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Poset embeddings of Hilbert functions and Betti numbers ☆

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ABSTRACT

We study inequalities between graded Betti numbers of ideals in a standard graded algebra over a field and their images under embedding maps, defined earlier by us in Caviglia and Kummini (2013) [5]. We show that if graded Betti numbers do not decrease when we replace ideals in an algebra by their embedded versions, then the same behaviour is carried over to ring extensions. As a corollary we give alternative inductive proofs of earlier results of Bigatti, Hulett, Pardue, Mermin, Peeva, Stillman and Murai. We extend a hypersurface restriction theorem of Herzog and Popescu to the situation of embeddings. We show that we can obtain the Betti table of an ideal in the extension ring from the Betti table of its embedded version by a sequence of consecutive cancellations. We further show that the lex-plus-powers conjecture of Evans reduces to the Artinian situation.

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1. Introduction

This paper is part of our study of embeddings of the poset of Hilbert series into the poset of ideals, in a standard-graded algebra R over a field $\Bbbk$. In our earlier paper [5], we looked into ways of embedding the set $\mathcal{H}_R$ of all Hilbert series of graded R-ideals (partially ordered by comparing the coefficients) into the set $\mathcal{I}_R$ of graded R-ideals (partially ordered by inclusion). Here we look at the behaviour of graded Betti numbers when we replace an ideal by an ideal (with the same Hilbert function) that is in the image of the embedding.

Examples of such embeddings, studied classically, are polynomial rings and quotient rings of polynomial rings by regular sequences generated by powers of the variables. If R is a polynomial ring, then for every R-ideal I, there exists a lex-segment ideal L in R such that the Hilbert functions of I and L are identical (a theorem of F.S. Macaulay, see [1, Section 4.2]) and, moreover, each of the graded Betti numbers of L is at least as large as the corresponding graded Betti numbers of I (A.M. Bigatti [2], H.A. Hulett [19] and K. Pardue [29]). Similarly, if $R = \Bbbk[x_1, \dots, x_n]/(x_1^{e_1}, \dots, x_n^{e_n})$ for some integers $2 \leq e_1 \leq \cdots \leq e_n$, then for every R-ideal I there exists a lex-segment $\Bbbk[x_1, \dots, x_n]$-ideal L such that I and LR have identical Hilbert functions (J.B. Kruskal [22] and G. Katona [21] for the case $e_1 = \cdots = e_n = 2$, and G.F. Clements and B. Lindström [6], in general). Again, as $\Bbbk[x_1, \dots, x_n]$-modules, each graded Betti number of R/LR is at least as large as the corresponding graded Betti number of R/I; this was proved by J. Mermin, I. Peeva and M. Stillman [27] for the case $e_1 = \cdots = e_n = 2$ in characteristic zero, by S. Murai [28] in the general case for strongly stable ideals and by Mermin and Murai [24] in full generality. In both these cases, mapping the Hilbert series of I to L (or to LR in the second case) gives an embedding of $\mathcal{H}_R$ into $\mathcal{I}_R$, such that the graded Betti numbers do not decrease after the embedding. See [13,14,20] for comparing the graded Betti numbers of R/I and R/LR over $\Bbbk[x_1, \dots, x_n]$ and [26] for comparing the graded Betti numbers over R.

Before we spell out what is done in this paper, let us fix some notation. By z we mean an indeterminate over R. Set $S = R[z]/(z^t)$, where $t \geq 1 \in \mathbb{N}$ or $t = \infty$; if $t = \infty$, we mean that $z^t = 0$. By A we denote a standard-graded polynomial ring over $\Bbbk$ that minimally presents R, i.e., $\ker(A \longrightarrow R)$ does not have any linear forms. Let $B = A[z]$. The *graded Betti numbers* of a finitely generated graded R-module M are $\beta^R_{i,j}(M) = \dim_{\Bbbk} \operatorname{Tor}^R_i(M, \Bbbk)_j$. The *Betti table* of M, denoted $\beta^R(M)$, is $\sum_{i,j} \beta^R_{i,j}(M)\mathbf{e}_{i,j}$, where $\{\mathbf{e}_{i,j}, i \in \mathbb{N}, j \in \mathbb{Z}\}$ is the standard basis of $\mathbb{Z}^{\mathbb{N}\times\mathbb{Z}}$.

In this paper, we show that if graded Betti numbers do not decrease when we replace R-ideals by their embedded versions, then the same behaviour extends to S-ideals; see Theorem 3.1 for the precise statement. This generalizes analogous results of Bigatti, Hulett, Pardue and Murai mentioned above. Moreover, our technique gives short and self-contained proofs of [27, Theorem 1.1] (more precisely, the key intricate step for proving it for strongly-stable-plus-squares ideals — see [27, Section 3 and 4]) and [28, Theorem 1.1] (which generalizes the above theorem to arbitrary powers); see Corollary 3.7.

