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Visible actions on flag varieties of exceptional groups and a generalization of the Cartan decomposition ☆

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ABSTRACT

We give a generalization of the Cartan decomposition for connected compact exceptional Lie groups motivated by the work on visible actions of T. Kobayashi [T. Kobayashi, J. Math. Soc. Japan 59 (2007) 669–691] for type A groups. This paper extends his results to the exceptional groups. First, we classify a pair of Levi subgroups (L, H) of any compact exceptional simple Lie group G such that $G = LG^{\sigma}H$ where σ is a Chevalley–Weyl involution. This implies that the natural L-action on the generalized flag variety G/H is strongly visible, and likewise the H-action on G/L and the G-action on $(G \times G)/(L \times H)$ are strongly visible. Second, we find a generalized Cartan decomposition $G = LBH$ with B in G^{σ} by using the herringbone stitch method which was introduced by Kobayashi. Applications to multiplicity-free representations are also discussed.

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1. Introduction and statement of main results

We give a classification of a pair of Levi subgroups (L, H) of any connected compact exceptional Lie group G such that $G = LG^\sigma H$ holds where σ is a Chevalley–Weyl involution of G. This can be interpreted as a generalization of the Cartan decomposition to the non-symmetric setting. (We refer the reader to [2–4,9,13] and references therein for some aspects of the Cartan decomposition from geometric and group theoretic viewpoints.) The motivation for considering such kind of decomposition comes from the notion of *visible action* on complex manifolds, which was introduced by T. Kobayashi. It is a geometric condition for the propagation theorem of the multiplicity-freeness property, and classification theory of (strongly) visible action has been recently made in the linear case [14], symmetric case [10] and some other non-symmetric cases [15].

A generalization of the Cartan decomposition for symmetric pairs has been used in various contexts including analysis on symmetric spaces, however, there was no analogous result for non-symmetric cases before Kobayashi's paper [9]. Motivated by visible actions on flag varieties of type A, he introduced a generalization of the Cartan decomposition for the unitary group $\mathrm{U}(n)$ taking the form:

$$G = LBH,$$

where B is a subset of the orthogonal group $\mathrm{O}(n)$. He completely classified a pair of Levi subgroups (L, H) satisfying $\mathrm{U}(n) = L\mathrm{O}(n)H$, and gave a *slice* B explicitly for each of such pairs (L, H) by the herringbone stitch method [9].

More generally, we consider the following problem: Let G be a connected compact Lie group, T a maximal torus, and σ a Chevalley–Weyl involution of G with respect to T. (We recall that an involutive automorphism σ of a connected compact Lie group G is said to be a Chevalley–Weyl involution if there exists a maximal torus T of G such that $\sigma(t) = t^{-1}$ for any $t \in T$ [18].)

1) Classify all the pairs of Levi subgroups L and H with respect to $\mathfrak{t}$ such that the multiplication map $\psi : L \times G^\sigma \times H \to G$ is surjective.
2) Find a "good" representative $B \subset G^\sigma$ such that $G = LBH$ in the case ψ is surjective.

Here $G^\sigma = \{g \in G\colon\ \sigma(g) = g\}$. We call such a decomposition $G = LBH$ a *generalized Cartan decomposition*. Let us note that the role of H and L is symmetric. The surjectivity of ψ implies that the subgroup L acts on G/H in a (strongly) visible fashion (see Definition 6.1). At the same time the H-action on G/L, and the diagonal G-action on $(G \times G)/(L \times H)$ are strongly visible. Then the propagation theorem of multiplicity-freeness property [8, Theorem 4.3] leads us to three multiplicity-free theorems (*triunity* à la [6]):

$$\begin{aligned}
&\text{Restriction } G \downarrow L\colon && \mathrm{Ind}_H^G(\mathbb{C}_\lambda)|_L,\\
&\text{Restriction } G \downarrow H\colon && \mathrm{Ind}_L^G(\mathbb{C}_\lambda)|_H,\\
&\text{Tensor product:} && \mathrm{Ind}_H^G(\mathbb{C}_\lambda) \otimes \mathrm{Ind}_L^G(\mathbb{C}_\mu).
\end{aligned}$$

Here $\mathrm{Ind}_H^G(\mathbb{C}_\lambda)$ denotes a holomorphically induced representation of G from a unitary character $\mathbb{C}_\lambda$ of H by the Borel–Weil theorem. See [6–8,10,11] for the general theory on the application of visible actions (including the vector bundle setting), and also Section 6 for an application of our results to the exceptional groups.

For compact classical Lie groups of types B, C and D, the aforementioned problems will be treated in separated papers (partly by using invariant theory for quivers). In this article we deal with exceptional Lie groups, and thus give a complete solution to the above problems for all compact Lie groups. Here we remark that a classification of a triple (G, L, H) satisfying $G = LG^\sigma H$ for any connected compact simple Lie group G has already been published in [17] with a sketch of the proof. This is now the detailed version for the exceptional case.

In order to state our main results, we label the Dynkin diagrams of type E_6 and type E_7 following Bourbaki [1] (see Figs. 3.2 and 3.3). For a subset Π' of a simple system Π, we denote by $\mathrm{L}_{\Pi'}$ the Levi subgroup whose root system is generated by Π', and by $(\Pi')^c$ for the complement of Π' in Π.

Theorem 1.1. *Let G be a connected compact Lie group with an exceptional simple Lie algebra $\mathfrak{g}$, Π a simple system of the root system $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ with respect to a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, and σ a Chevalley–Weyl involution of G with respect to $\mathfrak{t}$. Take proper subsets Π' and Π'' of Π. Then the following two conditions are equivalent.*

(i) $G = \mathrm{L}_{\Pi'} G^\sigma \mathrm{L}_{\Pi''}$.
(ii) *Up to switch of Π' and Π'', one of the below conditions is satisfied.*

I. $\mathfrak{g} = \mathfrak{e}_6$, $(\Pi')^c = \{\alpha_i\}$, $(\Pi'')^c = \{\alpha_1, \alpha_6\}$, $i = 1$ *or* 6.
II. $\mathfrak{g} = \mathfrak{e}_6$, $(\Pi')^c = \{\alpha_i\}$, $(\Pi'')^c = \{\alpha_j\}$, $i = 1$ *or* 6, $j \neq 4$.
III. $\mathfrak{g} = \mathfrak{e}_7$, $(\Pi')^c = \{\alpha_7\}$, $(\Pi'')^c = \{\alpha_i\}$, $i = 1, 2$ *or* 7.

In particular, there are no such pair (Π', Π'') for $\mathfrak{g} = \mathfrak{e}_8$, $\mathfrak{f}_4$ or $\mathfrak{g}_2$.

Cases I, II and III amount to

I. $\mathfrak{g} = \mathfrak{e}_6$, $\mathfrak{l}_{\Pi'} = \mathfrak{so}(10) \oplus \mathbb{R}$, $\mathfrak{l}_{\Pi''} = \mathfrak{so}(8) \oplus \mathbb{R}^2$.
II. $\mathfrak{g} = \mathfrak{e}_6$, $\mathfrak{l}_{\Pi'} = \mathfrak{so}(10) \oplus \mathbb{R}$, $\mathfrak{l}_{\Pi''} = \mathfrak{so}(10) \oplus \mathbb{R}$, $\mathfrak{su}(6) \oplus \mathbb{R}$ or $\mathfrak{su}(2) \oplus \mathfrak{su}(5) \oplus \mathbb{R}$.
III. $\mathfrak{g} = \mathfrak{e}_7$, $\mathfrak{l}_{\Pi'} = \mathfrak{e}_6 \oplus \mathbb{R}$, $\mathfrak{l}_{\Pi''} = \mathfrak{so}(12) \oplus \mathbb{R}$, $\mathfrak{su}(7) \oplus \mathbb{R}$ or $\mathfrak{e}_6 \oplus \mathbb{R}$.

As a corollary of Theorem 1.1, we obtain three kinds of multiplicity-free theorems for representations of exceptional Lie groups (see Corollary 6.4 for the restrictions to Levi subgroups and Corollary 6.5 for the tensor products). In the course of the proof, we find explicitly a slice B that gives a generalized Cartan decomposition $G = \mathrm{L}_{\Pi'} B \mathrm{L}_{\Pi''}$ (see Propositions 3.2, 3.3, 3.5, 3.6, 3.8 and 3.10) by using the herringbone stitch method [9]. The 'slice' B plays an important role in dealing with more delicate cases (vector bundle cases) in the application to representation theory, which is not discussed in this article.

Special feature of exceptional Lie groups. Here we mention some new features in dealing with exceptional groups, which arise both in the proof and in the main results:

- (In the proof.) In order to find explicit generalized Cartan decompositions in the exceptional case, our argument relies on the root systems rather than matrix computations that were effectively used in the classical case.
- (In the main results.) For all classical compact groups G, there exist pairs of proper Levi subgroups $\mathrm{L}_{\Pi'}$ and $\mathrm{L}_{\Pi''}$ such that the multiplication mapping $\mathrm{L}_{\Pi'} \times G^\sigma \times \mathrm{L}_{\Pi''} \to G$ is surjective [9,17]. However, none of the exceptional compact Lie groups G_2, F_4 or E_8 admits such a pair of proper Levi subgroups. This corresponds to the representation theoretic fact (cf. [12,16] and Section 6 in this article) that $\#\mathrm{MF}_f(G, L)$ is finite for any Levi subgroup L of G if and only if G is of type E_8, F_4 or G_2, where $\mathrm{MF}_f(G, L)$ is the set of equivalence classes of finite dimensional irreducible representations π of G such that the restrictions $\pi|_L$ to L are multiplicity-free.

