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On the structure of the fiber cone of ideals with analytic spread one

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Abstract

For a given a local ring $(A, \mathfrak{m})$, we study the fiber cone of ideals in A with analytic spread one. In this case, the fiber cone has a structure as a module over its Noether normalization which is a polynomial ring in one variable over the residue field. One may then apply the structure theorem for modules over a principal domain to get a complete description of the fiber cone as a module. We analyze this structure in order to study and characterize in terms of the ideal itself the arithmetical properties and other numerical invariants of the fiber cone as multiplicity, reduction number or Castelnuovo–Mumford regularity.

Keywords: Fiber cone; Cohen–Macaulay; Buchsbaum; Gorenstein; Multiplicity; Castelnuovo–Mumford regularity

1. Introduction

Let $(A, \mathfrak{m})$ be a Noetherian local ring and let I be an ideal of A. The *fiber cone of I* (or the special fiber of the Rees algebra $A[It]$) is the ring

$$F(I) = \bigoplus_{n \geqslant 0} I^n/\mathfrak{m}I^n \cong A[It] \otimes_A A/\mathfrak{m}.$$

Its Krull dimension is called the *analytic spread of I* and we will denote it by $l(I)$.

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An ideal $J \subseteq I$ is called a *reduction of* I if there exists an integer n such that $I^{n+1} = JI^n$. Phrased otherwise, J is a reduction of I if

$$A[Jt] \hookrightarrow A[It]$$

is a finite morphism of graded algebras. Equivalently, it is known that J is a reduction of I if and only if I is integral over J.

A reduction J of I is a minimal reduction if J is minimal with respect to inclusion among reductions of I. By Northcott and Rees [30] minimal reductions always exist. Let J be a reduction of I and assume in addition that the residue field of A is infinite. Then, J is a minimal reduction of I if, and only if, J is minimally generated by $l(I)$ elements if, and only if, J is generated by a family of analytically independent elements in I. Therefore, given J a minimal reduction of I, the ring $F(J)$ is isomorphic to a polynomial ring in $l(I)$ variables over $A/\mathfrak{m}$ and the equalities $\mathfrak{m}I^n \cap J^n = \mathfrak{m}J^n$ are satisfied for all n. That is, the graded morphism

$$F(J) \hookrightarrow F(I)$$

is a Noether normalization.

For $a \in I$, we will denote by a^0 the class of a in $I/\mathfrak{m}I$. Minimal reductions also provide homogeneous systems of parameters of $F(I)$. Concretely, if the residue field of A is infinite, a family of elements $a_1, \dots, a_l \in I$ is a minimal set of generators of a minimal reduction of I if and only if $a_1^0, \dots, a_l^0$ is a homogeneous system of parameters of $F(I)$.

Assume now that the residue field is infinite and $l(I) = 1$. If $J = (a)$ is a minimal reduction of I, then $F(J)$ is isomorphic to a polynomial ring in one variable over $A/\mathfrak{m}$ and $F(I)$ is a graded finite module over $F(J)$. So we may apply the structure theorem of finitely generated graded modules over a principal ideal graded domain to get a set of invariants describing the precise structure of $F(I)$ as $F(J)$-module.

Our purpose in this paper is to analyze in detail the information provided by this set of invariants in order to study the properties of fiber cones of dimension one. In particular, the Cohen–Macaulay, Gorenstein or Buchsbaum properties, and other numerical information such as Castelnuovo–Mumford regularity, multiplicity, Hilbert function, reduction number or postulation number. As we will see, although the structure of $F(I)$ as $F(J)$-module is less rich than the structure of $F(I)$ as $F(J)$-algebra, it suffices in this case to characterize all the above properties in terms of the ideal itself.

The fiber cone of an ideal I is one of the so-called *blow up algebras of* I and its *Proj* represents the fiber of the maximal ideal $\mathfrak{m}$ by the blow up of A with center I. Moreover, it provides interesting information about the ideal itself: The Hilbert function of the fiber cone describes the minimal number of generators of the powers of I and, when the residue field is infinite, its dimension coincides with the minimal number of generators of any minimal reduction of I. For the maximal ideal itself, the fiber cone coincides with the associated graded ring, and so in this particular situation it has been extensively studied, the case of analytic spread one being the tangent cones of curve singularities. But for a general ideal, the properties of the fiber cone are much less known. Nevertheless, in recent years some effort has been done by several authors in order to understand its behavior.

With respect to the arithmetical properties of the fiber cone, one of the first known results was given by Huneke and Sally [25] who proved that, if A is Cohen–Macaulay, the fiber cone of any $\mathfrak{m}$-primary ideal of reduction number one is Cohen–Macaulay. This result was later extended by

K. Shah [33,34] to equimultiple ideals of reduction number one, giving also some conditions for the Cohen–Macaulayness of the fiber cone of equimultiple ideals of reduction number two. Subsequent results by Cortadellas and Zarzuela [6,7], D'Cruz, Raghavan and Verma [12], and D'Cruz and Verma [13] completed the results of Shah for more general families of ideals. Also, the fiber cone of the defining ideal of a monomial curve in $\mathbb{P}^3$ lying on a quadric was proven to be Cohen–Macaulay by Morales and Simis [29]. This result was later extended by P. Gimenez [16] and Barile and Morales [1] to the defining ideal of a projective monomial variety of codimension two.

On the other hand, motivated by work of R. Hübl [20], Hübl and Huneke [21] studied the Cohen–Macaulay property of the fiber cone of special ideals in connection with the theory of evolutions introduced by Eisenbud and Mazur [15], which is related to A. Wiles's work on Fermat's Last Theorem [39]. Hübl and Swanson [22] have also made some concrete computations on fiber cones in this context. More recent work concerning the properties of the fiber cones (multiplicity, Hilbert function, Cohen–Macaulayness, Gorensteiness, depth...) has been done by Corso, Ghezzi, Polini and Ulrich [4], Corso, Polini and Vasconcelos [3], T. Cortadellas [5], D'Cruz and Puthenpurakal [11], Heinzer and Kim [18], Heinzer, Kim and Ulrich [19], Jayanthan and Verma [26,27], Jayanthan, Puthenpurakal and Verma [28], or D.Q. Viêt [38] and others.

The case of ideals having a principal reduction has also been considered in some detail by several authors. S. Huckaba [23] studied the reduction number and observed that, for a regular ideal of analytic spread one, the reduction number does not depend on the minimal reduction. And more recently, D'Anna, Guerrieri and Heinzer [8,9] have also considered several aspects of these ideals, such as their fiber cone, the relation type or the Ratliff–Rush closure. On the other hand, one can also find the case of analytic spread one ideals in induction arguments, such as the so-called Sally machine for fiber cones, see Jayanthan and Verma [27].

Next, we briefly explain the content and structure of this paper. In Section 2 we describe the structure of $F(I)$ as $F(J)$-module, introducing the set of invariants provided by this structure. We relate them to several other numerical invariants of the ideal such as reduction number or minimal number of generators, and of the fiber cone such as multiplicity, regularity or postulation number. Then, we give some formulas which allow to compute this set of invariants in terms of lengths of annihilator ideals. In particular, we prove the invariance with respect to the chosen reduction J of a distinguished subset of this set of invariants. Section 3 is devoted to the study of the Gorenstein, Cohen–Macaulay and Buchsbaum properties of the fiber cone; We give several characterizations of all these properties, both in terms of the set of invariants coming form the structure of $F(I)$ as $F(J)$-module and the corresponding lengths of annihilator ideals introduced in the previous section. We point out that the Buchsbaum property of the fiber cone is equivalent to the fact that its structure as a module over $F(J)$ is independent of the chosen minimal reduction J. In Section 4 we give some applications and explicit examples, which explain the results obtained in Sections 2 and 3. In particular, we get that the fiber cone of any regular ideal with analytic spread one and reduction number two is Buchsbaum, and give examples showing that this is no more true for reduction number three. Finally, in Section 5 we use induction arguments to extend some of the previous results to ideals of higher analytic spread, recovering several known results for which we give an alternative and easier proof.

Throughout this paper we will assume that $(A, \mathfrak{m})$ is a local Noetherian ring with an infinite residue field. For all unexplained terminology one may use Bruns and Herzog [10].

2. The structure of $F(I)$ as $F(J)$-module

Let I be an ideal of A with analytic spread $l(I)=l$. Let $J \subseteq I$ be a minimal reduction. Then, the least integer r such that $I^{r+1}=JI^r$ is the *reduction number* of I with respect to J and it is denoted by $r_J(I)$. Let $Y_1,\dots,Y_s$ be a minimal set of homogeneous generators of $F(I)$ as $F(J)$-module. Then, by lifting the equality $F(I)=\sum F(J)Y_i$ to $A[It]$ and by Nakayama's lemma one gets that

$$r_J(I)=\max\bigl\{\deg(Y_i),\, 1\leqslant i\leqslant s\bigr\}.$$

Recall that given a finitely generated graded module M over a polynomial ring $k[x_1,\dots,x_n]$ over a field k and a minimal graded free resolution of M

$$0\to F_s\to\cdots\to F_1\to F_0\to M\to 0,$$

the *Castelnuovo–Mumford regularity* of M is the number

$$\operatorname{reg}(M):=\max\bigl\{b_i(M)-i \bigm| i=0,\dots,s\bigr\},$$

where $b_i(M)$ denotes the maximum of the degrees of the generators of F_i.

More in general, let $S=\bigoplus_{n\geqslant 0}S_n$ be a finitely generated standard graded algebra over a Noetherian commutative ring S_0 and let $S_+=\bigoplus_{n>0}S_n$ be the irrelevant ideal of S. Given $M=\bigoplus_{n\in\mathbb{Z}}M_n$ a finitely generated graded S-module, let $H^i_{S_+}(M)$ be the ith graded local cohomology module of M with respect to S_+. For any graded S-module N we consider

$$\operatorname{end}(N):=\begin{cases}\sup\{n\mid N_n\neq 0\} & \text{if } N\neq 0,\\ -\infty & \text{if } N=0\end{cases}$$

and denote by $a_i(M):=\operatorname{end}(H^i_{S+}(M))$. Then, the *Castelnuovo–Mumford regularity* of M is the number

$$\operatorname{reg}_S(M):=\max\bigl\{a_i(M)+i \bigm| i\geqslant 0\bigr\}.$$

It is well known that this definition extends the classical definition of the Castelnuovo–Mumford regularity for modules over a polynomial ring.

Observe that since $\operatorname{rad}(F(J)_+F(I))=F(I)_+$, then for every graded $F(I)$-module M one has $H^i_{F(I)_+}(M)=H^i_{F(J)_+}(M)$ and so

$$\operatorname{reg}\bigl(F(I)\bigr):=\operatorname{reg}_{F(I)}\bigl(F(I)\bigr)=\operatorname{reg}_{F(J)}\bigl(F(I)\bigr).$$

Let S be a graded standard algebra over a field k. We shall denote the *length* of a graded S-module M by $\lambda(M)=\lambda_S(M)=\lambda_k(M)=\sum\lambda_k(M_n)$. Then, $\lambda(F(I))=\lambda_{F(I)}(F(I))=\lambda_{F(J)}(F(I))=\sum\lambda_{A/\mathfrak{m}}(I^n/\mathfrak{m}I^n)$. Let $H(F(I),n)=\lambda_{A/\mathfrak{m}}(I^n/\mathfrak{m}I^n)=\mu(I^n)$ be the *Hilbert function* of $F(I)$. Then, $H(F(I),n)$ is of polynomial type of degree $l-1$. The unique polynomial $P_{F(I)}(x)\in\mathbb{Q}[x]$ for which $H(F(I),n)=P_{F(I)}(n)$ for all n large enough is the *Hilbert polynomial* of $F(I)$ and has the form

$$P_{F(I)}(x)=\sum_{i=0}^{l-1}(-1)^{l-1-i}e_{l-1-i}\binom{x+i}{i}.$$

The *multiplicity* of $F(I)$ is defined as

$$e\big(F(I)\big) = \begin{cases} e_0 & \text{if } l > 0, \\ \lambda(F(I)) & \text{if } l = 0 \end{cases}$$

and the *fiber postulation number* $\mathrm{fp}(I)$ of I as the largest integer n such that $P_{F(I)}(n) \neq H(F(I), n) = \mu(I^n)$.

Let $H_{F(I)}(x) = \sum_{n \geqslant 0} \mu(I^n)x^n$ the *Hilbert series* of $F(I)$. Then

$$H_{F(I)}(x) = \frac{Q_{F(I)}(x)}{(1-x)^l}$$

for an unique $Q_{F(I)}(x) \in \mathbb{Z}[x, x^{-1}]$ and $Q_{F(I)}(1) = e(F(I))$.

