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Height-zero characters and normal subgroups in p-solvable groups

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ABSTRACT

Fix a prime number p, and let N be a normal subgroup of a finite p-solvable group G. Let b be a p-block of N and suppose B is a p-block of G covering b. Let D be a defect group for the Fong–Reynolds correspondent of B with respect to b and let $\widehat{B}$ be the unique p-block of $NN_G(D)$ having defect group D and inducing B. Suppose, further, that $\mu \in \mathrm{Irr}(b)$, and let $\mathrm{Irr}_0(B|\mu)$ be the set of irreducible characters in B of height zero that lie over μ. We show that the number of characters in $\mathrm{Irr}_0(B|\mu)$ is equal to the number of characters in $\bigcup_t \mathrm{Irr}_0(\widehat{B}|\mu^t)$, where t runs through the inertial group T of b in G. This result generalizes a theorem of T. Okuyama and M. Wajima, which confirms the Alperin–McKay conjecture for p-solvable groups.

1. Introduction

Fix a prime p and let G be an arbitrary finite group. The McKay conjecture claims that, if P is a Sylow p-subgroup of G, then G and $N_G(P)$ have equal numbers of irreducible (complex) characters of degree not divisible by p. Although this conjecture has been verified for many families of groups, no general proof has yet been discovered. Perhaps, the most significant achievement concerning this conjecture in recent years is a work by I.M. Isaacs, G. Malle and G. Navarro [6], in which they reduced the McKay conjecture to a question about simple groups. Also, several strengthenings of the McKay conjecture due to Isaacs and Navarro [5], Navarro [14] and A. Turull [18] have been proposed with the hope of gaining some insight into the deeper underlying reason behind the conjecture.

Let now B be a p-block of G. Denote by $\mathrm{Irr}_0(B)$ the set of irreducible characters in B of height zero. The Alperin–McKay conjecture asserts that, if D is a defect group of B and $\widehat{B}$ is the p-block of $N_G(D)$ which corresponds to B under the Brauer correspondence, then the numbers of characters in

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$\mathrm{Irr}_0(B)$ and $\mathrm{Irr}_0(\widehat{B})$ are equal. Note that the Alperin–McKay conjecture implies the McKay conjecture by simply summing over all of the p-blocks of maximal defect. The Alperin–McKay conjecture has been shown to be valid for many families of groups. It was first proved for all p-solvable groups by T. Okuyama and M. Wajima in [15]. In [17], B. Späth proves a reduction for the Alperin–McKay conjecture in the same spirit as that for the McKay conjecture in [6].

Let N be a normal subgroup of G and let b be a p-block of N covered by B. Following [10], a defect group D of B is an *inertial defect group* of B (with respect to b) if it is a defect group for the Fong–Reynolds correspondent of B with respect to b.

Now let $\mu \in \mathrm{Irr}(b)$. Write $\mathrm{Irr}_0(B|\mu)$ for the intersection $\mathrm{Irr}_0(B) \cap \mathrm{Irr}(G|\mu)$, where $\mathrm{Irr}(G|\mu)$ is the set of irreducible characters of G lying over μ. By [10, Theorem 4.4(i)], $\mathrm{Irr}_0(B|\mu) \neq \emptyset$ if and only if μ is of height zero and μ extends to DN for some inertial defect group D of B. Also, when $N = 1$, then μ is the trivial character of N and $\mathrm{Irr}_0(B|\mu)$ is the set of irreducible characters in B of height zero. The aim of this paper is to prove the following generalization of Okuyama–Wajima's result [15].

Theorem A. *Let N be a normal subgroup of a p-solvable group G, and let B and b be p-blocks of G and N respectively such that B covers b. Write T for the inertial group of b in G and let D be an inertial defect group of B (with respect to b). If $\mu \in \mathrm{Irr}(b)$ and $\widehat{B}$ is the unique p-block of $NN_G(D)$ with defect group D such that $\widehat{B}^G = B$, then $|\mathrm{Irr}_0(B|\mu)| = |\bigcup_{t\in T} \mathrm{Irr}_0(\widehat{B}|\mu^t)|$.*

Of course, by taking $N = 1$ in Theorem A, we recover Okuyama–Wajima's theorem. We should, nevertheless, mention that our proof of Theorem A depends on this result of Okuyama and Wajima.

The equality in Theorem A involves a union of sets of characters which is certainly not in general a disjoint union. A disjoint union can be obtained as follows. Let $NN_T(D)$ act by conjugation on the set $\{\mu^t\colon\ t \in T\}$, and choose $\mu_1 = \mu, \ldots, \mu_n$, a complete set of representatives for the resulting orbits. Then it is not hard to check that $\bigcup_{i=1}^n \mathrm{Irr}_0(\widehat{B}|\mu_i)$ is a disjoint union, which equals $\bigcup_{t\in T} \mathrm{Irr}_0(\widehat{B}|\mu^t)$.

Finally, we mention that the analogue of Theorem A for Brauer characters is proved in [8].

2. Navarro nuclei and vertices

Let π be a set of primes and let π' be the complementary set of primes. Let G be a π-separable group. Following Isaacs [3, Section 2], a character $\chi \in \mathrm{Irr}(G)$ is called π-*special* provided that $\chi(1)$ is a π-number and that for every $S \lhd\lhd G$ and every irreducible constituent θ of χ_S, the determinantal order $o(\theta)$ of θ is a π-number. π-Special characters were first introduced and studied by D. Gajendragadkar in [1].