Organization of this article. This article is organized as follows. In Section 2 we discuss a slice for symmetric pairs. In Section 3, we give a proof of the implication (ii) $\Rightarrow$ (i) together with a generalized Cartan decomposition $G = L_{\Pi'} B L_{\Pi''}$ by postponing the proofs of some technical lemmas to Section 4. In Section 4, we deal with double coset decompositions of classical Lie groups, and complete the proof for the implication (ii) $\Rightarrow$ (i). The converse implication (i) $\Rightarrow$ (ii) is proved in Section 5 by using the fact that a strongly visible action gives rise to multiplicity-free representations, and classifications of multiplicity-free tensor product representations by P. Littelmann [12] for the maximal parabolic case and J.R. Stembridge [16] for the general case. An application to multiplicity-free representations is discussed in Section 6.

In the following, we denote Lie groups by capital Latin letters and their Lie algebras by corresponding small German letters. Also, for a given real Lie algebra $\mathfrak{g}$, we denote its complexification by $\mathfrak{g}_{\mathbb{C}}$.

2. Construction of a slice for the symmetric case

Let $\mathfrak{g}$ be a compact Lie algebra and τ an involution of $\mathfrak{g}$. Then we take a τ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and write $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ where $\mathfrak{t} = \mathfrak{h}^{\tau}$ and $\mathfrak{a} = \mathfrak{h}^{-\tau}$. Here, $\mathfrak{h}^{-\tau}$ is defined by $\mathfrak{h}^{-\tau} := \{X \in \mathfrak{h}\colon \tau(X) = -X\}$. In this section, we shall see how to construct a maximal abelian subspace of $\mathfrak{g}^{-\tau}$, which is fixed by σ. We begin by the following proposition.

Proposition 2.1. *Let us suppose that there exists an automorphism σ of $\mathfrak{g}$, which preserves $\mathfrak{a}$ and acts on $\mathfrak{t}$ as the multiplication by (-1), and that the Cartan subalgebra $\mathfrak{t} \oplus \sqrt{-1}\mathfrak{a}$ of the non-compact dual $\mathfrak{g}^{\tau} \oplus \sqrt{-1}\mathfrak{g}^{-\tau}$ is not maximally non-compact. Then for any root vector $X_\beta \in \mathfrak{g}_{\mathbb{C}}$ of any imaginary non-compact root β, there exists $Z \in \mathfrak{t}$ such that $\mathrm{Ad}(\exp(Z))(X_\beta + \overline{X_\beta})$ is fixed by σ. Here we extend σ to $\mathfrak{g}_{\mathbb{C}}$ holomorphically, and $\overline{X}$ denotes the conjugate element with respect to $\mathfrak{g}$ for any $X \in \mathfrak{g}_{\mathbb{C}}$.*

Proof. Since both $\overline{X_\beta}$ and $\sigma(X_\beta)$ belong to the root subspace $\mathfrak{g}_{-\beta}$ of $-\beta$, $\sigma(X_\beta) = e^{\sqrt{-1}\theta}\overline{X_\beta}$ for some $\theta \in \mathbb{R}$. Then we take $Z \in \mathfrak{t}$ satisfying $\beta(Z) = -\sqrt{-1}\theta/2$. (Here we note that β is imaginary.) For this $Z \in \mathfrak{t}$, we have

$$\begin{aligned}
\sigma\bigl(\mathrm{Ad}\bigl(\exp(Z)\bigr)(X_\beta + \overline{X_\beta})\bigr) &= \sigma\bigl(e^{-\frac{\sqrt{-1}\theta}{2}} X_\beta + \overline{e^{\frac{-\sqrt{-1}\theta}{2}} X_\beta}\bigr) \\
&= e^{-\frac{\sqrt{-1}\theta}{2}}\bigl(e^{\sqrt{-1}\theta}\overline{X_\beta}\bigr) + e^{\frac{\sqrt{-1}\theta}{2}}\bigl(e^{-\sqrt{-1}\theta} X_\beta\bigr) \\
&= e^{\frac{\sqrt{-1}\theta}{2}}\overline{X_\beta} + e^{-\frac{\sqrt{-1}\theta}{2}} X_\beta \\
&= \mathrm{Ad}\bigl(\exp(Z)\bigr)(X_\beta + \overline{X_\beta}).
\end{aligned}$$

This completes the proof. $\square$

By using Proposition 2.1, we can give a simpler proof of the following result, which was originally proved in [10] by Berger's classification of semisimple symmetric pairs.

Corollary 2.2. *Let us suppose that $(\mathfrak{g}, \mathfrak{g}^{\tau})$ is a Hermitian symmetric pair and that $\mathfrak{a} = \{0\}$, i.e., $\mathfrak{h} = \mathfrak{t}$. Let σ be a Chevalley–Weyl involution of $\mathfrak{g}$ with respect to $\mathfrak{t}$. Then there exists a maximal abelian subspace of $\mathfrak{g}^{-\tau}$, which is fixed by σ.*

Proof. Consider the non-compact dual $\mathfrak{g}^{\tau} \oplus \sqrt{-1}\mathfrak{g}^{-\tau}$ of $\mathfrak{g}$ with respect to τ. Since we can construct a maximally non-compact Cartan subalgebra by a succession of the Cayley transforms from $\mathfrak{t}$, the corollary follows from Proposition 2.1. Here we note that we can choose strongly orthogonal roots in a succession of the Cayley transforms if $(\mathfrak{g}, \mathfrak{g}^{\tau})$ is a Hermitian symmetric pair. $\square$

Remark 2.3. From the proofs of Proposition 2.1 and Corollary 2.2, we can see the following: Retain the setting of the proof of Corollary 2.2. We let $\{\beta_1, \dots, \beta_r\}$ be a set of imaginary non-compact roots (with respect to $\mathfrak{t} \otimes \mathbb{C}$), which may be used in a succession of the Cayley transforms for obtaining a σ-fixed maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{g}^{-\tau}$. If some roots (with respect to $\mathfrak{t} \otimes \mathbb{C}$) $\alpha_1, \dots, \alpha_s$ are strongly orthogonal to β_i for any i $(1 \leqslant i \leqslant r)$, then a semisimple subalgebra whose set of simple roots is given by $\{\alpha_1, \dots, \alpha_s\}$ centralizes $\mathfrak{a}$.

Remark 2.3 can help us to determine a set of simple roots of the centralizer of $\mathfrak{a}$ in the next section.

3. Generalized Cartan decomposition

The aim of this section is to prove that (ii) implies (i) in Theorem 1.1 (by postponding some technical lemmas to the next section). The idea is to use the herringbone stitch method [9] that reduces unknown decompositions for non-symmetric pairs to the known Cartan decomposition for symmetric pairs (Fact 3.1). We divide the proof to four parts (Sections 3.1–3.4).

In the following, $\mathfrak{k}_{ss}$ denotes the semisimple part of $\mathfrak{k}$ and K_{ss} the analytic subgroup of K with Lie algebra $\mathfrak{k}_{ss}$ for a compact Lie group K. Also, we write $G_1 \approx G_2$ if two Lie groups G_1 and G_2 are locally isomorphic.

3.1. Decompositions for the symmetric case

In this subsection, we recall a well-known fact ([4, Theorem 6.10], [13, Theorem 1]) on the Cartan decomposition for the symmetric case, and deal with Case II with $j = 1$ or 6 and Case III with $i = 7$ in Theorem 1.1.

Fact 3.1. Let K be a connected compact Lie group with Lie algebra $\mathfrak{k}$, and τ and τ' two involutions of K. Let H and L be subgroups of K such that

$$\left(K^{\tau}\right)_0 \subset L \subset K^{\tau} \quad \text{and} \quad \left(K^{\tau'}\right)_0 \subset H \subset K^{\tau'}.$$

Here F_0 denotes the connected component of F containing the identity element for a Lie group F.

We take a maximal abelian subspace $\mathfrak{a}$ in

$$\mathfrak{k}^{-\tau,-\tau'} := \left\{X \in \mathfrak{k}\colon\ \tau(X) = \tau'(X) = -X\right\}.$$

Suppose that $\tau\tau'$ is semisimple on the center $\mathfrak{z}$ of $\mathfrak{k}$. Then we have

$$K = L \exp(\mathfrak{a}) H.$$

By combining this fact with Corollary 2.2, we immediately obtain the following (cf. [10]).

Proposition 3.2. (*Case II with $j = 1$ or 6 and Case III with $i = 7$.*) *Let G be a connected compact Lie group, $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$, σ a Chevalley–Weyl involution of G with respect to $\mathfrak{t}$ and L, H Levi subgroups of G with respect to a simple system of the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Suppose that both L and H are Hermitian symmetric subgroups of G. Then we have*

$$G = L \exp(\mathfrak{a}) H$$

where $\mathfrak{a}$ is an abelian subspace of $\mathfrak{g}^{\sigma}$.

Since the surjectivity of the multiplication mapping $\mathrm{L}_{\Pi'} \times G^\sigma \times \mathrm{L}_{\Pi''} \to G$ depends on neither the coverings of the group G nor the choice of Cartan subalgebras and Chevalley–Weyl involutions, we may and do work with connected simply connected compact exceptional Lie groups, and fix a Cartan subalgebra and a Chevalley–Weyl involution in each of the subsections below.

3.2. *Decompositions for the type E_6 (non-maximal parabolic type)*

In this subsection, we deal with Case I in Theorem 1.1. (See Fig. 3.1.)

Proposition 3.3. (*Case I.*) *Let G be the connected simply connected compact simple Lie group of type* E_6, $\mathfrak{t}$ *a Cartan subalgebra of $\mathfrak{g}$ and σ a Chevalley–Weyl involution of G with respect to $\mathfrak{t}$. Take two subsets Π' and Π'' of a simple system Π as $(\Pi')^c = \{\alpha_i\}$ $(i = 1$ or $6)$ and $(\Pi'')^c = \{\alpha_1, \alpha_6\}$. (We label the Dynkin diagram following Bourbaki [1]. See Fig. 3.2 in Section 3.3.) Then we have*

$$G = \mathrm{L}_{\Pi'} B \mathrm{L}_{\Pi''}$$

for a subset $B \subset G^\sigma$.