Assume now that $l(I) = 1$ and let $J = (a)$ be a minimal reduction of I. Then, $F(J)$ is isomorphic to a polynomial ring in one variable over the residue field and so a graded principal ideal domain. In this way, we can consider the graded decomposition of $F(I)$ as direct sum of cyclic graded $F(J)$-modules, see also W.V. Vasconcelos [37, 9.3],

$$F(I) \simeq \bigoplus_{i=1}^{e} F(J)(-b_i) \bigoplus_{j=1}^{f} \big(F(J)/a^{c_j} F(J)\big)(-d_j) \tag{$*$}$$

where we may assume $b_1 \leqslant \cdots \leqslant b_e$, $d_1 \leqslant \cdots \leqslant d_f$. In particular one immediately has

$$H_{F(I)}(x) = \frac{x^{b_1} + \cdots + x^{b_e} + (1 - x^{c_1})x^{d_1} + \cdots + (1 - x^{c_f})x^{d_f}}{1-x}.$$

Moreover, in this case the Hilbert polynomial $P_{F(I)}(x) = e(F(I))$ is a constant. As a consequence, for these ideals we have that $F(I)$ (as a $F(J)$-module) satisfies

$$\begin{gathered} \mu_{F(J)}\big(F(I)\big) = e + f, \\ r_J(I) = \max\{b_e, d_f\}, \\ \operatorname{reg}\big(F(I)\big) = \max\{b_e, c_j + d_j - 1\}, \\ e\big(F(I)\big) = e. \end{gathered}$$

Assume moreover that I contains a regular element: These ideals are usually called *regular ideals*. One immediately gets that a must be a regular element. Put $r := r_J(I)$. If $n \geqslant r$ then

$$I^n/\mathfrak{m}I^n = a^{n-r}I^r/a^{n-r}\mathfrak{m}I^r \cong I^r/\mathfrak{m}I^r$$

and $\mu(I^n) = \mu(I^r)$. So, the postulation number $\mathrm{fp}(I) \leqslant r - 1$ and the Hilbert series is in this case

$$H_{F(I)}(x) = \sum_{n \geqslant 0} \mu\big(I^n\big)x^n = \frac{1 + (\mu(I) - 1)x + \cdots + (\mu(I^r) - \mu(I^{r-1}))x^r}{1-x}.$$

Comparing both expressions of the Hilbert series it follows that

$$c_j + d_j \leqslant r$$

and so

$$d_f \leqslant r - 1.$$

In particular,

$$r_J(I) = b_e.$$

Now, for regular ideals with analytic spread one we have

$$\begin{aligned} e\big(F(I)\big) &= \mu\big(I^r\big) = e, \\ \operatorname{reg}\big(F(I)\big) &= r_J(I) = b_e, \\ \mu_{F(J)}\big(F(I)\big) &= \mu\big(I^r\big) + f. \end{aligned}$$

Observe that, in this case, the reduction number $r_J(I)$ turns out to be independent of the chosen minimal reduction J, as was already noted by S. Huckaba in [23]. Also that

$$\begin{aligned} \mu_{F(J)}\big(F(I)\big) &= \dim_{F(J)/F(J)_+}\big(F(I)/\big(F(J)_+F(I)\big)\big) \\ &= \sum_{n=0}^{r} \lambda_{A/\mathfrak{m}}\big(I^n/\mathfrak{m}I^n + JI^{n-1}\big). \end{aligned}$$

In order to make a deeper analysis of the decomposition of $F(I)$ as $F(J)$-module we can rewrite it in the form

$$F(I) \cong \bigoplus_{i=0}^{r}\big(F(J)(-i)\big)^{\alpha_i} \bigoplus_{i=1}^{r-1}\bigoplus_{j=1}^{r-i}\big(\big(F(J)/a^jF(J)\big)(-i)\big)^{\alpha_{i,j}}. \qquad (**)$$

Note that $\alpha_0 = 1$ and $\alpha_r \neq 0$ since $r =$ the biggest possible degree among the generators of $F(I)$ as a graded $F(J)$-module. Also that

$$f = \sum_{\substack{1 \leqslant i \leqslant r-1 \\ 1 \leqslant j \leqslant r-i}} \alpha_{i,j}.$$

From now on we shall denote by $T(F(I))$ the $F(J)$-torsion submodule of $F(I)$ and assume that I is a regular ideal. Observe that $T(F(I)) = 0$ if $r(I) = 0, 1$ and so in both cases $F(I)$ is a Cohen–Macaulay ring.

Lemma 1. *Let k, l and n be natural numbers. Then*:

(1) $(\mathfrak{m}I^{k+l} : a^l) = \mathfrak{m}I^k$, *for $k = 0$ or $k \geqslant r$.*
(2) $(\mathfrak{m}I^{k+1} : a) \subseteq \cdots \subseteq (\mathfrak{m}I^r : a^{r-k}) = (\mathfrak{m}I^{r+n} : a^{r-k+n})$, *for $k = 1, \ldots, r-1$.*

Proof. For $k=0$ we have $(\mathfrak{m}I^l : a^l) = \mathfrak{m}$ since a is analytically independent on I. Now, let $k \geqslant r$ and x be an element such that $xa^l \in \mathfrak{m}I^{k+l} = a^l \mathfrak{m}I^k$. Then $x \in \mathfrak{m}I^k$ since a^l is a nonzero divisor in A.

The only nontrivial inclusion in (2) is $(\mathfrak{m}I^{r+n} : a^{r-k+n}) \subseteq (\mathfrak{m}I^r : a^{r-k})$. Let x be an element such that $xa^{r-k+n} \in \mathfrak{m}I^{r+n} = a^n \mathfrak{m}I^r$. Then $xa^{r-k}a^n \in a^n \mathfrak{m}I^r$ and now, the regularity of a^n gives that $xa^{r-k} \in \mathfrak{m}I^r$. $\square$

Proposition 2. *We have*

$$T\big(F(I)\big) = H^0_{F(I)_+}\big(F(I)\big) = \big(0 :_{F(I)} \big(a^0\big)^{r-1}\big) = \bigoplus_{k=1}^{r-1} \big(I^k \cap \big(\mathfrak{m}I^r : a^{r-k}\big)\big)/\mathfrak{m}I^k.$$

Proof. We have that

$$T\big(F(I)\big) = H^0_{F(J)_+}\big(F(I)\big) = H^0_{F(I)_+}\big(F(I)\big).$$

On the other hand, $H^0_{F(J)_+}(F(I)) = \bigcup_{l \geqslant 0}(0 :_{F(I)} (a^0)^l) = \bigcup_{l \geqslant r-1}(0 :_{F(I)} (a^0)^l)$. Thus, by the above lemma we get

$$\begin{aligned}
\big(0 :_{F(I)} \big(a^0\big)^l\big) &= \bigoplus_{k \geqslant 0} \big(I^k \cap \big(\mathfrak{m}I^{l+k} : a^l\big)\big)/\mathfrak{m}I^k \\
&= \bigoplus_{k=1}^{r-1} \big(I^k \cap \big(\mathfrak{m}I^{l+k} : a^l\big)\big)/\mathfrak{m}I^k \\
&= \bigoplus_{k=1}^{r-1} \big(I^k \cap \big(\mathfrak{m}I^{r+(l+k-r)} : a^{r-k+(l+k-r)}\big)\big)/\mathfrak{m}I^k \\
&= \bigoplus_{k=1}^{r-1} \big(I^k \cap \big(\mathfrak{m}I^r : a^{r-k}\big)\big)/\mathfrak{m}I^k
\end{aligned}$$

for all $l \geqslant r-1$. In particular,

$$H^0_{F(J)_+}\big(F(I)\big) = \big(0 :_{F(I)} \big(a^0\big)^{r-1}\big) = \bigoplus_{k=1}^{r-1} \big(I^k \cap \big(\mathfrak{m}I^r : a^{r-k}\big)\big)/\mathfrak{m}I^k. \qquad \square$$

Given the two decompositions of the torsion of $F(I)$

$$T\big(F(I)\big) = \bigoplus_{k=1}^{r-1} \big(I^k \cap \big(\mathfrak{m}I^r : a^{r-k}\big)\big)/\mathfrak{m}I^k \cong \bigoplus_{i=1}^{r-1} \bigoplus_{j=1}^{r-i} \big(\big(F(J)/a^j F(J)\big)(-i)\big)^{\alpha_{i,j}},$$

we will consider the numbers

$$f_{k,l} := \lambda\big(\big(I^k \cap \big(\mathfrak{m}I^{k+l} : a^l\big)\big)/\mathfrak{m}I^k\big).$$

Then, it is clear that $f_{k,1} \leqslant \cdots \leqslant f_{k,r-k} = \lambda([T(F(I))]_k)$ and $\lambda(T(F(I))) = \sum_{k=1}^{r-1} f_{k,r-k}$. Also, that the extremal numbers $f_{k,r-k}$ are independent of the chosen minimal reduction J. In addition, note that being a a nonzero divisor in A one has isomorphisms

$$\bigl(I^k \cap \bigl(\mathfrak{m}I^{k+l} : a^l\bigr)\bigr)/\mathfrak{m}I^k \cong \bigl(a^l I^k \cap \mathfrak{m}I^{k+l}\bigr)/a^l\mathfrak{m}I^k.$$

If Y is an homogeneous element of $F(I)$ of degree n we will denote by y an element of A such that $Y = y^0 \in I^n/\mathfrak{m}I^n \hookrightarrow F(I)$.

Proposition 3. *For* $1 \leqslant k \leqslant r-1$ *and* $1 \leqslant l \leqslant r-k$ *we have*

$$f_{k,l} = \sum_{(i,j)\in\Lambda} \alpha_{i,j},$$

where $\Lambda := \{(i,j) \mid 1 \leqslant i \leqslant k,\ k-i+1 \leqslant j \leqslant k-i+l\}$.

Proof. Let $\{Y_1^{i,j}, \ldots, Y_{\alpha_{i,j}}^{i,j}\}_{\{1\leqslant i\leqslant r-1,\,1\leqslant j\leqslant r-i\}}$ be a minimal system of homogeneous generators of $T(F(I))$. That is,

$$T\bigl(F(I)\bigr) = \bigoplus_{i=1}^{r-1}\bigoplus_{j=1}^{r-i}\bigl(F(J)Y_1^{i,j} \oplus \cdots \oplus F(J)Y_{\alpha_{i,j}}^{i,j}\bigr)$$

with $F(J)Y_*^{i,j} \cong (F(J)/a^j F(J))(-i)$. So,

$$\bigl[F(J)Y_*^{i,j}\bigr]_k = \begin{cases} ((a^{k-i}y_*^{i,j}) + \mathfrak{m}I^k)/\mathfrak{m}I^k \cong A/\mathfrak{m} & \text{for } i \leqslant k \leqslant i+j-1,\\ 0 & \text{otherwise.}\end{cases}$$

Now, fixed k, we have $[F(J)Y_*^{i,j}]_k \neq 0$ for $i \leqslant k$ and $j \geqslant k-i+1$. Moreover, $a^l a^{k-i} y_*^{i,j} \in \mathfrak{m}I^{k+l}$ if, and only if, $l+k-i \geqslant j$. Therefore, we may conclude $f_{k,l} := \lambda((I^k \cap (\mathfrak{m}I^{k+l} : a^l))/\mathfrak{m}I^k) = \sum_{(i,j)\in\Lambda}\alpha_{i,j}$. □

Corollary 4.

$$f = \sum_{\substack{1\leqslant i\leqslant r-1\\ 1\leqslant j\leqslant r-i}} \alpha_{i,j} = \sum_{k=1}^{r-1} f_{k,1} = \sum_{k=1}^{r-1}\lambda\bigl(\bigl(aI^k \cap \mathfrak{m}I^{k+1}\bigr)/a\mathfrak{m}I^k\bigr),$$

and $\lambda(F(I)/aF(I)) = \mu(I^r) + \sum_{k=1}^{r-1}\lambda((aI^k \cap \mathfrak{m}I^{k+1})/a\mathfrak{m}I^k)$.

Remark 5. The invariants $\alpha_{i,j}$ are univocally related by the $f_{k,l}$ (and vice versa): In fact, if we write

$$\alpha = (\alpha_{1,1}, \ldots, \alpha_{1,r-1}, \alpha_{2,1}, \ldots, \alpha_{r-1,1}),$$

$$F = (f_{1,1}, \ldots, f_{1,r-1}, f_{2,1}, \ldots, f_{r-1,1}),$$

it is then easy to see that there exists by Proposition 3 an invertible inferior triangular matrix $B \in M_{r(r-1)/2}$ such that $F = B\alpha$.

Remark 6. Observe that $f_{k,l} = 0$ if $k \notin \{1, \ldots, r-1\}$.