In Section 2 of [3], an irreducible character χ is said to be π-*factorable* if it can be written in the form $\alpha\beta$, where α is π-special and β is π'-special. If $\chi \in \mathrm{Irr}(G)$ is π-factorable, we write χ_π and $\chi_{\pi'}$ for the π-special and π'-special factors, respectively.

A π-*factorable normal pair* in G is a pair (N, θ), where $N \lhd G$ and θ is π-factorable. We order the set $\mathcal{E}(G)$ of π-factorable normal pairs by setting $(N, \theta) \leqslant (M, \eta)$ if $N \subseteq M$ and θ lies under η. The set of maximal elements of $\mathcal{E}(G)$ is denoted by $\mathcal{E}^*(G)$.

Now let $\chi \in \mathrm{Irr}(G)$. By Corollary 2.3 in [12], there is, up to G-conjugacy, a unique pair $(L, \zeta) \in \mathcal{E}^*(G)$ such that χ lies over ζ and that if $(N, \theta) \in \mathcal{E}(G)$ with θ under χ, there exists an element $g \in G$ such that $(N, \theta^g) \leqslant (L, \zeta)$. We call such a pair a *maximal p-factorable normal pair under* χ.

The *nucleus via normal pairs* (W, γ) of χ is constructed by Navarro in [12] in the following manner. If χ is π-factorable, then (W, γ) is just (G, χ). If χ is not π-factorable, select a maximal π-factorable normal pair (L, ζ) under χ. Now if I is the inertial group of ζ in G, [12, Corollary 2.4] implies that $I < G$. Let $\psi \in \mathrm{Irr}(I|\zeta)$ be the Clifford correspondent of χ. We recursively define the nucleus via normal pairs of χ to be any G-conjugate of any nucleus via normal pairs of ψ. For convenience, we will simply refer to (W, γ) as a normal nucleus of χ.

Note that the set of normal nuclei of χ is a G-conjugacy class of pairs. Also, if (W, γ) is a normal nucleus for χ, then $\gamma^G = \chi$ and γ is π-factorable.

We now let $\pi = \{p\}$ (a single prime), so that G is p-solvable. Suppose that χ has normal nucleus (W, γ). If $Q \in \mathrm{Syl}_p(W)$ and $\delta = (\gamma_p)_Q$, then $\delta \in \mathrm{Irr}(Q)$ by [1, Proposition 6.1]. The pair (Q, δ) is said to be a *vertex* for χ. Since (W,γ) is unique up to conjugacy, then so is (Q,δ). This concept of vertex was introduced and studied by Navarro in Section 3 of [12].

If P is a p-subgroup of G and $\epsilon \in \mathrm{Irr}(P)$, we denote by $\mathrm{Irr}(G|P,\epsilon)$ the set of irreducible characters of G having (P,ϵ) as a vertex.

3. Proof of the main theorem

We begin this section by fixing a prime number p. In order to prove the main theorem we need a number of preliminary results. The first one is quite general and easy.

Lemma 3.1. *Let H be a subgroup of a finite group G, b a p-block of H and $\theta \in \mathrm{Irr}(b)$. Suppose θ^G is irreducible and let B be the p-block of G to which θ^G belongs. Then b^G is defined and equals B. Moreover, if θ^G is of height zero, then θ is of height zero and B and b have a common defect group.*

Proof. The fact that b^G is defined and $b^G = B$ is immediate from [11, Lemma 5.3.1]. Now suppose θ^G is of height zero. Let P be a defect group for b. Then by Lemma 5.3.3 in [11] P is contained in some defect group D of B. Next $\theta(1)_p \geqslant |H:P|_p$ and, as θ^G has height zero, we have $|G:H|_p\theta(1)_p = |G:D|_p$. Then

$$|G:D|_p \geqslant |G:H|_p|H:P|_p = |G:P|_p \geqslant |G:D|_p.$$

It follows that $P = D$ and that θ is of height zero. $\square$

Lemma 3.2. *Let B be a p-block of a p-solvable group G and let $\chi \in \mathrm{Irr}(B)$ with vertex (Q,δ). Then:*

(a) *Q is contained in some defect group of B.*
(b) *$\chi \in \mathrm{Irr}_0(B)$ if and only if Q is a defect group for B and δ is linear.*

Proof. Let (W,γ) be a normal nucleus of χ such that $Q \in \mathrm{Syl}_p(W)$ and $\delta = (\gamma_p)_Q$. Now let b be the block of W to which γ belongs. Since γ is p-factorable, [16, Lemma 2.10] tells us that Q is a defect group for b. Also, as $\gamma^G = \chi$, we have that b^G is defined and equals B by Lemma 3.1. Then Q is contained in some defect group D of B by [11, Lemma 5.3.3]. This proves (a).

Next, we have

$$\begin{aligned}\chi(1)_p &= |G:W|_p\gamma(1)_p\\ &= |G:Q|_p\gamma_p(1)\\ &= |G:Q|_p\delta(1).\end{aligned}$$

Since $Q \subseteq D$, it is clear that $|G:D|_p \leqslant |G:Q|_p\delta(1)$. It follows that $\chi \in \mathrm{Irr}_0(B)$ if and only if $Q = D$ and $\delta(1) = 1$. This clearly proves (b). $\square$

Our proof of the main theorem relies on a result of Navarro [13]. In order to state Navarro's theorem, we need some terminology and notation.

