_k = 1}} R^{s_{\mu_k}\lambda,\epsilon}_{\mu_1,\dots,\mu_{k-1}} \mathcal{O}^H_{k,\epsilon} + \sum_{\substack{\epsilon \in \{0,1\}^k \\ \epsilon_k = 0}} \sum_{j=0}^{\lambda(\mu_k^\vee)-1} R^{\lambda - j\mu_k,\epsilon}_{\mu_1,\dots,\mu_{k-1}} \mathcal{O}^H_{k,\epsilon}.$$

We obtain the statement since by definition

$$R^{\lambda,\epsilon}_{\mu_1,\dots,\mu_k} = \begin{cases} R^{s_{\mu_k}\lambda,\epsilon}_{\mu_1,\dots,\mu_{k-1}} & \text{if } \epsilon_k = 1, \\ \sum_{j=0}^{\lambda(\mu_k^\vee)-1} R^{\lambda - j\mu_k,\epsilon}_{\mu_1,\dots,\mu_{k-1}} & \text{if } \epsilon_k = 0. \end{cases}$$

The argument works in the same way in the cases $\lambda(\mu_k^\vee) = 0$ and $\lambda(\mu_k^\vee) < 0$. $\square$

**Example 1.** In type $A_2$ ($G = SL(3,\mathbb{C})$), we decompose $[\mathcal{L}^\Gamma_{\rho_1}]^H$ in $K(H,\Gamma)$ where $\Gamma = \Gamma(\alpha_2, \alpha_1, \alpha_2)$.

Since $\rho_1(\alpha_2^\vee) = 0$, we get

$$T^0_{\alpha_2}\left(e^{\rho_1}\right) = 0 \quad \text{and} \quad T^1_{\alpha_2}\left(e^{\rho_1}\right) = e^{\rho_1}.$$

Since $\rho_1(\alpha_1^\vee)=1$, we find

$$T^0_{\alpha_1}T^1_{\alpha_2}\left(e^{\rho_1}\right)=e^{\rho_1}\quad\text{and}\quad T^1_{\alpha_1}T^1_{\alpha_2}\left(e^{\rho_1}\right)=e^{\rho_1-\alpha_1}=e^{-\rho_1+\rho_2},$$

since $\alpha_1=2\rho_1-\rho_2$.

Since $\rho_1(\alpha_2^\vee)=0$ and $(-\rho_1+\rho_2)(\alpha_2^\vee)=1$, we find

$$T^0_{\alpha_2}T^0_{\alpha_1}T^1_{\alpha_2}\left(e^{\rho_1}\right)=0,\qquad T^1_{\alpha_2}T^0_{\alpha_1}T^1_{\alpha_2}\left(e^{\rho_1}\right)=e^{\rho_1},$$

$$T^0_{\alpha_2}T^1_{\alpha_1}T^1_{\alpha_2}\left(e^{\rho_1}\right)=e^{-\rho_1+\rho_2},\qquad T^1_{\alpha_2}T^1_{\alpha_1}T^1_{\alpha_2}\left(e^{\rho_1}\right)=e^{-\rho_1+\rho_2-\alpha_2}=e^{-\rho_2},$$

since $\alpha_2=-\rho_1+2\rho_2$.

Then Theorem 2 gives us the following relation in $K(H,\Gamma)$

$$\left[\mathcal{L}^\Gamma_{\rho_1}\right]^H=e^{-\rho_2}\mathcal{O}^H_{3,(1,1,1)}+e^{-\rho_1+\rho_2}\mathcal{O}^H_{3,(0,1,1)}+e^{\rho_1}\mathcal{O}^H_{3,(1,0,1)}.$$

**Example 2.** In the case $G_2$, we decompose $[\mathcal{L}^\Gamma_{\rho_2}]^H$ in $K(H,\Gamma)$ where we take $\Gamma=\Gamma(\alpha_1,\alpha_2,\alpha_1,\alpha_2)$.

We compute $T^1_{\alpha_1}T^0_{\alpha_2}T^0_{\alpha_1}T^1_{\alpha_2}(e^{\rho_2})$.