We consider now the free part of $F(I)$ as $F(J)$-module:

$$F(I)/T\big(F(I)\big) = A/\mathfrak{m} \bigoplus_{i=1}^{r-1} I^i/\big(I^i \cap \big(\mathfrak{m}I^r : a^{r-i}\big)\big) \bigoplus_{n \geqslant r} I^n/\mathfrak{m}I^n$$
$$\cong \bigoplus_{i=0}^{r} F(J)(-i)^{\alpha_i}.$$

By convention, we put $\mu(I^0) = 1$ and $\mu(I^n) = 0$ if $n < 0$ for the rest of the paper.

Proposition 7. *For $1 \leqslant i \leqslant r$ we have*

$$\alpha_i = \mu\big(I^i\big) - \mu\big(I^{i-1}\big) - \big(f_{i,r-i} - f_{i-1,r-(i-1)}\big).$$

Proof. Put

$$\alpha_i' := \lambda\big(I^i/\big(I^i \cap \big(\mathfrak{m}I^r : a^{r-i}\big) + aI^{i-1}\big)\big), \qquad \alpha_r' := \lambda\big(I^r/\big(\mathfrak{m}I^r + aI^{r-1}\big)\big),$$

for $1 \leqslant i \leqslant r-1$, and let

$$\Omega = \big\{1, \big\{y_{i,1}^0, \ldots, y_{i,\alpha_i'}^0\big\}\big\}$$

with $y_{i,j}^0 \in [F(I)/T(F(I))]_i$, for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant \alpha_i'$, such that

$$\overline{y_{i,1}^0}, \ldots, \overline{y_{i,\alpha_i'}^0} \in I^i/\big(I^i \cap \big(\mathfrak{m}I^r : a^{r-i}\big) + aI^{i-1}\big), \qquad \overline{y_{r,1}^0}, \ldots, \overline{y_{r,\alpha_r'}^0} \in I^r/\big(\mathfrak{m}I^r + aI^{r-1}\big)$$

form an $A/\mathfrak{m}$-basis. Then, Ω is a system of homogeneous generators of $F(I)/T(F(I))$ as $F(J)$-module.

On the other hand, for $1 \leqslant i \leqslant r-1$ we have the exact sequences

$$0 \to aI^{i-1}/\big(aI^{i-1} \cap \big(\mathfrak{m}I^r : a^{r-i}\big)\big) \to I^i/\big(I^i \cap \big(\mathfrak{m}I^r : a^{r-i}\big)\big)$$
$$\to I^i/\big(I^i \cap \big(\mathfrak{m}I^r : a^{r-i}\big) + aI^{i-1}\big) \to 0,$$

and

$$0 \to \big(I^i \cap \big(\mathfrak{m}I^r : a^{r-i}\big)\big)/\mathfrak{m}I^i \to I^i/\mathfrak{m}I^i \to I^i/\big(I^i \cap \big(\mathfrak{m}I^r : a^{r-i}\big)\big) \to 0,$$

and isomorphisms

$$aI^{i-1}/\big(aI^{i-1}\cap\big(\mathfrak{m}I^r : a^{r-i}\big)\big)\cong I^{i-1}/\big(I^{i-1}\cap\big(\mathfrak{m}I^r : a^{r-i+1}\big)\big).$$

From them we obtain $\alpha'_i=\mu(I^i)-f_{i,r-i}-\mu(I^{i-1})+f_{i-1,r-(i-1)}$, for $2\leqslant i\leqslant r-1$, and $\alpha'_1=\mu(I)-f_{1,r-1}-1$.

Also we have the exact sequence

$$0\to aI^{r-1}/\big(aI^{r-1}\cap\mathfrak{m}I^r\big)\to I^r/\mathfrak{m}I^r\to I^r/\big(\mathfrak{m}I^r+aI^{r-1}\big)\to 0$$

which gives $\alpha'_r=\mu(I^r)-\mu(I^{r-1})+f_{r-1,1}$.

Now, $1+\sum_{i=1}^r\alpha'_i=\mu(I^r)=\lambda(F(I)/T(F(I)))$ and hence Ω is a basis of the free $F(J)$-module $F(I)/T(F(I))$. As a consequence, $\alpha'_i=\alpha_i$ for all $1\leqslant i\leqslant r$. $\square$

Since $\alpha_r\neq 0$, the following corollary extends the invariance of the reduction number with respect to the chosen minimal reduction.

Corollary 8. *The invariants α_i, for $0\leqslant i\leqslant r$, are independent of the choice of the minimal reduction.*

Proof. The assertion follows from Proposition 7 and Proposition 2. $\square$

We finish this section with a lemma expressing the difference between the minimal number of generators of the powers of I in terms of certain lengths involving minimal reductions.

Lemma 9. *For all n and $0\leqslant i\leqslant n-1$ we have*

$$\lambda\big(I^n/\big(\mathfrak{m}I^n+a^{n-i}I^i\big)\big)=\mu\big(I^n\big)-\mu\big(I^i\big)+\lambda\big(\big(a^{n-i}I^i\cap\mathfrak{m}I^n\big)/a^{n-i}\mathfrak{m}I^i\big).$$

In particular,

$$\lambda\big(I^n/\big(\mathfrak{m}I^n+aI^{n-1}\big)\big)=\mu\big(I^n\big)-\mu\big(I^{n-1}\big)+\lambda\big(\big(aI^{n-1}\cap\mathfrak{m}I^n\big)/a\mathfrak{m}I^{n-1}\big).$$

Proof. The exact sequence

$$0\to\big(a^n\big)/\big(\mathfrak{m}I^n\cap\big(a^n\big)\big)\to I^n/\mathfrak{m}I^n\to I^n/\big(\mathfrak{m}I^n+\big(a^n\big)\big)\to 0,$$

and the equality $\mathfrak{m}I^n\cap(a^n)=a^n\mathfrak{m}$ gives $\lambda(I^n/(\mathfrak{m}I^n+(a^n)))=\mu(I^n)-1$.

For $1\leqslant i\leqslant n-1$ we consider the exact sequences

$$0\to a^{n-i}I^i/\big(a^{n-i}I^i\cap\mathfrak{m}I^n\big)\to I^n/\mathfrak{m}I^n\to I^n/\big(\mathfrak{m}I^n+a^{n-i}I^i\big)\to 0,$$
$$0\to\big(a^{n-i}I^i\cap\mathfrak{m}I^n\big)/a^{n-i}\mathfrak{m}I^i\to a^{n-i}I^i/a^{n-i}\mathfrak{m}I^i\to a^{n-i}I^i/\big(a^{n-i}I^i\cap\mathfrak{m}I^n\big)\to 0$$

and the isomorphism

$$a^{n-i}I^i/a^{n-i}\mathfrak{m}I^i\cong I^i/\mathfrak{m}I^i.$$

Then, the result follows from the additivity of $\lambda(\cdot)$. $\square$

3. Buchsbaum, Cohen–Macaulay and Gorenstein properties

Let I be a regular ideal of A with analytic spread one and reduction number r and $J = (a) \subseteq I$ be a minimal reduction. Consider the *Hilbert–Samuel* function

$$HS\big(F(I), n\big) = \lambda\big(F(I)/a^{n+1}F(I)\big)$$

of $aF(I)$ with respect to $F(I)$. Then, $HS(F(I), n)$ is of polynomial type of degree one and has the form

$$e_0\big(aF(I), F(I)\big)(n+1) + e_1\big(aF(I), F(I)\big)$$

for n big enough. We shall write

$$e\big(aF(I), F(I)\big) := e_0\big(aF(I), F(I)\big).$$

In the following remark we consider in our case several well-known characterizations of the Buchsbaum property, see for instance Stückrad–Vogel [35].

Remark 10. The following conditions are equivalent:

(1) $F(I)$ is a Buchsbaum ring.
(2) $(0 :_{F(I)} F(I)_+) = (0 :_{F(I)} a^0)$ for any $(a) \subseteq I$ minimal reduction of I.
(3) There exists a natural number C such that any $(a) \subseteq I$ minimal reduction of I satisfies

$$C = \lambda\big(F(I)/aF(I)\big) - e\big(aF(I), F(I)\big).$$

(4) $F(I)_+ \cdot H^0_{F(I)_+}(F(I)) = 0$.
(5) $(0 : a^0) = (0 : (a^0)^2)$ for any minimal reduction (a) of I.

In this case, $C = \lambda(T(F(I)))$.

Lemma 11. *We have*

(1) $\lambda(F(I)/aF(I)) = \mu(I^r) + \sum_{n=1}^{r-1} \lambda((aI^n \cap \mathfrak{m}I^{n+1})/a\mathfrak{m}I^n)$.
(2) $\lambda(F(I)/a^{n+1}F(I)) = \mu(I^r)(n+1) + \sum_{k=1}^{r-1} \lambda((a^{r-k}I^k \cap \mathfrak{m}I^r)/a^{r-k}\mathfrak{m}I^k)$ *for all* $n \geqslant r$.
(3) $e(aF(I), F(I)) = \mu(I^r)$.
(4) $\lambda(F(I)/aF(I)) - e(aF(I), F(I)) = \sum_{n=1}^{r-1} \lambda((aI^n \cap \mathfrak{m}I^{n+1})/a\mathfrak{m}I^n)$.

Proof. (1) is proved in Corollary 4. On the other hand, for $s \geqslant 1$ we have

$$\begin{aligned} F(I)/a^{r+s}F(I) = A/\mathfrak{m} \oplus \cdots \oplus I^r/\mathfrak{m}I^r \oplus I^{r+1}/\mathfrak{m}I^{r+1} \oplus \cdots \oplus I^{r+s-1}/\mathfrak{m}I^{r+s-1} \\ \oplus I^{r+s}/\big(\big(a^{r+s}\big) + \mathfrak{m}I^{r+s}\big) \oplus I^{r+s+1}/\big(a^{r+s}I + \mathfrak{m}I^{r+s+1}\big) \oplus \cdots \\ \oplus I^{2r+s-1}/\big(a^{r+s}I^{r-1} + \mathfrak{m}I^{2r+s-1}\big). \end{aligned}$$

Observe that there are isomorphisms

$$I^{r+i}/\mathfrak{m}I^{r+i} \cong I^r/\mathfrak{m}I^r$$

for $1 \leqslant i \leqslant s-1$, and that

$$I^{r+s+i}/\big(a^{r+s}I^i + \mathfrak{m}I^{r+s+i}\big) \cong I^r/\big(a^{r-i}I^i + \mathfrak{m}I^r\big)$$

for $0 \leqslant i \leqslant r-1$. Thus,

$$\lambda\big(F(I)/a^{r+s}F(I)\big) = 1 + \mu(I) + \cdots + \mu\big(I^r\big) + (s-1)\mu\big(I^r\big) + \sum_{i=0}^{r-1} \lambda\big(I^r/\big(a^{r-i}I^i + \mathfrak{m}I^r\big)\big).$$

Now,

$$\lambda\big(I^r/\big(a^{r-i}I^i + \mathfrak{m}I^r\big)\big) = \mu\big(I^r\big) - \mu\big(I^i\big) + \lambda\big(\big(a^{r-i}I^i \cap \mathfrak{m}I^r\big)/a^{r-i}\mathfrak{m}I^i\big)$$

by Lemma 9 and

$$\lambda\big(F(I)/a^{r+s}F(I)\big) = (r+s)\mu\big(I^r\big) + \sum_{i=1}^{r-1} \lambda\big(\big(a^{r-i}I^i \cap \mathfrak{m}I^r\big)/a^{r-i}\mathfrak{m}I^i\big)$$

which gives (2). Finally, (3) and (4) are immediate consequences of (1) and (2). □

Theorem 12. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with an infinite residue field and let I be a regular ideal of A with analytic spread one and reduction number r. The following conditions are then equivalent*:

(1) *$F(I)$ is a Buchsbaum ring.*

(2) *For any minimal reduction (a) of I one has*

$$\big(I^n \cap \big(\mathfrak{m}I^{n+1} : a\big)\big)/\mathfrak{m}I^n = \big(I^n \cap \big(\mathfrak{m}I^{n+1} : I\big)\big)/\mathfrak{m}I^n$$

for $1 \leqslant n \leqslant r-1$.

(3) *There exists an integer $C \geqslant 0$ such that*

$$C = \sum_{n=1}^{r-1} \lambda\big(\big(aI^n \cap \mathfrak{m}I^{n+1}\big)/a\mathfrak{m}I^n\big)$$

for any minimal reduction (a) of I.