Let G be a p-solvable group and let N be a normal subgroup of G. Let Q be a p-subgroup of G and $\delta \in \mathrm{Irr}(Q)$.

Assume that $Q \cap N \in \mathrm{Syl}_p(N)$ and that δ extends to QN. Then $Q \in \mathrm{Syl}_p(QN)$ and so by [4, Theorem F] δ extends to a p-special character $\widetilde{\delta}$ of QN. Furthermore, by [1, Proposition 6.1], $\widetilde{\delta}$ is the unique p-special extension of δ to QN.

Let μ be a p-factorable character of N. In Section 5 of [13], (N,μ,Q,δ) is called a *quadruple* provided that $Q \cap N \in \mathrm{Syl}_p(N)$, $\mu_{p'}$ is Q-invariant, δ extends to QN and μ_p lies under the unique p-special extension $\widetilde{\delta}$ of δ to QN.

Suppose (N,μ,Q,δ) is a quadruple. The character μ is said to be *good* for $\chi\in\operatorname{Irr}(G|Q,\delta)$ if χ has a normal nucleus (W,γ) such that $Q\in\operatorname{Syl}_p(W)$, $(\gamma_p)_Q=\delta$, $N\subseteq W$ and μ lies under γ. We write $\operatorname{Irr}(G|Q,\delta,\mu)$ for the set of all $\chi\in\operatorname{Irr}(G|Q,\delta)$ for which μ is good. Also, for a block B of G, we denote by $\operatorname{Irr}(B|Q,\delta,\mu)$ the intersection $\operatorname{Irr}(B)\cap\operatorname{Irr}(G|Q,\delta,\mu)$. We can now state Navarro's result. (See [13, Theorem 5.5].)

Theorem 3.3. *Let $N\lhd G$, where G is p-solvable and let Q be a p-subgroup of G and $\delta\in\operatorname{Irr}(Q)$. Write $N_G(Q,\delta)$ for the inertial group of δ in $N_G(Q)$ and suppose μ is a p-factorable character of N. If B is a p-block of G and (N,μ,Q,δ) is a quadruple, then*

$$\left|\operatorname{Irr}(B|Q,\delta,\mu)\right|\leqslant\sum_b\left|\operatorname{Irr}(b|Q,\delta,\mu)\right|,$$

where b runs over the p-blocks of $NN_G(Q,\delta)$ inducing B.

Proposition 3.4. *Let $N\lhd G$, where G is p-solvable and let B and b be p-blocks of G and N respectively such that B covers b. Let D be a defect group of B and suppose $\mu\in\operatorname{Irr}(b)$ is G-invariant and p-factorable. If $\widehat{B}$ is the p-block of $NN_G(D)$ with defect group D such that $\widehat{B}^G=B$, then $|\operatorname{Irr}_0(B|\mu)|\leqslant|\operatorname{Irr}_0(\widehat{B}|\mu)|$.*

Proof. Since μ is G-invariant and p-factorable, [1, Proposition 7.1] implies that both μ_p and $\mu_{p'}$ are invariant in G. Also, b is G-stable and so by Proposition 4.2 in [7], $D\cap N$ is a defect group of b. Now by [16, Lemma 2.10], it follows that $D\cap N\in\operatorname{Syl}_p(N)$.

We may clearly assume that $\operatorname{Irr}_0(B|\mu)\neq\emptyset$. Let $\chi\in\operatorname{Irr}_0(B|\mu)$. Then by Lemma 3.2(b), χ has vertex (D,δ), where δ is linear. Let (W,γ) be a normal nucleus for χ such that $D\in\operatorname{Syl}_p(W)$ and $(\gamma_p)_D=\delta$. As μ is p-factorable and G-invariant, it follows by the construction of the normal nucleus that $N\subseteq W$ and μ lies under γ. So, in particular, μ_p lies under γ_p. Now since γ_p is linear, we must have $(\gamma_p)_N=\mu_p$.

We have $DN\subseteq W$. Then, in light of [1, Proposition 6.1], the restriction $(\gamma_p)_{DN}$ is the unique p-special extension $\widetilde{\delta}$ of δ to DN. Now $\widetilde{\delta}_N=\mu_p$ and we have that (N,μ,D,δ) is a quadruple. Also, notice that μ is good for χ. Hence $\chi\in\operatorname{Irr}(B|D,\delta,\mu)$.

