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On Hermite's invariant for binary quintics

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Abstract

Let $\mathcal{H} \subseteq \mathbb{P}^5$ denote the hypersurface of binary quintics in involution, with defining equation given by the Hermite invariant $\mathbb{H}$. In Section 2 we find the singular locus of $\mathcal{H}$, and show that it is a complete intersection of a linear covariant of quintics. In Section 3 we show that $\mathcal{H}$ is canonically isomorphic to its own projective dual via an involution. The Jacobian ideal of $\mathbb{H}$ is shown to be perfect of height two in Section 4, moreover we describe its SL_2-equivariant minimal free resolution. The last section develops a general formalism for evectants of covariants of binary forms, which is then used to calculate the evectant of $\mathbb{H}$.

Keywords: classical invariant theory; covariant; evectant; Hermite invariant; Hilbert–Burch theorem; involution; Morley form; transvectant

1. Introduction

This paper analyses the geometry and invariant theory of the Hermite invariant for binary quintics. We begin by recalling the elementary properties of this invariant; the main results are summarised on p. 332 after the required notation is available.

We refer to [9,11,22] for foundational notions in the classical invariant theory of binary forms, as well as the symbolic method. Modern treatments of this material may be found in [5,12,17,21]. The encyclopædia article [19] contains a very readable introduction to the classical theory. We will use [7, Lecture 11] and [23, §4.2] for the basic representation theory of SL_2. The discovery of the Hermite invariant was first reported in [14, Première Partie, §IV–VII].

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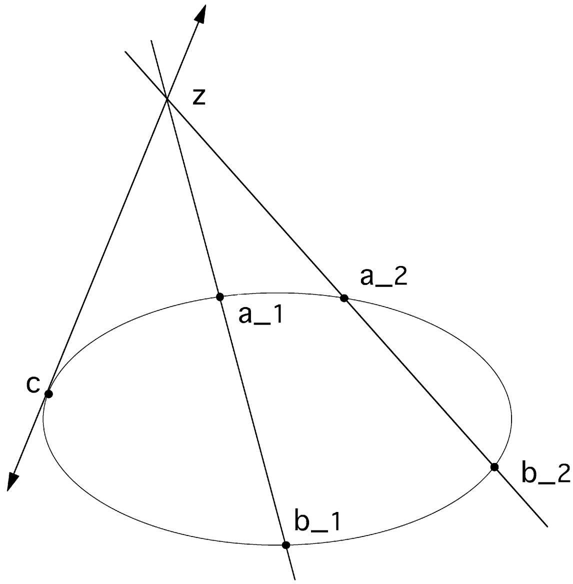

Fig. 1.

The results in Lemma 1.1 and Proposition 1.4 below are classical; I have included them for completeness of treatment.

1.1. The base field will be **C**. Let V denote a two-dimensional complex vector space with basis $\mathbf{x} = \{x_1, x_2\}$ and a natural action of $SL(V)$. For $m \geqslant 0$, let $S_m = \operatorname{Sym}^m V$ denote the $(m+1)$-dimensional irreducible $SL(V)$-representation consisting of binary m-ics in $\mathbf{x}$. Consider the quadratic Veronese imbedding

$$\phi : \mathbb{P}V \longrightarrow \mathbb{P}S_2, \quad [c_1x_1 + c_2x_2] \longrightarrow \left[(c_1x_1 + c_2x_2)^2\right],$$

whose image is a smooth conic $\phi(\mathbb{P}^1) = C \subseteq \mathbb{P}^2$. We identify $\mathbb{P}^5$ with $\operatorname{Sym}^5 C \simeq \mathbb{P}S_5$, i.e., a point in $\mathbb{P}^5$ is alternately seen as a degree 5 effective divisor on C, or as a binary quintic in $\mathbf{x}$ distinguished up to scalars.

Let z be a point of $\mathbb{P}^2 \setminus C$, and L_1, L_2 two lines through z intersecting C in $\mathsf{a}_1, \mathsf{b}_1; \mathsf{a}_2, \mathsf{b}_2$ (see Fig. 1). Let $\mathsf{c} \in C$ be one of the two points such that the line $\overline{\mathsf{cz}}$ is tangent to C, and now define a divisor $\mathsf{a}_1 + \mathsf{b}_1 + \mathsf{a}_2 + \mathsf{b}_2 + \mathsf{c} \in \mathbb{P}^5$. As z, L_1, L_2 move, let $\mathcal{H} \subseteq \mathbb{P}^5$ denote the closure of the set of all such divisors. (The closure includes all divisors of the form $3\mathsf{z} + \mathsf{a} + \mathsf{b}$ for arbitrary points $\mathsf{z}, \mathsf{a}, \mathsf{b}$ in C.)

There are ∞^2 possible positions for z, and then ∞^1 positions for each of the L_i once z is fixed; hence $\dim \mathcal{H} = 4$. By construction $\mathcal{H}$ is an irreducible variety. The action of $SL(V)$ on $\mathbb{P}S_2$ induces an action on C, moreover it takes a tangent line to C to another tangent line, hence $SL(V)$ acts on the imbedding $\mathcal{H} \subseteq \mathbb{P}^5$. Consequently the equation of $\mathcal{H}$ is an invariant of binary quintics, usually called the Hermite invariant $\mathbb{H}$. This defines $\mathbb{H}$ only up to a multiplicative constant; but see formula (11) below.

A point $\mathsf{z} \in \mathbb{P}^2 \setminus C$ defines an order 2 automorphism of C, sending $\mathsf{a} \in C$ to the other intersection of $\overline{\mathsf{za}}$ with C. The divisor $\mathsf{a}_1 + \mathsf{b}_1 + \mathsf{a}_2 + \mathsf{b}_2 + \mathsf{c}$ is said to be *in involution* with respect to z since it is fixed by this automorphism.

Lemma 1.1. *The degree of* $\mathcal{H}$ *is* 18.

Proof. For $\mathsf{p} \in C$, let $\Gamma_{\mathsf{p}} \subseteq \mathbb{P}^5$ denote the hyperplane defined by all the divisors containing p. Given general points $\mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3, \mathsf{p}_4$ in C, consider the intersection $\Sigma = \mathcal{H} \cap \Gamma_{\mathsf{p}_1} \cap \cdots \cap \Gamma_{\mathsf{p}_4}$. The three points

$$\overline{\mathsf{p}_1\mathsf{p}_2} \cap \overline{\mathsf{p}_3\mathsf{p}_4}, \qquad \overline{\mathsf{p}_1\mathsf{p}_3} \cap \overline{\mathsf{p}_2\mathsf{p}_4}, \qquad \overline{\mathsf{p}_1\mathsf{p}_4} \cap \overline{\mathsf{p}_2\mathsf{p}_3}, \tag{1}$$

give 6 elements in Σ (since two tangents to C can be drawn from each). Alternately, let the tangent to C at p_1 intersect $\overline{\mathsf{p}_2\mathsf{p}_3}$ at z, and let $\overline{\mathsf{zp}_4}$ intersect C in the additional point q; which gives $\mathsf{p}_1 + \cdots + \mathsf{p}_4 + \mathsf{q} \in \Sigma$. This construction produces $4 \times 3 = 12$ more elements in Σ, hence $|\Sigma| = 18$. $\square$

1.2. With notation as in the diagram, after a change of variables write $\mathsf{c} = [\phi(x_1)]$ and $\mathsf{z} = [x_1 x_2]$. Then $\mathsf{a}_1, \mathsf{b}_1$ must equal $\phi([\alpha_1 x_1 + \alpha_2 x_2])$, $\phi([\alpha_1 x_1 - \alpha_2 x_2])$ for some $[\alpha_1, \alpha_2] \in \mathbb{P}^1$, and similarly for $\mathsf{a}_2, \mathsf{b}_2$. Hence $\mathsf{a}_1 + \mathsf{a}_2 + \mathsf{b}_1 + \mathsf{b}_2 + \mathsf{c}$ corresponds to the quintic

$$\mathcal{F}_Q = x_1\bigl(q_0 x_1^4 + 2q_1 x_1^2 x_2^2 + q_2 x_2^4\bigr) \tag{2}$$

for some $Q = [q_0, q_1, q_2] \in \mathbb{P}^2$. This 'canonical form' will prove most useful for computations. Since any $[F] \in \mathcal{H}$ lies in the SL_2-orbit of some $[\mathcal{F}_Q]$, any 'equivariant' calculation which is valid for $\mathcal{F}_Q$ is valid generally.

In the next few sections we will gather some needed preliminaries from classical invariant theory; we will take up $\mathbb{H}$ once more on p. 331.

1.3. Transvectants

Given integers $m, n \geqslant 0$, we have a decomposition of $SL(V)$-representations

$$S_m \otimes S_n \simeq \bigoplus_{r=0}^{\min(m,n)} S_{m+n-2r}. \tag{3}$$

Let A, B denote binary forms in $\mathbf{x}$ of respective orders m, n. The rth transvectant of A with B, written $(A, B)_r$, is defined to be the image of $A \otimes B$ via the projection map

$$\pi_r : S_m \otimes S_n \longrightarrow S_{m+n-2r}.$$

It is given by the formula

$$(A, B)_r = \frac{(m-r)!(n-r)!}{m!n!} \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{\partial^r A}{\partial x_1^{r-i} \partial x_2^i} \frac{\partial^r B}{\partial x_1^i \partial x_2^{r-i}}. \tag{4}$$

(Some authors choose the initial scaling factor differently, cf. [21, Chapter 5].) By convention $(A, B)_r = 0$ if $r > \min(m, n)$. If we symbolically write $A = \alpha_{\mathbf{x}}^m$, $B = \beta_{\mathbf{x}}^n$, then $(A, B)_r = (\alpha\beta)^r \alpha_{\mathbf{x}}^{m-r} \beta_{\mathbf{x}}^{n-r}$. There is a canonical isomorphism of representations

$$S_m \xrightarrow{\sim} S_m^* \quad \bigl(= \mathrm{Hom}_{SL(V)}(S_m, S_0)\bigr) \tag{5}$$

which sends $A \in S_m$ to the functional $B \longrightarrow (A, B)_m$. Hence if A is an order m form such that $(A, B)_m = 0$ for all $B \in S_m$, then A must be zero.

1.4. Gordan series

Introduce a parallel set of letters $\mathbf{y} = (y_1, y_2)$, and define Cayley's Omega operator

$$\Omega_{\mathbf{xy}} = \frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial x_2 \partial y_1}.$$

If we represent an element in $S_m \otimes S_n$ as a bihomogeneous form G of orders m, n in $\mathbf{x}, \mathbf{y}$, then

$$\pi_r(G) = \frac{(m-r)!(n-r)!}{m!n!} \bigl\{\Omega_{\mathbf{xy}}^r \circ G\bigr\}_{\mathbf{y}:=\mathbf{x}.}$$

A splitting to π_r is given by the map

$$\iota_r : S_{m+n-2r} \longrightarrow S_m \otimes S_n, \quad \alpha_{\mathbf{x}}^{m+n-2r} \longrightarrow g(m, n; r)(\mathbf{xy})^r \alpha_{\mathbf{x}}^{m-r} \alpha_{\mathbf{y}}^{n-r},$$

where $(\mathbf{xy}) = x_1 y_2 - x_2 y_1$, and $g(m, n; r) = \frac{\binom{m}{r}\binom{n}{r}}{\binom{m+n-r+1}{r}}$. (It follows from Lemma 5.2 below that $\pi_r \circ \iota_r$ is the identity map.) The decomposition $G = \sum_r \iota_r \circ \pi_r(G)$ is called the Gordan series for G. In its general form it may be symbolically written as

$$\alpha_{\mathbf{x}}^m \beta_{\mathbf{y}}^n = \sum_{r=0}^{\min(m,n)} g(m, n; r)(\mathbf{xy})^r \theta_{(r)}{}_{\mathbf{x}}^{m-r} \theta_{(r)}{}_{\mathbf{y}}^{n-r},$$

where $\theta_{(r)}{}_{\mathbf{x}}^{m+n-2r}$ stands for $(\alpha\beta)^r \alpha_{\mathbf{x}}^{m-r} \beta_{\mathbf{x}}^{n-r}$ (see [11, p. 55] or [12, §24.4]).

1.5. Wronskians

Let $m, n \geqslant 0$ be integers such that $m \leqslant n + 1$. Consider the following composite morphism of representations

$$w : \bigwedge^m S_n \xrightarrow{\sim} S_m(S_{n-m+1}) \longrightarrow S_{m(n-m+1)},$$

where the first map is an isomorphism (see [1, §2.5]) and the second is the natural surjection.

Given a sequence of binary n-ics $A_1, \dots, A_m$, define their Wronskian $W(A_1, \dots, A_m)$ to be the determinant

$$(i, j) \longrightarrow \frac{\partial^{m-1} A_i}{\partial x_1^{m-j} \partial x_2^{j-1}} \quad (1 \leqslant i, j \leqslant m).$$

It equals the image $w(A_1 \wedge \cdots \wedge A_m)$. We have $W(A_1, \dots, A_m) = 0$, iff the A_i are linearly dependent over $\mathbf{C}$. (The 'if' part is obvious. For the converse, see [20, §1.1].)