Since $\rho_2=3\alpha_1+2\alpha_2$, and $\rho_2(\alpha_2^\vee)=1$, we get

$$T^1_{\alpha_2}\left(e^{\rho_2}\right)=e^{3\alpha_1+\alpha_2}.$$

Since $(3\alpha_1+\alpha_2)(\alpha_1^\vee)=3$, we find

$$T^0_{\alpha_1}T^1_{\alpha_2}\left(e^{\rho_2}\right)=e^{3\alpha_1+\alpha_2}+e^{2\alpha_1+\alpha_2}+e^{\alpha_1+\alpha_2}.$$

Since $(3\alpha_1+\alpha_2)(\alpha_2^\vee)=-1$, $(2\alpha_1+\alpha_2)(\alpha_2^\vee)=0$, $(\alpha_1+\alpha_2)(\alpha_2^\vee)=1$, we get

$$T^0_{\alpha_2}T^0_{\alpha_1}T^1_{\alpha_2}\left(e^{\rho_2}\right)=-e^{3\alpha_1+2\alpha_2}+e^{\alpha_1+\alpha_2}.$$

Since $(3\alpha_1+2\alpha_2)(\alpha_1^\vee)=0$, $(\alpha_1+\alpha_2)(\alpha_1^\vee)=-1$, we find

$$T^1_{\alpha_1}T^0_{\alpha_2}T^0_{\alpha_1}T^1_{\alpha_2}\left(e^{\rho_2}\right)=-e^{3\alpha_1+2\alpha_2}+e^{2\alpha_1+\alpha_2}.$$

We compute the other terms in the same way, and we get

$$\begin{aligned}\left[\mathcal{L}^\Gamma_{\rho_2}\right]^H&=e^{-3\alpha_1-\alpha_2}\mathcal{O}^H_{4,(1,1,1,1)}+\left(e^{-\alpha_2}+e^{-\alpha_1-\alpha_2}+e^{-2\alpha_1-\alpha_2}\right)\mathcal{O}^H_{4,(0,1,1,1)}\\&\quad+\left(e^{3\alpha_1+\alpha_2}+1\right)\mathcal{O}^H_{4,(1,0,1,1)}-\left(e^{\alpha_1+\alpha_2}+e^{2\alpha_1+\alpha_2}+e^{3\alpha_1+\alpha_2}\right)\mathcal{O}^H_{4,(0,0,1,1)}\\&\quad+\left(e^{3\alpha_1+2\alpha_2}+e^{\alpha_1+\alpha_2}+e^{-\alpha_1}\right)\mathcal{O}^H_{4,(1,1,0,1)}+\left(e^{2\alpha_1+\alpha_2}+e^{\alpha_1}+1\right)\mathcal{O}^H_{4,(0,1,0,1)}\\&\quad+\left(-e^{3\alpha_1+2\alpha_2}+e^{2\alpha_1+\alpha_2}\right)\mathcal{O}^H_{4,(1,0,0,1)}-e^{2\alpha_1+\alpha_2}\mathcal{O}^H_{4,(0,0,0,1)}\\&\quad+e^{\alpha_2}\mathcal{O}^H_{4,(1,1,1,0)}+\left(e^{3\alpha_1+\alpha_2}+e^{2\alpha_1+\alpha_2}+e^{\alpha_1+\alpha_2}\right)\mathcal{O}^H_{4,(0,1,1,0)}+e^{3\alpha_1+2\alpha_2}\mathcal{O}^H_{4,(1,0,1,0)}.\end{aligned}$$

## 6. Equivariant $K$-theory of flag varieties

### 6.1. Definitions

Since $X$ is a complex irreducible smooth $H$-variety, we denote by $K(H,X)$ its $H$-equivariant $K$-theory.

For $w \in W$, we set $\mathcal{O}_w^H = [\mathcal{O}_{\overline{X}_w}]^H \in K(H,X)$.

Since $X = \coprod_{w\in W} X_w$ is a cell decomposition of $X$, the family $\{\mathcal{O}_w^H\}_{w\in W}$ is a $R[H]$-basis of the module $K(H,X)$. Moreover, $X^H$ is finite and isomorphic to $W$. Thus we have the following proposition.