We show that we can obtain the Betti table of an S-ideal from the Betti table of its embedded version by a sequence of consecutive cancellations. This is analogous to, and motivated by, the similar result proved by Peeva comparing the Betti table of an arbitrary homogeneous ideal in a polynomial ring with the Betti table of the lex-segment ideal with the same Hilbert function [30, Theorem 1.1]. Additionally, we show that a minimal graded B-free resolution of an ideal in the image of the embedding map can be seen as an iterated mapping cone. The simplest instance of this is the Eliahou–Kervaire resolution (see [10,31]) of strongly-stable monomial ideals in polynomial rings. Similar iterated mapping cones have been considered for ideals containing monomial regular sequences, and have been used to obtain exact expressions for graded Betti numbers; see [27,28,13]. As an application of Theorem 3.1, we show that the lex-plus-powers conjecture of E.G. Evans reduces to the Artinian situation (Theorem 4.1).

A word about the proofs. Problems on finding bounds for Betti numbers such as those studied in much of the body of work above can be, inductively, reduced to studying polynomial extensions by one variable, or their quotients. This is what we address in Theorem 3.1. Its proof makes crucial use of [5, Theorem 3.10] (reproduced as Theorem 2.1(ii) below) which is an analogue of the Hyperplane Restriction Theorem of M. Green [15], and of similar results of J. Herzog and D. Popescu [18] and of Gasharov [12]. Caviglia and E. Sbarra [8, Theorem 3.1] use Theorem 2.1(ii), again, to prove a result similar to Theorem 3.1, where graded Betti numbers are replaced by Hilbert series of local cohomology modules. Also see [4] for a proof of the aforementioned result of Bigatti and Hulett, in characteristic zero, using Green's Hyperplane Restriction Theorem.

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2. Embeddings

We recall some notation and definitions from [5]. Let $\Bbbk$ be a field and A a finitely generated polynomial ring over $\Bbbk$. We treat A as standard graded, i.e., the indeterminates have degree one. Let $\mathfrak{a}$ be a homogeneous A-ideal and $R = A/\mathfrak{a}$. Let $\mathcal{I}_R = \{J : J \text{ is a homogeneous } R\text{-ideal}\}$, considered as a poset under inclusion. For a finitely generated graded R-module $M = \bigoplus_{t\in\mathbb{Z}} M_t$, the *Hilbert series* of M is the formal power series

$$H_M(\mathfrak{z}) = \sum_{t\in\mathbb{Z}} (\dim_{\Bbbk} M_t)\mathfrak{z}^t \in \mathbb{Z}[\![\mathfrak{z}]\!].$$

The *poset of Hilbert series of homogeneous R-ideals* is the set $\mathcal{H}_R = \{H_J : J \in \mathcal{I}_R\}$ endowed with the following partial order: $H \succcurlyeq H'$ (or $H' \preccurlyeq H'$) if, for all $t \in \mathbb{Z}$, the coefficient of $\mathfrak{z}^t$ in H is at least as large as that in H'. (We point out that by H_I we mean the Hilbert series of I and not of R/I.)

In [5] we studied the following question: given such a standard-graded $\Bbbk$-algebra R, is there an (order-preserving) embedding $\epsilon : \mathcal{H}_R \hookrightarrow \mathcal{I}_R$ as posets, such that $\mathrm{H} \circ \epsilon = \mathrm{id}_{\mathcal{H}_R}$,

where $\mathrm{H} : \mathcal{I}_R \longrightarrow \mathcal{H}_R$ is the function $J \mapsto H_J$? We will say that $\mathcal{H}_R$ *admits an embedding into* $\mathcal{I}_R$ (and often, by abuse of terminology, merely that $\mathcal{H}_R$ *admits an embedding*) if this question has an affirmative answer.

Recall (from Section 1) that $R = A/\mathfrak{a}$, $B = A[z]$ and $S = R[z]/(z^t) = B/(\mathfrak{a}B + (z^t))$ where t is a positive integer or is ∞. Let $\mathfrak{I} = \{i \in \mathbb{N} : i < t\}$. Treat B as *multigraded*, with $\deg x_i = (1, 0)$ and $\deg z = (0, 1)$ and let the grading on S be the one induced by this choice. (In order to study embeddings of $\mathcal{H}_S$, we think of S as standard-graded, but we will use its multigraded structure, which is a refinement of the standard grading, to construct them.)