Proof. Let us explicitly write the root system Δ and the simple system Π of type E_6 as follows (see Plate V of [1]):

$$\begin{aligned}
&\Delta = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \left\{ \pm\varepsilon_i \pm \varepsilon_j, \frac{1}{2}\left(\varepsilon_8 - \varepsilon_7 - \varepsilon_6 + \sum_{k=1}^{5} (-1)^{\nu_k} \varepsilon_k \right) : 1 \leqslant i < j \leqslant 5, \sum_{k=1}^{5} \nu_k \text{ is even} \right\}, \\
&\Pi = \{\alpha_i : 1 \leqslant i \leqslant 6\}, \\
&\quad \text{where } \alpha_1 = \frac{1}{2}(\varepsilon_1 + \varepsilon_8) - \frac{1}{2}(\varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7),\ \alpha_2 = \varepsilon_1 + \varepsilon_2,\ \alpha_3 = \varepsilon_2 - \varepsilon_1, \\
&\quad\ \alpha_4 = \varepsilon_3 - \varepsilon_2,\ \alpha_5 = \varepsilon_4 - \varepsilon_3,\ \alpha_6 = \varepsilon_5 - \varepsilon_4.
\end{aligned}$$

We may and do assume that $i = 6$ since $\mathrm{L}_{\{\alpha_1\}^c}$ is conjugate to $\mathrm{L}_{\{\alpha_6\}^c}$ under the action of the Weyl group, and hence that of G^σ [5, Theorem 6.57]. Let τ denote the involution of G, which corresponds to $\mathrm{L}_{\Pi'}$. By using two non-compact imaginary roots $\varepsilon_5 - \varepsilon_4$ and $\varepsilon_5 + \varepsilon_4$ for the Cayley transforms of the compact Cartan subalgebra $\mathfrak{t}$ of the non-compact dual $\mathfrak{g}^\tau \oplus \sqrt{-1}\mathfrak{g}^{-\tau}$ of $\mathfrak{g}$, we obtain a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{g}^{-\tau}$, which is fixed by σ (Corollary 2.2). We apply Fact 3.1 to (G, τ, τ) as follows.

$$G = G^\tau \exp(\mathfrak{a}) G^\tau. \tag{3.1}$$

Since the pair $(\mathfrak{g}, \mathfrak{g}^\tau)$ is Hermitian of non-tube type, there exists $X \in \mathfrak{g}^\tau_{ss}$ such that $\mathbb{R}(Z + X)$ is the center of $Z_{\mathfrak{g}^\tau}(\mathfrak{a})$ where Z is a non-zero element of the center of $\mathfrak{g}^\tau$. Then we have the following lemma on a representative of the double coset decomposition of $G^\tau_{ss} (\approx \mathrm{SO}(10))$ by $\exp(\mathbb{R}X) \cdot M_{ss}$ and $G^\tau_{ss} \cap \mathrm{L}_{\Pi''} (\approx \mathrm{U}(1) \times \mathrm{SO}(8))$, where $M (\approx \mathrm{U}(4))$ is the analytic subgroup of G^τ with Lie algebra $Z_{\mathfrak{g}^\tau}(\mathfrak{a})$.

Lemma 3.4. *There exists a subset B' of G^σ such that the multiplication mapping*

$$\bigl(\exp(\mathbb{R}X) M_{ss}\bigr) \times B' \times \bigl(G^\tau_{ss} \cap \mathrm{L}_{\Pi''}\bigr) \to G^\tau_{ss}$$

is surjective.

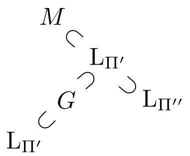

Fig. 3.1. Herringbone stitch used for $\mathrm{L}_{\Pi'} \setminus G/\mathrm{L}_{\Pi''}$ in Case I.

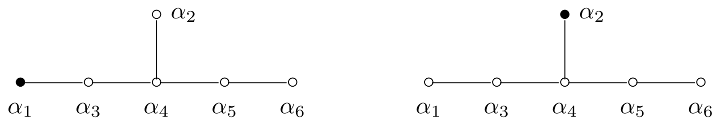

Fig. 3.2. Vogan diagrams of type E III and type E II.

We postpone the proof of this lemma to Lemma 4.1 in Section 4. The above surjection implies that $G^\tau_{ss} = (\exp(\mathbb{R}X)M_{ss})B'(G^\tau_{ss} \cap \mathrm{L}_{\Pi''})$, and thus we obtain

$$G^\tau = MB'\mathrm{L}_{\Pi''}. \tag{3.2}$$

Then we put $B = \exp(\mathfrak{a})B'$, and substitute (3.2) to (3.1) as follows.

$$G = G^\tau \exp(\mathfrak{a})\bigl(MB'\mathrm{L}_{\Pi''}\bigr) = G^\tau M \exp(\mathfrak{a})B'\mathrm{L}_{\Pi''} = \mathrm{L}_{\Pi'}B\mathrm{L}_{\Pi''}.$$

This completes the proof since $B = \exp(\mathfrak{a})B'$ is contained in G^σ. $\square$

3.3. Decompositions for type E_6 (maximal parabolic type)

In this subsection, we discuss Case II with $j = 2, 3$ or 5 in Theorem 1.1. Let G denote the connected simply connected compact simple Lie group of type E_6, $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$ and σ a Chevalley–Weyl involution of G with respect to $\mathfrak{t}$. We take a simple system Π of the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, and two commuting involutions τ and τ' of $\mathfrak{g}_{\mathbb{C}}$, which preserve $\mathfrak{g}$ and correspond to the Vogan diagrams of type E III and type E II respectively (see Appendix C of [5]).

Then the fixed part of the involution $\tau\tau'$ is given by $\mathfrak{g}^{\tau\tau'} = \mathbb{R} \oplus \mathfrak{so}(10)$. Since the root system $\Delta(\mathfrak{g}^{\tau\tau'}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ of $\mathfrak{g}^{\tau\tau'}$ is contained in $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, there exists $\gamma \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ such that $\{\alpha_3, \alpha_4, \alpha_5, \alpha_6, \gamma\}$ gives rise to a simple system of $\mathfrak{g}^{\tau\tau'}$. We may and do assume that γ is connected to α_4. We take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{g}^{-\tau,-\tau'}$ as follows: Let us explicitly write the simple system $\Pi(\mathfrak{g}^{\tau\tau'}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ and the root system $\Delta(\mathfrak{g}^{\tau\tau'}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ of $\mathfrak{g}^{\tau\tau'}_{\mathbb{C}}$ (see Plate IV of [1]).

$$\begin{aligned}
&\Pi\bigl(\mathfrak{g}^{\tau\tau'}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}\bigr) = \{\alpha_i,\ \gamma\colon 3 \leqslant i \leqslant 6\},\\
&\Delta\bigl(\mathfrak{g}^{\tau\tau'}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}\bigr) = \{\pm f_i \pm f_j\colon 1 \leqslant i < j \leqslant 5\},\\
&\quad \text{where } \alpha_3 = f_4 - f_5,\ \alpha_4 = f_3 - f_4,\ \alpha_5 = f_2 - f_3,\ \alpha_6 = f_1 - f_2,\ \gamma = f_4 + f_5.
\end{aligned}$$

Using two non-compact imaginary roots $f_2 + f_3$ and $f_4 + f_5$ for the Cayley transforms of the compact Cartan subalgebra $\mathfrak{t}$ of the non-compact dual $\mathfrak{g}^{\tau,\tau'} \oplus \sqrt{-1}\mathfrak{g}^{-\tau,-\tau'}$ of $\mathfrak{g}^{\tau\tau'}$, we obtain a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{g}^{-\tau,-\tau'}$, which is fixed by σ (Corollary 2.2). Then we consider the centralizer $Z_{\mathfrak{g}^{\tau,\tau'}}(\mathfrak{a})$ of $\mathfrak{a}$ in $\mathfrak{g}^{\tau,\tau'}$. For simplicity, we put $\mathfrak{m} = Z_{\mathfrak{g}^{\tau,\tau'}}(\mathfrak{a})$. We note the following decomposition of $\mathfrak{g}^{\tau\tau'}$.

$$\mathfrak{g}^{\tau\tau'} = \mathfrak{g}^{\tau,\tau'} \oplus \mathfrak{g}^{-\tau,-\tau'} = \mathbb{R}K_1 \oplus \bigl(\bigl(\mathfrak{g}^{\tau\tau'}\bigr)_{ss}\bigr)^\tau \oplus \mathfrak{g}^{-\tau,-\tau'} = \mathbb{R}K_1 \oplus \mathbb{R}K_2 \oplus \bigl(\mathfrak{g}^{\tau,\tau'}\bigr)_{ss} \oplus \mathfrak{g}^{-\tau,-\tau'},$$

where K_1 is a non-zero element of the center of $\mathfrak{g}^{\tau\tau'}$, and K_2 that of the center of $((\mathfrak{g}^{\tau\tau'})_{ss})^{\tau} = \mathfrak{u}(5)$. Since the pair $((\mathfrak{g}^{\tau\tau'})_{ss}, ((\mathfrak{g}^{\tau\tau'})_{ss})^{\tau}) = (\mathfrak{so}(10), \mathfrak{u}(5))$ is Hermitian of non-tube type, there exists $K_3 \in (\mathfrak{g}^{\tau,\tau'})_{ss} \cap \mathfrak{t}$ such that $\mathbb{R}(K_2 + K_3) \oplus \mathfrak{m}_{ss}$ gives rise to the centralizer of $\mathfrak{a}$ in $((\mathfrak{g}^{\tau\tau'})_{ss})^{\tau}$. Hence we obtain

$$\mathfrak{m} = \mathbb{R}K_1 \oplus \mathbb{R}(K_2 + K_3) \oplus \mathfrak{m}_{ss}. \tag{3.3}$$