**Proposition 4.**

(i) *The $H$-equivariant $K$-theory of $X^H$ can be identified with $F(W;R[H])$.*
(ii) $K(H,X) = \bigoplus_{w\in W} R[H]\mathcal{O}_w^H$.
(iii) *The restriction to fixed points $i_H^* : K(H,X) \to F(W;R[H])$ is injective.*

For all $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$, we denote by $\mathcal{L}_\lambda^X$ the $B$-equivariant line bundle over $X$ defined as the space of orbits of $B$ acting on $G \times \mathbb{C}$ by

$$(g,v)b = \bigl(gb, e^\lambda\bigl(b^{-1}\bigr)v\bigr), \quad b \in B,\ g \in G,\ v \in \mathbb{C}.$$

### 6.2. Link with Bott–Samelson varieties

Let $\mu_1,\dots,\mu_N$ be a sequence of $N$ simple roots. We set $\Gamma = \Gamma(\mu_1,\dots,\mu_N)$ and we define an $H$-equivariant map $g$ from $\Gamma$ to $X$ by multiplication

$$g\bigl([g_1,\dots,g_N]\bigr) = g_1 \times \cdots \times g_N\ [B].$$

For all $\epsilon \in \{0,1\}^N$, we define

$$\underline{v}(\epsilon) = \prod_{\substack{1\leqslant j\leqslant N\\ \epsilon_j=1}} \underline{s}_{\mu_j} \in \underline{W}.$$

For all $\epsilon \in \{0,1\}^N$, we set $\mathcal{O}_\epsilon^H = \mathcal{O}_{N,\epsilon}^H \in K(H,\Gamma)$ (see Section 5.1). We have the following theorem.

**Theorem 3.** *For all $\epsilon \in \{0,1\}^N$,*

$$g_*\bigl(\mathcal{O}_\epsilon^H\bigr) = \mathcal{O}_{T^{-1}(\underline{v}(\epsilon))}^H,$$

*where $T$ is defined in Section* 2.3.

**Proof.** By Theorem 8.1.13 and Corollary 8.2.3 of [9], the image of $g$ is a Schubert variety $\overline{X}_w$, where $w \in W$, and $g_*(\mathcal{O}_\epsilon^H) = \mathcal{O}_w^H$. By Lemma 2.3 of [14], $w = T^{-1}(\underline{v}(\epsilon))$. □

**Example 3.** In the case $A_2$, if we take $\Gamma = \Gamma(\alpha_2, \alpha_1, \alpha_2)$, we get the following relations:

$$g_*\big(\mathcal{O}^H_{(0,0,0)}\big) = \mathcal{O}^H_1,$$
$$g_*\big(\mathcal{O}^H_{(0,0,1)}\big) = \mathcal{O}^H_{s_2},$$
$$g_*\big(\mathcal{O}^H_{(0,1,0)}\big) = \mathcal{O}^H_{s_1},$$
$$g_*\big(\mathcal{O}^H_{(1,0,0)}\big) = \mathcal{O}^H_{s_2},$$
$$g_*\big(\mathcal{O}^H_{(1,0,1)}\big) = \mathcal{O}^H_{s_2},$$
$$g_*\big(\mathcal{O}^H_{(0,1,1)}\big) = \mathcal{O}^H_{s_1 s_2},$$
$$g_*\big(\mathcal{O}^H_{(1,1,0)}\big) = \mathcal{O}^H_{s_2 s_1},$$
$$g_*\big(\mathcal{O}^H_{(1,1,1)}\big) = \mathcal{O}^H_{s_2 s_1 s_2}.$$

**Lemma 2.** *Let $w = s_{\mu_1} \cdots s_{\mu_N}$ be a reduced decomposition of $w \in W$ $(N = l(w))$. For all $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$,*

$$g_*\big(\big[\mathcal{L}^\Gamma_\lambda\big]^H\big) = \big[\mathcal{L}^X_\lambda\big]^H \times \mathcal{O}^H_w.$$