Let $\Delta=\{(D,\delta)\colon\ (D,\delta)$ is a vertex of some $\chi\in\operatorname{Irr}_0(B|\mu)\}$. It is clear that $N_G(D)$ acts by conjugation on Δ. Let $\{(D,\delta_1),\dots,(D,\delta_m)\}$ be a complete set of representatives of the orbits of this action. Then $\operatorname{Irr}(B|D,\delta_i,\mu)\neq\emptyset$ for each i, and $\operatorname{Irr}_0(B|\mu)\subseteq\bigcup_{i=1}^m\operatorname{Irr}(B|D,\delta_i,\mu)$. So, in particular,

$$\left|\operatorname{Irr}_0(B|\mu)\right|\leqslant\sum_{i=1}^m\left|\operatorname{Irr}(B|D,\delta_i,\mu)\right|.$$

For each i, write E_i for $\bigcup_\beta\operatorname{Irr}(\beta|D,\delta_i,\mu)$, where β runs over the blocks of $NN_G(D,\delta_i)$ inducing B. Then, by Theorem 3.3, we have $|\operatorname{Irr}(B|D,\delta_i,\mu)|\leqslant|E_i|$, for all i. Therefore

$$\left|\operatorname{Irr}_0(B|\mu)\right|\leqslant\sum_{i=1}^m|E_i|.\tag{1}$$

Next, we have that $\mu_{p'}$ is invariant in DN. Then by [1, Proposition 4.3], there exists a unique p'-special character $\widetilde{\mu_{p'}}$ of DN that lies over $\mu_{p'}$, and, in fact, $\widetilde{\mu_{p'}}$ extends $\mu_{p'}$.

Let $i\in\{1,\dots,m\}$. Then if $\widetilde{\delta_i}$ is the unique p-special extension of δ_i to DN, we have that the character $\widetilde{\delta_i}\widetilde{\mu_{p'}}$ is p-factorable. Also, in view of Theorem 5.4 in [13], $E_i\subseteq\operatorname{Irr}(NN_G(D,\delta_i)|\widetilde{\delta_i}\widetilde{\mu_{p'}})$. Now we claim that $NN_G(D,\delta_i)$ is the inertial group I_i of $\widetilde{\delta_i}\widetilde{\mu_{p'}}$ in $NN_G(D)$.

Since $\mu_{p'}$ is G-invariant and $\widetilde{\mu_{p'}}$ is the unique p'-special character of DN over $\mu_{p'}$, note that $\widetilde{\mu_{p'}}$ is invariant in $NN_G(D)$. Then, in light of [1, Proposition 7.1], it follows that I_i equals the inertial group J_i of $\widetilde{\delta_i}$ in $NN_G(D)$. Now to prove our claim, as $N\subseteq J_i$, it suffices to show that $J_i\cap N_G(D)=N_G(D,\delta_i)$.

Let $x \in N_G(D)$. Then $\widetilde{\delta_i}^x \in \operatorname{Irr}(DN)$ and $(\widetilde{\delta_i}^x)_D = ((\widetilde{\delta_i})_D)^x = \delta_i^x$. If $x \in J_i$, we get $\delta_i = \delta_i^x$ and so $x \in N_G(D, \delta_i)$. Next assume $x \in N_G(D, \delta_i)$. Then $(\widetilde{\delta_i}^x)_D = \delta_i$. But since $\widetilde{\delta_i}$ is the unique p-special extension of δ_i to DN, we are forced to have $\widetilde{\delta_i}^x = \widetilde{\delta_i}$. Hence $x \in J_i$. We have thus proved our claim.

Now let $F_i = \{\theta^{NN_G(D)}\colon\ \theta \in E_i\}$. By Theorem 6.11 in [2], $F_i \subseteq \operatorname{Irr}(NN_G(D))$ and character induction defines a bijection from E_i onto F_i. Consequently

$$|F_i| = |E_i|. \tag{2}$$

Next, we show that $F_i \subseteq \operatorname{Irr}_0(\widehat{B}|\mu)$.

Let β be a block of $NN_G(D, \delta_i)$ such that $\beta^G = B$ and $\operatorname{Irr}(\beta|D, \delta_i, \mu) \neq \emptyset$. Next let $\theta \in \operatorname{Irr}(\beta|D, \delta_i, \mu)$. Then $\theta^{NN_G(D)}$ is irreducible. Let β' be the block of $NN_G(D)$ to which $\theta^{NN_G(D)}$ belongs. By Lemma 3.1, $\beta^{NN_G(D)}$ is defined and equals β'. Also, since θ has vertex (D, δ_i) and $\beta^G = B$, it follows by Lemma 3.2(a) and [11, Lemma 5.3.3] that D is a defect group for β. In addition, as $\beta' = \beta^{NN_G(D)}$, another application of [11, Lemma 5.3.3] gives us that D is contained in some defect group Q of β'.

Since $\beta^{NN_G(D)} = \beta'$ and $\beta^G = B$, then Lemma 5.3.4 in [11] says that β'^G is defined and equals B. Now, in view of Lemma 5.3.3 of [11], since $D \subseteq Q$ and B has defect group D, we must have $Q = D$. As $\widehat{B}$ is the unique block of $NN_G(D)$ having defect group D and such that $\widehat{B}^G = B$, we conclude that $\beta' = \widehat{B}$. Hence $\theta^{NN_G(D)} \in \operatorname{Irr}(\widehat{B})$.

Next, as β has defect group D and θ has vertex (D, δ_i) with δ_i linear, we have $\theta \in \operatorname{Irr}_0(\beta)$ by Lemma 3.2(b). Then

$$\begin{aligned}\theta^{NN_G(D)}(1)_p &= \bigl|NN_G(D) : NN_G(D, \delta_i)\bigr|_p \bigl|NN_G(D, \delta_i) : D\bigr|_p \\ &= \bigl|NN_G(D) : D\bigr|_p.\end{aligned}$$

Thus $\theta^{NN_G(D)} \in \operatorname{Irr}_0(\widehat{B})$.

