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Stanley conjecture in small embedding dimension ☆

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Abstract

We show that Stanley's conjecture holds for a polynomial ring over a field in four variables. In the case of polynomial ring in five variables, we prove that the monomial ideals with all associated primes of height two, are Stanley ideals.

Keywords: Monomial ideals; Prime filtrations; Pretty clean filtrations; Stanley ideals

1. Introduction

Let $S = K[x_1, x_2, \dots, x_n]$ be a polynomial ring in n variables over a field K and $I \subset S$ a monomial ideal. In this paper a *prime filtration* of I is assumed to be a monomial prime filtration, that is a monomial filtration

$$\mathcal{F}: \quad I = I_0 \subset I_1 \subset \cdots \subset I_r = S$$

with $I_j/I_{j-1} \cong S/P_j(-a_j)$ for some monomial prime ideals P_j of S, $a_j \in \mathbb{N}^n$ and $j = 1, 2, \dots, r$. Set $\mathrm{Supp}(\mathcal{F}) = \{P_1, \dots, P_r\}$. After [4], the prime filtration $\mathcal{F}$ is called *pretty clean*,

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if for all $i < j$ for which $P_i \subseteq P_j$, it follows that $P_i = P_j$. The monomial ideal I is called *pretty clean*, if it has a pretty clean filtration.

Let $I \subset S$ be a monomial ideal, any decomposition of S/I as a direct sum of K-vector spaces of the form $uK[Z]$ where u is a monomial in S, and $Z \subseteq \{x_1, x_2, \ldots, x_n\}$ is called *Stanley decomposition*. Stanley [9] conjectured that there always exists a Stanley decomposition

$$S/I = \bigoplus_{i=1}^{r} u_i K[Z_i]$$

such that $|Z_i| \geqslant \operatorname{depth}(S/I)$ for all i, $1 \leqslant i \leqslant r$. If this is the case, we call I a *Stanley ideal*. Sometimes Stanley decompositions of S/I arise from prime filtrations. In fact, if $\mathcal{F}$ is a prime filtration of S/I with factors $(S/P_i)(-a_i)$ for $i = 1, 2, \ldots, r$ then set $u_i = \prod_{j=1}^{n} x_j^{a_{ij}}$ and $Z_i = \{x_j\colon x_j \notin P_i\}$ and we have

$$S/I = \bigoplus_{i=1}^{r} u_i K[Z_i].$$

If $\mathcal{F}$ is a pretty clean filtration of S/I, then by [4, Corollary 3.4]

$$\operatorname{Ass}(S/I) = \operatorname{Supp}(\mathcal{F}).$$

The converse is not always true (see [8, Example 4.4]). However not all Stanley decompositions arise from the prime filtrations (see [5, Example 3.8]). A prime filtration $\mathcal{F}$ is a *Stanley filtration* if the Stanley decomposition arising from $\mathcal{F}$ satisfies the Stanley conjecture. In [8, Proposition 2.2] it is shown that all prime filtrations $\mathcal{F}$ for which $\operatorname{Ass}(S/I) = \operatorname{Supp}(\mathcal{F})$ are Stanley filtrations, in particular all monomial ideals $I \subset S$ for which S/I is pretty clean, are Stanley (see [4, Theorem 6.5]). In case $n = 3$, for any monomial ideal $I \subset S$ we have S/I is pretty clean by [8, Theorem 1.10] and so I is Stanley. This result was first obtained by different methods in [2]. Recently, Herzog, Soleyman Jahan, Yassemi [3, Proposition 1.4] showed that if I is a monomial ideal of S (for any n) such that S/I is Cohen–Macaulay of codimension two then I is a Stanley ideal.

It is the purpose of our note to describe the Stanley ideals of the polynomial ring $S = K[x_1, x_2, \ldots, x_n]$, $n \leqslant 5$. If $n = 4$ we show that all monomial ideals of S are Stanley (see Theorem 2.4). This extends [1, Corollary 1.4], which says that a sequentially Cohen–Macaulay monomial ideal $I \subset K[x_1, \ldots, x_4]$ is Stanley. If $n = 5$ we show that all monomial ideals $I \subset S$ having all the associated prime ideals of height 2 are Stanley ideals (see Corollary 2.3).

2. Stanley's conjecture in small embedding dimension

We start with a very elementary lemma.

Lemma 2.1. *Let $S = K[x_1, x_2, \ldots, x_n]$, $T = K[x_1, x_2, \ldots, x_r]$ for some $1 \leqslant r \leqslant n$ and $\mathcal{J} \subset T$ a monomial ideal. Then $T/\mathcal{J}$ is pretty clean if and only if $S/\mathcal{J}S$ is pretty clean.*

The following lemma is a key result in this note.