(4) $(0 : a^0) = (0 : (a^0)^{r-1})$ *for any minimal reduction (a) of I.*

(5) $\lambda((aI^n \cap \mathfrak{m}I^{n+1})/a\mathfrak{m}I^n) = \lambda((a^{r-n}I^n \cap \mathfrak{m}I^r)/a\mathfrak{m}I^n)$ *for any minimal reduction (a) of I and $1 \leqslant n \leqslant r-1$.*

(6) $\lambda((I^n \cap (\mathfrak{m}I^{n+1} : a))/\mathfrak{m}I^n)$ *is independent of the choice of the minimal reduction (a) of I, for $1 \leqslant n \leqslant r-1$.*

(7) *There exists a natural number C such that if $J=(a)$ is any minimal reduction of I and*

$$F(I)\cong\bigoplus_{i=0}^{r}\big(F(J)(-i)\big)^{\alpha_i}\bigoplus_{i=1}^{r-1}\bigoplus_{j=1}^{r-i}\big(\big(F(J)/a^jF(J)\big)(-i)\big)^{\alpha_{i,j}}$$

is the decomposition of $F(I)$ as $F(J)$-module, then

$$C=\sum_{\substack{1\leqslant i\leqslant r-1\\ 1\leqslant j\leqslant r-i}}\alpha_{i,j}.$$

(8) *There exist integers $\alpha_0,\dots,\alpha_r,\alpha_{1,1},\dots,\alpha_{r-1,1}$ such that for every $J=(a)\subset I$ minimal reduction of I, the decomposition of $F(I)$ as $F(J)$-module has the form*

$$F(I)\cong\bigoplus_{i=0}^{r}\big(F(J)(-i)\big)^{\alpha_i}\bigoplus_{i=1}^{r-1}\big(\big(F(J)/aF(J)\big)(-i)\big)^{\alpha_{i,1}}.$$

Proof. The equivalence (1) $\Leftrightarrow$ (2) is the corresponding one in Remark 10. Now, by Lemma 11 we get the equivalence (1) $\Leftrightarrow$ (3). And by Corollary 4 we have (3) $\Leftrightarrow$ (7).

On the other hand, (1) $\Leftrightarrow$ (4) easily follows from (1) $\Leftrightarrow$ (5) in Remark 10, and taking components and their lengths in (4), we get condition (5). Now, the isomorphisms $(a^{r-n}I^n\cap\mathfrak{m}I^r)/a\mathfrak{m}I^n\cong[H^0_{F(I)_+}(F(I))]_n$ give (5) $\Rightarrow$ (6). And it is clear that (6) $\Rightarrow$ (3).

Finally, to get (7) $\Leftrightarrow$ (8) observe first that if we have such a decomposition of $F(I)$ as in (7), $F(I)$ is Buchsbaum and $C=\lambda(T(F(I)))$ by Remark 10. Thus $\alpha_{i,j}=0$ for any $j\geqslant 2$ and, by Proposition 3, for any $1\leqslant k\leqslant r-1$ it holds that $f_{k,r-k}=\alpha_{k,1}$. Since the numbers $f_{k,r-k}$ are independent of the chosen minimal reduction J this implies that the invariants $\alpha_{i,1}$, for $1\leqslant i\leqslant r-1$, are also independent of J. □

Remark 13. Observe that as a consequence of the above theorem we have that the Buchsbaum property of $F(I)$ is equivalent to the invariance of the structure of $F(I)$ as a $F(J)$-module with respect to the chosen minimal reduction J.

Now, we recall in the following remark several characterizations of the Cohen–Macaulay property translated to the fiber cone in this case.

Remark 14. The following conditions are equivalent:

(1) $F(I)$ is a Cohen–Macaulay ring.
(2) $(0:_{F(I)}a^0)=0$ for all (some) $(a)\subseteq I$ minimal reduction.
(3) $\lambda(F(I)/aF(I))-e(aF(I),F(I))=0$ for every (some) $(a)\subseteq I$ minimal reduction.
(4) $H^0_{F(I)_+}(F(I))=0$.
(5) $F(I)$ is a free $F(J)$-module, for every (some) minimal reduction J of I.

Theorem 15. *Let $(A,\mathfrak{m})$ be a Noetherian local ring with an infinite residue field and let I be a regular ideal of A with analytic spread one and reduction number r. The following conditions are then equivalent*:

(1) *$F(I)$ is a Cohen–Macaulay ring.*
(2) *For every (some) minimal reduction (a) of I, $I^n \cap (\mathfrak{m}I^{n+1} : a) = \mathfrak{m}I^n$ for $1 \leqslant n \leqslant r-1$.*
(3) *For every (some) minimal reduction (a) of I, $\lambda(F(I)/aF(I)) = \mu(I^r)$.*
(4) *For every (some) minimal reduction (a) of I, $I^k \cap (\mathfrak{m}I^r : a^{r-k}) = \mathfrak{m}I^k$ for $1 \leqslant k \leqslant r-1$.*
(5) *For every (some) minimal reduction (a) of I, $aI^n \cap \mathfrak{m}I^{n+1} = a\mathfrak{m}I^n$ for $1 \leqslant n \leqslant r-1$.*
(6) *For every (some) minimal reduction (a) of I, $\lambda(I^n/(\mathfrak{m}I^n + aI^{n-1})) = \mu(I^n) - \mu(I^{n-1})$ for $1 \leqslant n \leqslant r$.*
(7) *For every (some) $J = (a)$ minimal reduction of I the decomposition of $F(I)$ as $F(J)$ module has the form*

$$F(I) \cong F(J) \oplus \bigl(F(J)(-1)\bigr)^{\mu(I)-1} \oplus \cdots \oplus \bigl(F(J)(-r)\bigr)^{\mu(I^r)-\mu(I^{r-1})}$$
$$= \bigoplus_{i=0}^{r} F(J)(-i)^{\mu(I^i)-\mu(I^{i-1})}.$$

Proof. Sentences (1), (2), (3) and (4) correspond to the same ones in Remark 14 and so the equivalences. The equivalence (2) $\Leftrightarrow$ (5) follows for the regularity of a in A, while Lemma 9 gives (5) $\Leftrightarrow$ (6).

On the other hand, if there exists a minimal reduction J of I as in (7) then $F(I)$ is a free $F(J)$-module and so $F(I)$ is a Cohen–Macaulay ring. Conversely, if $F(I)$ is Cohen–Macaulay then it is a free $F(J)$-module for every J minimal reduction, and it has a decomposition as a direct sum of simple free $F(J)$-modules

$$F(I) \cong \bigoplus_{i=0}^{r-1} F(I)(-i)^{\alpha_i}$$

where, by Proposition 7, $\alpha_i = \mu(I^i) - \mu(I^{i-1}) - f_{i,r-i} + f_{i-1,r-i+1}$ and $f_{k,r-k} = \lambda((I^k \cap (\mathfrak{m}I^r : a^{r-k}))/\mathfrak{m}I^k) = 0$ for $1 \leqslant k \leqslant r-1$. □

Corollary 16. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with an infinite residue field and let I be a regular ideal of A with analytic spread one and reduction number r. If $F(I)$ is Cohen–Macaulay then $\mu(I^{n-1}) < \mu(I^n)$ for $1 \leqslant n \leqslant r$ and the postulation number* $\mathrm{fp}(I) = r - 1$.

Proof. Let $J = (a)$ be a minimal reduction of I. If $F(I)$ is Cohen–Macaulay then, for $1 \leqslant n \leqslant r$, we have by the Nakayama lemma and the previous result that $0 < \lambda(I^n/(\mathfrak{m}I^n + aI^{n-1})) = \mu(I^n) - \mu(I^{n-1})$. Thus, $1 < \mu(I) < \mu(I^2) < \cdots < \mu(I^{r-1}) < e(F(I)) = \mu(I^r) = \mu(I^m)$ for all $m \geqslant r$. (See also D'Anna–Guerrieri–Heinzer [8, Proposition 3.2] for the computation of postulation number.) □

To conclude this section we study the Gorenstein property of $F(I)$.

Remark 17. The following conditions are equivalent:

(1) $F(I)$ is a Gorenstein ring.
(2) $F(I)$ is a Cohen–Macaulay ring and $\mathrm{type}(F(I)) = 1$.

(3) For every (some) minimal reduction J of I, $F(I)$ is a free $F(J)$-module and the canonical module $\omega_{F(I)} \cong F(I)(k)$, for some $k \in \mathbb{Z}$.

Lemma 18. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with an infinite residue field and let I be a regular ideal of A with analytic spread one and reduction number r. Assume that $F(I)$ is a Cohen–Macaulay ring and let $J = (a) \subseteq I$ be a minimal reduction. Then*

$$\operatorname{type}\big(F(I)\big) = \sum_{i=1}^{r-1} \lambda\big(\big(I^i \cap \big(aI^i + \mathfrak{m}I^{i+1} : I\big)\big)/\big(aI^{i-1} + \mathfrak{m}I^i\big)\big) + \lambda\big(I^r/\big(aI^{r-1} + \mathfrak{m}I^r\big)\big).$$

Proof. Let (a) be a minimal reduction of I. Since $F(I)$ is Cohen–Macaulay, a^0 is a regular element in $F(I)$ and

$$\operatorname{type}\big(F(I)\big) = \lambda\big(\operatorname{Ext}^1\big(A/\mathfrak{m}, F(I)\big)\big) = \lambda\big(\operatorname{Socle}\big(F(I)/aF(I)\big)\big) = \lambda\big((0 :_{F'} F'_+)\big)$$

where $F' = F(I)/aF(I)$. The statement then follows from the equality

$$(0 :_{F'} F'_+) = \bigoplus_{i=1}^{r-1} \big(\big(I^i \cap \big(aI^i + \mathfrak{m}I^{i+1} : I\big)\big)/\big(aI^{i-1} + \mathfrak{m}I^i\big)\big) \oplus I^r/\big(aI^{r-1} + \mathfrak{m}I^r\big). \qquad \square$$

Assume that $F(I)$ is Cohen–Macaulay. In the following lemma we describe the structure as a $F(J)$-module of the canonical module of $F(I)$. Recall that, since $F(J)$ is a polynomial ring in one variable over a field, the a-invariant of $F(J)$ is -1 and $\omega_{F(J)} \simeq F(J)(-1)$.

Lemma 19. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with an infinite residue field and let I be a regular ideal of A with analytic spread one and reduction number r. Assume that $F(I)$ is a Cohen–Macaulay ring and let J be a minimal reduction of I. Then,*

$$\omega_{F(I)} \simeq \operatorname{Hom}_{F(J)}\big(F(I), F(J)(-1)\big) \cong \bigoplus_{i=0}^{r} \big(F(J)(i-1)\big)^{\mu(I^i)-\mu(I^{i-1})}$$

and the a-invariant of $F(I)$ is $r - 1$.

Proof. We may write $F(I) \cong \bigoplus_{i=0}^{r} (F(J)(-i))^{\mu(I^i)-\mu(I^{i-1})}$. Then, by local duality,

$$\begin{aligned}
\omega_{F(I)} &\cong \operatorname{Hom}_{F(J)}\big(F(I), F(J)(-1)\big) \\
&\cong \operatorname{Hom}_{F(J)}\left(\bigoplus_{i=0}^{r} \big(F(J)(-i)\big)^{\mu(I^i)-\mu(I^{i-1})}, F(J)(-1)\right) \\
&\cong \bigoplus_{i=0}^{r} \operatorname{Hom}_{F(J)}\big(\big(F(J)(-i)\big)^{\mu(I^i)-\mu(I^{i-1})}, F(J)(-1)\big) \\
&\cong \bigoplus_{i=0}^{r} \big(\operatorname{Hom}_{F(J)}\big(F(J)(-i), F(J)(-1)\big)\big)^{\mu(I^i)-\mu(I^{i-1})}
\end{aligned}$$

$$\cong \bigoplus_{i=0}^{r} \bigl(\mathrm{Hom}_{F(J)}\bigl(F(J), F(J)\bigr)(i-1)\bigr)^{\mu(I^i)-\mu(I^{i-1})}$$
$$\cong \bigoplus_{i=0}^{r} \bigl(F(J)(i-1)\bigr)^{\mu(I^i)-\mu(I^{i-1})}. \qquad \square$$

Theorem 20. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with an infinite residue field and let I be a regular ideal of A with analytic spread one and reduction number r. Then, the following conditions are equivalent*:

(1) *$F(I)$ is a Gorenstein ring.*
(2) *$\mu(I^r) = \mu(I^{r-1}) + 1$, and for every (some) minimal reduction $J = (a)$ of I the following equalities hold*

$$I^n \cap \bigl(\mathfrak{m}I^{n+1} : a\bigr) = \mathfrak{m}I^n \quad \textit{and} \quad I^n \cap \bigl(aI^n + \mathfrak{m}I^{n+1} : I\bigr) = aI^{n-1} + \mathfrak{m}I^n$$

for $1 \leqslant n \leqslant r-1$.

In this case, the decomposition of $F(I)$ as the direct sum of cyclic $F(J)$-module has the form

$$F(I) \cong \bigoplus_{i=0}^{r} F(J)(-i)^{\mu(I^i)-\mu(I^{i-1})},$$

with $\mu(I^i) - \mu(I^{i-1}) = \mu(I^{r-i}) - \mu(I^{r-i-1})$ for $0 \leqslant i \leqslant r$.