**Proof.** Since $\mathcal{O}^H_{(\mathbf{1})} = 1 \in K(H, \Gamma)$ where $(\mathbf{1}) = (1, 1, \ldots, 1) \in \{0, 1\}^N$, and for all $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$, $g^*([\mathcal{L}^X_\lambda]^H) = [\mathcal{L}^\Gamma_\lambda]^H$, we have

$$g_*\big(\big[\mathcal{L}^\Gamma_\lambda\big]^H\big) = g_*\big(g^*\big(\big[\mathcal{L}^X_\lambda\big]^H\big) \times \mathcal{O}^H_{(\mathbf{1})}\big) = \big[\mathcal{L}^X_\lambda\big]^H \times g_*\big(\mathcal{O}^H_{(\mathbf{1})}\big) = \big[\mathcal{L}^X_\lambda\big]^H \times \mathcal{O}^H_w,$$

where the last equality is a consequence of Theorem 3.

### 6.3. A Chevalley formula

Theorems 2, 3 and Lemma 2 give us the following theorem.

**Theorem 4.** *Let $w = s_{\mu_1} \cdots s_{\mu_N}$ be a reduced decomposition of $w \in W$. For all $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$,*

$$\big[\mathcal{L}^X_\lambda\big]^H \times \mathcal{O}^H_w = \sum_{\epsilon \in \{0,1\}^N} R^{\lambda,\epsilon}_{\mu_1,\ldots,\mu_N} \mathcal{O}^H_{T^{-1}(\underline{v}(\epsilon))}.$$

**Example 4.** In the case $A_2$, if we take $w = s_2 s_1 s_2$ and $\lambda = \rho_1$, Examples 1, 3 and Theorem 4 give us the following relation in $K(H, X)$

$$\big[\mathcal{L}^X_{\rho_1}\big]^H \times \mathcal{O}^H_{s_2 s_1 s_2} = \mathcal{L}^X_{\rho_1} = e^{-\rho_2} \mathcal{O}^H_{s_2 s_1 s_2} + e^{-\rho_1 + \rho_2} \mathcal{O}^H_{s_1 s_2} + e^{\rho_1} \mathcal{O}^H_{s_2}.$$

**Example 5.** In the case $G_2$, if we take $w = s_1 s_2 s_1 s_2$ and $\lambda = \rho_2$, Example 2 and Theorem 4 give us the following relation in $K(H, X)$

$$\begin{aligned}
\big[\mathcal{L}^X_{\rho_2}\big]^H \times \mathcal{O}^H_{s_1s_2s_1s_2} &= e^{-3\alpha_1-\alpha_2}\mathcal{O}^H_{s_1s_2s_1s_2} + \big(e^{-\alpha_2} + e^{-\alpha_1-\alpha_2} + e^{-2\alpha_1-\alpha_2}\big)\mathcal{O}^H_{s_2s_1s_2} \\
&\quad + \big(e^{3\alpha_1+\alpha_2} + 1\big)\mathcal{O}^H_{s_1s_2} + \big(-e^{\alpha_1+\alpha_2} - e^{2\alpha_1+\alpha_2} - e^{3\alpha_1+\alpha_2}\big)\mathcal{O}^H_{s_1s_2} \\
&\quad + \big(e^{3\alpha_1+2\alpha_2} + e^{\alpha_1+\alpha_2} + e^{-\alpha_1}\big)\mathcal{O}^H_{s_1s_2} + \big(e^{2\alpha_1+\alpha_2} + e^{\alpha_1} + 1\big)\mathcal{O}^H_{s_2} \\
&\quad + \big(-e^{3\alpha_1+2\alpha_2} + e^{2\alpha_1+\alpha_2}\big)\mathcal{O}^H_{s_1s_2} - e^{2\alpha_1+\alpha_2}\mathcal{O}^H_{s_2} \\
&\quad + e^{\alpha_2}\mathcal{O}^H_{s_1s_2s_1} + \big(e^{3\alpha_1+\alpha_2} + e^{2\alpha_1+\alpha_2} + e^{\alpha_1+\alpha_2}\big)\mathcal{O}^H_{s_2s_1} + e^{3\alpha_1+2\alpha_2}\mathcal{O}^H_{s_1} \\
&= e^{-3\alpha_1-\alpha_2}\mathcal{O}^H_{s_1s_2s_1s_2} + \big(e^{-\alpha_2} + e^{-\alpha_1-\alpha_2} + e^{-2\alpha_1-\alpha_2}\big)\mathcal{O}^H_{s_2s_1s_2} \\
&\quad + e^{\alpha_2}\mathcal{O}^H_{s_1s_2s_1} + \big(e^{3\alpha_1+\alpha_2} + e^{2\alpha_1+\alpha_2} + e^{\alpha_1+\alpha_2}\big)\mathcal{O}^H_{s_2s_1} + \big(e^{-\alpha_1} + 1\big)\mathcal{O}^H_{s_1s_2} \\
&\quad + \big(e^{\alpha_1} + 1\big)\mathcal{O}^H_{s_2} + e^{3\alpha_1+2\alpha_2}\mathcal{O}^H_{s_1}.
\end{aligned}$$