The subalgebra $\mathfrak{g}^{\tau'}$ has two simple factors $\mathfrak{su}(2)$ and $\mathfrak{su}(6)$. So, we write $\mathfrak{g}^{\tau'} = \mathfrak{g}' \oplus \mathfrak{g}''$ where $\mathfrak{g}' = \mathfrak{su}(2)$ and $\mathfrak{g}'' = \mathfrak{su}(6)$. Then we decompose $\mathfrak{g}^{\tau'}$ as follows.

$$\begin{aligned}\mathfrak{g}^{\tau'} &= \mathfrak{g}^{\tau',\tau} \oplus \mathfrak{g}^{\tau',-\tau} = \left(\mathfrak{g}'\right)^{\tau} \oplus \left(\mathfrak{g}''\right)^{\tau} \oplus \mathfrak{g}^{\tau',-\tau} = \mathbb{R}Z_1 \oplus \left(\mathfrak{g}''\right)^{\tau} \oplus \mathfrak{g}^{\tau',-\tau} \\ &= \mathbb{R}Z_1 \oplus \mathbb{R}Z_2 \oplus \left(\mathfrak{g}^{\tau,\tau'}\right)_{ss} \oplus \mathfrak{g}^{\tau',-\tau}.\end{aligned}$$

Here Z_1 is a non-zero element of $(\mathfrak{g}')^{\tau}$, and Z_2 that of the center of $(\mathfrak{g}'')^{\tau} = \mathfrak{u}(5)$. Then we give generalized Cartan decompositions for Case II with $j = 2$ and $j = 3$ or 5 separately.

3.3.1. Case II with $j = 2$

Proposition 3.5. (*Case II with $j = 2$.*) *Let $G, \mathfrak{g}, \mathfrak{t}, \sigma, \Pi, \tau$ and τ' be as in the beginning of this subsection. Take two subsets Π' and Π'' of the simple system Π of the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ as $(\Pi')^c = \{\alpha_i\}$ and $(\Pi'')^c = \{\alpha_2\}$ where $i = 1$ or 6. Then we have*

$$G = \mathrm{L}_{\Pi'} B \mathrm{L}_{\Pi''}$$

for a subset $B \subset G^{\sigma}$.

Proof. Let G' and G'' be the analytic subgroups of $G^{\tau'}$ with Lie algebras $\mathfrak{g}' = \mathfrak{su}(2)$ and $\mathfrak{g}'' = \mathfrak{su}(6)$ respectively. We apply Fact 3.1 to (G, τ, τ'):

$$G = G^{\tau} \exp(\mathfrak{a}) G^{\tau'} = G^{\tau} \exp(\mathfrak{a}) G' G''. \tag{3.4}$$

Here $\mathfrak{a}$ is the maximal abelian subspace of $\mathfrak{g}^{-\tau,-\tau'}$, which is constructed in the above. Since $(\mathfrak{g}', \mathbb{R}Z_1)$ is also a symmetric pair, we can again use Fact 3.1 as follows.

$$G' = \exp(\mathbb{R}Z_1) \exp\left(\mathfrak{a}'\right) \exp(\mathbb{R}Z_1), \tag{3.5}$$

where $\mathfrak{a}'$ is the σ-fixed one-dimensional subspace of $\mathfrak{g}'$. Since the vector space $\mathbb{R}Z_1 \oplus \mathbb{R}Z_2$ coincides with $\mathbb{R}K_1 \oplus \mathbb{R}K_2$, there are real numbers a and b such that $Z_1 = aK_1 + bK_2$. Then we have the following equality.

$$\left(\exp(\mathbb{R}Z_1) \exp\left(\mathfrak{a}'\right) \exp(\mathbb{R}Z_1)\right) G'' = \left(\exp\left(\mathbb{R}\left(aK_1 + b(K_2 + K_3)\right)\right) \exp\left(\mathfrak{a}'\right) \exp(\mathbb{R}Z_1)\right) G''. \tag{3.6}$$

Put $B = \exp(\mathfrak{a}) \exp(\mathfrak{a}')$. By combining (3.5) and (3.6) with (3.4), we obtain

$$\begin{aligned}G &= G^{\tau} \exp(\mathfrak{a}) G' G'' \quad \text{by (3.4)} \\ &= G^{\tau} \exp(\mathfrak{a}) \left(\exp(\mathbb{R}Z_1) \exp\left(\mathfrak{a}'\right) \exp(\mathbb{R}Z_1)\right) G'' \quad \text{by (3.5)} \\ &= G^{\tau} \exp(\mathfrak{a}) \left(\exp\left(\mathbb{R}\left(aK_1 + b(K_2 + K_3)\right)\right) \exp\left(\mathfrak{a}'\right) \exp(\mathbb{R}Z_1)\right) G'' \quad \text{by (3.6)} \\ &= G^{\tau} \exp\left(\mathbb{R}\left(aK_1 + b(K_2 + K_3)\right)\right) \exp(\mathfrak{a}) \exp\left(\mathfrak{a}'\right) \exp(\mathbb{R}Z_1) G'' \quad \text{by (3.3)} \\ &= G^{\tau} B \exp(\mathbb{R}Z_1) G''.\end{aligned}$$

Since $\exp(\mathbb{R}Z_1)G''$ coincides with $\mathrm{L}_{\Pi''}$ and G^τ is conjugate to $\mathrm{L}_{\Pi'}$ under the Weyl group and hence under G^σ ([5, Theorem 6.57]), we have shown the proposition. □

3.3.2. Case II with $j = 3$ or 5

Proposition 3.6. (*Case II with $j = 3$ or 5.*) *Let $G, \mathfrak{g}, \mathfrak{t}, \sigma, \Pi, \tau$ and τ' be as in the beginning of this subsection. Take two subsets Π' and Π'' of the simple system Π of the root system $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ as $(\Pi')^c = \{\alpha_i\}$ and $(\Pi'')^c = \{\alpha_j\}$ where $(i, j) = (1, 3), (1, 5), (6, 3)$ or $(6, 5)$. Then we have*

$$G = \mathrm{L}_{\Pi'} B \mathrm{L}_{\Pi''}$$

for a subset $B \subset G^\sigma$.

Proof. Retain the notations $G', G'', \mathfrak{a}, Z_1, Z_2, K_1, K_2, a$ and b in the proof of Proposition 3.5. We have the following lemma on a representative of the double coset of $G''(\approx \mathrm{SU}(6))$ by $\exp(\mathbb{R}Z_2)M_{ss}(\approx \mathrm{U}(1) \times \mathrm{SU}(2)^2)$ and $(G'')^\tau_0(\approx \mathrm{U}(5))$, where M is the analytic subgroup of $G^{\tau'}$ with Lie algebra $\mathfrak{m} = Z_{\mathfrak{g}^{\tau,\tau'}}(\mathfrak{a})$.

Lemma 3.7. *There exists a subset B' of G^σ such that the multiplication mapping*

$$\bigl(\exp(\mathbb{R}Z_2)M_{ss}\bigr) \times B' \times \bigl(G''\bigr)^\tau_0 \to G''$$

is surjective.

We postpone the proof of this lemma to Lemma 4.2 in Section 4. As in the proof of Proposition 3.5, there are real numbers c and d such that $Z_2 = cK_1 + dK_2$. Hence we have

$$\begin{aligned} G'G'' &= G'\bigl(\exp\bigl(\mathbb{R}(cK_1 + dK_2)\bigr)M_{ss}B'\bigl(G''\bigr)^\tau_0\bigr) \quad \text{by Lemma 3.7} \\ &= \exp\bigl(\mathbb{R}(cK_1 + dK_2)\bigr)G'M_{ss}B'\bigl(G''\bigr)^\tau_0 \\ &= \exp\left(\mathbb{R}\left(cK_1 + dK_2 - \frac{d}{b}Z_1\right)\right)G'M_{ss}B'\bigl(G''\bigr)^\tau_0 \quad \text{by } Z_1 \in \mathfrak{g}' \\ &= \exp(\mathbb{R}K_1)G'M_{ss}B'\bigl(G''\bigr)^\tau_0 \quad \text{by } Z_1 = aK_1 + bK_2. \end{aligned} \tag{3.7}$$

Here, we note that a direct computation shows $b \neq 0$. Put $B = \exp(\mathfrak{a})B'$. Substituting (3.7) to (3.4), we obtain

$$\begin{aligned} G &= G^\tau \exp(\mathfrak{a})G'G'' \quad \text{by (3.4)} \\ &= G^\tau \exp(\mathfrak{a})\bigl(\exp(\mathbb{R}K_1)G'M_{ss}B'\bigl(G''\bigr)^\tau_0\bigr) \quad \text{by (3.7)} \\ &= G^\tau \exp(\mathbb{R}K_1)M_{ss}\exp(\mathfrak{a})B'G'\bigl(G''\bigr)^\tau_0 \quad \text{by } \mathbb{R}K_1, \mathfrak{m}_{ss} \subset Z_\mathfrak{g}(\mathfrak{a}) \\ &= G^\tau BG'\bigl(G''\bigr)^\tau_0. \end{aligned}$$

This completes the proof since G^τ and $G'(G'')^\tau_0$ are conjugate to $\mathrm{L}_{\Pi'}$ and $\mathrm{L}_{\Pi''}$ respectively by elements of G^σ. □

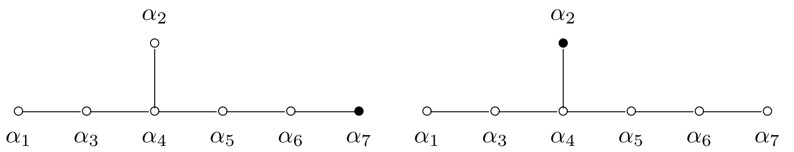

Fig. 3.3. Vogan diagrams of type E VII and type E V.