Lemma 1.2. *Let $A_1, \dots, A_m$ be linearly independent forms of order m. Then $W = W(A_1, \dots, A_m)$ is (up to scalar) the unique form of order m such that $(W, A_i)_m = 0$ for all i.*

Proof. Consider the composite morphism

$$g : \bigwedge^{m+1} S_m \longrightarrow \bigwedge^{m} S_m \otimes S_m \xrightarrow{\sim} S_m \otimes S_m \longrightarrow \mathbf{C},$$

where the first map is dual to the exterior product. For any i, we have $(W, A_i)_m = g(A_1 \wedge \cdots \wedge A_m \wedge A_i) = 0$. The pairing

$$S_m \times S_m \longrightarrow \mathbf{C}, \quad (A, B) \longrightarrow (A, B)_m$$

is nondegenerate, hence such a form is unique up to scalar. □

1.6. Covariants

Reviving an old notation due to Cayley, we will write $(\alpha_0, \dots, \alpha_n \between u, v)^n$ for the expression

$$\sum_{i=0}^{n} \binom{n}{i} \alpha_i u^{n-i} v^i.$$

In particular $\mathbb{F} = (a_0, \dots, a_d \between x_1, x_2)^d$ denotes the *generic* d-ic, which we identify with the natural trace form in $S_d \otimes S_d^*$. Using the duality in (5), this amounts to the identification of $a_i \in S_d^*$ with $\frac{1}{d!} x_2^{d-i} (-x_1)^i$. Let R denote the symmetric algebra

$$\bigoplus_{m \geqslant 0} S_m\big(S_d^*\big) = \bigoplus_{m \geqslant 0} R_m = \mathbf{C}[a_0, \dots, a_d],$$

and $\mathbb{P}^d = \mathbb{P} S_d = \operatorname{Proj} R$.

By definition, a *covariant* of degree-order (m, q) of binary d-ics is an $SL(V)$-equivariant imbedding $S_0 \hookrightarrow S_m(S_d) \otimes S_q$. Let Φ denote the image of 1 via this map, then we may write $\Phi = (\varphi_0, \dots, \varphi_q \between x_1, x_2)^q$ where each φ_i is a homogeneous degree m form in the $\{a_i\}$. The weight of Φ is defined to be $\frac{1}{2}(dm - q)$ (which is always a nonnegative integer). A covariant of order 0 is called an invariant. E.g., $(\mathbb{F}, \mathbb{F})_2$ is a covariant of degree-order $(2, 2d - 4)$, and for $d = 4$, the compound transvectant $((\mathbb{F}, \mathbb{F})_2, \mathbb{F})_4$ is an invariant of degree 3. If $\mathbb{F}$ is specialized to $F \in S_d$, then Φ gets specialized to $\Phi_F \in S_q$.

1.7. Let Φ denote a covariant of degree-order (m, q). Let a, b denote nonnegative integers, and let $r = (a + q - b)/2$. For every $F \in S_d$, we have a map

$$h_F : S_a \longrightarrow S_b, \quad G \longrightarrow (\Phi_F, G)_r.$$

Since the entries of the matrix describing h_F are degree m forms in the $\{a_i\}$, we may see it as an SL_2-equivariant map of graded R-modules

$$R \otimes S_a \longrightarrow R(m) \otimes S_b. \tag{6}$$

Conversely, every equivariant map of the form (6) arises from such a covariant. (Indeed, in degree zero it reduces to a map of representations $S_a \longrightarrow S_m(S_d) \otimes S_b$.) The numerical conditions are assumed to be such that the transvection is possible, i.e., we must have $a + q - b$ nonnegative and even, and $r \leqslant \min(a, q)$.

If $a \leqslant b$, then by the Wronskian of the map h we mean

$$W\big(h_{\mathbb{F}}\big(x_1^a\big), h_{\mathbb{F}}\big(x_1^{a-1}x_2\big), \ldots, h_{\mathbb{F}}\big(x_2^a\big)\big),$$

which is a covariant of degree $m(a+1)$ and order $(a+1)(b-a)$. Its coefficients are (up to signs) the maximal minors of $h_{\mathbb{F}}$.

1.8. We will let $\mathfrak{I}(\Phi) \subseteq R$ denote the ideal generated by the coefficients of a covariant Φ. E.g., if $d = 3$, then $\mathfrak{I}((\mathbb{F}, \mathbb{F})_2)$ is the defining ideal of the twisted rational cubic curve.

If $\mathbb{I}(a_0, \ldots, a_d)$ is an invariant of degree m, then its *evectant* is defined to be

$$\mathcal{E}_{\mathbb{I}} = \frac{1}{m} \sum_{i=0}^{d} \frac{\partial \mathbb{I}}{\partial a_i} (-x_2)^{d-i} x_1^i, \tag{7}$$

which is a covariant of degree-order $(m - 1, d)$. By Euler's formula we have an identity $(\mathcal{E}_{\mathbb{I}}, \mathbb{F})_d = \frac{1}{m} \sum_i a_i \frac{\partial \mathbb{I}}{\partial a_i} = \mathbb{I}$.

Let $\mathcal{A} \subseteq \mathbf{Q}[a_0, \ldots, a_d; x_1, x_2]$ denote the subring of covariants, which is naturally bigraded by (m, q). By a fundamental theorem of Gordan, $\mathcal{A}$ is finitely generated. A minimal set of generators of $\mathcal{A}$ is called a *fundamental system* for d-ics. Moreover $\mathcal{A}$ is a unique factorization domain, and each of the minimal generators is a prime element of $\mathcal{A}$.

The number of linearly independent covariants of d-ics of degree-order (m, q) is given by the Cayley–Sylvester formula (see [23, Corollary 4.2.8]). For integers n, k, l, let $p(n, k, l)$ denote the number of partitions of n into k parts such that no part exceeds l. Then

$$\zeta_{m,q} = \dim \mathcal{A}_{m,q} = p\left(\frac{dm - q}{2}, d, m\right) - p\left(\frac{dm - q - 2}{2}, d, m\right). \tag{8}$$

Example 1.3. Let $d = 5$, then $\zeta_{4,8} = p(6, 5, 4) - p(5, 5, 4) = 2$. A basis for the space $\mathcal{A}_{4,8}$ is given by

$$(\mathbb{F}, \mathbb{F})_2 (\mathbb{F}, \mathbb{F})_4, \qquad \mathbb{F}\big(\mathbb{F}, (\mathbb{F}, \mathbb{F})_4\big)_2.$$

1.9. Quintics

The following table (adapted from [11, p. 131]) lists the degree-orders of the elements in the fundamental system for quintics. For instance, there is one generator in degree-order $(5, 3)$ and none in $(3, 7)$.

We will frequently need the following covariants:

$$\begin{aligned} &\vartheta_{22} = (\mathbb{F}, \mathbb{F})_4, && \vartheta_{26} = (\mathbb{F}, \mathbb{F})_2, && \vartheta_{33} = (\vartheta_{22}, \mathbb{F})_2, \\ &\vartheta_{39} = (\mathbb{F}, \vartheta_{26})_1, && \vartheta_{40} = (\vartheta_{22}, \vartheta_{22})_2, && \vartheta_{44} = (\vartheta_{22}, \vartheta_{26})_2, \\ &\vartheta_{51} = \left(\vartheta_{22}^2, \mathbb{F}\right)_4, && \vartheta_{80} = \left(\vartheta_{22}^3, \vartheta_{26}\right)_6. \end{aligned} \tag{9}$$

The notation is so set up that ϑ_{mq} is a generator in degree-order (m, q). (The comma is omitted for ease of reading.)

degree \ order	0	1	2	3	4	5	6	7	9
1						1			
2			1				1		
3				1		1			1
4	1				1		1		
5		1		1				1	
6			1		1				
7		1				1			
8	1		1						
9				1					
11		1							
12	1								
13		1							
18	1								

The computations which go into constructing such tables are generally very laborious, and of course the classical invariant theorists carried them out without the aid of machines. Hence, it is not unreasonable to worry about their correctness (also see the footnote on [11, pp. 131–132]). In the case of binary quintics however, I have thoroughly checked that the table above is entirely correct.

Here is a typical instance of how the table is used: we have

$$\zeta_{9,5} = p(20, 5, 9) - p(19, 5, 9) = 98 - 93 = 5,$$

i.e., $\mathcal{A}_{9,5}$ is 5-dimensional. Notice that

$$B = \left\{\vartheta_{51}\vartheta_{22}^2, \vartheta_{51}\vartheta_{44}, \vartheta_{40}\vartheta_{33}\vartheta_{22}, \vartheta_{40}^2\mathbb{F}, \vartheta_{80}\mathbb{F}\right\} \tag{10}$$

are all of degree-order $(9, 5)$. Since they are linearly independent over $\mathbf{Q}$ (this can be checked by specializing to $F = x_1^5 + x_2^5 + (x_1 + x_2)^5$ and solving a system of linear equations), B is a basis of $\mathcal{A}_{9,5}$. This basis will be used in Section 4.2.

Since $\zeta_{18,0} = p(45, 5, 18) - p(44, 5, 18) = 967 - 966 = 1$, up to scalar, quintics have a unique invariant of degree 18. Hence, following [11, p. 131], we will define

$$\mathbb{H} = \left(\vartheta_{22}^7, \mathbb{F}\vartheta_{39}\right)_{14}. \tag{11}$$

(This merely requires checking that the transvectant is not identically zero, which can be done by specializing $\mathbb{F}$ and calculating directly.) Usually $\mathbb{H}$ is called a skew-invariant (since it is of odd weight). Indeed, $\mathbb{H}$ was the first discovery of a skew-invariant for any d. (They do not occur for $d \leqslant 4$.) For what it is worth, a MAPLE computation shows that $\mathbb{H}$ is a linear combination of 848 monomials in $a_0, \dots, a_5$.

1.10. Let $u \in S_2$ be a nonzero vector. The duality in (5) identifies the point $[u] \in \mathbb{P}S_2$ with its polar line $\{[v] \in \mathbb{P}^2\colon (u, v)_2 = 0\} \in \mathbb{P}S_2^*$. The point lies on its own polar iff $(u, u)_2 = 0$, which happens iff $[u] \in C$. If $[u]$ lies on the polar of $[v]$, then $[v]$ lies on the polar of $[u]$. The pole of the line joining two points $[u], [v]$ is given by $[(u, v)_1]$. Three points $[u], [v], [w]$ are collinear iff $((u, v)_1, w)_2 = 0$.

If $l \in S_1$, then the tangent to $\phi(l) \in C$ is the line $\{[lm]\colon m \in S_1\}$. The line joining $\phi([l]), \phi([m])$ is (the polar of) $[lm]$.

1.11. The following proposition will be needed in Section 2. Let G denote a binary quartic identified with four points $\Pi = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}\} \subseteq C$. Consider the three pairwise intersections $\overline{\mathsf{ab}} \cap \overline{\mathsf{cd}}$, $\overline{\mathsf{ac}} \cap \overline{\mathsf{bd}}$, $\overline{\mathsf{ad}} \cap \overline{\mathsf{bc}}$, regarding each as a form in S_2.

Proposition 1.4. *The product of the three points is given* (*of course up to scalar*) *by the order* 6 *covariant* $\mathbb{T}(G) = (G, (G, G)_2)_1$.

Proof. Let us write $G = a_{\mathbf{x}} b_{\mathbf{x}} c_{\mathbf{x}} d_{\mathbf{x}}$, where $a_{\mathbf{x}} = a_1 x_1 + a_2 x_2$ and $\mathsf{a} = \phi([a_{\mathbf{x}}])$, etc. By Section 1.10, the intersection $\overline{\mathsf{ab}} \cap \overline{\mathsf{cd}}$ corresponds to

$$\left(\left(a_{\mathbf{x}}^2, b_{\mathbf{x}}^2\right)_1, \left(c_{\mathbf{x}}^2, d_{\mathbf{x}}^2\right)_1\right)_1 = (ab)(cd)(a_{\mathbf{x}} b_{\mathbf{x}}, c_{\mathbf{x}} d_{\mathbf{x}})_1,$$

where $(ab) = a_1 b_2 - a_2 b_1$, etc. Hence, up to a factor, the product corresponds to

$$(a_{\mathbf{x}} b_{\mathbf{x}}, c_{\mathbf{x}} d_{\mathbf{x}})_1 (a_{\mathbf{x}} c_{\mathbf{x}}, b_{\mathbf{x}} d_{\mathbf{x}})_1 (a_{\mathbf{x}} d_{\mathbf{x}}, b_{\mathbf{x}} c_{\mathbf{x}})_1. \tag{12}$$

The last expression is of degree 3 in the coefficients of G (since each of the letters $a, \dots, d$ occurs thrice), moreover it is a covariant since the underlying geometric construction is compatible with the $SL(V)$-action. However, $\zeta_{3,6} = 1$ for binary quartics, hence $\mathbb{T}(G)$ and (12) are equal up to a scalar. □

The result remains true if Π contains one double point, say $\mathsf{a} = \mathsf{b}$, with $\overline{\mathsf{ab}}$ interpreted as the tangent to C at a. By [9, §3.5.2], the covariant $\mathbb{T}(G)$ vanishes identically iff Π consists of two (possibly coincident) double points, say $\mathsf{a} = \mathsf{b}, \mathsf{c} = \mathsf{d}$. In this case the geometric construction collapses, since $\overline{\mathsf{ac}} \cap \overline{\mathsf{bd}}$ is no longer a determinate point.