Since μ is good for θ, we have that θ lies over μ. Hence $\theta^{NN_G(D)}$ lies over μ and we now have $\theta^{NN_G(D)} \in \operatorname{Irr}_0(\widehat{B}|\mu)$. This shows that $F_i \subseteq \operatorname{Irr}_0(\widehat{B}|\mu)$, as needed.

Next, we want to show that the F_i's are mutually disjoint. So let $i, j \in \{1, \dots, m\}$ and suppose $F_i \cap F_j \neq \emptyset$. Then $\widetilde{\delta_j}\widetilde{\mu_{p'}}$ is $NN_G(D)$-conjugate to $\widetilde{\delta_i}\widetilde{\mu_{p'}}$ and hence (in view of [1, Proposition 7.1]) $\widetilde{\delta_j}$ is $NN_G(D)$-conjugate to $\widetilde{\delta_i}$. Therefore $\widetilde{\delta_j} = \widetilde{\delta_i}^y$ for some $y \in N_G(D)$. Now $\delta_j = (\widetilde{\delta_j})_D = ((\widetilde{\delta_i})_D)^y = \delta_i^y$, and so $(D, \delta_j) = (D, \delta_i)^y$. We conclude then that $i = j$. Therefore the F_i's are mutually disjoint, as wanted.

By (1) and (2), we have $|\operatorname{Irr}_0(B|\mu)| \leqslant \sum_{i=1}^m |F_i|$. But $\sum_{i=1}^m |F_i| = |\bigcup_{i=1}^m F_i|$ and $\bigcup_{i=1}^m F_i \subseteq \operatorname{Irr}_0(\widehat{B}|\mu)$. It follows that $|\operatorname{Irr}_0(B|\mu)| \leqslant |\operatorname{Irr}_0(\widehat{B}|\mu)|$, which clearly ends the proof of the proposition. $\square$

The proof of the following result is the same as that of [9, Lemma 5.3(i)].

Lemma 3.5. *Let $N \lhd G$, where G is p-solvable and suppose $\mu \in \operatorname{Irr}(N)$ is invariant in G. Let (W, γ) be a normal nucleus of μ. If S is the stabilizer of (W, γ) in G, then $G = SN$, $W = S \cap N$ and character induction defines a bijection of $\operatorname{Irr}(S|\gamma)$ onto $\operatorname{Irr}(G|\mu)$.*

In the following, the assumption in Proposition 3.4 that μ is p-factorable is removed.

Proposition 3.6. *Let $N \lhd G$, where G is p-solvable and let B and b be p-blocks of G and N respectively such that B covers b. Suppose B has defect group D and $\mu \in \operatorname{Irr}(b)$ is G-invariant. If $\widehat{B}$ is the unique p-block of $NN_G(D)$ with defect group D such that $\widehat{B}^G = B$, then $|\operatorname{Irr}_0(B|\mu)| \leqslant |\operatorname{Irr}_0(\widehat{B}|\mu)|$.*

Proof. We may assume that $\operatorname{Irr}_0(B|\mu) \neq \emptyset$. Let (W, γ) be a normal nucleus for μ, and let S be the stabilizer of (W, γ) in G. Choose $\chi_0 \in \operatorname{Irr}_0(B|\mu)$. By Lemma 3.5, there exists $\theta_0 \in \operatorname{Irr}(S|\gamma)$ such that $\chi_0 = \theta_0^G$. Then, in view of Lemma 3.1, if β_0 is the block of S to which θ_0 belongs, we have that β_0^G

is defined, $\beta_0^G = B$ and some G-conjugate of D is a defect group for β_0. Since μ is G-invariant, any G-conjugate of (W, γ) is also a normal nucleus of μ. Then, by replacing (W, γ) by a conjugate if necessary, we may assume that D is a defect group for β_0. So, in particular, $D \subseteq S$.

Let b' be the block of W to which γ belongs. Since θ_0 lies over γ, note that β_0 covers b'. Now let $\beta_0, \dots, \beta_m$ be all the (distinct) blocks of S covering b', having defect group D, and such that $\beta_i^G = B$ for every $i \in \{0, \dots, m\}$.

In light of Lemma 3.5 and Lemma 3.1, character induction defines an injection ι from $\bigcup_{i=0}^m \mathrm{Irr}_0(\beta_i|\gamma)$ into $\mathrm{Irr}_0(B|\mu)$. We claim that ι is onto.

Let χ be any element of $\mathrm{Irr}_0(B|\mu)$. Then, as for χ_0, there exists $\theta \in \mathrm{Irr}(S|\gamma)$ such that $\theta^G = \chi$. Let β be the block of S to which θ belongs. It is clear that β covers b'. Also, by Lemma 3.1, $\beta^G = B$, θ has height zero and there exists $g \in G$ for which D^g is a defect group of β. Now, to prove our claim, it suffices to show that β has defect group D.