3.4. *Decompositions for type E_7*

In this subsection we discuss Case III with $i = 1$ or 2 in Theorem 1.1. Let G denote the connected simply connected compact simple Lie group of type E_7, $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$, and σ a Chevalley–Weyl involution of G with respect to $\mathfrak{t}$. We fix a simple system Π of the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. (For the labeling of the Dynkin diagram, see Fig. 3.3.) We give the proofs for Case III with $i = 1$ and with $i = 2$ separately.

3.4.1. *Case III with $i = 2$*

Proposition 3.8. (*Case III with $i = 2$.*) *Let G, Π and σ be as in the beginning of this subsection. Take two subsets Π' and Π'' of Π as $(\Pi')^c = \{\alpha_7\}$ and $(\Pi'')^c = \{\alpha_2\}$. Then we have*

$$G = \mathrm{L}_{\Pi'} B \mathrm{L}_{\Pi''}$$

for a subset $B \subset G^\sigma$.

Proof. We take two commuting involutions τ and τ' of $\mathfrak{g}_{\mathbb{C}}$, which preserve $\mathfrak{g}$ and correspond to the Vogan diagrams of type E VII and type E V respectively (see Appendix C of [5]).

Then the fixed part of the involution $\tau\tau'$ is given by $\mathfrak{g}^{\tau\tau'} = \mathfrak{su}(2) \oplus \mathfrak{so}(12)$. Let $\tilde{\alpha}$ denote the smallest root of $\mathfrak{g} = \mathfrak{e}_7$, and $\tilde{\beta}$ that of $\mathfrak{g}^{\tau'} = \mathfrak{su}(8)$. Since the root system $\Delta(\mathfrak{g}_{\mathbb{C}}^{\tau\tau'}, \mathfrak{t}_{\mathbb{C}})$ of $\mathfrak{g}^{\tau\tau'}$ is contained in $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, there exists $\gamma \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ such that $\{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \tilde{\beta}, \gamma\}$ gives rise to a simple system of $\mathfrak{g}^{\tau\tau'}$. We note that γ is connected to α_3 or α_5. We may and do assume that γ is connected to α_5. Then we take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{g}^{-\tau,-\tau'}$ as follows: Let us explicitly write the simple system $\Pi(\mathfrak{g}_{\mathbb{C}}^{\tau\tau'}, \mathfrak{t}_{\mathbb{C}})$ and the root system $\Delta(\mathfrak{g}_{\mathbb{C}}^{\tau\tau'}, \mathfrak{t}_{\mathbb{C}})$ of $\mathfrak{g}_{\mathbb{C}}^{\tau\tau'}$ (see Plate I and Plate IV of [1]).

$$\Pi\big(\mathfrak{g}_{\mathbb{C}}^{\tau\tau'}, \mathfrak{t}_{\mathbb{C}}\big) = \{\tilde{\beta}\} \cup \{\alpha_i, \gamma \colon 1 \leqslant i \neq 2 \leqslant 6\},$$
$$\Delta\big(\mathfrak{g}_{\mathbb{C}}^{\tau\tau'}, \mathfrak{t}_{\mathbb{C}}\big) = \{\pm\tilde{\beta}\} \cup \{\pm\varepsilon_i \pm \varepsilon_j \colon 1 \leqslant i < j \leqslant 6\},$$
$$\text{where } \alpha_1 = \varepsilon_1 - \varepsilon_2,\ \alpha_3 = \varepsilon_2 - \varepsilon_3,\ \alpha_4 = \varepsilon_3 - \varepsilon_4,\ \alpha_5 = \varepsilon_4 - \varepsilon_5,\ \alpha_6 = \varepsilon_5 - \varepsilon_6,\ \gamma = \varepsilon_5 + \varepsilon_6.$$

By using three non-compact imaginary roots $\varepsilon_1 + \varepsilon_2, \varepsilon_3 + \varepsilon_4$ and $\varepsilon_5 + \varepsilon_6$ for the Cayley transforms of the compact Cartan subalgebra $\mathfrak{t}$ of the non-compact dual $\mathfrak{g}^{\tau,\tau'} \oplus \sqrt{-1}\mathfrak{g}^{-\tau,-\tau'}$ of $\mathfrak{g}^{\tau\tau'}$, we obtain a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{g}^{-\tau,-\tau'}$, which is fixed by σ (Corollary 2.2). We apply Fact 3.1 to (G, τ, τ'):

$$G = G^\tau \exp(\mathfrak{a}) G^{\tau'}. \tag{3.8}$$

We define a subgroup $M(\approx \mathrm{SU}(2)^4)$ to be the analytic subgroup of G with Lie algebra $Z_{\mathfrak{g}^{\tau,\tau'}}(\mathfrak{a})$, the centralizer of $\mathfrak{a}$ in $\mathfrak{g}^{\tau,\tau'}$. Then we have the following lemma on the double coset of $G^{\tau'}(\approx \mathrm{SU}(8))$ by $M(\approx \mathrm{SU}(2)^4)$ and $\mathrm{L}_{\Pi''}(\approx \mathrm{U}(7))$.

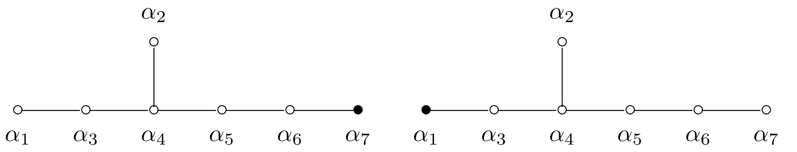

Fig. 3.4. Vogan diagrams of type E VII and type E VI.

Lemma 3.9. *There exists a subset B' of G^σ such that the multiplication mapping*

$$M \times B' \times \mathrm{L}_{\Pi''} \to G^{\tau'}$$

is surjective.

We postpone the proof of this lemma to Lemma 4.3 in Section 4. Put $B = \exp(\mathfrak{a})B'$. Combining Lemma 3.9 with (3.8), we obtain

$$G = G^\tau \exp(\mathfrak{a})\big(MB'\mathrm{L}_{\Pi''}\big) = G^\tau M\big(\exp(\mathfrak{a})B'\big)\mathrm{L}_{\Pi''} = G^\tau B\mathrm{L}_{\Pi''}.$$

This completes the proof since $G^\tau = \mathrm{L}_{\Pi'}$. $\square$

3.4.2. Case III with $i = 1$

Proposition 3.10. (*Case III with $i = 1$.*) *Let G, Π and σ be as in the beginning of this subsection. Take two subsets Π' and Π'' of the simple system Π as $(\Pi')^c = \{\alpha_7\}$ and $(\Pi'')^c = \{\alpha_1\}$. Then we have*

$$G = \mathrm{L}_{\Pi'}B\mathrm{L}_{\Pi''}$$

for a subset $B \subset G^\sigma$.

Proof. We take two commuting involutions τ and τ' of $\mathfrak{g}_{\mathbb{C}}$, which preserve $\mathfrak{g}$ and correspond to the Vogan diagrams of type E VII and type E VI respectively (see Appendix C of [5]). (See Fig. 3.4.)

Then the fixed part of the involution $\tau\tau'$ is given by $\mathfrak{g}^{\tau\tau'} = \mathbb{R} \oplus \mathfrak{e}_6$. Let us take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{g}^{-\tau,-\tau'}$, which is fixed by σ as in the proof of Proposition 3.3. We also take the σ-fixed one-dimensional subspace $\mathfrak{a}'$ of the normal subalgebra $\mathfrak{su}(2)$ of $\mathfrak{g}^{\tau'}$. Put $B = \exp(\mathfrak{a})\exp(\mathfrak{a}')$. By the same argument as in the proof of Proposition 3.5 (we note that $((\mathfrak{g}^{\tau\tau'})_{ss}, ((\mathfrak{g}^{\tau\tau'})_{ss})^\tau) = (\mathfrak{e}_6, \mathbb{R} \oplus \mathfrak{so}(10))$ is Hermitian of non-tube type), we obtain a generalized Cartan decomposition for Case III with $i = 1$:

$$G = \mathrm{L}_{\Pi'}B\mathrm{L}_{\Pi''}. \qquad \square$$

4. Completion of the proofs in Section 3

We have postponed the proofs of double coset decompositions for some subgroups of the exceptional compact simple Lie groups E_6 and E_7, which were used in the herringbone stitch method in the previous section. This section gives the proofs of Lemmas 3.4, 3.7 and 3.9.

All of the compact Lie groups which appear in this section are of classical type. Thus we work on (non-symmetric) generalized Cartan decompositions in the classical case. However, we have to be careful how they are embedded in exceptional Lie groups.

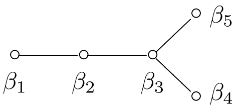

Fig. 4.1. Dynkin diagram of type D_5.

4.1. Proof of Lemma 3.4

Retain the setting in the proof of Proposition 3.3. We note that simple systems of $(\mathfrak{g}^\tau)_{ss}$, $(\mathfrak{g}^\tau)_{ss} \cap \mathfrak{l}_{\Pi''}$ and $Z_{\mathfrak{g}^\tau}(\mathfrak{a})_{ss}$ are given by $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$, $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ and $\{\alpha_2, \alpha_3, \alpha_4\}$ respectively (Remark 2.3), and that $X \in (\mathfrak{g}^\tau)_{ss}$ centralizes $Z_{\mathfrak{g}^\tau}(\mathfrak{a})_{ss}$. Let $\{H_i\}_{i=1}^6 \subset \mathfrak{t}_{\mathbb{C}}$ denotes the dual basis of $\{\alpha_i\}_{i=1}^6$ with respect to the Killing form. Then a direct computation shows that $\sqrt{-1}H_1$ has a non-zero coefficient in $X = \sum_{1 \leqslant i \leqslant 5} a_i \sqrt{-1} H_i$, i.e., $a_1 \neq 0$. Now we find that Lemma 3.4 follows from the lemma below.