Let J be a multigraded S-ideal. It is isomorphic, as an R-module, to $\bigoplus_{i\in\mathfrak{I}} J_{\langle i\rangle} z^i$, where for every $i \in \mathfrak{I}$, $J_{\langle i\rangle} = (J :_S z^i) \cap R$. Notice that for all $i \in \mathfrak{I}$, $i > 0$, $J_{\langle i-1\rangle} \subseteq J_{\langle i\rangle}$. We say that J is *z-stable* if $\mathfrak{m}_R J_{\langle i\rangle} \subseteq J_{\langle i-1\rangle}$, for all $i \in \mathfrak{I}, i > 0$ where $\mathfrak{m}_R$ denotes the homogeneous maximal ideal of R. We denote the set $\{H_J : J \text{ is a } z\text{-stable } S\text{-ideal}\}$ by $\mathcal{H}_S^{\mathrm{stab}}$. The main results from our earlier work [5] are:

Theorem 2.1. *Let t be a positive integer or ∞. Let $S = R[z]/(z^t)$. Suppose that $\mathcal{H}_R$ admits an embedding ϵ_R. Then:*

(i) There exists an embedding of posets $\epsilon_S : \mathcal{H}_S^{\mathrm{stab}} \hookrightarrow \mathcal{I}_S$ such that for all $H \in \mathcal{H}_S^{\mathrm{stab}}$, the Hilbert function of $\epsilon(H)$ is H.
(ii) If J is a z-stable S-ideal and $L = \epsilon_S(H_J)$, then $H_{(J+(z^i))} \succcurlyeq H_{(L+(z^i))}$ for all $i \in \mathfrak{I}$.

Part (i) is the content of [5, Theorem 3.3], where we further assume that for all S-ideals J, there is a z-stable S-ideal with the same Hilbert function (i.e., $\mathcal{H}_S^{\mathrm{stab}} = \mathcal{H}_S$) and conclude that $\mathcal{H}_S$ admits an embedding. If $t = \infty$, then $\mathcal{H}_S^{\mathrm{stab}} = \mathcal{H}_S$ [5, Lemma 4.1]. When t is finite, if $\Bbbk$ contains a primitive tth root of unity, $\mathfrak{a}$ is a monomial ideal in some $\Bbbk$-basis $\mathcal{B}$ of A_1, $l^t \in \mathfrak{a}$ for all $l \in \mathcal{B}$ then, again, $\mathcal{H}_S^{\mathrm{stab}} = \mathcal{H}_S$ [5, Lemma 4.2]. Part (ii), which can be viewed as a hypersurface restriction theorem, is [5, Theorem 3.10]; see Discussion 2.4.

Remark 2.2. Let J be a multigraded S-ideal. Write $J = \bigoplus_{i\in\mathfrak{I}} J_{\langle i\rangle} z^i$. Then $J' := \bigoplus_{i\in\mathfrak{I}} \epsilon_R(H_{J_{\langle i\rangle}}) z^i$ is an S-ideal. If J is additionally z-stable, then so is J'. To see this, we note that, since J is z-stable, $H_{J_{\langle i-1\rangle}} \succcurlyeq H_{\mathfrak{m}_R J_{\langle i\rangle}}$, and therefore $\epsilon_R(H_{J_{\langle i-1\rangle}}) \supseteq \epsilon_R(H_{\mathfrak{m}_R J_{\langle i\rangle}}) \supseteq \mathfrak{m}_R \epsilon_R(H_{J_{\langle i\rangle}})$, where the last inclusion follows by using embedding filtrations [5, Definition 2.3]. □

Remark 2.3. The map $\epsilon_S : \mathcal{H}_S^{\mathrm{stab}} \hookrightarrow \mathcal{I}_S$ constructed in the proof of Theorem 2.1 has additional properties inherited from ϵ_R, namely, if an S-ideal L is in the image of ϵ_S, it is multigraded, z-stable and, when written as $L = \bigoplus_{i\in\mathfrak{I}} L_{\langle i\rangle} z^i$, the ideal $L_{\langle i\rangle}$ is in the image of ϵ_R for all $i \in \mathfrak{I}$, i.e. $\epsilon_R(H_{L_{\langle i\rangle}}) = L_{\langle i\rangle}$. □

Discussion 2.4 *(Restriction to general hypersurfaces).* The Hyperplane Restriction Theorem of Green [15] (see, also, [16, Theorem 3.4]) asserts that if $\Bbbk$ is an infinite field, J is

a B-ideal and L the lex-segment B-ideal with $H_I = H_L$, then for any general linear form $f \in S$, $H_{J+(f)} \succcurlyeq H_{L+(z)}$. (In the lexicographic order on B, z is the last variable.) Herzog and Popescu [18, Theorem in Introduction] (in characteristic zero) and Gasharov [12, Theorem 2.2(2)] (in arbitrary characteristic) generalized this to forms of arbitrary degree: $H_{J+(f)} \succcurlyeq H_{L+(z^d)}$ for all general forms f of degree d for all $d \geq 1$. Restated in the language of embeddings, this is $H_{J+(f)} \succcurlyeq H_{\epsilon_B(H_J)+(z^d)}$. Therefore, we may wonder whether this is true more generally than for polynomial rings. More precisely, putting ourselves in the context of Theorem 2.1, we show that if $\operatorname{char} \Bbbk = 0$, then