Since γ is S-invariant, the block b' is S-stable. Then by [7, Proposition 4.2], as β_0 covers b', $D \cap W$ is a defect group for b'. Now, since γ is p-factorable, [16, Lemma 2.10] tells us that $D \cap W \in \mathrm{Syl}_p(W)$. We deduce, in particular, that $D \in \mathrm{Syl}_p(DW)$. As $G = NS$ by Lemma 3.5, we may assume that $g \in N$. Let $d \in D$. Then $d^g d^{-1} \in N \cap S$. Since $N \cap S = W$ by Lemma 3.5, we conclude that $D^g \subseteq DW$. But we know that $D \in \mathrm{Syl}_p(DW)$. It follows that $D^g \in \mathrm{Syl}_p(DW)$, and hence $D^g = D^w$ for some $w \in W$. As $W \subseteq S$, we get that D is a defect group of β, as needed.

Now ι is a bijection and so, in particular,

$$\sum_{i=0}^m \left|\mathrm{Irr}_0(\beta_i|\gamma)\right| = \left|\mathrm{Irr}_0(B|\mu)\right|. \tag{1}$$

Next for each $i \in \{0, \dots, m\}$, denote by $\widehat{\beta_i}$ the block of $WN_S(D)$ with defect group D such that $\widehat{\beta_i}^S = \beta_i$. By Proposition 3.4 (with S, W in place of G and N, respectively) we have

$$\left|\mathrm{Irr}_0(\beta_i|\gamma)\right| \leqslant \left|\mathrm{Irr}_0(\widehat{\beta_i}|\gamma)\right|, \tag{2}$$

for all i. Also, by the remark following Lemma 3.1 in [8] and [11, Lemma 5.5.7], it is easy to see that every $\widehat{\beta_i}$ covers b', as b' is S-stable and β_i covers b'.

The stabilizer of (W, γ) in $NN_G(D)$ is clearly $S \cap NN_G(D)$. We claim that $S \cap NN_G(D) = WN_S(D)$. First, it is clear that $WN_S(D) \subseteq S \cap NN_G(D)$. Next let $x \in S \cap NN_G(D)$. If d is any element of D, we have $d^x = d'^y$ for some $d' \in D$ and $y \in N$. Then $d^x d'^{-1} = d'^y d'^{-1} \in N$. Also, $d^x d'^{-1} \in S$ as $D \subseteq S$ and $x \in S$. Hence $d^x d'^{-1} \in W$. It follows that $D^x \subseteq DW$. Now since $D \in \mathrm{Syl}_p(DW)$, we have $D^x = D^v$ for some $v \in W$. Therefore $xv^{-1} \in N_S(D)$ and it follows that $x \in WN_S(D)$. We have thus shown that $S \cap NN_G(D) = WN_S(D)$, as claimed.

Let $i \in \{0, \dots, m\}$. Since $\widehat{\beta_i}$ covers b', there exists a character $\tau \in \mathrm{Irr}(\widehat{\beta_i}|\gamma)$ by [11, Lemma 5.5.8]. Then by Lemma 3.5, $\tau^{NN_G(D)}$ is irreducible, and hence $\widehat{\beta_i}^{NN_G(D)}$ is defined by Lemma 3.1. Next, we have $\widehat{\beta_i}^S = \beta_i$ and $\beta_i^G = B$. Then, in view of [11, Lemma 5.3.4], $\widehat{\beta_i}^G$ is defined and equals B. Now another application of the same lemma gives us that $(\widehat{\beta_i}^{NN_G(D)})^G$ is defined and equals B. Also, since both $\widehat{\beta_i}$ and B have defect group D, we deduce by Lemma 5.3.3 in [11] that D is a defect group for $\widehat{\beta_i}^{NN_G(D)}$. Now by the uniqueness of $\widehat{B}$, we are forced to have $\widehat{\beta_i}^{NN_G(D)} = \widehat{B}$.

Now, taking into account Lemma 3.5 and Lemma 3.1, character induction defines an injective map from $\bigcup_{i=0}^m \mathrm{Irr}_0(\widehat{\beta_i}|\gamma)$ into $\mathrm{Irr}_0(\widehat{B}|\mu)$. Since $\beta_0, \dots, \beta_m$ are distinct, note that the blocks $\widehat{\beta_0}, \dots, \widehat{\beta_m}$ are distinct. It follows that

$$\sum_{i=0}^m \left|\mathrm{Irr}_0(\widehat{\beta_i}|\gamma)\right| \leqslant \left|\mathrm{Irr}_0(\widehat{B}|\mu)\right|.$$

Then, using (1) and (2), we finally get that $|\mathrm{Irr}_0(B|\mu)| \leqslant |\mathrm{Irr}_0(\widehat{B}|\mu)|$. The proof of the proposition is now complete. $\square$

We are now ready to prove the main theorem.

Proof of Theorem A. *Step* 1. Our objective in this step is to show that

$$\left|\mathrm{Irr}_0(B|\mu)\right| \leqslant \left|\bigcup_{t\in T}\mathrm{Irr}_0\big(\widehat{B}|\mu^t\big)\right|.$$

We may clearly assume that $\mathrm{Irr}_0(B|\mu)\neq\emptyset$. Let I be the inertial group of μ in G and let B_T be the Fong–Reynolds correspondent of B with respect to b.