Proof. Let $J = (a) \subseteq I$ be a minimal reduction. We know that $F(I)$ is Cohen–Macaulay if and only if $I^n \cap (\mathfrak{m}I^{n+1} : a) = \mathfrak{m}I^n$ for $1 \leqslant n \leqslant r-1$. Moreover, in this case $\lambda(I^r/(aI^{r-1} + \mathfrak{m}I^r)) = \mu(I^r) - \mu(I^{r-1}) > 0$. Thus, we may assume that $F(I)$ is Cohen–Macaulay and so it is Gorenstein if, and only if, $\mathrm{type}(F(I)) = 1$. By Lemma 18 this is equivalent to $1 = \lambda(I^r/(aI^{r-1} + \mathfrak{m}I^r)) = \mu(I^r) - \mu(I^{r-1})$ and $I^n \cap (aI^n + \mathfrak{m}I^{n+1} : I) = aI^{n-1} + \mathfrak{m}I^n$, for $1 \leqslant n \leqslant r-1$.

Now, by Theorem 15 $F(I)$ is Cohen–Macaulay if, and only if,

$$F(I) \simeq \bigoplus_{i=0}^{r} F(J)(-i)^{\mu(I^i)-\mu(I^{i-1})}.$$

And by Lemma 19,

$$\omega_{F(I)} \cong \mathrm{Hom}_{F(J)}\bigl(F(I), F(J)(-1)\bigr) \cong \bigoplus_{i=0}^{r} \bigl(F(J)(i-1)\bigr)^{\mu(I^i)-\mu(I^{i-1})}$$

with $a(F(I)) = r-1$. On the other hand, $F(I)(k) \cong \bigoplus_{i=0}^{r} (F(J)(k-i))^{\mu(I^i)-\mu(I^{i-1})}$. Hence, if $F(I)$ is Gorenstein, $\omega_{F(I)} \cong F(I)(r-1)$ and just comparing we get $\mu(I^i) - \mu(I^{i-1}) = \mu(I^{r-i}) - \mu(I^{r-i-1})$. (Observe that one may also obtain these equalities from the well-known fact that the h-vector of a Gorenstein graded algebra is symmetric.) $\square$

4. Applications and examples

We may first apply the results in the above section to the case of ideals with small reduction number.

Proposition 21. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with an infinite residue field and let I be a regular ideal of A with analytic spread one and reduction number* 1. *Then,*

(1) *For any J minimal reduction of I, $F(I) \cong F(J) \oplus (F(J)(-1))^{\mu(I)-1}$.*
(2) *$F(I)$ is a Cohen–Macaulay ring.*
(3) *$F(I)$ is a Gorenstein ring if and only if $\mu(I) = 2$.*

Proof. We have already noted in Section 2 that $F(I)$ is Cohen–Macaulay if $r(I) = 1$. Then, apply Theorem 5 to get (1). Finally, (3) is a consequence of Theorem 20. □

Proposition 22. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with an infinite residue field and let I be a regular ideal of A with analytic spread one and reduction number* 2. *Then,*

(1) *For any minimal reduction $J = (a)$ of I,*

$$F(I) \cong F(J) \oplus \big(F(J)(-1)\big)^{\mu(I)-1-\alpha} \oplus \big(F(J)(-2)\big)^{\mu(I^2)-\mu(I)+\alpha} \oplus \big(F(J)/JF(J)(-1)\big)^{\alpha},$$

where $\alpha = \lambda((aI \cap \mathfrak{m}I^2)/a\mathfrak{m}I)$.
(2) *$F(I)$ is a Buchsbaum ring.*
(3) *The following conditions are equivalent*:
  (a) *$F(I)$ is a Cohen–Macaulay ring.*
  (b) *$aI \cap \mathfrak{m}I^2 = a\mathfrak{m}I$ for every* (*some*) *minimal reduction* (a) *of I.*
  (c) *$\mu(I^2) - \mu(I) = \lambda(I^2/(\mathfrak{m}I^2 + aI))$ for every* (*some*) *minimal reduction* (a) *of I.*
(4) *The following conditions are equivalent*:
  (a) *$F(I)$ is a Gorenstein ring.*
  (b) *$\mu(I^2) - \mu(I) = \lambda(I^2/(\mathfrak{m}I^2 + aI)) = 1$ for every* (*some*) *minimal reduction* (a) *of I and $I \cap (aI + \mathfrak{m}I^2 : I) = (a) + \mathfrak{m}I$.*

Proof. Let $J = (a)$ be a minimal reduction of I. Then, (1) is a direct consequence of Corollary 4 and Proposition 7. By Theorem 12, $F(I)$ is Buchsbaum if and only if $I \cap (\mathfrak{m}I^2 : a) = I \cap (\mathfrak{m}I^2 : I)$ for every minimal reduction (a) of I. Let $x \in I$ such that $xa \in \mathfrak{m}I^2$. Then, for any $y \in I$ one has $axy \in \mathfrak{m}I^3 = a\mathfrak{m}I^2$ and so $xy \in \mathfrak{m}I^2$ since a is regular, that is, $x \in I \cap (\mathfrak{m}I^2 : I)$. Now, (3) and (4) are direct consequence of Theorem 15 and Theorem 20, respectively, for $r = 2$. □

Assume in addition that $I = \mathfrak{m}$. Then, condition (3), (b) in the above proposition trivially holds since $\mathfrak{m}^3 = a\mathfrak{m}^2$. One can also see in this case that condition (4), (b) holds if A is Gorenstein (see, for instance, Proposition 3.3 and the final part of the proof of Theorem 3.4 in J. Sally [32]). As for the Buchsbaum property, it is known that $F(I) = G(\mathfrak{m})$ is Buchsbaum if A is Buchsbaum of $\dim A = 1$ and $r(\mathfrak{m}) = 2$, see S. Goto [17, Proposition 7.4].

It is easy to see that if $I = \mathfrak{m}$ and $r(\mathfrak{m}) = 3$, then condition (2) in Theorem 12 holds and so $F(I) = G(\mathfrak{m})$ is Buchsbaum in this case, see also S. Goto [17, Proposition 7.7]. Nevertheless, this result cannot be extended to more general fiber cones as the following examples will show:

D'Anna–Guerrieri–Heinzer describe in [8, Example 2.3] a family $(R_n, \mathfrak{m}_n)$ (for $n \geqslant 3$) of one-dimensional local Cohen–Macaulay rings and $\mathfrak{m}_n$-primary ideals I_n for which $\mu(I_n) = n$, $r(I_n) = n - 1$ and the fiber cone $F(I_n)$ is not Cohen–Macaulay. Moreover, $\mu((I_n)^j) = \mu(I_n)$, for all $j \geqslant 1$. For our purposes, we are going to consider the particular cases $n = 3, 4$.

Example 23. Consider $R_3 = K[\![t^6, t^{11}, t^{15}, t^{31}]\!]$ and $I_3 = I = (t^6, t^{11}, t^{31})$. Since $r(I) = 2$, the fiber cone $F(I)$ is a Buchsbaum ring (and not Cohen–Macaulay). For any minimal reduction $J = (a)$ of I, the structure of $F(I)$ as $F(J)$-module is, by Proposition 22,

$$F(I) \cong F(J) \oplus F(J)(-1) \oplus F(J)(-2) \oplus \big(F(J)/JF(J)\big)(-1),$$

since in this case $\mu(I) = \mu(I^2) = 3$, and $\alpha = \lambda((aI \cap \mathfrak{m}I^2)/\lambda\mathfrak{m}I) = \lambda(I^2/(\mathfrak{m}I^2 + aI)) = \lambda((t^{12}, t^{17}, t^{22})/\mathfrak{m}I^2 + (t^{12}, t^{17}, t^{37})) = 1$.

Example 24. Consider $R_4 = K[\![t^8, t^{15}, t^{28}, t^{50}, t^{57}]\!]$ and $I_4 = I = (t^8, t^{15}, t^{50}, t^{57})$. We claim that $F(I)$ is not a Buchsbaum ring.

By [8], (t^8) is a minimal reduction of I, $\mu(I) = 4$ and $r(I) = 3$. In order to prove that $F(I)$ is not Buchsbaum we will show that there exists an element $x \in I \setminus \mathfrak{m}I$ such that $x \cdot (t^8)^2 \in \mathfrak{m}I^3$ and $x \cdot t^8 \notin \mathfrak{m}I^2$; This implies that $x^0 \in (0 : (t^{8^0})^2)$, $0 \neq t^{8^0} \cdot x^0 = \overline{t^8 x} \in I^2/\mathfrak{m}I^2 \subset F(I)$ and so $F(I)_+ \cdot (0 : (t^{8^0})^2) \neq 0$. Take $x = t^{57}$: Then x fulfills the conditions since $t^{16}t^{57} = t^{73} = t^{28}(t^{15})^3 \in \mathfrak{m}I^3$ and $t^8 t^{57} = t^{65} \notin \mathfrak{m}I^2$.

In order to describe the structure of $F(I)$ as $F(t^8)$-module we observe that, since $\mu(I^4) = \mu(I) = 4$ and $f_{11} < f_{12}$,

$$\begin{aligned}
f_{11} &= \lambda\big(I^2/\big(\mathfrak{m}I^2 + t^8 I\big)\big) = \lambda\big(\big(t^{16}, t^{23}, t^{30}, t^{65}\big)/\big(\mathfrak{m}I^2 + \big(t^{16}, t^{23}, t^{58}, t^{65}\big)\big)\big) = 1,\\
f_{12} &= \lambda\big(I^3/\big(\mathfrak{m}I^3 + t^{16} I\big)\big) = \lambda\big(\big(t^{24}, t^{31}, t^{45}, t^{38}\big)/\big(\mathfrak{m}I^3 + \big(t^{24}, t^{31}, t^{66}, t^{73}\big)\big)\big) = 2,\\
f_{21} &= \lambda\big(I^3/\big(\mathfrak{m}I^3 + t^8 I^2\big)\big) = \lambda\big(\big(t^{24}, t^{31}, t^{45}, t^{38}\big)/\big(\mathfrak{m}I^3 + \big(t^{24}, t^{31}, t^{38}, t^{73}\big)\big)\big) = 1.
\end{aligned}$$

So, $\alpha_1 = \mu(I) - 1 - f_{12} = 1$, $\alpha_2 = f_{12} - f_{21} = 1$, $\alpha_3 = f_{21} = 1$, $\alpha_{11} = f_{11} = 1$, $\alpha_{12} = f_{12} - f_{11} = 1$, $\alpha_{21} = f_{21} - f_{12} + f_{11} = 0$ and

$$\begin{aligned}
F(I) \cong{}& F\big(t^8\big) \oplus F\big(t^8\big)(-1) \oplus F\big(t^8\big)(-2) \oplus F\big(t^8\big)(-3)\\
&\oplus \big(F\big(t^8\big)/\big(t^8 F\big(t^8\big)\big)(-1)\big) \oplus \big(F\big(t^8\big)/\big(t^{16} F\big(t^8\big)\big)(-1)\big).
\end{aligned}$$

On the other hand, $t^{57} I^3 \subseteq \mathfrak{m}I^4$. Thus, writing $t^8 x = (t^8 + t^{57})x - t^{57}x$ for any $x \in I^3$ one gets that $I^4 = t^8 I^3 \subseteq (t^8 + t^{57})I^3 + \mathfrak{m}I^4$. By Nakayama's lemma we obtain that $(t^8 + t^{57})$ is also a minimal reduction of I such that

$$\lambda\big(\big(I \cap \big(\mathfrak{m}I^2 : \big(t^8 + t^{57}\big)\big)\big)/\mathfrak{m}I\big) = \lambda\big(I^2/\big(\mathfrak{m}I^2 + \big(t^6 + t^{57}\big)I\big)\big) = 2.$$

In this case, the structure of $F(I)$ as $F(t^8 + t^{57})$-module is given by

$$\begin{aligned} F(I) \cong{}& F\big(t^8+t^{57}\big) \oplus F\big(t^8+t^{57}\big)(-1) \oplus F\big(t^8+t^{57}\big)(-2) \oplus F\big(t^8+t^{57}\big)(-3) \\ &\oplus \big(\big(F\big(t^8+t^{57}\big)/\big(\big(t^8+t^{57}\big)F\big(t^8+t^{57}\big)\big)(-1)\big)\big)^2 \\ &\oplus \big(F\big(t^8+t^{57}\big)/\big(\big(t^8+t^{57}\big)F\big(t^8+t^{57}\big)\big)(-2)\big). \end{aligned}$$

This shows that the structure of $F(I)$ as $F(J)$-module may depend on the chosen reduction J when $F(I)$ is not Buchsbaum.