This proposition can be used to give an alternate definition of $\mathbb{H}$. Let $\mathfrak{R}$ denote the resultant $\mathrm{Res}(\mathbb{F}, \vartheta_{33})$, defined as the determinant of an 8×8 Sylvester matrix (see [18, Chapter V, §10]). By construction it is of degree $5 \times 3 + 3 \times 1 = 18$ in the $\{a_i\}$.

Proposition 1.5. *The hypersurface defined by $\mathfrak{R}$ coincides with $\mathcal{H}$.*

We will avoid using the fact that $\zeta_{18,0} = 1$.

Proof. Let us first show that $\mathfrak{R}$ is not identically zero. Specialize to $F = x_1^5 + 2x_2^5 + (x_1 + x_2)^5$. Then $\vartheta_{33} = -12x_1x_2(x_1 + x_2)$, which has no common factor with F, hence $\mathfrak{R} \not\equiv 0$. Now assume that F and $\vartheta_{33}(F)$ have a common linear factor, we may take it to be x_1 after a change of variables. Let $F = x_1 G$, with $G = (a_0, a_1, a_2, a_3, a_4 \between x_1, x_2)^4$. Calculating directly, we have

$$\vartheta_{33}(F)|_{x_1:=0} = \frac{24}{125} x_2^3 \big(2a_3^3 + a_1 a_4^2 - 3a_2 a_3 a_4\big), \tag{13}$$

which vanishes by hypothesis. Hence

$$\mathbb{T}(G)|_{x_1:=0} = -x_2^6 \big(2a_3^3 + a_1 a_4^2 - 3a_2 a_3 a_4\big)$$

must also vanish, i.e., x_1 must divide one of the three intersection points coming from G. Denote this point by $\mathsf{z} = [x_1(\alpha x_1 + \beta x_2)]$. It is now immediate that the divisor corresponding to F is in involution with respect to z, hence $[F] \in \mathcal{H}$. Thus we have an inclusion of hypersurfaces $\{[F] \in \mathbb{P}^5\colon \mathfrak{R} = 0\} \subseteq \mathcal{H}$. Since the latter is irreducible, they must be equal. $\square$

1.12. It will prove useful to introduce the following loci in $\mathbb{P}^5$. If $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a partition of 5, let X_λ denote the closed subvariety

$$\Big\{[F] \in \mathbb{P}^5\colon F = \prod l_i^{\lambda_i} \text{ for some } l_i \in S_1\Big\}.$$

In other words, the divisor of $[F] \in X_\lambda$ is of the form $\lambda_1 \mathsf{a}_1 + \cdots + \lambda_r \mathsf{a}_r$ with some of the a_i possibly coincident. The dimension of X_λ equals the number of (nonzero) parts in λ. There is an inclusion $X_\mu \subseteq X_\lambda$ iff λ is a refinement of μ. For instance, $X_{(5)}$ is the rational normal quintic curve, $X_{(2,1,1,1)}$ is the discriminant hypersurface, and $X_{(3,1,1)}$ is the locus of nullforms; with obvious inclusions $X_{(5)} \subseteq X_{(3,1,1)} \subseteq X_{(2,1,1,1)}$.

1.13. A summary of results

In Section 2 we will construct a desingularization of $\mathcal{H}$, and then show that its singular locus $\mathcal{B}$ consists of three components $\Omega_{(1)}$, $\Omega_{(2)}$ and $X_{(3,1,1)}$. They are respectively the SL_2-orbit closures of the forms

$$x_1^5 + x_2^5, \qquad x_1x_2(x_1 - x_2)\big(x_1^2 + x_1x_2 + x_2^2\big), \qquad x_1^3x_2(x_1 + x_2).$$

Their degrees are 6, 10 and 9, hence $\mathcal{B}$ is of degree 25 and pure codimension two. Next we show that the ideal $I_{\mathcal{B}} \subseteq R$ is a complete intersection, defined by the coefficients of ϑ_{51}.

In Section 3 it will be seen that $\mathcal{H}$ is naturally isomorphic to its own dual variety. The duality $S_5 \simeq S_5^*$ in (5) induces an isomorphism $\sigma : \mathbb{P}^5 \xrightarrow{\sim} (\mathbb{P}^5)^*$. Let $[F] \in \mathcal{H} \setminus \mathcal{B}$, with $T_{\mathcal{H},[F]}$ the tangent space to $\mathcal{H}$ at $[F]$. Then the point $\sigma^{-1}(T_{\mathcal{H},[F]})$ coincides with $[\mathcal{E}_{\mathbb{H}}(F)]$ (the value of the evectant at F). It turns out however, that this point also belongs to $\mathcal{H}$. Thus we get a morphism

$$\mathcal{H} \setminus \mathcal{B} \longrightarrow \mathcal{H} \setminus \mathcal{B}, \quad [F] \longrightarrow \big[\mathcal{E}_{\mathbb{H}}(F)\big].$$

This map is involutive, i.e., $\mathcal{E}_{\mathbb{H}}(\mathcal{E}_{\mathbb{H}}(F))$ equals F up to a scalar.

Let $J = (\frac{\partial \mathbb{H}}{\partial a_0}, \dots, \frac{\partial \mathbb{H}}{\partial a_5}) \subseteq R$ denote the Jacobian ideal of $\mathbb{H}$. In Section 4.4 we will show that J is a perfect ideal of height two, with an SL_2-equivariant minimal resolution

$$\begin{aligned} 0 \leftarrow R/J \leftarrow R \leftarrow R(-17) \otimes S_5 \\ \leftarrow R(-18) \otimes S_2 \oplus R(-22) \oplus R(-26) \leftarrow 0. \end{aligned}$$

During the course of the proof it will be seen that J naturally fits into a three-parameter family of perfect ideals.

The results of Section 4 allow us to identify the morphisms in this resolution up to three distinct possibilities, but no further. In order to resolve this ambiguity it would suffice to calculate the value of $\mathcal{E}_{\mathbb{H}}$ at $\mathcal{F}_Q$. A general formalism is developed in Section 5 to solve this problem. For any covariant Φ of d-ics, we construct a sequence of covariants $\mathcal{A}_\bullet$ called its evectants; this generalizes the classical construction from Section 1.8. Given two arbitrary covariants Φ, Ψ with evectants $\mathcal{A}_\bullet, \mathcal{B}_\bullet$, we deduce formulae for calculating the evectants of a general transvectant $(\Phi, \Psi)_r$. This iterative scheme is then applied to formula (11) to evaluate $\mathcal{E}_{\mathbb{H}}$. Nearly all of Section 5 can be read independently of the rest of the paper.

1.14. A note on computational procedures

Since I have used machine computations in several parts of this paper, their rôle should be specified. All the computations have been done in MAPLE. I have written routines to calculate the numbers $p(n, k, l)$ and $\zeta_{m,q}$ appearing in formula (8). I have also programmed formula (4) for calculating transvectants; hence identities such as (14) and (18) are machine-computed. I have also used MAPLE for some routine calculation in linear algebra, e.g., for evaluating Wronskian determinants and for solving systems of linear equations. None of the results depends upon calculating Gröbner bases in any guise (e.g., minimal free resolutions).

On the whole, I have not succeeded in bypassing heavy calculations entirely, and I doubt very much if this is at all possible. The Hermite invariant is a specific algebro-geometric object which is not a member of any natural 'family,' hence it seems unlikely that merely general considerations will enable us to prove much about it. Even so, I believe that none of the calculations done here by a machine are beyond the ambit of a patient and able human mathematician.[1]

2. The singular locus

2.1. First we will construct a natural desingularization of $\mathcal{H}$. Let

$$Y = \{(\mathsf{c}, \mathsf{z}) \in C \times \mathbb{P}^2\colon \text{ the tangent to } C \text{ at } \mathsf{c} \text{ passes through } \mathsf{z}\}.$$

The second projection $Y \xrightarrow{\alpha} \mathbb{P}^2$ is a double cover ramified along C. Let $\mathbb{P}T_{\mathbb{P}^2} \longrightarrow \mathbb{P}^2$ denote the projectivisation of the tangent bundle of $\mathbb{P}^2$, so that the fibre over $\mathsf{z} \in \mathbb{P}^2$ can be identified with the pencil of lines through z. Define the $\mathbb{P}^2$-bundle

$$\mathrm{Sym}^2(\mathbb{P}T_{\mathbb{P}^2}) \xrightarrow{\beta} \mathbb{P}^2,$$

[1] Paul Gordan and George Salmon come to mind; for instances, see [10] or the tables at the end of [22].

so that an element in $\beta^{-1}(\mathsf{z})$ is an unordered pair of (possibly coincident) lines L_1, L_2 through z. Consider the pullback square

$$\begin{tikzcd} \mathcal{Z} \arrow[r] \arrow[d] & \mathrm{Sym}^2(\mathbb{P}T_{\mathbb{P}^2}) \arrow[d, "\beta"] \\ Y \arrow[r, "\alpha"'] & \mathbb{P}^2. \end{tikzcd}$$

Define $\mathcal{Z} \xrightarrow{f} \mathcal{H}$ by sending $(\mathsf{c}, \mathsf{z}) \times (L_1, L_2)$ to the divisor

$$\mathsf{c} + L_1 \cap C + L_2 \cap C.$$

(Of course, $L_i \cap C$ are interpreted scheme-theoretically.) By construction f is a projective birational morphism which is a desingularization of $\mathcal{H}$. We will use it to detect the singularities of $\mathcal{H}$. Since Y is a rational variety (in fact isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$), so are $\mathcal{Z}$ and $\mathcal{H}$. Henceforth we will write $(\mathsf{c}, \mathsf{z}; L_1, L_2)$ for $(\mathsf{c}, \mathsf{z}) \times (L_1, L_2) \in \mathcal{Z}$.

Lemma 2.1. *The morphism* $\mathcal{Z} \setminus f^{-1}(X_{(5)}) \xrightarrow{f} \mathcal{H} \setminus X_{(5)}$ *is finite.*

Proof. Since the morphism is projective, it suffices to show that it has finite fibres (see [13, Lemma 14.8]). Let $[F] \in \mathcal{H} \setminus X_{(5)}$, and $(\mathsf{c}, \mathsf{z}; L_1, L_2) \in f^{-1}([F])$. There are finitely many choices for c. By hypothesis there is a point $\mathsf{a}(\neq \mathsf{c})$ appearing in $[F]$; hence for a given c there are only finitely many possibilities for z (because $\overline{\mathsf{za}} \cap C$ must be contained in $[F]$). Then for a given z, there are only finitely many possibilities for the L_i. □

This argument breaks down over $X_{(5)}$; in fact $f^{-1}(X_{(5)}) \longrightarrow X_{(5)}$ is a $\mathbb{P}^1$-bundle.

2.2. Define the forms

$$\begin{aligned} &\mathbb{U}_{(1)} = x_1^5 + x_2^5, && \mathbb{U}_{(2)} = x_1 x_2 (x_1 - x_2)\big(x_1^2 + x_1 x_2 + x_2^2\big), \\ &\mathbb{U}_{(3)} = x_1^3 x_2 (x_1 + x_2), && \mathbb{U}_{(4)} = x_1^3 x_2^2, \\ &\mathbb{U}_{(5)} = x_1^4 x_2, && \mathbb{U}_{(6)} = x_1^5. \end{aligned}$$

Let $\mathcal{B} \subseteq \mathcal{H}$ denote the union of the orbits of all the $U_{(i)}$. We claim that $\mathcal{B}$ is closed. Indeed, by [2, §2] the closure of any orbit is a union of orbits of forms of the type $x_1^a x_2^b$, and they have been already included.

Theorem 2.2. *The singular locus* $\mathrm{Sing}(\mathcal{H})$ *coincides with* $\mathcal{B}$.

The theorem will follow from the following proposition.

Proposition 2.3.

(1) *For $[F] \in \mathcal{H}$, the fibre $f^{-1}([F])$ consists of more than one point iff F lies in the orbit of one of the forms $\mathbb{U}_{(i)}$ for $1 \leqslant i \leqslant 6$, $i \neq 5$.*
(2) *Assume $[F] \in \mathcal{H} \setminus \mathcal{B}$, and $f^{-1}([F]) = \{\mathsf{w}\}$. Then the morphism on tangent spaces $T_{\mathcal{Z},\mathsf{w}} \longrightarrow T_{\mathcal{H},[F]}$ is injective.*

Let us show the theorem assuming the proposition. If $[F]$ lies in the orbit of one of $\mathbb{U}_{(1)}, \ldots, \mathbb{U}_{(4)}$, then the fibre $f^{-1}([F])$ is disconnected, hence $[F]$ is not a normal point. Since $\mathbb{U}_{(5)}, \mathbb{U}_{(6)}$ lie in the orbit closure of $\mathbb{U}_{(3)}$, we deduce that $\mathcal{B} \subseteq \mathrm{Sing}(\mathcal{H})$. If $[F] \in \mathcal{H} \setminus \mathcal{B}$, then by [13, Theorem 14.9] the map f is a local isomorphism in a neighbourhood of w, hence $[F]$ is a nonsingular point. $\square$

2.3. Let us prove part (1) of the proposition. Define

$$\mathcal{S} = \{[F] \in \mathcal{H}\colon\ f^{-1}([F]) \text{ consists of at least two points}\}.$$

Evidently $\mathbb{U}_{(6)} \in \mathcal{S}$. Assume that $[F] = 3\mathsf{c} + \mathsf{a}_1 + \mathsf{a}_2$, where $\mathsf{a}_1, \mathsf{a}_2$ are (possibly coincident) points each different from c. Let z denote the intersection $\overline{\mathsf{cc}} \cap \overline{\mathsf{a}_1\mathsf{a}_2}$, then $(\mathsf{c}, \mathsf{z}; \overline{\mathsf{cc}}, \overline{\mathsf{a}_1\mathsf{a}_2})$ and $(\mathsf{c}, \mathsf{c}; \overline{\mathsf{ca}_1}, \overline{\mathsf{ca}_2})$ both map to $[F]$; this shows that $\mathbb{U}_{(3)}, \mathbb{U}_{(4)} \in \mathcal{S}$. It is equally clear that $\mathbb{U}_{(5)} \notin \mathcal{S}$.