Let $\chi\in\mathrm{Irr}_0(B|\mu)$. Then there exists a unique irreducible character $\chi_{(I)}$ of I lying over μ such that $\chi=(\chi_{(I)})^G$. Now let $\chi_{(T)}$ be the irreducible character $(\chi_{(I)})^T$. Then since $\chi_{(T)}$ lies over μ and $\mu\in\mathrm{Irr}(b)$, Theorem 5.5.10 in [11] implies that $\chi_{(T)}$ belongs to B_T. Also, as B_T has defect group D and χ has height zero, note that $\chi_{(T)}$ is of height zero. Let β be the block of I to which $\chi_{(I)}$ belongs. Then Lemma 3.1 says that $\beta^T=B_T$ (and hence $\beta^G=B$), $\chi_{(I)}$ has height zero and some T-conjugate of D is a defect group for β. Furthermore, note that β covers b.

Choose now a minimal subset U of T such that for each $\chi\in\mathrm{Irr}_0(B|\mu)$, there is a unique $u\in U$ for which D^u is a defect group for the block of I to which $\chi_{(I)}$ belongs. Next for each $u\in U$, write E_u for the set of all $\chi\in\mathrm{Irr}_0(B|\mu)$ such that $\chi_{(I)}$ belongs to a block of I having D^u as a defect group. By our choice of the set U, it is clear that $\{E_u\colon u\in U\}$ is a partition of $\mathrm{Irr}_0(B|\mu)$. Hence

$$\left|\mathrm{Irr}_0(B|\mu)\right| = \sum_{u\in U}|E_u|. \tag{1}$$

Let $u\in U$ and let $\beta_{u,1},\dots,\beta_{u,m_u}$ be all the (distinct) blocks of I covering b, having defect group D^u and such that $(\beta_{u,i})^G=B$ for all $i\in\{1,\dots,m_u\}$. By the above discussion, if $\chi\in E_u$, then we have $\chi_{(I)}\in\mathrm{Irr}_0(\beta_{u,i}|\mu)$ for some i. It follows that $|E_u|\leqslant\sum_{i=1}^{m_u}|\mathrm{Irr}_0(\beta_{u,i}|\mu)|$.

Next, for each i, denote by $\widehat{\beta_{u,i}}$ the unique block of $NN_I(D^u)$ having defect group D^u and such that $\widehat{\beta_{u,i}}^I=\beta_{u,i}$. By Proposition 3.6, we have $|\mathrm{Irr}_0(\beta_{u,i}|\mu)|\leqslant|\mathrm{Irr}_0(\widehat{\beta_{u,i}}|\mu)|$ for every i. Since $\widehat{\beta_{u,1}},\dots,\widehat{\beta_{u,m_u}}$ are distinct blocks of $NN_I(D^u)$, it follows that

$$|E_u| \leqslant \left|\bigcup_{i=1}^{m_u}\mathrm{Irr}_0(\widehat{\beta_{u,i}}|\mu)\right|. \tag{2}$$

Let $i\in\{1,\dots,m_u\}$. In light of the remark following Lemma 3.1 of [8], since b is I-stable and $\beta_{u,i}$ covers b, one can easily see that $\widehat{\beta_{u,i}}$ covers b. So, in particular, there exists an irreducible character θ of $\widehat{\beta_{u,i}}$ that lies over μ. Then, as $NN_I(D^u)$ is the inertial group of μ in $NN_G(D^u)$, we have $\theta^{NN_G(D^u)}\in\mathrm{Irr}(NN_G(D^u))$ by [2, Theorem 6.11]. Now Lemma 3.1 tells us that $\widehat{\beta_{u,i}}^{NN_G(D^u)}$ is defined.

As $\widehat{\beta_{u,i}}^I=\beta_{u,i}$ and $(\beta_{u,i})^G=B$, we have that $\widehat{\beta_{u,i}}^G$ is defined and equals B by [11, Lemma 5.3.4]. Next, since $\widehat{\beta_{u,i}}^{NN_G(D^u)}$ is defined, a second application of the same lemma gives us that $(\widehat{\beta_{u,i}}^{NN_G(D^u)})^G$ is defined and equals B. Also, we know that both $\widehat{\beta_{u,i}}$ and B have D^u as a defect group. Then by Lemma 5.3.3 in [11], we deduce that $\widehat{\beta_{u,i}}^{NN_G(D^u)}$ has defect group D^u. However, $\widehat{B}^u$ is the only block of $NN_G(D^u)$ that has defect group D^u and such that $(\widehat{B}^u)^G=B$. We are then forced to have $\widehat{\beta_{u,i}}^{NN_G(D^u)}=\widehat{B}^u$.

By [2, Theorem 6.11], character induction defines an injection from the set $\bigcup_{i=1}^{m_u}\mathrm{Irr}_0(\widehat{\beta_{u,i}}|\mu)$ into $\mathrm{Irr}(NN_G(D^u)|\mu)$. Now suppose $\eta\in\bigcup_{i=1}^{m_u}\mathrm{Irr}_0(\widehat{\beta_{u,i}}|\mu)$. Then $\eta^{NN_G(D^u)}\in\mathrm{Irr}(\widehat{B}^u)$ by Lemma 3.1. Moreover, since D^u is a defect group for $\widehat{B}^u$ and for each $\widehat{\beta_{u,i}}$, the character $\eta^{NN_G(D^u)}$ is of height zero. It follows that

$$\left|\bigcup_{i=1}^{m_u}\mathrm{Irr}_0(\widehat{\beta_{u,i}}|\mu)\right| \leqslant \left|\mathrm{Irr}_0\big(\widehat{B}^u|\mu\big)\right|.$$