Lemma 4.1. *Let L be a connected compact simple Lie group of type D_5, $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{l}$ and σ a Chevalley–Weyl involution of L with respect to $\mathfrak{t}$. We label the Dynkin diagram of type D_5 as in Fig. 4.1.*

We take two subsets Φ' and Φ'' of the simple system $\Phi = \{\beta_i\colon 1 \leqslant i \leqslant 5\}$ of $\mathfrak{l}$ as $(\Phi')^c = \{\beta_1\}$ and $(\Phi'')^c = \{\beta_1, \beta_5\}$, and define a one-dimensional abelian subgroup U by $U := \exp(\mathbb{R}(\sum_{i=1}^5 a_i \sqrt{-1} H_i))$ with $a_1 \neq 0$ where $\{H_i\}_{i=1}^5$ denotes the dual basis of $\{\beta_i\}_{i=1}^5$ with respect to the Killing form. Then we have

$$L = U(\mathrm{L}_{\Phi''})_{ss} B' \mathrm{L}_{\Phi'},$$

for a subset B' of L^σ.

Proof. It suffices to consider the case where $L = \mathrm{SO}(10)$. We give a matrix realization of L as follows:

$$L = \mathrm{SO}(10) = \left\{ g \in \mathrm{SL}(10, \mathbb{C})\colon {}^t g J_{10} g = J_{10},\ {}^t \bar{g} g = I_{10} \right\},$$

where I_m denotes the identity matrix and J_m is defined by a bilinear form given by

$$\mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}, \quad (x, y) \mapsto {}^t x J_m y := \sum_{i=1}^m x_i y_{m+1-i}.$$

Here x_i and y_i denote the i-th entries in x and y respectively. We take a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{l}$ as diagonal matrices:

$$\mathfrak{t} = \bigoplus_{1 \leqslant i \leqslant 5} \mathbb{R}\sqrt{-1} A_i,$$

where $A_i := E_{i,i} - E_{11-i,11-i}$.

We define an involutive automorphism σ of L by

$$\sigma : L \to L, \quad g \mapsto \bar{g}, \tag{4.1}$$

where $\bar{g}$ denotes the complex conjugate of $g \in L$. Then σ is a Chevalley–Weyl involution of L with respect to $\mathfrak{t}$. Note that Lemma 4.1 is independent of the choice of a Chevalley–Weyl involution since $\mathrm{L}_{\Phi'}$ contains $\exp(\mathfrak{t})$, and both U and $(\mathrm{L}_{\Phi''})_{ss}$ are stable under the conjugation by any element of $\exp(\mathfrak{t})$.

We let $\{\varepsilon_i\}_{1\leqslant i\leqslant 5} \subset (\mathfrak{t}\otimes_{\mathbb{R}}\mathbb{C})^*$ be the dual basis of $\{A_i\}_{1\leqslant i\leqslant 5}$. Then we define a set of simple roots $\Phi := \{\beta_1,\ldots,\beta_5\}$ by

$$\beta_i := \varepsilon_i - \varepsilon_{i+1} \quad (1\leqslant i\leqslant 4), \qquad \beta_5 := \varepsilon_4+\varepsilon_5.$$

Since the sets of the simple roots of $\mathrm{L}_{\Phi''}$ and $\mathrm{L}_{\Phi'}$ are given by $\{\beta_2,\beta_3,\beta_4\}$ and $\{\beta_2,\beta_3,\beta_4,\beta_5\}$ respectively, $\mathrm{L}_{\Phi''}$ and $\mathrm{L}_{\Phi'}$ take the forms:

$$\mathrm{L}_{\Phi''} = \left\{ \begin{pmatrix} e^{\sqrt{-1}\theta} & & & \\ & A & & \\ & & J_4\bar{A}J_4 & \\ & & & e^{-\sqrt{-1}\theta} \end{pmatrix} \in \mathrm{SO}(10)\colon \theta\in\mathbb{R},\ A\in \mathrm{U}(4) \right\},$$

$$\mathrm{L}_{\Phi'} = \left\{ \begin{pmatrix} e^{\sqrt{-1}\theta} & & \\ & A & \\ & & e^{-\sqrt{-1}\theta} \end{pmatrix} \in \mathrm{SO}(10)\colon \theta\in\mathbb{R},\ A\in \mathrm{SO}(8) \right\}.$$

Here, all the entries in the blank space are zero. We give a proof of the lemma by the herringbone stitch method [9]. First, we show that $L = \mathrm{L}_{\Phi''}B'\mathrm{L}_{\Phi'}$ for a subset B' of L^σ. Next, we prove that $\mathrm{L}_{\Phi''}B'\mathrm{L}_{\Phi'}$ coincides with $U\cdot(\mathrm{L}_{\Phi''})_{ss}B'\mathrm{L}_{\Phi'}$. Then we can see that $L = U\cdot(\mathrm{L}_{\Phi''})_{ss}B'\mathrm{L}_{\Phi'}$ holds.

Let us show the first assertion that the group L can be written as $\mathrm{L}_{\Phi''}B'\mathrm{L}_{\Phi'}$ with $B'\subset L^\sigma$. We define an abelian subgroup B_1 by

$$B_1 := \exp\left(\bigoplus_{i=1,2}\mathbb{R}(E_{1,4+i}-E_{4+i,1}-E_{7-i,10}+E_{10,7-i})\right).$$

Then we have the following decomposition of L by Fact 3.1.

$$L = \mathrm{L}_{\Phi'}B_1\mathrm{L}_{\Phi'}. \tag{4.2}$$

We define a symmetric subgroup K of $(\mathrm{L}_{\Phi'})_{ss}$ and an abelian subgroup B_2 by

$$\begin{aligned} K &:= \mathrm{SO}(6)\times\mathrm{SO}(2)\\ &= \left\{ \begin{pmatrix} 1 & & & & & 0\\ & A & & & B & \\ & & e^{\sqrt{-1}\theta} & & & \\ & & & e^{-\sqrt{-1}\theta} & & \\ & C & & & D & \\ 0 & & & & & 1 \end{pmatrix} \in \mathrm{SO}(10)\colon \begin{pmatrix} A & B\\ C & D\end{pmatrix}\in\mathrm{SO}(6),\ \theta\in\mathbb{R} \right\},\\ B_2 &:= \exp\big(\mathbb{R}(E_{2,6}-E_{6,2}-E_{5,9}+E_{9,5})\big). \end{aligned}$$

Then we obtain the following decomposition of $\mathrm{L}_{\Phi'}$ by using Fact 3.1.

$$\mathrm{L}_{\Phi'} = \mathrm{L}_{\Phi''}B_2K.$$

It is easy to see that K and K_{ss} satisfy $\mathrm{L}_{\Phi''}B_2K = \mathrm{L}_{\Phi''}B_2K_{ss}$. Thus we have

$$\mathrm{L}_{\Phi'} = \mathrm{L}_{\Phi''}B_2K_{ss}. \tag{4.3}$$

Let us set $B' := B_2B_1$. The following is a proof of the first assertion.

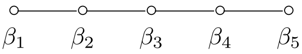

Fig. 4.2. Dynkin diagram of type A_5.

$$\begin{aligned} L &= \mathrm{L}_{\Phi'} B_1 \mathrm{L}_{\Phi'} \quad \text{by (4.2)} \\ &= (\mathrm{L}_{\Phi''} B_2 K_{ss}) B_1 \mathrm{L}_{\Phi'} \quad \text{by (4.3)} \\ &= \mathrm{L}_{\Phi''} B_2 B_1 K_{ss} \mathrm{L}_{\Phi'} \quad \text{by } K_{ss} \subset Z_L(B_1) \\ &= \mathrm{L}_{\Phi''} B' \mathrm{L}_{\Phi'}. \end{aligned}$$

Then we give a proof of the second assertion, that is, we shall prove that $\mathrm{L}_{\Phi''} B' \mathrm{L}_{\Phi'}$ coincides with $U \cdot (\mathrm{L}_{\Phi''})_{ss} B' \mathrm{L}_{\Phi'}$. We define one-dimensional abelian subgroup T_1 by

$$T_1 := \exp\bigl(\mathbb{R}\sqrt{-1}(E_{3,3} - E_{8,8})\bigr) \subset \mathrm{L}_{\Phi''}.$$

Since T_1 centralizes B', $U \cdot (\mathrm{L}_{\Phi''})_{ss} B' \mathrm{L}_{\Phi'}$ is equal to $U \cdot ((\mathrm{L}_{\Phi''})_{ss} \cdot T_1) B' \mathrm{L}_{\Phi'}$, and hence to $U \cdot ((\mathrm{L}_{\Phi'})_{ss} \cap \mathrm{L}_{\Phi''}) B' \mathrm{L}_{\Phi'}$. Further, $U \cdot ((\mathrm{L}_{\Phi'})_{ss} \cap \mathrm{L}_{\Phi''})$ is equal to $\mathrm{L}_{\Phi''}$ because $a_1 \neq 0$ (we recall that $U = \exp(\mathbb{R} \sum_{i=1}^{5} a_i \sqrt{-1} H_i)$). Consequently we have

$$U (\mathrm{L}_{\Phi''})_{ss} B' \mathrm{L}_{\Phi'} = U \cdot \bigl((\mathrm{L}_{\Phi'})_{ss} \cap \mathrm{L}_{\Phi''}\bigr) B' \mathrm{L}_{\Phi'} = \mathrm{L}_{\Phi''} B' \mathrm{L}_{\Phi'} = L.$$

We have finished the proof. □

4.2. Proof of Lemma 3.7

Retain the setting of Section 3.3. We note that simple systems of $\mathfrak{g}''$, $(\mathfrak{g}'')^{\tau}$ and $Z_{\mathfrak{g}^{\tau,\tau'}}(\mathfrak{a})$ are given by $\{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$, $\{\alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ and $\{\alpha_3, \alpha_5\}$ respectively (Remark 2.3), and that $\mathbb{R}Z_2$ is the center of $(\mathfrak{g}'')^{\tau}$. Now we can see that Lemma 3.7 follows from the lemma below.