If a point of the form $(\mathsf{c}, \mathsf{c}, L_1, L_2)$ belongs to $f^{-1}([F])$, then $[F]$ must have a point of multiplicity $\geqslant 3$ at c, which is already considered above. Hence assume that $[F] \in \mathcal{S} \setminus X_{(3,1,1)}$, and $(\mathsf{c}, \mathsf{z}; L_1, L_2)$, $(\mathsf{c}', \mathsf{z}'; L_1', L_2')$ are two distinct points in $f^{-1}([F])$. Since $\mathsf{c} \neq \mathsf{z}$, we may write $\mathsf{c} = \phi([x_1])$, $\mathsf{z} = [x_1 x_2]$ after a change of variables. Then $[F] = [\mathcal{F}_Q]$ for some $Q \in \mathbb{P}^2$ (see Section 1.2).

If $q_0 = 0$, then both q_1, q_2 must be nonzero (otherwise $[\mathcal{F}_Q] \in X_{(3,1,1)}$). But then $[\mathcal{F}_Q]$ is in the orbit of $A = x_1 x_2^2 (x_1 + x_2)(x_1 - x_2)$, and it is clear from the geometry that $[A] \notin \mathcal{S}$.

Hence we may assume $q_0 = 1$, and then

$$F = x_1(x_1 - \alpha x_2)(x_1 + \alpha x_2)(x_1 - \beta x_2)(x_1 + \beta x_2)$$

for some α, β, such that $\mathsf{c}' = \phi([x_1 - \alpha x_2])$. By assumption z' is one of the diagonal intersection points (see Section 1.11) coming from the quartic form $G = x_1(x_1 + \alpha x_2)(x_1 - \beta x_2)(x_1 + \beta x_2)$. The quadratic form corresponding to z' must divide $\mathbb{T}(G)$, and hence $x_1 - \alpha x_2$ must divide $\mathbb{T}(G)$. By a direct calculation,

$$\mathbb{T}(G)|_{x_1 := \alpha x_2} = \frac{1}{32} x_2^6 \alpha^3 (\alpha^2 + 3\beta^2)(\alpha^2 + 4\alpha\beta - \beta^2)(\alpha^2 - 4\alpha\beta - \beta^2), \tag{14}$$

which must vanish. Now $\alpha \neq 0$, since $[F] \notin X_{(3,1,1)}$. Hence we have two cases

$$\frac{q_0 q_2}{q_1^2} = \frac{4\alpha^2\beta^2}{(\alpha^2 + \beta^2)^2} = \begin{cases} 1/5 & \text{if } \alpha^2 \pm 4\alpha\beta - \beta^2 = 0, \\ -3 & \text{if } \alpha^2 + 3\beta^2 = 0. \end{cases}$$

A form satisfying the first case is in the orbit of $\mathcal{F}_{[1,5,5]} = x_1 (1, 5, 5 \between x_1^2, x_2^2)^2$. By the transformation $(x_1, x_2) \longrightarrow (x_1 + x_2, x_1 - x_2)$ it can be brought into the more manageable form

$$\mathbb{U}_{(1)} = x_1^5 + x_2^5. \tag{15}$$

Similarly in the second case $\mathcal{F}_{[1,1,-3]}$ can be brought into the form

$$\mathbb{U}_{(2)} = x_1x_2(x_1 - x_2)\big(x_1^2 + x_1x_2 + x_2^2\big), \tag{16}$$

via $(x_1, x_2) \longrightarrow (x_1 - x_2, x_1 + x_2)$. We have shown that any form in $\mathcal{S} \setminus X_{(3,1,1)}$ belongs to the orbit of either $\mathbb{U}_{(1)}$ or $\mathbb{U}_{(2)}$. It remains to show that the latter two belong to $\mathcal{S}$, this can be done by an explicit construction as follows:

Let $\omega = \exp(\frac{\pi\sqrt{-1}}{5})$, and $y = \omega^r x_2$. Define points $\mathsf{c} = \phi([x_1 - y]), \mathsf{z} = [(x_1 + y)(x_1 - y)]$, and L_i to be the line joining $\phi([x_1 - \omega^i y])$ and $\phi([\omega^i x_1 - y])$ for $i = 1, 2$. This gives a point of $f^{-1}([\mathbb{U}_{(1)}])$ for every $1 \leqslant r \leqslant 5$.

Let $\nu = \exp(\frac{2\pi\sqrt{-1}}{3})$, and $y = \nu^r x_2$. Define points $\mathsf{c} = \phi([x_1 - y]), \mathsf{z} = [(x_1 - y)(x_1 + y)]$. Let L_1 be the line joining $\phi([x_1]), \phi([x_2])$, and L_2 joining $\phi([x_1 - \nu y])$ and $\phi([\nu x_1 - y])$. This gives a point of $f^{-1}([\mathbb{U}_{(2)}])$ for every $1 \leqslant r \leqslant 3$.

This completes the proof of part (1). $\square$

2.4. We will prove part (2) by introducing a local parametrisation of the affine version of f, and directly calculating the map on tangent spaces. Since $[F] \notin X_{(3,1,1)}$, after a change of variables we may write $F = x_1(1, \xi, 1 \between x_1^2, x_2^2)^2$ for some $\xi \in \mathbf{C}$.

Let $\mathbb{A} = S_1 \times S_1 \times \mathbf{C}$, and define a morphism from $\mathbb{A}$ to $\mathcal{Z}$ by sending $(l_1, l_2, \xi) \in \mathbb{A}$ to $([l_1^2], [l_1l_2], L_1, L_2)$, where L_1, L_2 correspond to the solutions of the equation

$$(1, \xi, 1 \between l_1^2, l_2^2)^2 = 0.$$

Since the morphism is smooth, for a local parametrisation of f we may use the map

$$\hat{f} : \mathbb{A} \longrightarrow \mathrm{Cone}(\mathcal{H}), \quad (l_1, l_2, \xi) = l_1(1, \xi, 1 \between l_1^2, l_2^2)^2.$$

The image of an arbitrary tangent vector (m_1, m_2, η) via $d\hat{f}$ is given by the limit

$$\tau(m_1, m_2, \eta) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\big[\hat{f}(l_1 + \epsilon m_1, l_2 + \epsilon m_2, \xi + \epsilon\eta) - \hat{f}(l_1, l_2, \xi)\big].$$

Writing $\mathsf{w} = (x_1, x_2, \xi)$, the image of the map $T_{\mathbb{A},\mathsf{w}} \longrightarrow T_{\mathrm{Cone}(\mathcal{H}),F}$ is spanned by the five vectors

$$\begin{aligned} &\tau(x_1, 0, 0) = x_1(5, 3\xi, 1 \between x_1^2, x_2^2)^2, && \tau(x_2, 0, 0) = x_2(5, 3\xi, 1 \between x_1^2, x_2^2)^2, \\ &\tau(0, x_1, 0) = 4x_1^2x_2\big(\xi x_1^2 + x_2^2\big), && \tau(0, x_2, 0) = 4x_1x_2^2\big(\xi x_1^2 + x_2^2\big), \\ &\tau(0, 0, 1) = 2x_1^3x_2^2. \end{aligned}$$

In order to verify that they are linearly independent, we calculate their Wronskian

$$\begin{aligned} &\begin{vmatrix} 600x_1 & 72\xi x_2 & 72\xi x_1 & 24x_2 & 24x_1 \\ 120x_2 & 120x_1 & 72\xi x_2 & 72\xi x_1 & 120x_2 \\ 96\xi x_2 & 96\xi x_1 & 48x_2 & 48x_1 & 0 \\ 0 & 48\xi x_2 & 48\xi x_1 & 96x_2 & 96x_1 \\ 0 & 24x_2 & 24x_1 & 0 & 0 \end{vmatrix} \\ &= 2^{18}3^5 5^3 x_1\big(1 - \tfrac{6}{5}\xi^2, \xi, -1 \between x_1^2, x_2^2\big)^2. \end{aligned} \tag{17}$$

This is nonzero for any ξ, which proves part (2) of the proposition. The proof of Theorem 2.2 is complete. $\square$

One can restate the theorem as follows: $\mathcal{F}_Q$ is a singular point of $\mathcal{H}$, iff one of the expressions $q_2, q_0q_2 + 3q_1^2, 5q_0q_2 - q_1^2$ is zero.

2.5. For $i = 1, 2$, let $\Omega_{(i)}$ denote the orbit closure of $[\mathbb{U}_{(i)}]$, and let $\mathcal{G}_i \subseteq SL(V)$ denote the stabilizer subgroup of $[\mathbb{U}_{(i)}]$. By [2, §0], we have a formula

$$\deg \Omega_{(i)} = \frac{5.4.3}{|\mathcal{G}_i|}.$$

Since an element of $\mathcal{G}_i$ must permute the linear factors of $\mathbb{U}_{(i)}$, it is easy to determine all symmetries by mere inspection. The group $\mathcal{G}_1$ is the dihedral group D_5 of order 10, generated by the transformations

$$(x_1, x_2) \longrightarrow \begin{cases} (x_2, x_1), \\ (x_1, \exp(\frac{2\pi\sqrt{-1}}{5})x_2). \end{cases}$$

Similarly $\mathcal{G}_2$ is isomorphic to D_3, generated by

$$(x_1, x_2) \longrightarrow \begin{cases} (x_2, x_1), \\ (\exp(\frac{2\pi\sqrt{-1}}{3})x_1, x_2). \end{cases}$$

Hence $\Omega_{(1)}, \Omega_{(2)}$ are of degrees 6 and 10, respectively. The degree of $X_{(3,1,1)}$ is 9, as given by a formula due to Hilbert [15].

2.6. Let $\mathfrak{p}_{(i)} \subseteq R$ denote the homogeneous ideal of $\Omega_{(i)}$. The variety $\Omega_{(1)}$ is the closure of the union of secant lines to $X_{(5)}$, and it is known (as an instance of a more general result) that $\mathfrak{p}_{(1)}$ is a perfect ideal of height two (see [16, Theorem 1.56]). We briefly recapitulate the proof. Given $F \in S_5$, define

$$\alpha_F : S_2 \longrightarrow S_3, \quad G \longrightarrow (F, G)_2,$$

and let

$$\alpha : S_2 \otimes R(-1) \longrightarrow S_3 \otimes R$$

denote the corresponding morphism of graded R-modules (Section 1.7).

Lemma 2.4. *The map α_F is injective for a general F, moreover* $\ker \alpha_F$ *is nonzero iff* $[F] \in \Omega_{(1)}$.

Proof. It is easily verified from formula (4) that $\ker \alpha_F = 0$ for $F = x_1^5 + x_2^5 + (x_1 + x_2)^5$. Assume $G(\neq 0) \in \ker \alpha_F$, then after a change of variables G can be written as either x_1^2 or x_1x_2. In the former case $F = x_1^4(c_1x_1 + c_2x_2)$ and in the latter case $F = c_1x_1^5 + c_2x_2^5$. The ‘if’ part is equally clear. $\square$

By the Porteous formula (see [4, Chapter II.4]) the scheme-theoretic degeneracy locus $\{\operatorname{rank} \alpha_F \leqslant 2\}$ has degree 6 (it is the coefficient of h^2 in the Maclaurin expansion of $(1 + h)^{-3}$), and so does $\Omega_{(1)}$. Hence the ideal of maximal minors of α coincides with $\mathfrak{p}_{(1)}$, and we get a Hilbert–Burch resolution (see [6, §20.4])

$$0 \longleftarrow R/\mathfrak{p}_{(1)} \longleftarrow R \xleftarrow{\delta_0} R(-3) \otimes S_3 \xleftarrow{\delta_1} R(-4) \otimes S_2 \longleftarrow 0.$$

Now consider the complex

$$R \xrightarrow{\delta_0^\vee} R(3) \otimes S_3 \xrightarrow{\delta_1^\vee} R(4) \otimes S_2.$$

To describe the first map, let $\mathcal{W}_{(1)}$ denote the Wronskian of $\alpha_{\mathbb{F}}$, i.e., the determinant of the 3×3 matrix of linear forms

$$(i, j) \longrightarrow \frac{\partial^2(\mathbb{F}, x_1^{3-i} x_2^{i-1})_2}{\partial x_1^{3-j} \partial x_2^{j-1}} \quad (1 \leqslant i, j \leqslant 3).$$

Now $\mathcal{W}_{(1)}$ is a covariant of degree-order $(3, 3)$, and $\zeta_{3,3} = 1$ for quintics, hence it must coincide with ϑ_{33} up to a scalar. Thus $\mathfrak{p}_{(1)} = \mathfrak{I}(\vartheta_{33})$.

Up to a scalar, the map $\delta_1^\vee$ must be given by $S_3 \longrightarrow S_2, G \longrightarrow (F, G)_3$. From $\delta_1^\vee \circ \delta_0^\vee = 0$ we deduce the identity $(\vartheta_{33}, \mathbb{F})_3 = 0$.