Lemma 2.2. *Let $S = K[x_1, x_2, \dots, x_n]$, $n \leqslant 5$, be a polynomial ring and $I \subset S$ a monomial ideal having all the associated primes of height 2. Then there exists a prime filtration $\mathcal{F}$ of S/I such that $ht(P) \leqslant 3$ for all $P \in \operatorname{Supp}(\mathcal{F})$.*

Proof. We use induction on $\mathbf{s}(I)$, where $\mathbf{s}(I)$ denotes the number of irreducible monomial ideals appearing in the unique decomposition of I as an intersection of irreducible monomial ideals (see [10, Theorem 5.1.17]), let us say

$$I = \bigcap_{i=1}^{\mathbf{s}(I)} Q_i,$$

where Q_i's are irreducible monomial ideals of codimension 2. If $\mathbf{s}(I) = 1$, then the result follows because S/I is clean. If $\mathbf{s}(I) \geqslant 1$, then set

$$\mathcal{J} = \bigcap_{i=2}^{\mathbf{s}(I)} Q_i.$$

Therefore $I = \mathcal{J} \cap Q_1$. We may suppose $Q_1 = (x_1^{d_1}, x_2^{d_2})$ after renumbering of variables, with d_1 the largest power of x_1 in $\bigcup_{i=1}^{\mathbf{s}(I)} G(Q_i)$, where $G(Q_i)$ is the set of minimal monomial generators of Q_i. We claim that

$$\mathcal{F}_0 = I \subset \mathcal{F}_1 = \left(I, x_2^{d_2}\right) \subset \mathcal{F}_2 = \left(x_1^{d_1}, x_2^{d_2}\right) \subset \mathcal{F}_3 = S$$

is a filtration of S/I, which will give the desired filtration by refining. Note that $\mathcal{F}_3/\mathcal{F}_2 = S/(x_1^{d_1}, x_2^{d_2})$ is a clean module, so $\mathcal{F}_3/\mathcal{F}_2$ has a prime filtration involving only the prime (x_1, x_2).

Now for $\mathcal{F}_2/\mathcal{F}_1 = (x_1^{d_1}, x_2^{d_2})/(I, x_2^{d_2}) \cong S/((I, x_2^{d_2}) : x_1^{d_1})$ we have

$$E := \left(\left(I, x_2^{d_2}\right) : x_1^{d_1}\right) = \left(I : x_1^{d_1}, x_2^{d_2}\right) = \left(\mathcal{J} : x_1^{d_1}, x_2^{d_2}\right)$$

and we get

$$E = \bigcap_{i=2}^{\mathbf{s}(I)} \left(\left(Q_i : x_1^{d_1}\right), x_2^{d_2}\right).$$

Set $T = K[x_2, \dots, x_n]$. Since $U_i := ((Q_i : x_1^{d_1}), x_2^{d_2})$ is either S, or an irreducible ideal of height 2 or 3 in the variables $x_2, \dots, x_n$, we note that $E = WS$ for a monomial ideal $W \subset T$ with all associated prime ideals of dimension $n-2$ or $n-3$. If $n = 4$ then $\dim T = 3$ and T/W is pretty clean by [8, Theorem 1.10] and so S/E is pretty clean by Lemma 2.1. If $n = 5$ then set $G := \bigcap_{i=2,\ ht(U_i)=2}^{\mathbf{s}(I)} U_i$ and consider the filtration $W \subset G \subset T$ (this is the dimension filtration of [7]). As $\mathbf{s}(G) < \mathbf{s}(I)$ we get by induction hypothesis a prime filtration of S/G involving just prime of height $\leqslant 3$. Since $\operatorname{Ass}(G/W)$ contains just prime ideals of height 3 we get G/W clean by [6, Corollary 2.2]. So we get a prime filtration of T/W and by extension of S/E, involving only prime ideals of height $\leqslant 3$. Therefore in both cases, $\mathcal{F}_2/\mathcal{F}_1 \cong S/E$ has a prime filtration involving only prime ideals of height at most 3.