Lemma 4.2. *Let L be a connected compact simple Lie group of type A_5. We take a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{l}$ and label the Dynkin diagram of $\mathfrak{l}$ as in Fig. 4.2.*

Let $\mathfrak{k}$ be a Levi subalgebra whose root system is generated by $\{\beta_2, \beta_3, \beta_4, \beta_5\}$. We also define a reductive subalgebra $\mathfrak{m}$ by $\mathfrak{m} := \mathbb{R} \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ where a simple system of $\mathfrak{m}$ is given by $\{\beta_2, \beta_4\}$ and the center of $\mathfrak{m}$ coincides with that of $\mathfrak{k}$. Denote by K and M the analytic subgroups of L with Lie algebras $\mathfrak{k}$ and $\mathfrak{m}$ respectively. Then we have

$$L = M B' K$$

for a subset B' of L^{σ} where σ is a Chevalley–Weyl involution of L with respect to $\mathfrak{t}$.

Proof. It suffices to consider the case where L is simply connected. We realize $L = SU(6)$ as a matrix group:

$$L = \bigl\{g \in SL(6, \mathbb{C}):\ g^t \bar{g} = I_6\bigr\}.$$

Let us take the diagonal matrices consisting of purely imaginary numbers as a Cartan subalgebra $\mathfrak{t}$, and the complex conjugation as a Chevalley–Weyl involution σ of L. Here we note that both K and M are stable under the conjugation by any element of the maximal torus $\exp(\mathfrak{t})$ (independence of the choice of a Chevalley–Weyl involution). We define a simple system Φ of L by $\Phi := \{\varepsilon_i - \varepsilon_{i+1}:\ 1 \leqslant i \leqslant 5\}$

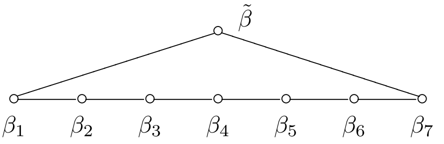

Fig. 4.3. Extended Dynkin diagram of type A_7.

where ε_i are given by $\varepsilon_i : \mathrm{diag}(a_1, \ldots, a_6) \mapsto a_i$. The Levi subgroup K and the closed subgroup M take the forms:

$$K = \left\{ \begin{pmatrix} \det(A)^{-1} & \\ & A \end{pmatrix} \in L\colon\ A \in \mathrm{U}(5) \right\},$$

$$M = \left\{ \begin{pmatrix} a^{-5} & & & \\ & aA & & \\ & & aB & \\ & & & a \end{pmatrix} \in L\colon\ a \in \mathrm{U}(1),\ A, B \in \mathrm{SU}(2) \right\}.$$

Then we define a subset B' of L^σ by $B' := B_1 B_2 B_3$ where

$$B_1 := \exp\bigl(\mathbb{R}(E_{1,2} - E_{2,1})\bigr), \qquad B_2 := \exp\bigl(\mathbb{R}(E_{1,4} - E_{4,1})\bigr), \qquad B_3 := \exp\bigl(\mathbb{R}(E_{1,6} - E_{6,1})\bigr).$$

We identify L/K with $\mathbb{C}P^5$ in the natural way. Through the identification, $B' \cdot K/K$ is identified with

$$\bigl\{[x_1 : x_2 : 0 : x_3 : 0 : x_4] \in \mathbb{C}P^5\colon\ x_i \in \mathbb{R}\ (1 \leqslant i \leqslant 4)\bigr\}.$$

For any $z = [z_1 : \cdots : z_6] \in L/K$, we may and do assume that $\arg z_1 + 5 \arg z_6 = 0$. Then there exists $g \in M$ such that

$$g \cdot z = \Bigl[|z_1| : \sqrt{|z_2|^2 + |z_3|^2} : 0 : \sqrt{|z_4|^2 + |z_5|^2} : 0 : |z_6|\Bigr] \in B' \cdot K/K.$$

Thus we obtain

$$M \cdot B' \cdot K/K = L/K. \qquad \square$$

4.3. Proof of Lemma 3.9

Retain the setting in the proof of Proposition 3.8. Since the set of simple roots of M is given by $\{\alpha_1, \alpha_4, \alpha_6, \tilde{\beta}\}$ (Remark 2.3) and that of $L_{\Pi'}$ by $\{\alpha_2\}^c$, we can see that Lemma 3.9 is followed by the lemma below.

Lemma 4.3. *Let L be a connected compact simple Lie group of type A_7, $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{l}$ and σ a Chevalley–Weyl involution of L with respect to $\mathfrak{t}$. We label the extended Dynkin diagram of $\mathfrak{l}$ as in Fig. 4.3 (see Plate I of [1]).*

Define a semisimple subalgebra $\mathfrak{m}$ by $\mathfrak{m} := \mathfrak{su}(2)^4$ whose simple system is given by $\{\beta_2, \beta_4, \beta_6, \tilde{\beta}\}$, and a Levi subalgebra $\mathfrak{k}$ by $\mathfrak{k} := \mathbb{R} \oplus \mathfrak{su}(7)$ whose simple system is given by $\{\beta_1\}^c$. Let M and K denote the analytic subgroups of L with Lie algebras $\mathfrak{m}$ and $\mathfrak{k}$ respectively. Then we have

$$L = M B' K$$

for a subset B' of L^σ.

Proof. It suffices to consider the case where L is simply connected. We realize $L = \mathrm{SU}(8)$ as a matrix group as follows.

$$L = \{g \in \mathrm{SL}(8, \mathbb{C}):\ g^t \bar{g} = I_8\}.$$

Let us take the diagonal matrices consisting of purely imaginary numbers as a Cartan subalgebra $\mathfrak{t}$, and the complex conjugation as a Chevalley–Weyl involution σ of L. Then we realize $K = \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(7))$ as follows:

$$K = \left\{ \begin{pmatrix} \det(A)^{-1} & \\ & A \end{pmatrix} \in L:\ A \in \mathrm{U}(7) \right\}.$$

We define a subgroup M' by

$$M' = \left\{ \begin{pmatrix} D_1 & & & \\ & D_2 & & \\ & & D_3 & \\ & & & D_4 \end{pmatrix} \in L:\ D_i \in \mathrm{SU}(2),\ 1 \leqslant i \leqslant 4 \right\} = \mathrm{SU}(2)^4,$$

and a subset B' of L^σ by $B' := B_1 B_2 B_3$ where

$$B_1 := \exp\bigl(\mathbb{R}(E_{1,3} - E_{3,1})\bigr), \qquad B_2 := \exp\bigl(\mathbb{R}(E_{1,5} - E_{5,1})\bigr), \qquad B_3 := \exp\bigl(\mathbb{R}(E_{1,7} - E_{7,1})\bigr).$$

We identify L/K with $\mathbb{C}P^7$ in the natural way. Since $\mathrm{SU}(2)$ acts on S^2 transitively, for any $z = [z_1 : \cdots : z_8] \in L/K$ there exists $m \in M'$ such that

$$m \cdot z = \Bigl[\sqrt{|z_1|^2 + |z_2|^2} : 0 : \sqrt{|z_3|^2 + |z_4|^2} : 0 : \sqrt{|z_5|^2 + |z_6|^2} : 0 : \sqrt{|z_7|^2 + |z_8|^2} : 0\Bigr] \in B' \cdot K/K.$$

Thus we obtain

$$M' \cdot B' \cdot K/K = L/K.$$

Since M' is conjugate to M by an element of $L^\sigma = \mathrm{SO}(8)$, the lemma follows. $\square$

Lemmas 4.1–4.3 complete the proofs in Section 3, and therefore we have finished the proof of the implication (ii) $\Rightarrow$ (i).

5. Proof of the implication (i) ⇒ (ii) of Theorem 1.1

In this section, we prove that the list in Theorem 1.1(ii) exhausts all the triple $(\mathfrak{g}, \mathfrak{l}_{\Pi'}, \mathfrak{l}_{\Pi''})$ satisfying the condition (i) in Theorem 1.1, and thus complete the proof of the remaining implication (i) $\Rightarrow$ (ii) of Theorem 1.1.

In the classical case (see [9] for type A), invariant theory for quivers was used in the proof for the classification of $G = \mathrm{L}_{\Pi'} G^\sigma \mathrm{L}_{\Pi''}$, however, it is not obvious if the method is applicable to exceptional groups. Instead we use the general theory that strongly visible actions give rise to multiplicity-free representations [8], and then apply the classification theorems of multiplicity-free tensor product representations by Littelmann [12] for the maximal parabolic case and Stembridge [16] for the general case.

Proof of the implication (i) ⇒ (ii) of Theorem 1.1. Let G be a connected simply connected compact simple Lie group, and $G_{\mathbb{C}}$ its complexification. We fix a Cartan subalgebra and a simple system Π

of $\mathfrak{g}$, and denote by B the corresponding Borel subgroup of $G_{\mathbb{C}}$. For a given subset Π' of Π, we write $P_{\Pi'} \supset B$ for a parabolic subgroup whose reductive part is given by the complexification of a Levi subgroup $\mathrm{L}_{\Pi'}$ of G (we recall that Π' is a simple system of $\mathrm{L}_{\Pi'}$). Also, we denote by ω_i a fundamental weight of G, which corresponds to a simple root α_i (we label the Dynkin diagrams of type E_6 and type E_7 following Bourbaki [1] as in Section 3), and by π_λ a finite dimensional irreducible representation with highest weight λ.

We let λ be a unitary character of $\mathrm{L}_{\Pi'}$, and extend it to a holomorphic character of $P_{\Pi'}$. By the Borel–Weil theory, we can realize the contragradient representation π_λ^* of π_λ as the space of holomorphic sections $\mathcal{O}(G_{\mathbb{C}}/P_{\Pi'}, \mathcal{L}_{-\lambda})$ of the line bundle $\mathcal{L}_{-\lambda}$ on $G_{\mathbb{C}}/P_{\Pi'}$.