**Remark 3.** Since $\rho_2$ is a dominant weight (i.e. $\rho_2(\alpha_i^\vee) \geqslant 0$ for all simple roots $\alpha_i$), we know that we must find positive coefficients (i.e. a linear combination of characters with positive coefficients, see [12]). Unfortunately, our formula is not positive. In this example, we see that we can find negative terms and then cancellations can occur. The formulas given by Pittie and Ram in [13] and Littelmann and Seshadri in [11] are positive. In [10] Lenart and Postnikov give a formula which works for all weights, and which is positive for dominant weights.

## 7. Ordinary $K$-theory

We denote by $\psi$ the forgetful map $K(H,X) \to K(X)$, where $K(X) \simeq K^0(X) \simeq K_0(X)$ is the ordinary $K$-theory of $X$, and by $ev : R[H] \to \mathbb{Z}$ the $\mathbb{Z}$-linear map defined by

$$ev\big(e^\alpha\big) = 1 \quad \text{for all characters } e^\alpha \in X(H).$$

For all $w \in W$, we denote by $\mathcal{O}_w \in K(X)$ the class of $\mathcal{O}_{\overline{X}_w}$ in $K(X)$, and for all $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$, we set $[\mathcal{L}^X_\lambda] = \psi([\mathcal{L}^X_\lambda]^H) \in K(X)$. Since $\psi(\mathcal{O}^H_w) = \mathcal{O}_w$, $\psi(e^\alpha) = ev(e^\alpha)$, and $\psi$ is a ring homomorphism, Theorem 4 gives us the following theorem.

**Theorem 5.** *Let $w = s_{\mu_1} \cdots s_{\mu_N}$ be a reduced decomposition of $w \in W$. For all $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$,*

$$\big[\mathcal{L}^X_\lambda\big] \times \mathcal{O}_w = \sum_{\epsilon \in \{0,1\}^N} ev\big(R^{\lambda,\epsilon}_{\mu_1,\ldots,\mu_N}\big)\mathcal{O}_{T^{-1}(\underline{v}(\epsilon))}.$$

**Example 6.** In the case $A_2$, if we take $w = s_2s_1s_2$ and $\lambda = \rho_1$, Example 4 and Theorem 5 give us the following relation in $K(X)$

$$\big[\mathcal{L}^X_{\rho_1}\big] \times \mathcal{O}_{s_2s_1s_2} = \big[\mathcal{L}^X_{\rho_1}\big] = \mathcal{O}_{s_2s_1s_2} + \mathcal{O}_{s_1s_2} + \mathcal{O}_{s_2}.$$

**Example 7.** In the case $G_2$, if we take $w = s_1s_2s_1s_2$ and $\lambda = \rho_2$, Example 5 and Theorem 5 give us the following relation in $K(X)$

$$\big[\mathcal{L}^X_{\rho_2}\big] \times \mathcal{O}_{s_1s_2s_1s_2} = \mathcal{O}_{s_1s_2s_1s_2} + 3\mathcal{O}_{s_2s_1s_2} + \mathcal{O}_{s_1s_2s_1} + 3\mathcal{O}_{s_2s_1} + 2\mathcal{O}_{s_1s_2} + 2\mathcal{O}_{s_2} + \mathcal{O}_{s_1}.$$

This example was computed by Pittie and Ram in [13] by using L–S paths.

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