$$H_{J+(f)} \succcurlyeq H_{\epsilon_S(H_J)+(z^d)} \tag{2.5}$$

for all general homogeneous elements $f \in S$ of degree d and for all $d \in \mathfrak{I}$. We also show that the conclusion fails in positive characteristic. In characteristic zero, the argument is as follows: Let g be a change of coordinates, fixing all the x_i and sending z to a sufficiently general linear form. Let w be the weight $w(x_i) = 1$ for all i and $w(z) = 0$. In characteristic zero, $\operatorname{in}_w(g \cdot J)$ is z-stable. (The proofs of [5, Lemmas 4.1 and 4.2] use many steps of distraction and taking initial ideals with respect to w, but in characteristic zero, they can be replaced by a single step.) Now, $H_{J+(f)} = H_{g\cdot J+(g\cdot f)}$. By [3, 7.1.2 and 7.1.3], $H_{g\cdot J+(g\cdot f)} \succcurlyeq H_{\operatorname{in}_w(g\cdot J)+(g\cdot f)}$. Had we chosen f to be $g^{-1}z^d$, we would have had the equality $H_{\operatorname{in}_w(g\cdot J)+(g\cdot f)} = H_{\operatorname{in}_w(g\cdot J)+(z^d)}$; however since f is general, we have $H_{\operatorname{in}_w(g\cdot J)+(g\cdot f)} \succcurlyeq H_{\operatorname{in}_w(g\cdot J)+(z^d)}$. Finally, using Theorem 2.1(ii), we see that $H_{\operatorname{in}_w(gJ)+(z^d)} \succcurlyeq H_{\epsilon_S(H_{\operatorname{in}_w(gJ)})+(z^d)}$. Note that $\epsilon_S(H_{\operatorname{in}_w(gJ)}) = \epsilon_S(H_J)$.

Now to show that the conclusion does not hold in positive characteristic, consider $R = \Bbbk[x,y]/(x^p, y^p)$ where $p = \operatorname{char} \Bbbk$. Let $S = R[z]$, l a general linear form in x, y and $J = z^p S$. Then, ϵ_S denoting the embedding induced by the lexicographic order on $\Bbbk[x,y,z]$, we have

$$H_{J+(z+l)} \preccurlyeq H_{\epsilon_S(H_J)+(z)} \quad \text{and} \quad H_{J+(z+l)} \neq H_{\epsilon_S(H_J)+(z)};$$

contrast this with (2.5). To see this, let us look at the corresponding quotients: $S/J + (z+l) \simeq R$ and, since $x^{p-1}y \in \epsilon_S(H_J)$, we have $S/\epsilon_S(H_J) + (z)$ is a homomorphic image of $R/(x^{p-1}y)$. □

Remark 2.6. Suppose that $\mathcal{H}_R$ admits an embedding ϵ. Let $I = \epsilon(H)$ for some $H \in \mathcal{H}_R$. Then $\mathcal{I}_{R/I} \simeq \{J \in \mathcal{I}_R : I \subseteq J\}$ and $\mathcal{H}_{R/I} \simeq \{H_J : J \in \mathcal{I}_R, I \subseteq J\}$. In particular, ϵ induces an embedding of $\mathcal{H}_{R/I}$ into $\mathcal{I}_{R/I}$ [5, Remark 2.8]. Let $S = R[z]$. Then $\mathcal{H}_S$ admits an embedding ϵ_S, by Theorem 2.1. (See, also, the paragraph following the theorem.) Let J be an S-ideal that is in the image of ϵ_S. By the above, $\mathcal{H}_{S/J}$ admits an embedding $\epsilon_{S/J}$. Moreover suppose that ϵ_R is given by images of lex-segment ideals of A, i.e., for all $H \in \mathcal{H}_R$, there is a lex-segment A-ideal L such that $\epsilon_R(H) = LR$. Then for all $H \in \mathcal{H}_{S/J}$, there exists a lex-segment $A[z]$-ideal L such that $\epsilon_{S/J}(H) = L(S/J)$; see [5, Theorem 3.12 and Proposition 2.16]. (In the lexicographic order on $A[z]$, z is the last variable.) We remark here that in [25, Theorems 4.1 and 5.1] Mermin and Peeva had

shown that if R is the quotient of A by a monomial ideal, then the lex-segment ideals of $A[z]$ give an embedding of $\mathcal{H}_{S/J}$. □