2.7. Using similar ideas we will find a free resolution of $\mathfrak{p}_{(2)}$. It is sensible to look for a 4×5 matrix of linear forms, since then by Porteous' formula the degeneracy locus $\{\text{rank} \leqslant 3\}$ has expected degree 10.

Proposition 2.5. *The ideal $\mathfrak{p}_{(2)}$ is perfect of height two.*

Proof. Consider the map

$$\beta_F : S_3 \longrightarrow S_4, \quad G \longrightarrow (F, G)_2,$$

and let $\mathcal{W}_{(2)}$ denote the corresponding 4×4 Wronskian determinant

$$(i, j) \longrightarrow \frac{\partial^3(\mathbb{F}, x_1^{4-i} x_2^{i-1})_2}{\partial x_1^{4-j} \partial x_2^{j-1}} \quad (1 \leqslant i, j \leqslant 4),$$

which is a covariant of degree-order $(4, 4)$. Let $\mathfrak{a} = \mathfrak{I}(\mathcal{W}_{(2)})$ denote the ideal of maximal minors; *a priori* we know it to be of height $\leqslant 2$. If it were to have height one, then an invariant would have to divide $\mathcal{W}_{(2)}$, which is impossible. Hence we get a free resolution

$$0 \longleftarrow R/\mathfrak{a} \longleftarrow R \longleftarrow R(-4) \otimes S_4 \longleftarrow R(-5) \otimes S_3 \longleftarrow 0.$$

Now a direct calculation shows that

$$\begin{aligned}\mathcal{W}_{(2)}(\mathcal{F}_Q) &= \begin{vmatrix} 24/5q_1x_1 & 12/5q_2x_2 & 12/5q_2x_1 & 0 \\ -2q_1x_2 & -2q_1x_1 & 2/5q_2x_2 & 2/5q_2x_1 \\ 8q_0x_1 & -4/5q_1x_2 & -4/5q_1x_1 & -16/5q_2x_2 \\ 6q_0x_2 & 6q_0x_1 & 18/5q_1x_2 & 18/5q_1x_1 \end{vmatrix} \\ &= \frac{1152}{125}\bigl(q_0q_2 + 3q_1^2\bigr)(5q_0q_2 + q_1^2, -2q_1q_2, 2q_2^2)(x_1^2, x_2^2)^2.\end{aligned}$$

Hence $\mathcal{W}_{(2)}$ vanishes on $\Omega_{(2)}$. Since the latter has degree 10, the scheme defined by $\mathfrak{a}$ coincides with $\Omega_{(2)}$ and $\mathfrak{p}_{(2)} = \mathfrak{a}$. $\square$

A basis for the space $\mathcal{A}_{4,4}$ is given by the two covariants $\vartheta_{22}^2, \vartheta_{44}$, hence $\mathcal{W}_{(2)}$ must be their linear combination. The actual coefficients can be easily found by specializing F and then solving a system of linear equations. This gives the relation $\mathcal{W}_{(2)} = 1/5760(7\vartheta_{22}^2 - 10\vartheta_{44})$. As before, we have an identity $(\mathcal{W}_{(2)}, \mathbb{F})_3 = 0$.

2.8. By a result of Weyman (see [24, Theorem 3]), the ideal of $X_{(3,1,1)}$ (say $\mathfrak{q}$) is generated in degrees $\leqslant 4$. If we specialize to $F = x_1^3x_2(x_1 + x_2)$ and search through all covariants in degrees $\leqslant 4$, then we find that only ϑ_{40} and $2\vartheta_{22}^2 + 15\vartheta_{44}$ vanish on F, hence their coefficients must generate $\mathfrak{q}$. One sees that $\mathfrak{q}$ is not perfect; indeed, it would have to arise as the ideal of maximal minors of a map

$$R \otimes (S_0 \oplus S_4) \longrightarrow \bigoplus_{i \geqslant 0} R(i) \otimes (S_{k_i} \oplus S_{k_i'} \oplus \cdots)$$

such that the target module has rank 5, the minors are of degree 4 and the Porteous degree is 9. However no such integers can be found.

2.9. Let $I_{\mathcal{B}} \subseteq R$ denote the defining ideal of the singular locus $\mathcal{B}$.

Proposition 2.6. *The ideal $I_{\mathcal{B}}$ is a complete intersection generated by the two coefficients of the covariant ϑ_{51}.*

Proof. The ideal $\mathfrak{e} = \mathfrak{I}(\vartheta_{51})$ is a complete intersection, since otherwise an invariant would have to divide both coefficients of ϑ_{51}. By a direct calculation,

$$\vartheta_{51}(\mathcal{F}_Q) = \frac{4}{625}q_2\bigl(q_0q_2 + 3q_1^2\bigr)\bigl(5q_0q_2 - q_1^2\bigr)x_1, \tag{18}$$

hence $\vartheta_{51}(F)$ vanishes on $\mathcal{B}$. Since $\deg \mathcal{B} = 25$, we must have $\mathfrak{e} = I_{\mathcal{B}}$. $\square$

The linear form ϑ_{51} evaluated at a nonsingular quintic in $\mathcal{H}$ 'detects' the point of tangency c in the configuration on p. 325. Indeed this is visibly true of $\mathcal{F}_Q$ from (18), and since ϑ_{51} is a covariant, it is true generally.

3. The dual variety

Let $\sigma : \mathbb{P}S_d \xrightarrow{\sim} \mathbb{P}S_d^*$ be the isomorphism induced by the duality in (5); it identifies $[A] \in \mathbb{P}S_d$ with the hyperplane $\{[B] \in \mathbb{P}S_d\colon (A, B)_d = 0\}$. Let $\mathbb{I}$ denote a degree m invariant of d-ics, defining a hypersurface $\mathcal{X} \subseteq \mathbb{P}S_d$.

Proposition 3.1. *Let $[F] \in \mathcal{X}$ be a nonsingular point, and let $T = T_{\mathcal{X},[F]}$ denote the tangent space to $\mathcal{X}$ at $[F]$, seen as a point in $\mathbb{P}S_d^*$. Then we have an equality*

$$\bigl[\mathcal{E}_{\mathbb{I}}(F)\bigr] = \sigma^{-1}(T).$$

Proof. Let $B = (b_0, \dots, b_d \between x_1, x_2)^d$. The point $[b_0, \dots, b_d]$ belongs to T iff

$$\sum_{i=0}^{d} b_i \left(\frac{\partial \mathbb{I}}{\partial a_i} \bigg|_F \right) = 0.$$

This condition can be rewritten as $(\mathcal{E}_{\mathbb{I}}(F), B)_d = 0$, hence the assertion. □

Now let $F = x_1(1, \xi, 1 \between x_1^2, x_2^2)^2$. By the proposition above, together with Lemma 1.2, the evectant $\mathcal{E}_{\mathbb{H}}(F)$ is given (up to scalar) by the Wronskian of a basis of $T_{\mathcal{H},[F]}$. But we have already calculated the latter in (17). After the substitution

$$(x_1, x_2, \xi) \longrightarrow \bigl(q_0^{1/5} x_1, q_2^{1/4} q_0^{-1/20} x_2, q_1 q_0^{-1/2} q_2^{-1/2}\bigr)$$

we get the expression

$$\mathcal{E}_{\mathbb{H}}(\mathcal{F}_Q) = \text{constant} \cdot \mathcal{F}_{Q'},$$

where

$$Q' = \left[q_0 q_2 - \frac{6}{5} q_1^2, q_1 q_2, -q_2^2 \right]. \tag{19}$$

Since $\mathcal{E}$ is a degree 17 covariant, the 'constant' must be a degree 15 polynomial in the q_i. Now $\mathcal{E}_{\mathbb{H}}(F)$ vanishes identically iff $[F] \in \mathcal{B}$, so we must have

$$\mathcal{E}_{\mathbb{H}}(\mathcal{F}_Q) = \Bbbk\, q_2^n \bigl(q_0 q_2 + 3 q_1^2\bigr)^{n'} \bigl(5 q_0 q_2 - q_1^2\bigr)^{n''} \mathcal{F}_{Q'}, \tag{20}$$

for some integers n, n', n'' such that $n + 2n' + 2n'' = 15$. Here (and subsequently) $\Bbbk$ stands for some *nonzero* rational number which need not be precisely specified. The indices n, n', etc. will be determined later in Section 5.5. Note the identity $(Q')' = [-q_2^3 q_0, -q_2^3 q_1, -q_2^4] = Q$. We have proved the following:

Theorem 3.2. *If $[F]$ is a nonsingular point in $\mathcal{H}$, then so is $[\mathcal{E}_{\mathbb{H}}(F)]$. The assignment*

$$\mathcal{H} \setminus \mathcal{B} \longrightarrow \mathcal{H} \setminus \mathcal{B}, \quad [F] \longrightarrow [\mathcal{E}_{\mathbb{H}}(F)]$$

is an involutive automorphism. In particular $\mathcal{H}$ is isomorphic to its own dual variety.

4. The Jacobian ideal

Let $J = \mathfrak{J}(\mathcal{E}_{\mathbb{H}}(\mathbb{F}))$ denote the Jacobian ideal of $\mathbb{H}$.

4.1. Let

$$0 \longleftarrow R/J \longleftarrow R \longleftarrow R(-17) \otimes S_5 \longleftarrow E_1 \longleftarrow E_2 \longleftarrow \cdots \tag{21}$$

denote the equivariant minimal resolution of J, i.e., E_i is the module of ith syzygies. Apply $\mathrm{Hom}_R(-, R)$ to (21) and consider the complex

$$0 \longrightarrow R \xrightarrow{\epsilon_0} R(17) \otimes S_5 \xrightarrow{\epsilon_1} E_1^\vee \longrightarrow \cdots$$

Write $E_1^\vee$ as a direct sum

$$\bigoplus_{r \geqslant 1} R(17 + r) \otimes M_r,$$

where each M_r is a finite direct sum of irreducible SL_2-representations.

By construction $\epsilon_0(1) = \mathcal{E}_{\mathbb{H}}(\mathbb{F})$. Let $S_p \subseteq M_r$ denote a direct summand, and consider the composite

$$\theta : R(17) \otimes S_5 \longrightarrow R(17 + r) \otimes M_r \longrightarrow R(17 + r) \otimes S_p.$$

It can be seen as a map $S_5 \longrightarrow S_p$ whose coefficients are degree r forms in the coefficients of $\mathbb{F}$. Hence, θ corresponds to a covariant Θ (determined up to a constant) of degree r and order (say) q, defining

$$S_5 \longrightarrow S_p, \quad G \longrightarrow (G, \Theta)_{\frac{1}{2}(5-p+q)}.$$

Altogether, the identity $\theta \circ \epsilon_0 = 0$ translates into

$$\big(\mathcal{E}_{\mathbb{H}}(\mathbb{F}), \Theta\big)_{\frac{1}{2}(5-p+q)} = 0.$$

4.2. First syzygies of J

We will enumerate some of the first syzygies of J by hand, and then show *a posteriori* that they are a complete list. Since a syzygy in a certain degree produces non-minimal syzygies in higher degrees, at each stage we should ensure that only 'new' syzygies are included.

(i) If $\mathbb{I}$ is any invariant of d-ics, then $(\mathcal{E}_{\mathbb{I}}(\mathbb{F}), \mathbb{F})_{d-1} = 0$ (see Corollary 5.9 below), hence S_2 is a summand in M_1.

(ii) The space $\mathcal{A}_{5,5}$ is 2-dimensional, and spanned by $\vartheta_{33}\vartheta_{22}$ and $\vartheta_{40}\mathbb{F}$. By construction $\widetilde{I} = (\mathcal{E}_{\mathbb{H}}, \vartheta_{33}\vartheta_{22})_5$ is an invariant of degree 22. Since $\zeta_{22,0} = 1$, we must have $\widetilde{I} = \alpha\vartheta_{40}\mathbb{H}$ for some $\alpha \in \mathbf{Q}$. Define

$$\mathcal{U} = \vartheta_{33}\vartheta_{22} - \alpha\vartheta_{40}\mathbb{F}, \tag{22}$$

so that $(\mathcal{E}_{\mathbb{H}}, \mathcal{U})_5 = 0$.