Let us suppose that the condition (i) holds. Then the diagonal action of G on $G_{\mathbb{C}}/P_{\Pi'} \times G_{\mathbb{C}}/P_{\Pi''}$ is strongly visible, and thus by Fact 6.3 below and the Borel–Weil theory,

the tensor product representation $\pi_\lambda^* \otimes \pi_\mu^*$ is multiplicity-free $\cdots \diamondsuit$

where λ and μ are any unitary characters of $\mathrm{L}_{\Pi'}$ and $\mathrm{L}_{\Pi''}$ respectively.

On the other hand, we can extract the following results from the classification theorems [12] and [16] on when $\pi_\lambda \otimes \pi_\mu$ is multiplicity-free for the maximal parabolic case and for the general case, respectively.

Fact 5.1. Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra of type E_6. Let I and J be non-empty subsets of $\{1,2,3,4,5,6\}$. Then the tensor product of $\mu = \sum_{i \in I} m_i \omega_i$ and $\nu = \sum_{j \in J} n_j \omega_j$ is multiplicity-free for arbitrary non-negative integers m_i $(i \in I)$ and n_j $(j \in J)$ if and only if one of the following conditions holds up to switch of the factors I and J.

(i) $I = \{1\}$ or $\{6\}$, $J = \{j\}$ with $j \neq 4$.
(ii) $I = \{1\}$ or $\{6\}$, $J = \{1,6\}$.

Fact 5.2. Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra of type E_7. Let I and J be non-empty subsets of $\{1,2,3,4,5,6,7\}$. Then the tensor product of $\mu = \sum_{i \in I} m_i \omega_i$ and $\nu = \sum_{j \in J} n_j \omega_j$ is multiplicity-free for arbitrary non-negative integers m_i $(i \in I)$ and n_j $(j \in J)$ if and only if the following condition holds up to switch of the factors I and J.

(i) $I = \{7\}$, $J = \{j\}$ with $j = 1, 2$ or 7.

Fact 5.3. Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra of type E_8, F_4 or G_2. Then there is no pair of non-empty subsets I and J of $\{1,\dots,r\}$ with $r = \operatorname{rank}\mathfrak{g}$, which satisfies the following:

The tensor product of $\mu = \sum_{i \in I} m_i \omega_i$ and $\nu = \sum_{j \in J} n_j \omega_j$ is multiplicity-free for arbitrary non-negative integers m_i $(i \in I)$ and n_j $(j \in J)$.

By the comparison of $\diamondsuit$ with Facts 5.1, 5.2 and 5.3, the triple $(\mathfrak{g}, \Pi', \Pi'')$ must be in the list given in Theorem 1.1(ii). Therefore the implication (i) $\Rightarrow$ (ii) holds. $\square$

6. Application to representation theory

In this section, we shall see a generalized Cartan decomposition leads to three kinds of multiplicity-free representations by using the framework of visible actions ("triunity" à la [6]). The notion of (strongly) visible actions on complex manifolds was introduced by T. Kobayashi. Let us recall the definition [7]

Definition 6.1. We say a biholomorphic action of a Lie group G on a complex manifold D is *strongly visible* if the following two conditions are satisfied:

(1) There exists a real submanifold S such that (we call S a "slice")

$$D' := G \cdot S \quad \text{is an open subset of } D.$$

(2) There exists an antiholomorphic diffeomorphism σ of D' such that

$$\sigma|_S = \mathrm{id}_S, \qquad \sigma(G \cdot x) = G \cdot x \quad \text{for any } x \in S.$$

Definition 6.2. In the above setting, we say the action of G on D is S-visible. This terminology will be used also if S is just a subset of D.

Let G be a connected compact Lie group and L, H its Levi subgroups. Then G/L, G/H and $(G \times G)/(L \times H)$ are complex manifolds. If the triple (G, L, H) satisfies $G = LG^\sigma H$, the following three group-actions are all strongly visible:

$$L \curvearrowright G/H, \qquad H \curvearrowright G/L, \qquad \Delta(G) \curvearrowright (G \times G)/(L \times H).$$

Here $\Delta(G)$ is defined by $\Delta(G) := \{(x, y) \in G \times G\colon\ x = y\}$. The following fact [8, Theorem 4.3] leads us to multiplicity-free representations:

Fact 6.3. Let G be a Lie group and $\mathcal{V}$ a G-equivariant Hermitian holomorphic vector bundle on a connected complex manifold D. If the following three conditions from (1) to (3) are satisfied, then any unitary representation that can be embedded in the vector space $\mathcal{O}(D, \mathcal{V})$ of holomorphic sections of $\mathcal{V}$ decomposes multiplicity-freely:

(1) The action of G on D is S-visible. That is, there exist a subset $S \subset D$ and an antiholomorphic diffeomorphism σ of D' satisfying the conditions given in Definition 6.1. Further, there exists an automorphism $\hat{\sigma}$ of G such that $\sigma(g \cdot x) = \hat{\sigma}(g) \cdot \sigma(x)$ for any $g \in G$ and $x \in D'$.
(2) For any $x \in S$, the fiber $\mathcal{V}_x$ at x decomposes as the multiplicity-free sum of irreducible unitary representations of the isotropy subgroup G_x. Let $\mathcal{V}_x = \bigoplus_{1 \leqslant i \leqslant n(x)} \mathcal{V}_x^{(i)}$ denote the irreducible decomposition of $\mathcal{V}_x$.
(3) σ lifts to an antiholomorphic automorphism $\tilde{\sigma}$ of $\mathcal{V}$ and satisfies $\tilde{\sigma}(\mathcal{V}_x^{(i)}) = \mathcal{V}_x^{(i)}$ for any i ($1 \leqslant i \leqslant n(x)$) for each $x \in S$.

By using the Borel–Weil theory together with Fact 6.3 and our generalized Cartan decompositions, we obtain the following two corollaries of Theorem 1.1. Let G be a connected compact exceptional simple Lie group and ω_i ($1 \leqslant i \leqslant \operatorname{rank} \mathfrak{g}$) its fundamental weights (we label the Dynkin diagrams following Bourbaki [1] as in Section 3).

Corollary 6.4. *If the triple (G, L, λ) is an entry in Table 6.1 or 6.2, then the restriction $\pi_\lambda|_L$ of the irreducible representation π_λ of G with highest weight λ to L decomposes multiplicity-freely.*

Corollary 6.5. *The tensor product representation $\pi_\lambda \otimes \pi_\mu$ of any two irreducible representations π_λ and π_μ of G with highest weights λ and μ listed in Table 6.3 or 6.4 decomposes as a multiplicity-free sum of irreducible representations of G.*

We note that the condition (2) of Fact 6.3 is automatically satisfied since the fiber of a holomorphic vector bundle is one-dimensional in the setting of the Borel–Weil theory.

Remark 6.6. Littelmann [12] classified for any simple algebraic group G over any algebraically closed field of characteristic zero, all the pairs of maximal parabolic subgroups P_ω and $P_{\omega'}$ corresponding

Table 6.1
Maximal parabolic type.

G	L	λ	Conditions
E_6	$L_{\{\alpha_i\}^c}$	$a\omega_j$	$i = 1$ or 6, $j \neq 4$ $i \neq 4$, $j = 1$ or 6
E_7	$L_{\{\alpha_i\}^c}$	$a\omega_j$	$i = 7$, $j = 1, 2$ or 7 $i = 1, 2$ or 7, $j = 7$

Here, a is arbitrary non-negative integer.

Table 6.2
Non-maximal parabolic type.

G	L	λ	Conditions
E_6	$L_{\{\alpha_1,\alpha_6\}^c}$	$a\omega_i$	$i = 1$ or 6
E_6	$L_{\{\alpha_i\}^c}$	$a\omega_1 + b\omega_6$	$i = 1$ or 6

Here, a and b are arbitrary non-negative integers.

Table 6.3
Maximal parabolic type.

G	(λ, μ)	Conditions
E_6	$(a\omega_i, b\omega_j)$	$i = 1$ or 6, $j \neq 4$
E_7	$(a\omega_i, b\omega_j)$	$i = 7$, $j = 1, 2$ or 7

Here, a and b are arbitrary non-negative integers.

Table 6.4
Non-maximal parabolic type.

G	(λ, μ)	Conditions
E_6	$(a\omega_1 + b\omega_6, c\omega_i)$	$i = 1$ or 6

Here, a, b and c are arbitrary non-negative integers.

to fundamental weights ω and ω' respectively such that the tensor product representation $\pi_{n\omega} \otimes \pi_{m\omega'}$ decomposes multiplicity-freely for arbitrary non-negative integers n and m. (His classification is exactly Table 6.3 and does not include Table 6.4 in the exceptional case.) Moreover, he found the branching rules of $\pi_{n\omega} \otimes \pi_{m\omega'}$ and the restriction of $\pi_{n\omega}$ to the maximal Levi subgroup $L_{\omega'}$ of $P_{\omega'}$ for any pair (ω, ω') that admits a G-spherical action on $G/P_\omega \times G/P_{\omega'}$.

Remark 6.7. Stembridge [16] gave a complete list of a pair (μ, ν) of highest weights such that the corresponding tensor product representation $\pi_\mu \otimes \pi_\nu$ is multiplicity-free for any complex simple Lie algebra. His method is combinatorial. He also classified all the pairs $(\mu, \mathfrak{l})$ of highest weights and Levi subalgebras with the restrictions $\pi_\mu|_{\mathfrak{l}}$ to Levi subalgebras multiplicity-free. Our approach has given a geometric proof of a part of his work based on generalized Cartan decompositions.

We hope that further applications of Theorem 1.1 and Fact 6.3 to representation theory will be discussed in a future paper.

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