Remark 2.7. Let $A = \Bbbk[x_1, \dots, x_n]$ and $B = A[x_{n+1}]$. Let $2 \leq e_1 \leq \cdots \leq e_{n+1} \leq \infty$. Let $\mathfrak{a} = (x_1^{e_1}, \dots, x_n^{e_n})A$, $\mathfrak{b} = (x_1^{e_1}, \dots, x_{n+1}^{e_{n+1}})B$, $R = A/\mathfrak{a}$ and $S = B/\mathfrak{b}$. Assume that we have inductively constructed an embedding ϵ_R of $\mathcal{H}_R$ such that for all R-ideals I, there exists a lex-segment A-ideal L such that $\epsilon_R(H_I) = LR$. Then $\mathcal{H}_S^{\mathrm{stab}}$ admits an embedding ϵ_S, by Theorem 2.1. We now argue that $\mathcal{H}_S^{\mathrm{stab}} = \mathcal{H}_S$. Let J be any B-ideal containing $\mathfrak{b}$. Replacing J by an initial ideal, we may assume that J is a monomial ideal. Therefore it suffices to prove that for all monomial B-ideals J containing $\mathfrak{b}$, there is a B-ideal J' such that $J'S$ is z-stable and $H_{JS} = H_{J'S}$. For this we may assume that $\Bbbk = \mathbb{C}$. Now, the remarks in the paragraph following Theorem 2.1 imply that $\mathcal{H}_S^{\mathrm{stab}} = \mathcal{H}_S$, so we have an embedding ϵ_S of $\mathcal{H}_S$. Again, by [5, Theorem 3.12 and Proposition 2.16] we see that for all S-ideals J, there exists a lex-segment B-ideal L such that $\epsilon_S(H_J) = LS$. As a corollary, we get the theorem of Clements–Lindström mentioned in Section 1. □

3. Graded Betti numbers

We are now ready to state and prove the main result of this paper. Let $\epsilon_R : \mathcal{H}_R \longrightarrow \mathcal{I}_R$ be an embedding and I be an R-ideal. Then $\beta_{1,j}^R(R/I) \leq \beta_{1,j}^R(R/\epsilon_R(H_I))$ [5, Remark 2.5]. We do not know whether $\beta_{i,j}^R(R/I) \leq \beta_{i,j}^R(R/\epsilon_R(H_I))$ for all i and j. We show that if $\beta_{i,j}^A(R/I) \leq \beta_{i,j}^A(R/\epsilon_R(H_I))$ for all i, j, then a similar inequality holds for the extension rings considered in Theorem 2.1. (In general, there are examples with $\beta_{i,j}^A(R/I) < \beta_{i,j}^A(R/\epsilon_R(H_I))$ [23, Proposition 3.2].)

Theorem 3.1. *Let t be a positive integer or ∞. Let $S = R[z]/(z^t)$. Suppose that $\mathcal{H}_R$ admits an embedding ϵ_R and that $\beta_{i,j}^A(R/I) \leq \beta_{i,j}^A(R/\epsilon_R(H_I))$ for all R-ideals I and for all i, j. Then for all i, j and for all z-stable S-ideals J, $\beta_{i,j}^B(S/J) \leq \beta_{i,j}^B(S/\epsilon_S(H_J))$.*

Suppose that t, $\Bbbk$ and A_1 satisfy the conditions discussed after Theorem 2.1 that ensure that $\mathcal{H}_S^{\mathrm{stab}} = \mathcal{H}_S$. Then for all S-ideals J, there exists a z-stable S-ideal J' such that $H_{J'} = H_J$ and $\beta_{i,j}^B(S/J) \leq \beta_{i,j}^B(S/J')$. This inequality of Betti numbers follows from the proofs of [5, Lemmas 4.1 and 4.2], and the upper-semi-continuity of graded Betti numbers in flat families. Hence, in these situations, we can conclude from Theorem 3.1 that for all i and j and for all S-ideals J, $\beta_{i,j}^B(S/J) \leq \beta_{i,j}^B(S/\epsilon_S(H_J))$.

We begin with a (somewhat inefficient) bound on the Hilbert series of Tor modules.

Lemma 3.2. *Let R' be any positively graded $\Bbbk$-algebra of finite type. Let M and N be finitely generated graded R'-modules. Then $H_{\mathrm{Tor}_i^{R'}(M,N)} \preccurlyeq H_M H_{\mathrm{Tor}_i^{R'}(N,\Bbbk)}$. In particular, if N is a graded R'-submodule of M or a graded homomorphic image of M, then $H_{\mathrm{Tor}_i^{R'}(M,\Bbbk)} \preccurlyeq H_{\mathrm{Tor}_i^{R'}(N,\Bbbk)} + (H_M - H_N) H_{\mathrm{Tor}_i^{R'}(\Bbbk,\Bbbk)}$.*

Proof. Let $F_\bullet$ be a minimal graded R'-free resolution of N. Then $H_{\mathrm{Tor}_i^{R'}(M,N)} \preccurlyeq H_{M \otimes_{R'} F_i} = H_M H_{\mathrm{Tor}_i^{R'}(N,\Bbbk)}$, proving the first assertion. For the second assertion, let

$L = \operatorname{coker}(N \longrightarrow M)$ or $L = \ker(M \longrightarrow N)$, as the case is. Then $H_{\operatorname{Tor}_i^{R'}(M,\Bbbk)} \preccurlyeq H_{\operatorname{Tor}_i^{R'}(N,\Bbbk)} + H_{\operatorname{Tor}_i^{R'}(L,\Bbbk)} \preccurlyeq H_{\operatorname{Tor}_i^{R'}(N,\Bbbk)} + H_L H_{\operatorname{Tor}_i^{R'}(\Bbbk,\Bbbk)}$ where the first inequality follows from the exact sequence of Tor and the second one follows from the first part of this proposition. Now note that $H_L = H_M - H_N$. □

The following lemma is perhaps well-known to many readers, but we give a proof for the sake of completeness.