Claim. *This syzygy cannot have arisen from the submodule $S_2 \subseteq M_1$.*

Proof. Otherwise it would correspond to a nonzero morphism $S_2 \otimes R_4 \longrightarrow S_0$. However $R_4 \simeq S_4(S_5)$ contains no copies of S_2 (or equivalently, $\zeta_{4,2} = 0$ for quintics), hence this is impossible. $\square$

(iii) By an analogous reasoning, if Φ is any covariant of degree-order $(9, 5)$, then

$$(\mathcal{E}_{\mathbb{H}}, \Phi)_5 = \text{some degree 8 invariant} \times \mathbb{H}.$$

Since $\mathcal{A}_{8,0}$ has $\{\vartheta_{40}^2, \vartheta_{80}\}$ as a basis, this would produce a syzygy of the form

$$\left(\mathcal{E}_{\mathbb{H}}, \Phi - \beta\vartheta_{40}^2\mathbb{F} - \gamma\vartheta_{80}\mathbb{F}\right)_5 = 0 \quad \text{for some } \beta, \gamma \in \mathbf{Q}. \tag{23}$$

However, we need to weed out those syzygies which come from earlier degrees. Broadly speaking, we have three syzygies in degree 9 which arise in this way, amongst which two come from earlier degrees and one will be new. The space $\mathcal{A}_{9,5}$ is 5-dimensional with a basis (see p. 330)

$$\vartheta_{51}\vartheta_{22}^2, \qquad \vartheta_{51}\vartheta_{44}, \qquad \vartheta_{40}\vartheta_{33}\vartheta_{22}, \qquad \vartheta_{40}^2\mathbb{F}, \qquad \vartheta_{80}\mathbb{F}. \tag{24}$$

The one-dimensional space $\mathcal{A}_{8,2}$ is spanned by ϑ_{82}. From part (i) we get the obvious identity $((\mathcal{E}_{\mathbb{H}}, \mathbb{F})_4, \vartheta_{82})_2 = 0$, which can be rewritten as $(\mathcal{E}_{\mathbb{H}}, (\mathbb{F}, \vartheta_{82})_1)_5 = 0$. (This is best seen symbolically. Writing $\mathcal{E} = e_{\mathbf{x}}^5, \mathbb{F} = f_{\mathbf{x}}^5, \vartheta_{82} = t_{\mathbf{x}}^2$, both compound transvectants evaluate to $(ef)^4(et)(ft)$.) Now $(\mathbb{F}, \vartheta_{82})_1$ is the following linear combination of the basis in (24):

$$-\frac{7}{10}\vartheta_{51}\vartheta_{22}^2 - \frac{1}{4}\vartheta_{51}\vartheta_{44} + \frac{5}{12}\vartheta_{40}\vartheta_{33}\vartheta_{22} - \frac{1}{20}\vartheta_{40}^2\mathbb{F} - \frac{1}{4}\vartheta_{80}\mathbb{F}.$$

From (ii) we have the obvious syzygy $(\mathcal{E}_{\mathbb{H}}, \vartheta_{40}\mathcal{U})_5 = 0$. Let us define $\beta, \gamma \in \mathbf{Q}$ such that the covariant

$$\mathcal{V} = \vartheta_{51}\vartheta_{22}^2 - \beta\vartheta_{40}^2\mathbb{F} - \gamma\vartheta_{80}\mathbb{F} \tag{25}$$

satisfies $(\mathcal{E}_{\mathbb{H}}, \mathcal{V})_5 = 0$. It is immediate that $\mathcal{V}$ cannot be a linear combination of $(\mathbb{F}, \vartheta_{82})_1$ and $\vartheta_{40}\mathcal{U}$, hence we have a new syzygy.

So far we have found three independent first syzygies of J corresponding to $S_2 \subseteq M_1$, $S_0 \subseteq M_5$, $S_0 \subseteq M_9$. The rational numbers α, β, γ are uniquely determined by the identities $(\mathcal{E}_{\mathbb{H}}, \mathcal{U})_5 = (\mathcal{E}_{\mathbb{H}}, \mathcal{V})_5 = 0$, but we do not yet know their values.

4.3. We will now construct the morphism whose Hilbert–Burch complex is expected to give a resolution of J. Let us change our approach somewhat, and let $\tau = (\alpha, \beta, \gamma)$ denote an *arbitrary* triplet in $\mathbf{Q}^3$. For $F \in S_5$, define

$$\sigma_\tau(F): S_2 \oplus S_0 \oplus S_0 \longrightarrow S_5,$$
$$(A, c_1, c_2) \longrightarrow (A, F)_1 + c_1\mathcal{U} + c_2\mathcal{V},$$

where $\mathcal{U}, \mathcal{V}$ are defined via formulae (22), (25). Let Γ_τ denote the Wronskian of $\sigma_\tau(\mathbb{F})$, which is a covariant of degree-order $(17, 5)$. Let $\mathfrak{b}_\tau \subseteq R$ denote the ideal generated by the coefficients of Γ_τ, and $V_\tau = V(\mathfrak{b}_\tau) \subseteq \mathbb{P}^5$ the corresponding subvariety. One knows *a priori* that each of the components of V_τ is of codimension $\leqslant 2$. We claim that $(\Gamma_\tau, \mathbb{F})_5 = \Bbbk\mathbb{H}$ for all τ. Indeed, the left-hand side is a degree 18 invariant, hence a numerical multiple of $\mathbb{H}$. It remains to check that it does not vanish identically, which is easily verified by specializing to $x_1^5 + x_2^5 + (x_1 + 2x_2)^5$. It follows that $(\mathbb{H}) \subseteq \mathfrak{b}_\tau$, hence $V_\tau \subseteq \mathcal{H}$. Since the latter contains no proper hypersurfaces, $\mathfrak{b}_\tau$ must be of pure height two. Hence the Eagon–Northcott complex (or what is the same, the Hilbert–Burch complex) of σ_τ is a minimal resolution of $\mathfrak{b}_\tau$.

By a direct calculation,

$$\Gamma_\tau(\mathcal{F}_Q) = -\frac{2^6 \cdot 3^2 \cdot 151 \cdot 293}{5^{15}} q_2^3\bigl(q_0 q_2 + 3q_1^2\bigr)\bigl(5q_0 q_2 - q_1^2\bigr) K_\tau \mathcal{F}_{Q'}, \tag{26}$$

where Q' is as in (19), and K_τ is the expression

$$\begin{aligned} &(75000\gamma + 28125)q_0^4 q_2^4 \\ &+ (520000\alpha + 42000\gamma - 22500 - 960000\beta)q_0^3 q_1^2 q_2^3 \\ &+ (-1344000\beta + 292800\gamma + 872000\alpha + 6750)q_0^2 q_1^4 q_2^2 \\ &+ (121200\gamma - 900 - 576000\beta + 408000\alpha)q_0 q_1^6 q_2 \\ &+ (12744\gamma - 69120\beta + 43200\alpha + 45)q_1^8. \end{aligned} \tag{27}$$

Since Γ_τ visibly vanishes on $\mathcal{B}$, we have $V_\tau \supseteq \mathcal{B}$. Now we should like to impose the condition on τ that $V_\tau = \mathcal{B}$. This will happen iff K_τ is nonzero at every point of $\mathcal{H} \setminus \mathcal{B}$, i.e., iff

$$K_\tau = \delta q_2^r \bigl(q_0 q_2 + 3q_1^2\bigr)^s \bigl(5q_0 q_2 - q_1^2\bigr)^t \tag{28}$$

for some $\delta \in \mathbf{Q}$, and nonnegative integers r, s, t satisfying $r + 2s + 2t = 8$. It is easy to see that if we fix the choice of the triple (r, s, t), then (28) is an inhomogeneous system of linear equations in the variables $\alpha, \beta, \gamma, \delta$. I solved this system in MAPLE, and found that it admits solutions only in the following cases, the solution being *unique* in every case.

(r, s, t)	$(\alpha, \beta, \gamma, \delta)$
$(0, 0, 4)$	$(0, 0, 0, 1/45)$
$(0, 2, 2)$	$(2/5, 14/75, -2/5, -1/75)$
$(0, 1, 3)$	$(1/6, 2/45, -1/3, 1/25)$.

Thus we have the following theorem.

Theorem 4.1.

(1) *For any $\tau \in \mathbf{Q}^3$, the ideal $\mathfrak{b}_\tau$ is perfect of height two with minimal resolution*

$$\begin{aligned} 0 \longleftarrow R/\mathfrak{b}_\tau \longleftarrow R \longleftarrow R(-17) \otimes S_5 \\ \longleftarrow R(-18) \otimes S_2 \oplus R(-22) \oplus R(-26) \longleftarrow 0. \end{aligned}$$

(2) *We have an inclusion of varieties $\mathcal{B} \subseteq V_\tau$, which is an equality iff τ is one of the following triples*:

$$(0,0,0), \qquad (2/5, 14/75, -2/5), \qquad (1/6, 2/45, -1/3). \tag{29}$$

4.4. Now let $\tau_\circ = (\alpha, \beta, \gamma)$ stand for the *specific* triple such that $(\mathcal{E}_{\mathbb{H}}, \mathcal{U})_5 = (\mathcal{E}_{\mathbb{H}}, \mathcal{V})_5 = 0$. We have shown that the complex

$$R \xrightarrow{\epsilon_0} R(17) \otimes S_5 \xrightarrow{\epsilon_1} R(18) \otimes S_2 \oplus R(22) \oplus R(26)$$

is exact in the middle. (Indeed, its middle cohomology is $\mathrm{Ext}^1_R(R/\mathfrak{b}_{\tau_\circ}, R)$, which is zero since $\mathfrak{b}_{\tau_\circ}$ is perfect of height 2.) Hence, up to scalar $\Gamma_{\tau_\circ}$ is the unique covariant of degree-order $(17, 5)$ whose image by ϵ_1 is zero. But $\mathcal{E}_{\mathbb{H}}$ also has this property, hence $\Gamma_{\tau_\circ} = \text{nonzero constant} \cdot \mathcal{E}_{\mathbb{H}}$. This forces $\mathfrak{b}_{\tau_\circ} = J$. Since $V_{\tau_\circ} = V(J) = \mathcal{B}$, we have the following result:

Proposition 4.2. *The ideal J is perfect. Moreover, $\tau_\circ$ is one amongst the three triples from* (29).

The value of $\tau_\circ$ will be found in Section 5.5.

4.5. The Cayley method

Initially I attempted to prove the perfection of J by using the Cayley method of calculating resultants (see [8, Chapter 2]). This attempt failed, but the outcome was yet another perfect ideal supported on $\mathcal{B}$. Since the details are similar to [3, §5], we will be brief.

Since $\mathbb{H}$ is the resultant of $\mathbb{F}$ and ϑ_{33}, it can be represented as the determinant of a complex. For a fixed $F \in S_5$, consider the Koszul complex

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-8) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(-5) \xrightarrow{u} \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0,$$

where u is defined on the fibres as the map $(A, B) \longrightarrow A\vartheta_{33}(F) + BF$. Form the tensor product with $\mathcal{O}_{\mathbb{P}^1}(5)$, and consider the resulting hypercohomology spectral sequence. This produces a morphism

$$g_F : S_2 \oplus S_0 \oplus S_1 \longrightarrow S_5,$$

such that $\det(g_{\mathbb{F}}) = \mathbb{H}$. The component maps $S_2 \longrightarrow S_5$, $S_0 \longrightarrow S_5$ are easily described, they are $A \longrightarrow A\vartheta_{33}(F)$, $B \longrightarrow BF$. The third map $\mu_F : S_1 \longrightarrow S_5$ (which is a d_2-differential in the spectral sequence) is given via the *Morley form*, described as follows. Symbolically write $F = f_{\mathbf{x}}^5$, $\vartheta_{33} = c_{\mathbf{x}}^3$, and define

$$\mathcal{M} = (f\,c)\big[f_{\mathbf{x}} f_{\mathbf{y}}^3 c_{\mathbf{y}}^2 + c_{\mathbf{x}} f_{\mathbf{y}}^4 c_{\mathbf{y}}\big].$$

This defines a *bivariate* covariant of $\mathbb{F}$, of orders 1 and 5 respectively in $\mathbf{x}, \mathbf{y}$. If $A \in S_1$, then $\mu_F(A) = (A, \mathcal{M})_1$. (The transvectant is with respect to $\mathbf{x}$-variables, so the result is an order 5 form in $\mathbf{y}$.) We may instead decompose $\mathcal{M}$ into its Gordan series, and write

$$\mu_F(A) = \big(A, (F, \vartheta_{33})_1\big)_1 + \frac{1}{6} A (F, \vartheta_{33})_2.$$

Now consider the truncated morphism $h_F : S_2 \oplus S_1 \longrightarrow S_5$, and let Λ denote the Wronskian of $h_{\mathbb{F}}$. By construction Λ is also a covariant of degree-order $(17, 5)$, moreover, $(\Lambda, \mathbb{F})_5 = \Bbbk\mathbb{H}$. (Compare the argument in the previous section.) By a direct calculation,

$$\Lambda(\mathcal{F}_Q) = -\frac{2^{16}3^9}{5^{14}} q_2^3 \left(q_0 q_2 + 3q_1^2\right)^2 \left(5q_0 q_2 - q_1^2\right)^5 x_1^5,$$

hence Λ differs from $\mathcal{E}_{\mathbb{H}}$ (or rather from any of the Γ_τ). However, Λ vanishes exactly over $\mathcal{B}$, hence by the usual argument we get the following result.

Proposition 4.3. *The ideal $\mathfrak{I}(\Lambda)$ is perfect of height two, with equivariant minimal resolution*

$$0 \longleftarrow R/\mathfrak{I}(\Lambda) \longleftarrow R \longleftarrow R(-17) \otimes S_5 \longleftarrow R(-20) \otimes S_2 \oplus R(-21) \otimes S_1 \longleftarrow 0.$$

5. Evectants

We could resolve the ambiguity about the correct value of $\tau_\circ$ (and hence about the maps in the resolution of J), if we could only derive an expression for $\mathcal{E}_{\mathbb{H}}$. It is certainly possible to compute the latter in MAPLE by a brute-force differentiation, but I have refrained from such a course due to æsthetic reasons. I hope that the general formalism developed here will prove useful elsewhere.