But $|\mathrm{Irr}_0(\widehat{B}^u|\mu)| = |\mathrm{Irr}_0(\widehat{B}|\mu^{u^{-1}})|$. Then we get

$$\left|\bigcup_{i=1}^{m_u} \mathrm{Irr}_0(\widehat{\beta_{u,i}}|\mu)\right| \leqslant \left|\mathrm{Irr}_0\left(\widehat{B}|\mu^{u^{-1}}\right)\right|. \tag{3}$$

Now by (1), (2) and (3), we have

$$\left|\mathrm{Irr}_0(B|\mu)\right| \leqslant \sum_{u\in U}\left|\mathrm{Irr}_0\left(\widehat{B}|\mu^{u^{-1}}\right)\right|. \tag{4}$$

Our next task is to show that the character sets $\mathrm{Irr}_0(\widehat{B}|\mu^{u^{-1}})$ are mutually disjoint. So let $u_1, u_2 \in U$ and assume $\mathrm{Irr}_0(\widehat{B}|\mu^{u_1^{-1}}) \cap \mathrm{Irr}_0(\widehat{B}|\mu^{u_2^{-1}}) \neq \emptyset$. Then $\mu^{u_2^{-1}} = (\mu^{u_1^{-1}})^g$ for some $g \in N_G(D)$. Therefore $u_1^{-1} g u_2 \in I$, and it follows that D^{u_1} is I-conjugate to D^{u_2}. Now by the choice of the set U, we must have $u_1 = u_2$. This shows that the sets $\mathrm{Irr}_0(\widehat{B}|\mu^{u^{-1}})$ are mutually disjoint.

Now $\sum_{u\in U}|\mathrm{Irr}_0(\widehat{B}|\mu^{u^{-1}})| = |\bigcup_{u\in U}\mathrm{Irr}_0(\widehat{B}|\mu^{u^{-1}})|$. Since $\bigcup_{u\in U}\mathrm{Irr}_0(\widehat{B}|\mu^{u^{-1}}) \subseteq \bigcup_{t\in T}\mathrm{Irr}_0(\widehat{B}|\mu^t)$, then in view of (4), we conclude that $|\mathrm{Irr}_0(B|\mu)| \leqslant |\bigcup_{t\in T}\mathrm{Irr}_0(\widehat{B}|\mu^t)|$, as needed to be shown.

Step 2. Here we show that $|\mathrm{Irr}_0(B|\mu)| = |\bigcup_{t\in T}\mathrm{Irr}_0(\widehat{B}|\mu^t)|$. Let T act by conjugation on $\mathrm{Irr}(b)$, and let $\mu_1 = \mu, \dots, \mu_n$ be a complete set of representatives for the orbits of this action.

Since B covers b, we have $\mathrm{Irr}_0(B) = \bigcup_{i=1}^n \mathrm{Irr}_0(B|\mu_i)$. Next let $1 \leqslant i, j \leqslant n$ and suppose $\mathrm{Irr}_0(B|\mu_i) \cap \mathrm{Irr}_0(B|\mu_j) \neq \emptyset$. Then μ_i is G-conjugate to μ_j. Now, as $\mu_i, \mu_j \in \mathrm{Irr}(b)$, it follows that μ_i is T-conjugate to μ_j. We must then have $i = j$. Now it holds that $|\mathrm{Irr}_0(B)| = \sum_{i=1}^n |\mathrm{Irr}_0(B|\mu_i)|$.

For each $i \in \{1, \dots, n\}$, denote by C_i the character set $\bigcup_{t\in T}\mathrm{Irr}_0(\widehat{B}|\mu_i^t)$. We claim that the C_i's are mutually disjoint. So let $1 \leqslant i, j \leqslant n$ and assume $C_i \cap C_j \neq \emptyset$. Then there exist $t_1, t_2 \in T$ such that $\mu_i^{t_1}$ is $NN_G(D)$-conjugate to $\mu_j^{t_2}$. Since $\mu_i^{t_1}, \mu_j^{t_2} \in \mathrm{Irr}(b)$, it follows that $\mu_i^{t_1}$ is $NN_T(D)$-conjugate to $\mu_j^{t_2}$, which forces $i = j$. This clearly proves our claim.

By Step 1, we have

$$\left|\mathrm{Irr}_0(B|\mu_i)\right| \leqslant |C_i| \tag{5}$$

for every i. Then

$$\left|\mathrm{Irr}_0(B)\right| = \sum_{i=1}^n \left|\mathrm{Irr}_0(B|\mu_i)\right| \leqslant \sum_{i=1}^n |C_i| = \left|\bigcup_{i=1}^n C_i\right|.$$

However, $\bigcup_{i=1}^n C_i \subseteq \mathrm{Irr}_0(\widehat{B})$, and by Okuyama and Wajima's theorem [15], $|\mathrm{Irr}_0(B)| = |\mathrm{Irr}_0(\widehat{B})|$. It follows that $\sum_{i=1}^n |\mathrm{Irr}_0(B|\mu_i)| = \sum_{i=1}^n |C_i|$, and hence, in view of (5), $|\mathrm{Irr}_0(B|\mu_i)| = |C_i|$ for every i. This clearly finishes the proof of the theorem. $\square$

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