Lemma 3.3. *Identify A with the quotient ring $B/(zB)$. Then for all B-ideals $\mathfrak{b}$, $\operatorname{Tor}_i^B(\mathfrak{b},\Bbbk) \simeq \operatorname{Tor}_i^A(\mathfrak{b}/z\mathfrak{b},\Bbbk)$.*

Proof. Let $F_\bullet$ be a graded free B-resolution of $\mathfrak{b}$. We can compute the Tor modules $\operatorname{Tor}_i^B(\mathfrak{b}, B/(zB))$ either as the homology of $F_\bullet \otimes_B B/(zB)$ or as the homology of

$$\big(0 \longrightarrow B(-1) \xrightarrow{z} B \longrightarrow 0\big) \otimes_B \mathfrak{b} = \big(0 \longrightarrow \mathfrak{b}(-1) \xrightarrow{z} \mathfrak{b} \longrightarrow 0\big).$$

Notice that z is a non-zerodivisor on $\mathfrak{b}$, so $\operatorname{Tor}_i^B(\mathfrak{b}, B/(zB)) = 0$ for all $i > 0$, i.e., $F_\bullet \otimes_B B/(zB)$ is a graded $(B/(zB))$-free resolution of $\mathfrak{b}/z\mathfrak{b}$. Now, $\operatorname{Tor}_i^B(\mathfrak{b},\Bbbk) = \mathrm{H}_i(F_\bullet \otimes_B \Bbbk) = \mathrm{H}_i((F_\bullet \otimes_B B/(zB)) \otimes_{B/(zB)} \Bbbk) = \operatorname{Tor}_i^{B/(z)}(\mathfrak{b}/z\mathfrak{b},\Bbbk) \simeq \operatorname{Tor}_i^A(\mathfrak{b}/z\mathfrak{b},\Bbbk)$. □

Proof of Theorem 3.1. We will work with B-ideals. Let J be a B-ideal containing $\mathfrak{a}B + (z^t)$ such that JS is z-stable. Write L for the preimage in B of the S-ideal $\epsilon_S(H_{JS})$. We need to show that $H_{\operatorname{Tor}_i^B(J,\Bbbk)} \preccurlyeq H_{\operatorname{Tor}_i^B(L,\Bbbk)}$. By Lemma 3.3, it suffices to show that $H_{\operatorname{Tor}_i^A(J/zJ,\Bbbk)} \preccurlyeq H_{\operatorname{Tor}_i^A(L/zL,\Bbbk)}$. Write $J = \bigoplus_{i\in\mathbb{N}} J_{\langle i\rangle} z^i$ and $L = \bigoplus_{i\in\mathbb{N}} L_{\langle i\rangle} z^i$ as A-modules. Let J' be the preimage of the S-ideal $\bigoplus_{i\in\mathbb{N}} \epsilon_R(H_{J_{\langle i\rangle}R}) z^i$.

Define graded A-modules

$$M_1 = \bigoplus_{j\in\mathfrak{I}} (J_{\langle j\rangle}/J_{\langle j-1\rangle})(-j) \qquad \text{and}$$

$$M_2 = \begin{cases} A/J_{\langle t-1\rangle}(-t), & \text{if } t \in \mathbb{N}, \\ 0, & \text{if } t = \infty. \end{cases}$$

Then, as an A-module, $J/zJ = J_{\langle 0\rangle} \oplus M_1 \oplus M_2$. Similarly define A-modules M_1' and M_2' for J' and N_1 and N_2 for L. Notice that $\mathfrak{m}_A M_1 = \mathfrak{m}_A M_1' = \mathfrak{m}_A N_1 = 0$, since JS, $J'S$ and LS are z-stable, by hypothesis, by Remark 2.2 and by Remark 2.3 respectively. (Here $\mathfrak{m}_A$ is the homogeneous maximal ideal of A.) Therefore M_1, M_1' and N_1 are (as A-modules) direct sums of shifted copies of $\Bbbk$, so

$$H_{\operatorname{Tor}_i^A(M_1,\Bbbk)} = H_{M_1} H_{\operatorname{Tor}_i^A(\Bbbk,\Bbbk)}, \qquad H_{\operatorname{Tor}_i^A(M_1',\Bbbk)} = H_{M_1'} H_{\operatorname{Tor}_i^A(\Bbbk,\Bbbk)} \quad \text{and}$$
$$H_{\operatorname{Tor}_i^A(N_1,\Bbbk)} = H_{N_1} H_{\operatorname{Tor}_i^A(\Bbbk,\Bbbk)}. \tag{3.4}$$