In this section the construction of evectants will be generalized as follows: given any covariant Φ of d-ics we will associate to it a sequence of covariants called the evectants of Φ. We will then deduce formulae for the evectants of $(\Phi, \Psi)_r$ in terms of those of Φ and Ψ. Finally this machinery will be applied to formula (11). We will heavily use the symbolic method, however the final result of the calculation can be understood (and used) without any reference to it. Additional variable-pairs $\mathbf{y}, \mathbf{z}$, etc. will be used as necessary, and then $\Omega_{\mathbf{yz}}$, etc. denote the corresponding Omega operators.

5.1. Evectants of a covariant

Let $\mathbb{F} = f_{\mathbf{x}}^d = \sum_{i=0}^d \binom{d}{i} a_i x_1^{d-i} x_2^i$ denote a generic binary d-ic, and let

$$\mathcal{E}(\mathbf{x}) = \sum_{i=0}^{d} \frac{\partial}{\partial a_i} x_1^i (-x_2)^{d-i}$$

denote the evectant operator. Let $\Phi = \varphi_{\mathbf{x}}^n$ be a covariant of degree-order (m, n) of d-ics. Define

$$\Gamma = \frac{1}{m}\left[\mathcal{E}(\mathbf{x}) \circ \varphi_{\mathbf{y}}^n\right], \tag{30}$$

which is a bihomogeneous form of orders d, n in $\mathbf{x}, \mathbf{y}$, respectively, so that

$$(\Gamma, \mathbb{F})_d = \frac{1}{m} \sum_{i=0}^{d} a_i \frac{\partial \Phi(\mathbf{y})}{\partial a_i} = \Phi(\mathbf{y}).$$

Expanding Γ into its Gordan series (Section 1.4), we may write

$$\Gamma = \sum_{i=0}^{\min(d,n)} (\mathbf{xy})^i \alpha_{(i)}{}_{\mathbf{x}}^{d-i} \alpha_{(i)}{}_{\mathbf{y}}^{n-i}, \tag{31}$$

where $\mathcal{A}_i = \alpha_{(i)}{}_{\mathbf{x}}^{d+n-2i}$ are a series of covariants of $f_{\mathbf{x}}^d$. Now apply $(-, f_{\mathbf{x}}^d)_d$ to each term in (31). Since

$$\big((\mathbf{xy})^i \alpha_{\mathbf{x}}^{d-i} \alpha_{\mathbf{y}}^{n-i}, f_{\mathbf{x}}^d\big)_d = (\alpha f)^{d-i} \alpha_{\mathbf{y}}^{n-i} f_{\mathbf{y}}^i = \big[\big(\alpha_{\mathbf{x}}^{d+n-2i}, f_{\mathbf{x}}^d\big)_{d-i}\big]_{\mathbf{x}:=\mathbf{y}},$$

we deduce the identity

$$\sum_i (\mathcal{A}_i, \mathbb{F})_{d-i} = \Phi. \tag{32}$$

The covariants $\mathcal{A}_\bullet = \{\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_{\min(d,n)}\}$ will be called the evectants of Φ. By construction $\mathcal{A}_i$ is of degree-order $(m-1, d+n-2i)$. If Φ is an invariant, then $\mathcal{A}_0$ (the only nonzero evectant) coincides with $\mathcal{E}_\Phi$ as defined in Section 1.8.

Lemma 5.1. *With notation as above,*

$$\mathcal{A}_i = \frac{(d+n-2i+1)!}{i!(d+n-i+1)!m} \big\{\Omega_{\mathbf{xy}}^i \circ \big[\mathcal{E}(\mathbf{x}) \circ \Phi(\mathbf{y})\big]\big\}_{\mathbf{y}:=\mathbf{x}.}$$

Proof. Apply $\Omega_{\mathbf{xy}}^\ell$ to each term in (31), and use Lemma 5.2 below. The terms with $\ell > i$ vanish because $\Omega_{\mathbf{xy}} \circ \alpha_{\mathbf{x}}^{d-i} \alpha_{\mathbf{y}}^{n-i} = 0$. Those with $\ell < i$ vanish after we set $\mathbf{y} := \mathbf{x}$, this leaves only the term $\ell = i$. $\square$

5.2. The evectants of a transvectant

Let $\Phi = \varphi_{\mathbf{x}}^n, \Psi = \psi_{\mathbf{x}}^{n'}$ denote two covariants with degree-orders (m, n), (m', n'), and evectants $\mathcal{A}_\bullet$, $\mathcal{B}_\bullet$, respectively. Their rth transvectant $\Theta = (\Phi, \Psi)_r$ is of degree-order $(m+m', n+n'-2r)$. We would like to deduce formulae for the evectants $\mathcal{C}_\bullet$ of Θ in terms of the data Φ, Ψ, $\mathcal{A}_\bullet$, $\mathcal{B}_\bullet$. Let us write

$$\Theta(\mathbf{y}) = \frac{(n-r)!(n'-r)!}{n!n'!} \big\{\Omega_{\mathbf{yz}}^r \circ \big[\Phi(\mathbf{y})\Psi(\mathbf{z})\big]\big\}_{\mathbf{z}:=\mathbf{y}}$$

(if we expand $\Omega_{\mathbf{yz}}^r$ by the binomial theorem, then this reduces to the definition in Section 1.3), and then

$$\mathcal{C}_s = \kappa \big\{\Omega_{\mathbf{xy}}^s \circ \underbrace{\big[\mathcal{E}(\mathbf{x}) \circ \big\{\Omega_{\mathbf{yz}}^r \circ \big[\Phi(\mathbf{y})\Psi(\mathbf{z})\big]\big\}_{\mathbf{z}:=\mathbf{y}}\big]}_{\langle a \rangle}\big\}_{\mathbf{y}:=\mathbf{x},}$$

where

$$\kappa = \frac{(n-r)!(n'-r)!(d+n+n'-2r-2s+1)!}{n!n'!s!(d+n+n'-2r-s+1)!(m+m')}. \tag{33}$$

It is understood that $r \leqslant \min(n, n')$ and $s \leqslant \min(d, n+n'-2r)$.

The operators $\mathcal{E}(\mathbf{x})$ and $\Omega_{\mathbf{yz}}$ commute, since they involve disjoint sets of variables. Hence

$$\langle a\rangle = \Big[\Omega^r_{\mathbf{yz}} \circ \underbrace{\big\{\mathcal{E}(\mathbf{x}) \circ \big[\Phi(\mathbf{y})\Psi(\mathbf{z})\big]\big\}}_{\langle b\rangle}\Big]_{\mathbf{z}:=\mathbf{y}}.$$

By the product rule for differentiation,

$$\langle b\rangle = \underbrace{\big[\mathcal{E}(\mathbf{x}) \circ \Phi(\mathbf{y})\big]\Psi(\mathbf{z})}_{\langle b_1\rangle} + \underbrace{\Phi(\mathbf{y})\big[\mathcal{E}(\mathbf{x}) \circ \Psi(\mathbf{z})\big]}_{\langle b_2\rangle}.$$

5.3. Writing $\mathcal{A}_i = {\alpha_{(i)}}_{\mathbf{x}}^{d+n-2i}$,

$$\langle b_1\rangle = m\sum_i (\mathbf{xy})^i {\alpha_{(i)}}_{\mathbf{x}}^{d-i} {\alpha_{(i)}}_{\mathbf{y}}^{n-i} \psi_{\mathbf{z}}^{n'}. \tag{34}$$

We have to apply $\Omega^r_{\mathbf{yz}}$ to each term in $\langle b_1\rangle$, and then set $\mathbf{z} := \mathbf{y}$. The recipe is best seen combinatorially (also see [9, §3.2.5]). From each summand in (34) we sequentially remove r symbolic factors involving $\mathbf{y}$, and pair them with similarly removed r factors involving $\mathbf{z}$. By pairing a factor of the type $\beta_{\mathbf{y}}$ with one of the type $\gamma_{\mathbf{z}}$, we get a new factor $(\beta\gamma)$.

The $\mathbf{z}$-factors are all necessarily equal to $\psi_{\mathbf{z}}$, on the other hand we may suppose that k of the $\mathbf{y}$-factors are $(\mathbf{xy})$ and the rest $r-k$ are ${\alpha_{(i)}}_{\mathbf{y}}$. It is convenient to see $(\mathbf{xy})$ as $h_{\mathbf{y}}$ with $(h_1, h_2) = (-x_2, x_1)$. Then the pairings produce factors $(h\psi)^k = (-1)^k \psi_{\mathbf{x}}^k$ and $(\alpha_{(i)}\psi)^{r-k}$, respectively. The r copies of $\Omega_{\mathbf{yz}}$ are seen as operating one after the other, so that the temporal sequence of removing the factors needs to be taken into account. At any stage, we may remove an $(\mathbf{xy})$, $\psi_{\mathbf{z}}$ pair or an ${\alpha_{(i)}}_{\mathbf{y}}$, $\psi_{\mathbf{z}}$ pair, hence there are $r!/(k!(r-k)!)$ ways of choosing this sequence. The $\psi_{\mathbf{z}}$ factors which have been removed can be sequentially ordered in $\frac{n'!}{(n'-r)!}$ ways (regarding them as mutually distinguishable), with a similar argument for other factors. This gives the expression

$$\begin{aligned}&\big[\Omega^r_{\mathbf{yz}} \circ \langle b_1\rangle\big]_{\mathbf{z}:=\mathbf{y}}\\ &\quad = m\sum_i\sum_k \lambda(i,k;n,n')\ \underbrace{(\mathbf{xy})^{i-k}(\alpha_{(i)}\psi)^{r-k}{\alpha_{(i)}}_{\mathbf{x}}^{d-i}{\alpha_{(i)}}_{\mathbf{y}}^{n-i-r+k}\psi_{\mathbf{x}}^k\,\psi_{\mathbf{y}}^{n'-r}}_{\langle c\rangle},\end{aligned}$$

where

$$\lambda(i,k;n,n') = (-1)^k \frac{r!}{k!(r-k)!}\,\frac{i!}{(i-k)!}\,\frac{(n-i)!}{(n-i-r+k)!}\,\frac{n'!}{(n'-r)!}. \tag{35}$$

The inner sum is quantified over $\max(0, r-n+i) \leqslant k \leqslant \min(i,r)$, which is exactly the possible range of removals. Our numerical assumptions imply that the range is always nonempty.

The reader who dislikes the combinatorial argument may verify the formula

$$\Omega_{\mathbf{yz}} \circ \alpha_{\mathbf{y}}^p \beta_{\mathbf{z}}^q = pq(\alpha\beta)\alpha_{\mathbf{y}}^{p-1}\beta_{\mathbf{z}}^{q-1}$$

by a direct calculation, and then proceed by induction.

5.4. The next task is to apply $\Omega_{\mathbf{xy}}^s$ to $\langle c\rangle$, and then set $\mathbf{y} := \mathbf{x}$. We need a preliminary lemma which describes how the operator $\Omega_{\mathbf{xy}}$ can be 'cancelled' against a factor of $(\mathbf{xy})$.

Lemma 5.2. *For integers $p, q, \ell, i \geqslant 0$, we have an equality*

$$\big[\Omega_{\mathbf{xy}}^{\ell} \circ (\mathbf{xy})^i a_{\mathbf{x}}^p b_{\mathbf{y}}^q\big]_{\mathbf{y}:=\mathbf{x}} = \begin{cases} \mu(p, q; \ell, i)[\Omega_{\mathbf{xy}}^{\ell-i} \circ a_{\mathbf{x}}^p b_{\mathbf{y}}^q]_{\mathbf{y}:=\mathbf{x}} & \text{if } \ell \geqslant i, \\ 0 & \text{otherwise}, \end{cases}$$

where

$$\mu(p, q; \ell, i) = \frac{\ell!}{(\ell-i)!} \frac{(p+q-\ell+2i+1)!}{(p+q-\ell+i+1)!}. \tag{36}$$

Proof. Let $\mathcal{G}$ denote an arbitrary bihomogeneous form of orders p, q in $\mathbf{x}, \mathbf{y}$, respectively. By straightforward differentiation,

$$\begin{aligned} \Omega_{\mathbf{xy}} \circ (\mathbf{xy})\mathcal{G} &= 2\mathcal{G} + \left(x_1 \frac{\partial \mathcal{G}}{\partial x_1} + x_2 \frac{\partial \mathcal{G}}{\partial x_2}\right) + \left(y_1 \frac{\partial \mathcal{G}}{\partial y_1} + y_2 \frac{\partial \mathcal{G}}{\partial y_2}\right) \\ &\quad + (x_1 y_2 - x_2 y_1)\left(\frac{\partial^2 \mathcal{G}}{\partial x_1 \partial y_2} - \frac{\partial^2 \mathcal{G}}{\partial x_2 \partial y_1}\right) \\ &= (p+q+2)\mathcal{G} + (\mathbf{xy})\Omega_{\mathbf{xy}} \circ \mathcal{G}. \end{aligned}$$

Now proceed by induction on ℓ, i, and observe that terms involving $(\mathbf{xy})$ vanish once we set $\mathbf{y} := \mathbf{x}$. □

Hence

$$\langle d\rangle = \big[\Omega_{\mathbf{xy}}^s \circ \langle c\rangle\big]_{\mathbf{y}:=\mathbf{x}}$$

vanishes if $s < i - k$. Assume $s \geqslant i - k$, then

$$\begin{aligned} \langle d\rangle &= \mu(d-i+k, n+n'-2r-i+k; s, i-k) \\ &\quad \times (\alpha_{(i)}\psi)^{r-k} \underbrace{\big[\Omega_{\mathbf{xy}}^{s-i+k} \circ \alpha_{(i)}{}_{\mathbf{x}}^{d-i} \alpha_{(i)}{}_{\mathbf{y}}^{n-i-r+k} \psi_{\mathbf{x}}^k \psi_{\mathbf{y}}^{n'-r}\big]_{\mathbf{y}:=\mathbf{x}}}_{\langle e\rangle}. \end{aligned} \tag{37}$$

Now $\langle e\rangle$ can be evaluated using the following lemma.