In the next lemma we prove a closed formula for the minimal number of generators of the powers I^n of a regular ideal with analytic spread one. It also provides an easy proof in this case of a well-known result of Eakin and Sathaye [14], see [37, 9.39] for a general proof, or the more recent by G. Caviglia [2].

Lemma 25. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with residue field infinite and let I be a regular ideal of A with analytic spread one and reduction number r. Let (a) be a minimal reduction if I. Then*

$$\mu\big(I^n\big) = 1 + \sum_{i=2}^{n} \lambda\left(\frac{\mathfrak{m}I^n + a^{i-1}I^{n-i+1}}{\mathfrak{m}I^n + a^i I^{n-i}}\right) + \lambda\left(\frac{I^n}{\mathfrak{m}I^n + aI^{n-1}}\right),$$

and, for $1 \leqslant n \leqslant r$, we have $\mu(I^n) \geqslant n+1$.

Proof. Put $J = (a)$. Fixed n, we consider the following exact sequences

$$\begin{gathered} 0 \to J^n/\big(\mathfrak{m}I^n \cap J^n\big) \to I^n/\mathfrak{m}I^n \to I^n/\big(\mathfrak{m}I^n + J^n\big) \to 0, \\ 0 \to \big(\mathfrak{m}I^n + J^{n-1}I\big)/\big(\mathfrak{m}I^n + J^n\big) \to I^n/\big(\mathfrak{m}I^n + J^n\big) \to I^n/\big(\mathfrak{m}I^n + J^{n-1}I\big) \to 0, \\ 0 \to \big(\mathfrak{m}I^n + J^{n-2}I^2\big)/\big(\mathfrak{m}I^n + J^{n-1}I\big) \to I^n/\big(\mathfrak{m}I^n + J^{n-1}I\big) \to I^n/\big(\mathfrak{m}I^n + J^{n-2}I^2\big) \to 0, \\ \vdots \\ 0 \to \big(\mathfrak{m}I^n + JI^{n-1}\big)/\big(\mathfrak{m}I^n + J^2I^{n-2}\big) \to I^n/\big(\mathfrak{m}I^n + J^2I^{n-2}\big) \to I^n/\big(\mathfrak{m}I^n + JI^{n-1}\big) \to 0. \end{gathered}$$

Then

$$\mu\big(I^n\big) = 1 + \sum_{i=2}^{n} \lambda\left(\frac{\mathfrak{m}I^n + J^{i-1}I^{n-i+1}}{\mathfrak{m}I^n + J^i I^{n-i}}\right) + \lambda\left(\frac{I^n}{\mathfrak{m}I^n + JI^{n-1}}\right).$$

Let now $n \leqslant r$. It is clear that $\lambda(I^n/(\mathfrak{m}I^n + JI^{n-1})) > 0$ by the Nakayama's lemma.

Claim: $\lambda\left(\dfrac{\mathfrak{m}I^n + J^{i-1}I^{n-i+1}}{\mathfrak{m}I^n + J^i I^{n-i}}\right) > 0.$

Assume that there exist n and i, $2 \leqslant i \leqslant n \leqslant r$, such that $\mathfrak{m}I^n + J^{i-1}I^{n-i+1} \subseteq \mathfrak{m}I^n + J^iI^{n-i}$. Then $J^{i-1}I^{n-i+1} \subseteq \mathfrak{m}I^n + J^iI^{n-i}$, and for all $k \geqslant n-i+1$ also have (multiplying by $I^{k-(n-i+1)}$) that $J^{i-1}I^k \subseteq \mathfrak{m}I^{k+i-1} + J^iI^{k-1}$. In particular, since $r \geqslant n-i+1$, $I^{r+i-1} = J^{i-1}I^r \subseteq \mathfrak{m}I^{r+i-1} + J^iI^{r-1}$. The Nakayama's lemma implies now $I^{r+i-1} = J^iI^{r-1}$. On the other hand, $I^{r+i-1} = J^{i-1}I^r$ $(i-1 \geqslant 1)$. Therefore $a^{i-1}I^r = a^i I^{r-1}$ with a regular, and so $I^r = aI^{r-1}$ which contradicts the definition of r. $\square$

Now we shall apply the above lemma to the case of ideals generated by exactly two elements.

Proposition 26. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with residue field infinite and let I be a regular ideal with analytic spread one. Assume that I is minimally generated by 2 elements. Let $J = (a)$ be a minimal reduction of I. Then, $F(I) \cong \bigoplus_{i=0}^{r} F(J)(-i)$ and $F(I)$ is a Gorenstein ring.*

Proof. Let (a) be a minimal reduction of I. Since $\mu(I) = 2$, it is easy to see that $\mu(I^n) \leqslant n+1$ for any $n \geqslant 1$. Thus, by the above lemma we have

$$\mu(I^n) = 1 + \sum_{i=2}^{n} \lambda\left(\frac{\mathfrak{m}I^n + a^{i-1}I^{n-i+1}}{\mathfrak{m}I^n + a^i I^{n-i}}\right) + \lambda\left(\frac{I^n}{\mathfrak{m}I^n + aI^{n-1}}\right) = n+1$$

for $1 \leqslant n \leqslant r$. This implies

$$\lambda\left(\frac{\mathfrak{m}I^n + a^{i-1}I^{n-i+1}}{\mathfrak{m}I^n + a^i I^{n-i}}\right) = \lambda\left(\frac{I^n}{\mathfrak{m}I^n + aI^{n-1}}\right) = 1$$

for any $1 \leqslant n \leqslant r$ and $2 \leqslant i \leqslant n$.

On the other hand, by Lemma 9

$$\lambda\big(I^n/\mathfrak{m}I^n + aI^{n-1}\big) = \mu\big(I^n\big) - \mu\big(I^{n-1}\big) + \lambda\big(\big(aI^{n-1} \cap \mathfrak{m}I^n\big)/a\mathfrak{m}I^{n-1}\big).$$

Thus $\lambda((aI^{n-1} \cap \mathfrak{m}I^n)/a\mathfrak{m}I^{n-1}) = 0$, and

$$F(I) \cong F(J) \oplus F(J)(-1) \oplus \cdots \oplus F(J)(-r) = \bigoplus_{i=0}^{r} F(J)(-i).$$

On the other hand,

$$\lambda\big(I^n \cap \big(aI^n + \mathfrak{m}I^{n+1} : I\big)/aI^{n-1} + \mathfrak{m}I^n\big) \leqslant \lambda\big(I^n/aI^{n-1} + \mathfrak{m}I^n\big) = 1$$

for any $1 \leqslant n \leqslant r-1$. Thus $(aI^n + \mathfrak{m}I^{n+1} : I) \neq aI^{n-1} + \mathfrak{m}I^n$ if, and only if, $\lambda(I^n \cap (aI^n + \mathfrak{m}I^{n+1} : I)/aI^{n-1} + \mathfrak{m}I^n) = 1$ if, and only if, $I^n \cap (aI^n + \mathfrak{m}I^{n+1} : I) = I^n$ if, and only if, $I^n \subset (aI^n + \mathfrak{m}I^{n+1} : I)$ if, and only if, $I^{n+1} \subset aI^n + \mathfrak{m}I^{n+1}$ if, and only if, $I^{n+1} \subset aI^n$, which is not possible for $n \leqslant r-1$. Thus, by Theorem 20, $F(I)$ is Gorenstein. $\square$

In fact, it is proven in D'Anna–Guerrieri–Heinzer [8, Proposition 3.5] that the fiber cone of a regular ideal minimally generated by two elements having a principal reduction is a complete intersection. This result has been extended by Heinzer–Kim [18, Theorem 5.6] to ideals of arbitrary analytic spread $l > 0$, minimally generated by $l+1$ elements and having a minimal reduction generated by a regular sequence, such that the associated graded ring has a homogeneous regular sequence of length at least $l-1$, see also Jayanthan–Puthenpurakal–Verma [28, Proposition 4.2].

Assume now that A is Cohen–Macaulay of dimension 1 and I is an $\mathfrak{m}$-primary ideal. Then

$$\lambda\big(I^{n+1}/aI^n\big) = \lambda\big(A/(a)\big) - \lambda\big(I^n/I^{n+1}\big)$$

for all $(a) \subseteq I$ minimal reduction. Since $\lambda(A/(a)) = e(I)$ the multiplicity of the ideal I, the lengths $\lambda(I^{n+1}/aI^n)$ are independent of (a). (An $\mathfrak{m}$-primary ideal I in a Cohen–Macaulay ring A such that $\lambda(I^2/JI) = 1$ for any minimal reduction J of I is called a Sally ideal in [28].)

Proposition 27. *(See also [28, Theorem 3.3].) Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension 1 with an infinite residue field and let I be an $\mathfrak{m}$-primary ideal with reduction number r. Let $J = (a) \subset I$ be a minimal reduction of I and Assume that $e(I) - \lambda(I/I^2) = 1$. Then*

(1) *$F(I)$ is a Buchsbaum ring and*

$$F(I) \simeq F(J) \oplus F(J)(-1)^{\mu(I^2)-2} \bigoplus_{i=2}^{r-1} F(J)(-i)^{\mu(I^{i+1})-\mu(I^i)}$$
$$\oplus\, F(J)(-r) \bigoplus_{i=1}^{r-1} \big(\big(F(J)/aF(J)\big)(-i)\big)^{\mu(I^i)-\mu(I^{i+1})+1}.$$

(2) *The following conditions are equivalent*
   (a) *$F(I)$ is a Cohen–Macaulay ring.*
   (b) *$\mu(I^{n+1}) = \mu(I^n) + 1$ for all $1 \leqslant n \leqslant r-1$.*
   (c) *$\mu(I^2) = \mu(I) + 1$.*
   (d) *$\mathfrak{m}I^2 = a\mathfrak{m}I$ for every (some) (a) minimal reduction of I.*
   (e) *For every (some) J minimal reduction of I there exists an isomorphism*

$$F(I) \cong F(J) \oplus F(J)(-1)^{\mu(I)-1} \oplus F(J)(-2) \oplus \cdots \oplus F(J)(-r).$$

   In this case, type$(F(I)) = \lambda((I \cap (aI : I))/((a) + \mathfrak{m}I)) + 1$.

(3) *If $r \geqslant 3$, $F(I)$ is Gorenstein if and only if $\mu(I) = 2$.*

Proof. Let (a) be a minimal reduction of I. Then $\lambda(I^2/aI) = 1$. This condition implies that $\mathfrak{m}I^{n+1} \subseteq aI^n$ for all n and $\lambda(I^{n+1}/aI^n) = 1$ for all $1 \leqslant n < r$ and so, applying Lemma 9 we obtain for $1 \leqslant n \leqslant r-1$ the equalities

$$\lambda\big(\mathfrak{m}I^{n+1}/a\mathfrak{m}I^n\big) = \lambda\big(\big(aI^n \cap \mathfrak{m}I^{n+1}\big)/a\mathfrak{m}I^n\big) = 1 + \mu\big(I^n\big) - \mu\big(I^{n+1}\big).$$

Therefore, $\lambda((aI^n \cap \mathfrak{m}I^{n+1})/a\mathfrak{m}I^n)$ is independent of the reduction and by Theorem 12 the ring $F(I)$ is Buchsbaum. Moreover, $\alpha_{i,j} = 0$ for $j \geqslant 2$ and by Proposition 3 $\alpha_{k,1} = f_{k,r-k} = \cdots = f_{k,1} = \lambda((aI^n \cap \mathfrak{m}I^{n+1})/a\mathfrak{m}I^n)$ for any $1 \leqslant k \leqslant r-1$. Then, by Proposition 7 we may get the values of α_i's for $0 \leqslant i \leqslant r$.

On the other hand, by Theorem 15 $F(I)$ is Cohen–Macaulay if and only if for $1 \leqslant n \leqslant r-1$ one has $\lambda(\mathfrak{m}I^{n+1}/a\mathfrak{m}I^n) = 0$ and (2) follows easily.

Assume now that $r \geqslant 3$ and $F(I)$ is Gorenstein. Then, by Theorem 20 $\mu(I) - 1 = \mu(I^{r-1}) - \mu(I^{r-2}) = 1$ and so $\mu(I) = 2$. Finally, by Proposition 26 we have the converse. $\square$

In [28, 6] one may find various interesting examples of Sally ideals with reduction number two. The following one is a Sally ideal of reduction number three whose fiber cone is not Cohen–Macaulay, see also [32, Example 2.2].

Let $A = k[[t^4, t^5, t^{11}]]$ where k is any field and t an indeterminate. Let $I = \mathfrak{m} = (t^4, t^5, t^{11})$ be the maximal ideal of A. Then, one can easily see that $J = (t^4)$ is a minimal reduction of I, the reduction number of I is 3, and $\lambda(I^2/t^4 I) = 1$. Moreover, $\mu(I) = \mu(I^2) = 3$ and $\mu(I^3) = 4$. Hence

$$F(I) \simeq F(J) \oplus F(J)(-1) \oplus F(J)(-2) \oplus F(J)(-3) \oplus F(J)/t^4 F(J)(-1).$$

5. Ideals of higher analytic spread

In this section we give some applications to ideals of higher analytic spread.