Moreover,

$$H_{\operatorname{Tor}_i^A(J/zJ,\Bbbk)} = H_{\operatorname{Tor}_i^A(J_{\langle 0\rangle},\Bbbk)} + H_{\operatorname{Tor}_i^A(M_1,\Bbbk)} + H_{\operatorname{Tor}_i^A(M_2,\Bbbk)}. \tag{3.5}$$

Similar expressions exist for J' and L. By the hypothesis and the fact that $H_{M_1} = H_{M_1'}$, we see that $H_{\operatorname{Tor}_i^A(J/zJ,\Bbbk)} \preccurlyeq H_{\operatorname{Tor}_i^A(J'/zJ',\Bbbk)}$. Therefore we may replace J by J' and assume that $J_{\langle i\rangle}R$ is in the image of ϵ_R for all $i \in \mathbb{N}$. Hence, by Theorem 2.1(ii), $L_{\langle 0\rangle} \subseteq J_{\langle 0\rangle}$, and, if $t < \infty$ then $J_{\langle t-1\rangle} \subseteq L_{\langle t-1\rangle}$. Hence, for all values of t, there is a surjective A-homomorphism $M_2 \longrightarrow N_2$. Since $H_J = H_L$, we see that

$$H_{J_{\langle 0\rangle}} + H_{M_1} + H_{M_2} = H_{J/zJ} = H_{L/zL} = H_{L_{\langle 0\rangle}} + H_{N_1} + H_{N_2}. \tag{3.6}$$

Therefore

$$\begin{aligned}
H_{\operatorname{Tor}_i^A(J/zJ,\Bbbk)} &= H_{\operatorname{Tor}_i^A(J_{\langle 0\rangle},\Bbbk)} + H_{\operatorname{Tor}_i^A(M_1,\Bbbk)} + H_{\operatorname{Tor}_i^A(M_2,\Bbbk)} \\
&\preccurlyeq H_{\operatorname{Tor}_i^A(L_{\langle 0\rangle},\Bbbk)} + H_{\operatorname{Tor}_i^A(M_1,\Bbbk)} + H_{\operatorname{Tor}_i^A(M_2,\Bbbk)} + (H_{J_{\langle 0\rangle}} - H_{L_{\langle 0\rangle}})H_{\operatorname{Tor}_i^A(\Bbbk,\Bbbk)} \\
&\preccurlyeq H_{\operatorname{Tor}_i^A(L_{\langle 0\rangle},\Bbbk)} + H_{\operatorname{Tor}_i^A(M_2,\Bbbk)} + (H_{J_{\langle 0\rangle}} - H_{L_{\langle 0\rangle}} + H_{M_1})H_{\operatorname{Tor}_i^A(\Bbbk,\Bbbk)} \\
&= H_{\operatorname{Tor}_i^A(L_{\langle 0\rangle},\Bbbk)} + H_{\operatorname{Tor}_i^A(M_2,\Bbbk)} + (H_{N_1} + H_{N_2} - H_{M_2})H_{\operatorname{Tor}_i^A(\Bbbk,\Bbbk)} \\
&\preccurlyeq H_{\operatorname{Tor}_i^A(L_{\langle 0\rangle},\Bbbk)} + H_{\operatorname{Tor}_i^A(N_2,\Bbbk)} + H_{N_1}H_{\operatorname{Tor}_i^A(\Bbbk,\Bbbk)} \\
&= H_{\operatorname{Tor}_i^A(L_{\langle 0\rangle},\Bbbk)} + H_{\operatorname{Tor}_i^A(N_1,\Bbbk)} + H_{\operatorname{Tor}_i^A(N_2,\Bbbk)} \\
&= H_{\operatorname{Tor}_i^A(L/zL,\Bbbk)}
\end{aligned}$$

where the second line uses Lemma 3.2, the third line uses (3.4), the fourth line uses (3.6), the fifth line uses Lemma 3.2 and the next line uses (3.4). □

The proof shows that, for fixed i and j, if $\beta_{i,j}^A(R/I) \leq \beta_{i,j}^A(R/\epsilon_R(H_I))$ and $\beta_{i-1,j}^A(R/I) \leq \beta_{i-1,j}^A(R/\epsilon_R(H_I))$ for all R-ideals I, then for all z-stable S-ideals J, $\beta_{i,j}^B(S/J) \leq \beta_{i,j}^B(S/\epsilon_S(H_J))$.