Lemma 5.3. *For integers $p_1, q_1, p_2, q_2, u \geqslant 0$, we have an equality*

$$\big[\Omega_{\mathbf{xy}}^u \circ a_{\mathbf{x}}^{p_1} a_{\mathbf{y}}^{q_1} b_{\mathbf{x}}^{p_2} b_{\mathbf{y}}^{q_2}\big]_{\mathbf{y}:=\mathbf{x}} = \nu(p_1, q_1, p_2, q_2; u) \times (ab)^u a_{\mathbf{x}}^{p_1+q_1-u}\, b_{\mathbf{x}}^{p_2+q_2-u},$$

where ν is given by the sum

$$\sum_t (-1)^{u-t} \frac{u!}{t!(u-t)!} \frac{p_1!}{(p_1-t)!} \frac{q_1!}{(q_1-u+t)!} \frac{p_2!}{(p_2-u+t)!} \frac{q_2!}{(q_2-t)!},$$

quantified over

$$\max\bigl(0, u - \min(q_1, p_2)\bigr) \leqslant t \leqslant \min(p_1, q_2, u).$$

(*The sum is understood to be zero if this range is empty.*)

Proof. This is essentially the same combinatorial argument as before. Note however that since $(aa) = 0$, we cannot pair $a_{\mathbf{x}}$ with $a_{\mathbf{y}}$, and similarly for b. Assume that we have removed respectively $t, u - t, u - t, t$ copies of $a_{\mathbf{x}}, a_{\mathbf{y}}, b_{\mathbf{x}}, b_{\mathbf{y}}$. Then pairings of $a_{\mathbf{x}}, b_{\mathbf{y}}$ produce $(ab)^t$, and those of $b_{\mathbf{x}}, a_{\mathbf{y}}$ produce $(ba)^{u-t} = (-1)^{u-t}(ab)^{u-t}$.

The range of t is exactly such that the removals are possible, e.g., $u - t$ cannot exceed q_1 or p_2, etc. □

It follows that

$$(\alpha_{(i)}\psi)^{r-k}\langle e\rangle = \nu(d-i, n-i-r+k, k, n'-r; s-i+k) \\ \times \underbrace{(\alpha_{(i)}\psi)^{r-i+s}\alpha_{(i)}{}_{\mathbf{x}}^{d+n-r-s-i}\psi_{\mathbf{x}}^{n'-r-s+i}}_{\langle f\rangle},$$

and of course $\langle f\rangle = (\mathcal{A}_i, \Psi)_{r-i+s}$. The calculation for $\langle b_2\rangle$ is essentially the same, hence we are done.

Theorem 5.4. *With notation as above,*

$$\mathcal{C}_s = \sum_{i=0}^{\min(d,n)} \xi_i(\mathcal{A}_i, \Psi)_{r-i+s} + \sum_{i=0}^{\min(d,n')} \eta_i(\mathcal{B}_i, \Phi)_{r-i+s}, \tag{38}$$

where

$$\xi_i = \kappa m \sum_k \bigl\{\lambda(i,k;n,n')\,\mu(d-i+k, n+n'-2r-i+k; s, i-k) \\ \times \nu(d-i, n-i-r+k, k, n'-r; s-i+k)\bigr\},$$

and

$$\eta_i = (-1)^r \kappa m' \sum_k \bigl\{\lambda(i,k;n',n)\,\mu(d-i+k, n+n'-2r-i+k; s, i-k) \\ \times \nu(d-i, n'-i-r+k, k, n-r; s-i+k)\bigr\}.$$

The sums are respectively quantified over

$$\max(0, r-n+i, i-s) \leqslant k \leqslant \min(i, r),$$

$$\max(0, r-n'+i, i-s) \leqslant k \leqslant \min(i, r).$$

5.5. Note the following classical proposition:

Proposition 5.5. *A degree m covariant Φ is a* $\mathbf{Q}$*-linear combination of compound transvectants*

$$\Big(\dots\big((\mathbb{F},\mathbb{F})_{r_1},\mathbb{F}\big)_{r_2},\dots,\mathbb{F}\Big)_{r_{m-1}}.$$

Proof. This is usually proved using the symbolic method, but it is easy to give an alternate proof. Write $\Phi=\sum_i(\mathcal{A}_i,\mathbb{F})_{d-i}$, and use the inductive hypothesis to write each $\mathcal{A}_i$ in terms of compound transvectants. □

The only nonzero evectant of $\Phi=\mathbb{F}$ is $\mathcal{A}_d=1$. Starting from this, in principle we can calculate the evectants of any covariant. I have programmed formula (38) in MAPLE, so that the calculations can be made seamlessly.

Example 5.6. Let $d=5$, $\Phi=(\mathbb{F},\mathbb{F})_2$, $\Psi=(\mathbb{F},\mathbb{F})_4$ and $\Theta=(\Phi,\Psi)_1$. Now Φ,Ψ have only one nonzero evectant each, namely $\mathcal{A}_3=\mathbb{F}$, $\mathcal{B}_1=\mathbb{F}$. Hence Θ has evectants

$$\begin{aligned}
&\mathcal{C}_0=\frac{1}{4}\mathbb{F}\Phi, && \mathcal{C}_1=\frac{2}{11}(\mathbb{F},\Phi)_1,\\
&\mathcal{C}_2=-\frac{1}{4}\mathbb{F}\Psi-\frac{5}{18}(\mathbb{F},\Phi)_2, && \mathcal{C}_3=\frac{2}{7}(\mathbb{F},\Psi)_1-\frac{10}{21}(\mathbb{F},\Phi)_3,\\
&\mathcal{C}_4=\frac{3}{20}(\mathbb{F},\Psi)_2-\frac{17}{56}(\mathbb{F},\Phi)_4, && \mathcal{C}_5=-\frac{2}{21}(\mathbb{F},\Phi)_5.
\end{aligned}$$

In fact $\mathcal{C}_5$ vanishes identically, since quintics have no covariant of degree-order (3, 1).

It is now a routine to calculate the evectants of ϑ_{22}^7 and $\mathbb{F}\vartheta_{39}$, and hence finally $\mathcal{E}_{\mathbb{H}}$. The result is

$$\mathcal{E}_{\mathbb{H}}(\mathcal{F}_Q)=-\frac{2^6}{3.5^{14}}q_2^3\big(q_0q_2+3q_1^2\big)^2\big(5q_0q_2-q_1^2\big)^4\mathcal{F}_{Q'}. \tag{39}$$

Corollary 5.7. *We have an equality* $J=\mathfrak{b}_{(1/6,2/45,-1/3)}$.

Proof. Comparing (26) and (28) with (39), we can read off the values $(r,s,t)=(0,1,3)$. □

It is unnecessary to make six iterations in order to calculate the evectants of ϑ_{22}^7. Instead observe that

$$14\Gamma=\mathcal{E}(\mathbf{x})\circ\vartheta_{22}(\mathbf{y})^7=7\vartheta_{22}(\mathbf{y})^6\underbrace{\mathcal{E}(\mathbf{x})\circ\vartheta_{22}(\mathbf{y})}_{(*)},$$

where $(*)=(\mathbf{xy})f_{\mathbf{x}}^4f_{\mathbf{y}}$. Now one can find the Gordan series of Γ directly.

5.6. Let Φ be a covariant of degree-order (m, n). The covariance property of Φ implies that its coefficients satisfy certain differential equations; this forces some identities between its evectants $\mathcal{A}_\bullet$. We now proceed to make them explicit. Let U denote an arbitrary quadratic form.

Proposition 5.8. *We have an equality*

$$\left(\left(\left[\mathcal{E}(\mathbf{x}) \circ \Phi(\mathbf{y})\right], \mathbb{F}(\mathbf{x})\right)_{d-1}, U(\mathbf{x})\right)_2 = \frac{n}{d}\left\{\left(\Phi(\mathbf{x}), U(\mathbf{x})\right)_1\right\}_{\mathbf{x}:=\mathbf{y}}. \tag{40}$$

By construction $\mathcal{E}(\mathbf{x}) \circ \Phi(\mathbf{y})$ has orders d, n in $\mathbf{x}, \mathbf{y}$. Its $(d-1)$th transvectant with $\mathbb{F}(\mathbf{x})$ has $\mathbf{x}$-order 2, and finally the second transvectant with $U(\mathbf{x})$ has no $\mathbf{x}$-variables remaining. Thus both sides are order n forms in $\mathbf{y}$.

Corollary 5.9. *If Φ is an invariant, then $(\mathcal{E}_\Phi, \mathbb{F})_{d-1} = 0$.*

Proof. The right-hand side of (40) vanishes, hence the second transvectant of an arbitrary quadratic form with $(\mathcal{E}_\Phi, \mathbb{F})_{d-1}$ is zero. This forces the latter to be zero. □

We sketch a proof of the proposition. Since both sides are linear in U, it suffices to check the identity for each of the basis elements $\{x_1^2, x_1x_2, x_2^2\}$. After unravelling the transvectants, we are reduced to the following differential equations known to be satisfied by any covariant (see [9, §1.2.12]):

$$\sum_{i=0}^{d-1}(d-i)a_{i+1}\frac{\partial\Phi}{\partial a_i} = x_1\frac{\partial\Phi}{\partial x_2}, \qquad \sum_{i=1}^{d} i a_{i-1}\frac{\partial\Phi}{\partial a_i} = x_2\frac{\partial\Phi}{\partial x_1},$$

$$\sum_{i=0}^{d}(d-2i)a_i\frac{\partial\Phi}{\partial a_i} = x_1\frac{\partial\Phi}{\partial x_1} - x_2\frac{\partial\Phi}{\partial x_2}. \qquad \Box \tag{41}$$

Broadly speaking, these equations express the fact that Φ is annihilated by the three generators $\left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right), \left(\begin{smallmatrix}0 & 0\\ 1 & 0\end{smallmatrix}\right), \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ of the Lie algebra $\mathfrak{sl}_2$.

5.7. It turns out that one can remove the reference to U from (40) and rephrase it as a set of three identities in the $\mathcal{A}_\bullet$. They will be of the form

$$\sum_i \omega_{i,q}(\mathcal{A}_i, \mathbb{F})_{d-i-1+q} = 0 \quad (q = 0, 1, 2),$$

where $\omega_{i,q}$ are certain rational numbers. The calculations are thematically similar to those we have just seen, so the derivation will only be sketched.

Write $\mathcal{E}(\mathbf{x}) \circ \Phi(\mathbf{y}) = p_\mathbf{x}^d q_\mathbf{y}^n$, $\mathbb{F} = f_\mathbf{x}^d$, $U = u_\mathbf{x}^2$, $\Phi = \varphi_\mathbf{x}^n$. Then the left- and right-hand sides of (40) respectively equal

$$\big(\underbrace{(pf)^{d-1} p_\mathbf{x} f_\mathbf{x} q_\mathbf{y}^n}_{(\star)}, u_\mathbf{x}^2\big)_2, \qquad \left(\underbrace{\frac{n}{d}(\mathbf{xy})\varphi_\mathbf{x}\varphi_\mathbf{y}^{n-1}}_{(\star\star)}, u_\mathbf{x}^2\right)_2.$$

The second transvectant of $Z = (\star) - (\star\star)$ with an arbitrary U is zero, so Z itself must be zero. Now substitute the sum $m \sum_i (\mathbf{xy})^i \alpha_{(i)}{}_{\mathbf{x}}^{d-i} \alpha_{(i)}{}_{\mathbf{y}}^{n-i}$ for $\mathcal{E}(\mathbf{x}) \circ \Phi(\mathbf{y})$, and expand Z into its Gordan series. It is of the form

$$Z = \mathbb{T}_0 + (\mathbf{xy})\mathbb{T}_1 + (\mathbf{xy})^2\mathbb{T}_2,$$

where $\mathbb{T}_q$ are of orders $2-q, n-q$ in $\mathbf{x}, \mathbf{y}$. We get the required identities by writing $\mathbb{T}_q|_{\mathbf{y}:=\mathbf{x}} = 0$. Although *a priori* $\mathbb{T}_1$ involves Φ, we can rewrite the latter in terms of $\mathcal{A}_\bullet$ using (32). In conclusion we have the following theorem:

Theorem 5.10. *With notation as above,*

$$\sum_{i=0}^{\min(d,n)} \omega_{i,q} (\mathcal{A}_i, \mathbb{F})_{d-i-1+q} = 0 \quad (q = 0, 1, 2), \tag{42}$$

where

$$\omega_{i,0} = d - i, \qquad \omega_{i,1} = \frac{(d-i)(2i-n)}{d(n+2)} + \frac{mi-n}{md},$$
$$\omega_{i,2} = i(n-i)(d-i+n+1).$$

These identities are collectively equivalent to (41), but in contrast to the latter, each of them is individually invariant under a change of coordinates.

If a sequence of covariants $\{\mathcal{A}_i\}$ of degree-orders $(m, d+n-2i)$ is to appear as the sequence of evectants of some Φ, it is necessary that they satisfy the identities above. It would be of interest to find a set of necessary and sufficient conditions.

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