Let I be an ideal of A. We will denote by $G(I)$ the associated graded ring of I. Given $a \in I$ we will set $a^* \in I/I^2 \hookrightarrow G(I)$ and $a^0 \in I/\mathfrak{m}I \hookrightarrow F(I)$. Let $a_1, \ldots, a_k$ be a family of elements in I. Then, by the well-known Valabrega–Valla criterium $a_1^*, \ldots, a_k^*$ is a regular sequence in $G(I)$ if and only if

(1) $a_1, \ldots, a_k$ is a regular sequence in A.
(2) $(a_1, \ldots, a_k) \cap I^n = (a_1, \ldots, a_k) I^{n-1}$, for all $n \geqslant 0$.

In this case, there are natural isomorphisms

$$G\big(I/(a_1, \ldots, a_k)\big) \simeq G(I)/\big(a_1^*, \ldots, a_k^*\big),$$
$$F\big(I/(a_1, \ldots, a_k)\big) \simeq F(I)/\big(a_1^0, \ldots, a_k^0\big).$$

Let S be a standard $\mathbb{N}$-graded algebra over a local ring. Recall that a sequence of homogeneous elements $x_1, \ldots, x_k$ is called *filter-regular* if for any $1 \leqslant i \leqslant k$, $[(x_1, \ldots, x_{i-1}) : x_i]_n = [(x_1, \ldots, x_{i-1})]_n$ for $n \gg 0$.

Let $a \in I$. If $a \in I \setminus I^2$, a is a superficial element for I if and only if a^* is filter regular in $G(I)$ (see for instance [5, Lemma 2.3]). In analogy to this situation, Jayanthan–Verma [27] define $a^0 \neq 0$ to be superficial in $F(I)$ if and only if a^0 is filter-regular in $F(I)$, and prove the so-called Sally-machine for the fiber cone. Namely, assume that a^0 is filter-regular in $F(I)$ and a^* is filter-regular in $G(I)$. Then

$$\operatorname{depth} F\big(I/(a)\big) > 0 \quad \Rightarrow \quad a^0 \text{ regular in } F(I).$$

Notice also that if in addition a^* is regular in $G(I)$ then $F(I)/(a^0) \cong F(I/(a))$ and so $\operatorname{depth} F(I) = \operatorname{depth} F(I/(a)) + 1$.

Now, we extend the Sally-machine for the fiber cone to sequences of arbitrary length.

Lemma 28. *Let $a_1, \ldots, a_k \in I$ such that $a_1^0, \ldots, a_k^0$ is a filter-regular sequence in $F(I)$ and $a_1^*, \ldots, a_k^*$ is a filter-regular sequence in $G(I)$. Assume that the sequence $a_1^*, \ldots, a_{k-1}^*$ is regular in $G(I)$. Then*

$$\operatorname{depth} F\big(I/(a_1,\dots,a_k)\big) > 0 \quad \Rightarrow \quad a_1^0,\dots,a_k^0 \textit{ regular in } F(I).$$

If in addition $a_1^*,\dots,a_k^*$ *is a regular sequence in* $G(I)$ *then*

$$\operatorname{depth} F(I) = \operatorname{depth} F\big(I/(a_1,\dots,a_k)\big) + k \geqslant 1 + k.$$

Proof. The case $k = 1$ is the above cited result as Sally-machine for the fiber cone. Assume now $k > 1$. For $i \leqslant k$ we will denote $A_i = A/(a_1,\dots,a_i)$, $\mathfrak{m}_i = \mathfrak{m}/(a_1,\dots,a_i)$ and $I_i = I/(a_1,\dots,a_i)$. Since $a_1^*,\dots,a_{k-1}^*$ is regular in $G(I)$ we have that

$$G(I_{k-1}) \simeq G(I)/\big(a_1^*,\dots,a_{k-1}^*\big),$$
$$F(I_{k-1}) \simeq F(I)/\big(a_1^0,\dots,a_{k-1}^0\big).$$

Therefore, putting $\overline{a_k} \in A_{k-1}$ we have that $\overline{a_k}^0 \in F(I_{k-1})$ and $\overline{a_k}^* \in G(I_{k-1})$ are filter regular.

Assume now that $\operatorname{depth} F(I_k) > 0$. Then, $F(I_k) \cong F(I_{k-1}/(\overline{a_k}))$ and by induction for $k = 1$ we get that $\overline{a_k}^0$ is regular in $F(I_{k-1})$. Hence, $\operatorname{depth} F(I_{k-1}) > 0$ and again by induction for $k-1$, $a_1^0,\dots,a_{k-1}^0$ is a regular sequence in $F(I)$, and $a_1^0,\dots,a_k^0$ is a regular sequence in $F(I)$ as well.

For the last assertion, if we assume that $a_1^*,\dots,a_k^*$ is a regular sequence in $G(I)$, then $F(I_k) \cong F(I)/(a_1^0,\dots,a_k^0)$ with $a_1^0,\dots,a_k^0$ a regular sequence in $F(I)$. So, $\operatorname{depth} F(I) = \operatorname{depth} F(I_k) + k \geqslant 1 + k$. $\square$

Suppose now that $l := l(I) \geqslant 1$ and let $J \subset I$ be a minimal reduction of I. By [27, Proposition 2.2] there always exist an element $a \in J \setminus \mathfrak{m}J$ such that a^* is filter-regular in $G(I)$ and a^0 is filter-regular in $F(I)$. Moreover, if $\operatorname{grade} G_+(I) > 0$, a^* is a regular element in $G(I)$ [5, Lemma 2.5]. Hence, if we assume that $\operatorname{grade} G(I)_+ \geqslant l - 1$, proceeding by induction one can always find $(a_1,\dots,a_l) = J$ such that $a_1^0,\dots,a_l^0$ is a filter-regular sequence in $F(I)$, $a_1^*,\dots,a_l^*$ is a filter regular sequence in $G(I)$, and $a_1^*,\dots,a_{l-1}^*$ is a regular sequence in $G(I)$. Set $A_{l-1} = A/(a_1,\dots,a_{l-1})$, $\mathfrak{m}_{l-1} = \mathfrak{m}/(a_1,\dots,a_{l-1})$ and $I_{l-1} = I/(a_1,\dots,a_{l-1}) \subset A_{l-1}$.

Lemma 29. *$F(I)$ is Cohen–Macaulay if and only if $F(I_{l-1})$ is Cohen–Macaulay.*

Proof. First note that since $a_1^*,\dots,a_{l-1}^*$ is a regular sequence in $G(I)$, then

$$F(I)/\big(a_1^0,\dots,a_{l-1}^0\big) \simeq F(I_{l-1})$$

and $\dim F(I_{l-1}) = 1$. Assume that $F(I)$ is Cohen–Macaulay. Then, $a_1^0,\dots,a_{l-1}^0$ is a regular sequence in $F(I)$ and so $F(I_{l-1})$ is Cohen–Macaulay too. Conversely, since $\operatorname{depth} F(I_{l-1}) > 0$ and $a_1^*,\dots,a_{l-1}^*$ is a regular sequence in $G(I)$ we have by Lemma 28 that $\operatorname{depth} F(I) = l - 1 + 1 = l$ and so $F(I)$ is Cohen–Macaulay. $\square$

Let I be an ideal with $l(I) = l \geqslant 1$ and $\operatorname{grade} G(I)_+ \geqslant l - 1$. If, in addition $\operatorname{grade}(I) = l$; that is, I is an equimultiple ideal with $\operatorname{grade}(I) = l(I)$, the reduction number $r_J(I)$ is independent of the choice of minimal reduction J (see [24, Theorem 2.1]).

Proposition 30. *(See [18, Theorem 5.6].) Let $(A, \mathfrak{m})$ be a Noetherian local ring with an infinite residue field and let I be an equimultiple ideal with analytic spread l,* $\operatorname{grade}(I) = l$ *and minimally generated by $l+1$ elements. Assume that* $\operatorname{grade}(G(I)_+) \geqslant l-1$. *Then $F(I)$ is Gorenstein.*

Proof. Consider the ideal I_{l-1} in the local ring A_{l-1} defined as above. This is a regular ideal of analytic spread one and minimally generated by 2 elements. So, its fiber cone $F(I_{l-1}) \simeq F(I)/(a_1^0, \dots, a_{l-1}^0)$ is Gorenstein by Proposition 26. Since depth $F(I_{l-1}) > 0$, $a_1^0, \dots, a_{l-1}^0$ is a regular sequence by Lemma 28 and $F(I)$ is also Gorenstein. $\square$

From now on, given $a_1, \dots, a_k \in A$ we will denote by $A_k = A/(a_1, \dots, a_k)$ and for an ideal L of A by $L_k = (L + (a_1, \dots, a_k))/(a_1, \dots, a_k)$.

Assume that $a_1, \dots, a_k \in I$ is such that $a_1^*, \dots, a_k^*$ is a regular sequence in $G(I)$ and $a_1^0, \dots, a_k^0$ is a regular sequence in $F(I)$. By the mixed Valabrega–Valla criterium, see Cortadellas–Zarzuela [6], these conditions are equivalent to

(1) $a_1, \dots, a_k$ is a regular sequence in A.
(2) $(a_1, \dots, a_k) \cap I^n = (a_1, \dots, a_k)I^{n-1}$, for all $n \geqslant 0$.
(3) $(a_1, \dots, a_k) \cap \mathfrak{m}I^n = (a_1, \dots, a_k)\mathfrak{m}I^{n-1}$ for all $n \geqslant 1$.

Lemma 31. *Let $a_1, \dots, a_k \in I$ such that $a_1^*, \dots, a_k^*$ is a regular sequence in $G(I)$ and $a_1^0, \dots, a_k^0$ is a regular sequence in $F(I)$. Then,*

$$\mu\big(I_k^n\big) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \mu\big(I^{n-i}\big).$$

Proof. We use induction on k. For $k = 1$, $\mu(I_1^n) = \lambda(I^n/(\mathfrak{m}I^n + a_1 I^n))$ by (2). Consider the exact sequence

$$0 \to a_1 I^{n-1}/\big(a_1 I^{n-1} \cap \mathfrak{m}I^n\big) \to I^n/\mathfrak{m}I^n \to I^n/\big(\mathfrak{m}I^n + a_1 I^{n-1}\big) \to 0.$$

By condition (3), $a_1 I^{n-1} \cap \mathfrak{m}I^n = a_1 \mathfrak{m} I^{n-1}$, and $a_1 I^{n-1}/a_1 \mathfrak{m} I^{n-1} \cong I^{n-1}/\mathfrak{m}I^{n-1}$ by condition (1). Thus $\mu(I_1^n) = \mu(I^n) - \mu(I^{n-1})$.

Let $1 < k$. Then, $\bar{a}_k$, $(\bar{a}_k)^*$ and $(\bar{a}_k)^0$ are regular elements, respectively, in the rings A_{k-1}, $G(I_{k-1})$ and $F(I_{k-1})$. Therefore, $\mu(I_k^n) = \mu(I_{k-1}^n) - \mu(I_{k-1}^{n-1})$ for the case $k = 1$, and by induction

$$\begin{aligned}
\mu\big(I_k^n\big) &= \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} \mu\big(I^{n-i}\big) - \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} \mu\big(I^{n-1-i}\big) \\
&= \sum_{i=0}^{k} (-1)^i \left(\binom{k-1}{i} + \binom{k-1}{i-1} \right) \mu\big(I^{n-i}\big) \\
&= \sum_{i=0}^{k} (-1)^i \binom{k}{i} \mu\big(I^{n-i}\big). \qquad \square
\end{aligned}$$

Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d and I be an $\mathfrak{m}$-primary ideal in A. By Valla [36],

$$\lambda(I/I^2) = e(I) + (d-1)\lambda(A/I) - \lambda(I^2/JI)$$

for any J minimal reduction of I where $e(I)$ denotes the multiplicity of the ideal I. In particular, the length of I^2/JI does not depend of the minimal reduction of I.

Proposition 32. *(See [28].) Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d with an infinite residue field and let I be an $\mathfrak{m}$-primary ideal. Assume that $e(I) + (d-1)\lambda(A/I) - \lambda(I/I^2) = 1$. Let J be a minimal reduction of I. Then $F(I)$ is a Cohen–Macaulay ring if and only if $\mathfrak{m}I^2 = J\mathfrak{m}I$. In this case,*

$$\sum_{i=0}^{d}(-1)^i\binom{d}{i}\mu(I^{n+1-i}) = 1$$

for $1 \leqslant n \leqslant r(I) - 1$.