Finally, $\mathcal{F}_1/\mathcal{F}_0 = (I, {x_2}^{d_2})/I \cong S/(I : x_2^{d_2})$, and we have

$$\bigl(I : x_2^{d_2}\bigr) = \bigl(\mathcal{J} : x_2^{d_2}\bigr) = \bigcap_{i=2}^{\mathbf{s}(I)} \bigl(Q_i : x_2^{d_2}\bigr),$$

where $(Q_i : x_2^{d_2})$ is either S, or irreducible of height 2. Since $\mathbf{s}(I : x_2^{d_2}) \leqslant \mathbf{s}(I) - 1 < \mathbf{s}(I)$, we get by induction hypothesis that $S/(I : x_2^{d_2})$ (and so $\mathcal{F}_1/\mathcal{F}_0$) has a prime filtration with prime ideals of height at most 3. Then by gluing together all these prime filtrations we get the desired one. $\square$

Corollary 2.3. *Let $S = K[x_1, x_2, \ldots, x_n]$, $n \leqslant 5$, be a polynomial ring and $I \subset S$ a monomial ideal having all the associated prime ideals of height 2. Then I is a Stanley ideal.*

Proof. If S/I is Cohen–Macaulay then I is a Stanley ideal by [3, Proposition 1.4]. Now if S/I is not Cohen–Macaulay then $\operatorname{depth}(S/I) \leqslant n-3$ because $\dim(S/I) = n-2$. Let $\mathcal{F}$ be the filtration given by Lemma 2.2. Then all the associated primes P of $\operatorname{Supp}(\mathcal{F})$ satisfy the condition,

$$\dim(S/P) \geqslant n - 3 \geqslant \operatorname{depth}(S/I).$$

Thus I is a Stanley ideal. $\square$

Theorem 2.4. *Any monomial ideal I of $S = K[x_1, x_2, x_3, x_4]$ is a Stanley ideal.*

Proof. Let $\mathbf{s}(I)$ be the number of irreducible monomial ideals appearing in the unique decomposition of I as an intersection of irreducible monomial ideals, let us say

$$I = \bigcap_{i=1}^{4} \bigcap_{j=1}^{\mathbf{s}(Q_i)} Q_{ij} \quad \text{and} \quad Q_i = \bigcap_{j=1}^{\mathbf{s}(Q_i)} Q_{ij}, \quad \mathbf{s}(I) = \sum_{i=1}^{4} \mathbf{s}(Q_i),$$

where Q_{ij}'s are irreducible monomial ideals of height i. If $ht(I) = t$, $1 \leqslant t \leqslant 4$, then $Q_k = S$ and $\mathbf{s}(Q_k) = 0$ for all $1 < k < t$. After Schenzel [7], the dimension filtration of I will be

$$\mathcal{F}_0 = I \subset \mathcal{F}_1 = Q_1 \cap Q_2 \cap Q_3 \subset \mathcal{F}_2 = Q_1 \cap Q_2 \subset \mathcal{F}_3 = Q_1 \subset \mathcal{F}_4 = S.$$

Now consider $\mathcal{F}_4/\mathcal{F}_3 \cong S/Q_1$, where $Q_1 = \bigcap_{j=1}^{\mathbf{s}(Q_1)} Q_{1j}$ with Q_{1j}'s principal ideals. Therefore $Q_1 = (u)$ for a monomial u in S (it is factorial ring) and so S/Q_1 is pretty clean (see e.g. the proof of [8, Lemma 1.9]). Hence $\mathcal{F}_3$ is a Stanley ideal.

Now take $\mathcal{F}_3/\mathcal{F}_2 \cong Q_1/(Q_1 \cap Q_2) \cong S/(Q_2 : u)$, where

$$(Q_2 : u) = \bigcap_{j=1}^{\mathbf{s}(Q_2)} (Q_{2j} : u).$$

Also $(Q_{2j} : u)$ is either S or of height 2 for all j. By Corollary 2.3, $(Q_2 : u)$ is a Stanley ideal and so is $\mathcal{F}_2$ because the clean filtrations of $S/(u)$ involve only prime ideals of depth 3. If $\mathbf{s}(Q_3) = \mathbf{s}(Q_4) = 0$ then we are done. If $\mathbf{s}(Q_3) \neq 0$ then $\operatorname{depth}(S/I) \leqslant 1$ and $\operatorname{Ass}(\mathcal{F}_2/\mathcal{F}_1)$ contains only prime ideals of height 3. Hence $\mathcal{F}_2/\mathcal{F}_1$ is pretty clean by [6, Corollary 2.2] and so $\mathcal{F}_2/\mathcal{F}_1$ has a prime filtration involving only prime ideals of height 3. But as above $S/\mathcal{F}_2$ has a prime filtration with prime ideals of height $\leqslant 3$. Gluing together these two prime filtrations we get a prime filtration with prime ideals P such that $\dim(S/P) \geqslant 1 \geqslant \operatorname{depth}(S/I)$. So $\mathcal{F}_1$ is a Stanley ideal. If $s(Q_4) \neq 0$ then $\operatorname{depth}(S/I) = 0$ and every prime filtration gives a Stanley filtration. $\square$

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