Proof. First, notice that the assumption $e(I) + (d-1)\lambda(A/I) - \lambda(I/I^2) = 1$ is equivalent to $\lambda(I^2/JI) = 1$. By Rossi [31] grade $G(I) \geqslant d-1$, hence there exist $(a_1, \ldots, a_d) = J$ such that $a_1^0, \ldots, a_d^0$ is a filter regular sequence in $F(I)$, $a_1^*, \ldots, a_d^*$ is a filter regular sequence in $G(I)$, and $a_1^*, \ldots, a_{d-1}^*$ is a regular sequence in $G(I)$. By Lemma 29 $F(I)$ is Cohen–Macaulay if and only if $F(I_{d-1})$ is Cohen–Macaulay. Moreover, by Huckaba [24, Lemma 1-1] $(\overline{a_d})$ is a minimal reduction of I_{d-1} with $r(I_{d-1}) = r(I)$, and $\lambda(I_{d-1}^2/(\overline{a_d})I_{d-1}) = \lambda(I^2/(JI + I^2 \cap (a_1, \ldots, a_{d-1}))) = \lambda(I^2/JI) = 1$.

Suppose that $\mathfrak{m}I^2 = J\mathfrak{m}I$. Then, $\mathfrak{m}_{d-1}I_{d-1}^2 = \overline{a_d}\mathfrak{m}_{d-1}I_{d-1}$ and $F(I_{d-1})$ is Cohen–Macaulay by Proposition 27 and so $F(I)$ is Cohen–Macaulay too. Conversely, if $F(I)$ is Cohen–Macaulay then $F(I_{d-1})$ is also Cohen–Macaulay, and by Proposition 27 $\mathfrak{m}_{d-1}I_{d-1}^2 = \overline{a_d}\mathfrak{m}_{d-1}I_{d-1}$. Hence, $\mathfrak{m}I^2 = J\mathfrak{m}I + \mathfrak{m}I^2 \cap (a_1, \ldots, a_{d-1}) = (a_1, \ldots, a_{d-1})\mathfrak{m}I = J\mathfrak{m}I$, since $a_1^*, \ldots, a_{d-1}^*$ and $a_1^0, \ldots, a_{d-1}^0$ are regular sequences in $G(I)$ and $F(I)$, respectively. Finally, if $F(I)$ is Cohen–Macaulay $F(I_{d-1})$ is Cohen–Macaulay too and by Proposition 27 $\mu(I_{d-1}^{n+1}) - \mu(I_{d-1}^n) = 1$ for $1 \leqslant n \leqslant r(I) - 1$. Therefore, we may apply Lemma 31 to obtain for $1 \leqslant n \leqslant r(I) - 1$ the equality

$$1 = \mu(I_{d-1}^{n+1}) - \mu(I_{d-1}^n) = \sum_{i=0}^{d}(-1)^i\binom{d}{i}\mu(I^{n+1-i}). \qquad \square$$

Theorem 33. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with an infinite residue field and let $I \subset A$ be an equimultiple ideal with analytic spread l and reduction number r. Assume that* grade$(I) = l$, grade $G(I) \geqslant l-1$ *and* depth $F(I) \geqslant l-1$. *Let $J \subset I$ be a minimal reduction of I. Then the following equalities hold*:

(1) $\operatorname{reg}(F(I)) = r$.
(2) $e(F(I)) = \sum_{i=0}^{l-1}\binom{l-1}{i}\mu(I^{r-i})$.

(3) *$F(I)$ is a Cohen–Macaulay ring if and only if*

$$\lambda\big(I^n/\big(\mathfrak{m}I^n + JI^{n-1}\big)\big) = \sum_{i=0}^{l}(-1)^i\binom{l}{i}\mu\big(I^{n-i}\big),$$

for $1 \leqslant n \leqslant r(I)$.

Proof. The results are true for $l = 1$ by Theorem 15. Assume that $l \geqslant 1$ and let $(a_1, \ldots, a_l) = J$ such that $a_1, \ldots, a_l$ is a regular sequence in A, $a_1^*, \ldots, a_{l-1}^*$ is a regular sequence in $G(I)$, and $a_1^0, \ldots, a_{l-1}^0$ is a regular sequence in $F(I)$. Observe that $I_{l-1} \subset A_{l-1}$ is a regular ideal with analytic spread 1 and the same reduction number as I. Then, $\operatorname{reg}(F(I)) = \operatorname{reg}(F(I)/(a_1^0, \ldots, a_{l-1}^0)) = \operatorname{reg}(F(I_{l-1})) = r$. Similarly, one has the equalities $e(F(I)) = e(F(I_{l-1})) = \mu(I_{l-1}^r) = \sum_{i=0}^{l-1}(-1)^i\binom{l-1}{i}\mu(I^{r-i})$ by Lemma 31.

Now, by Lemma 29 $F(I)$ is Cohen–Macaulay if, and only if, $F(I_{l-1})$ is Cohen–Macaulay, and by Theorem 15 this is equivalent to

$$\lambda\big(I_{l-1}^n/\big(\mathfrak{m}_{l-1}I_{l-1}^n + a_l I_{l-1}^{n-1}\big)\big) = \mu\big(I_{l-1}^n\big) - \mu\big(I_{l-1}^{n-1}\big),$$

for all $1 \leqslant n \leqslant r$. Then, we get (3) by the isomorphisms

$$\begin{aligned} I_{l-1}^n/\big(\mathfrak{m}I_{l-1}^n + a_l I_{l-1}^{n-1}\big) &\cong I^n/\big(\mathfrak{m}I^n + a_l I^{n-1} + I^n \cap (a_1, \ldots, a_{l-1})\big) \\ &\cong I^n/\big(\big(\mathfrak{m}I^n + (a_1, \ldots, a_l)\big)I^{n-1}\big), \end{aligned}$$

and taking into account that $\mu(I_{l-1}^n) - \mu(I_{l-1}^{n-1}) = \sum_{i=0}^{l}(-1)^i\binom{l}{i}\mu(I^{n-i})$ by Lemma 31. $\square$

References

[1] M. Barile, M. Morales, On certain algebras of reduction number one, J. Algebra 206 (1) (1998) 113–128.

[2] G. Caviglia, A theorem of Eakin and Sathaye and Green's hyperplane restriction theorem, in: Commutative Algebra. Geometric, Homological, Combinatorial and Computational Aspects, in: Lect. Notes Pure Appl. Math., vol. 244, Chapman & Hall/CRC, Boca Raton, 2006, pp. 1–4.

[3] A. Corso, C. Polini, W.V. Vasconcelos, Multiplicity of the special fiber of blowups, Math. Proc. Cambridge Philos. Soc. 140 (2) (2006) 207–219.

[4] A. Corso, L. Ghezzi, C. Polini, B. Ulrich, Cohen–Macaulayness of special fiber rings, in: Special Issue in Honor of Steven L. Kleiman, Comm. Algebra 31 (8) (2003) 3713–3734.

[5] T. Cortadellas, Fiber cones with almost maximal depth, Comm. Algebra 33 (3) (2005) 953–963.

[6] T. Cortadellas, S. Zarzuela, On the depth of the fiber cone of filtrations, J. Algebra 198 (2) (1997) 428–445.

[7] T. Cortadellas, S. Zarzuela, On the Cohen–Macaulay property of the fiber cone of ideals with reduction number at most one, in: Commutative Algebra, Algebraic Geometry, and Computational Methods, Hanoi, 1996, Springer-Verlag, Singapore, 1999, pp. 215–222.

[8] M. D'Anna, A. Guerrieri, W. Heinzer, Ideals having a one-dimensional fiber cone, in: Ideal Theoretic Methods in Commutative Algebra, Columbia, MO, 1999, in: Lect. Notes Pure Appl. Math., vol. 220, Dekker, New York, 2001, pp. 155–170.

[9] M. D'Anna, A. Guerrieri, W. Heinzer, Invariants of ideals having principal reductions, Comm. Algebra 29 (2) (2001) 889–906.

[10] W. Bruns, J. Herzog, Cohen–Macaulay Rings, in: Cambridge Stud. Adv. Math., vol. 39, Cambridge University Press, Cambridge, 1993, first paperback edition with revisions, 1998.

[11] C. D'Cruz, T.J. Puthenpurakal, The Hilbert coefficients of the fiber cone and the a-invariant of the associated graded ring, preprint, arXiv: math.AC/0601510, 2006.

[12] C. D'Cruz, K.N. Raghavan, J.K. Verma, Cohen–Macaulay fiber cones, in: Commutative Algebra, Algebraic Geometry, and Computational Methods, Hanoi, 1996, Springer-Verlag, Singapore, 1999, pp. 233–246.
[13] C. D'Cruz, J.K. Verma, Hilbert series of fiber cones of ideals with almost minimal mixed multiplicity, J. Algebra 251 (1) (2002) 98–109.
[14] P. Eakin, A. Sathaye, Prestable ideals, J. Algebra 41 (2) (1976) 439–454.
[15] D. Eisenbud, B. Mazur, Evolutions, symbolic squares, and Fitting ideals, J. Reine Angew. Math. 488 (1997) 189–201.
[16] P. Gimenez, Étude de la fibre spéciale de l'éclatement d'une varieté monomiale en codimension deux, Dissertation, Univ. of Grenoble, 1993.
[17] S. Goto, Noetherian local rings with Buchsbaum associated graded rings, J. Algebra 86 (1984) 336–384.
[18] W.J. Heinzer, M.-K. Kim, Properties of the fiber cone of ideals in local rings, Comm. Algebra 31 (7) (2003) 3529–3546.
[19] W. Heinzer, M.-K. Kim, B. Ulrich, Bernd, The Gorenstein and complete intersection properties of associated graded rings, J. Pure Appl. Algebra 201 (1–3) (2005) 264–283.
[20] R. Hübl, Evolutions and valuations associated to an ideal, J. Reine Angew. Math. 517 (1999) 81–101.
[21] R. Hübl, C. Huneke, Fiber cones and the integral closure of ideals, Collect. Math. 52 (1) (2001) 85–100.
[22] R. Hübl, I. Swanson, Normal cones of monomial primes, Math. Comp. 72 (241) (2003) 459–475 (electronic).
[23] S. Huckaba, Reduction numbers for ideals of analytic spread one, J. Algebra 108 (2) (1987) 503–512.
[24] S. Huckaba, Reduction numbers for ideals of higher analytic spread, Math. Proc. Cambridge Philos. Soc. 102 (1) (1987) 49–57.
[25] C. Huneke, J. Sally, Birational extensions in dimension two and integrally closed ideals, J. Algebra 115 (2) (1988) 481–500.
[26] A.V. Jayanthan, J.K. Verma, Fiber cones of ideals with almost minimal multiplicity, Nagoya Math. J. 177 (2005) 155–179.
[27] A.V. Jayanthan, J.K. Verma, Hilbert coefficients and depth of fiber cones, J. Pure Appl. Algebra 201 (1–3) (2005) 97–115.
[28] A.V. Jayanthan, T.J. Puthenpurakal, J.K. Verma, On fiber cones of $\mathfrak{m}$-primary ideals, Canad. J. Math. 59 (1) (2007) 109–126.
[29] M. Morales, A. Simis, Symbolic powers of monomial curves in P^3 lying on a quadric surface, Comm. Algebra 20 (4) (1992) 1109–1121.
[30] D.G. Northcott, D. Rees, Reductions of ideals in local rings, Proc. Cambridge Philos. Soc. 50 (1954) 145–158.
[31] M.E. Rossi, A bound on the reduction number of a primary ideal, Proc. Amer. Math. Soc. 128 (5) (1999) 1325–1332.
[32] J.D. Sally, Tangent cones at Gorenstein singularities, Compos. Math. 40 (2) (1980) 167–175.
[33] K. Shah, On the Cohen–Macaulayness of the fiber cone of an ideal, J. Algebra 143 (1) (1991) 156–172.
[34] K. Shah, On equimultiple ideals, Math. Z. 215 (1) (1994) 13–24.
[35] J. Stückrad, W. Vogel, Buchsbaum Rings and Applications. An Interaction Between Algebra, Geometry and Topology, Springer-Verlag, Berlin, 1986.
[36] G. Valla, On from rings which are Cohen–Macaulay, J. Algebra 58 (2) (1979) 247–250.
[37] W.V. Vasconcelos, Computational Methods in Commutative Algebra and Algebraic Geometry, with Chapters by David Eisenbud, Daniel R. Grayson, Jürgen Herzog and Michael Stillman, Algorithms and Computation in Mathematics, vol. 2, Springer-Verlag, Berlin, 1998.
[38] D.Q. Viêt, On the multiplicity of the fiber cones and when is the fiber cone Cohen–Macaulay?, preprint, 2004.
[39] A. Wiles, Modular elliptic curves and Fermat's last theorem, Ann. of Math. 141 (3) (1995) 443–551.