Now, as a corollary, we give a quick proof of a theorem of Mermin, Peeva and Stillman and its generalization by Murai. First, we relabel z as x_{n+1} and write $B = \Bbbk[x_1, \ldots, x_{n+1}]$. We say that a monomial B-ideal J is *strongly stable* if for all monomials $m \in I$ and for all $2 \leq j \leq n+1$ such that x_j divides m, $x_i(m/x_j) \in I$, for all $i \leq j$.

Corollary 3.7. *(See [27, Theorem 3.5], [28, Theorem 1.1].) Let J be a strongly stable monomial B-ideal. Let $\mathfrak{b} = (x_1^{e_1}, \ldots, x_{n+1}^{e_{n+1}})$, with $2 \leq e_1 \leq \cdots \leq e_{n+1} \leq \infty$. Then:*

(i) For all i, j, the value of $\beta_{i,j}^B(J + \mathfrak{b})$ does not depend on $\Bbbk$.
(ii) For all i, j, $\beta_{i,j}^B(J + \mathfrak{b}) \leq \beta_{i,j}^B(L + \mathfrak{b})$ where L is the lex-segment B-ideal such that $H_{L+\mathfrak{b}} = H_{J+\mathfrak{b}}$.

Proof. (i): We may compute the Betti numbers using Lemma 3.3 and (3.5). Note that $J_{\langle 0\rangle}$ and $J_{\langle e_n - 1\rangle}$ are strongly-stable-plus-$(x_1^{e_1}, \ldots, x_n^{e_n})$. The assertion now follows by induction. (ii): From Remark 2.7, whose notation we adopt, we have embeddings ϵ_R and ϵ_S of $\mathcal{H}_R$ and $\mathcal{H}_S^{\mathrm{stab}} = \mathcal{H}_S$ respectively. By induction on the number of variables, we

see that the hypothesis of Theorem 3.1 is satisfied. Now $\epsilon_S(H_{JS}) = LS$, so Theorem 3.1 completes the proof. □

Starting with a ring whose poset of Hilbert functions admits an embedding, it is possible to construct new examples of rings with the same property, by using Theorem 2.1 and Remark 2.6 recursively. We thus recover the following result of D.A. Shakin.

Corollary 3.8. *(See [32, Theorems 3.10 and 4.1].) Let $\mathfrak{a}_i$ be a lex-segment ideal in the ring $\Bbbk[x_1, \dots, x_i]$, $i = 1 \leq n$, then let $\mathfrak{a} = \sum_{i=1}^n \mathfrak{a}_i A$, with $A = \Bbbk[x_1, \dots, x_n]$. Let $R = A/\mathfrak{a}$. Then $\mathcal{H}_R$ admits an embedding ϵ_R induced by the lexicographic order on A. Moreover $\beta_{i,j}^A(R/I) \leq \beta_{i,j}^A(R/\epsilon_R(H_I))$, for all R-ideals I and for all i, j.*

Consecutive cancellation in Betti tables. We say that a Betti table $\beta^R(M)$ is obtained by *consecutive cancellations* from $\beta^R(N)$ if there exists a collection Λ of triples $(i, j, n_{i,j}) \in \mathbb{N} \times \mathbb{Z} \times \mathbb{N}$ such that $\beta^R(N) = \beta^R(M) + \sum_\Lambda n_{i,j}(\mathbf{e}_{i,j} + \mathbf{e}_{i+1,j})$. (Note that $\mathbf{e}_{i,j} + \mathbf{e}_{i+1,j}$ is the Betti table of a complex $R(-j) \xrightarrow{1} R(-j)$ concentrated in homological degrees $i+1$ and i.) Now revert to the situation of Theorem 3.1. We have the following:

Proposition 3.9. *If $\beta^A(R/I)$ can be obtained from $\beta^A(R/\epsilon_R(H_I))$ by consecutive cancellations for all R-ideals I, then $\beta^B(S/J)$ can be obtained from $\beta^B(S/\epsilon_S(H_J))$ by consecutive cancellations for all z-stable S-ideals J.*

Proof. As in the proof of Theorem 3.1, which we follow closely, we work with B-ideals. Let J be a B-ideal containing $\mathfrak{a}B + (z^t)$ such that JS is z-stable. Write L for the preimage of the S-ideal $\epsilon_S(H_{JS})$. Let J' be the preimage of the S-ideal $\bigoplus_{i\in\mathbb{N}} \epsilon_R(H_{J_{\langle i\rangle}R})z^i$, as in the proof of the theorem. We need to show that $\beta^B(J)$ can be obtained from $\beta^B(L)$ by consecutive cancellations. Note that, by our hypothesis, $\beta^B(J)$ can be obtained from $\beta^B(J')$ by consecutive cancellations, since these Betti tables are equal to $\beta^A(J/zJ)$ and $\beta^A(J'/zJ')$ respectively. Therefore, we may replace J by J' and assume that $J_{\langle i\rangle}R$ is in the image of ϵ_R for all $i \in \mathbb{N}$. Then $L_{\langle 0\rangle} \subseteq J_{\langle 0\rangle}$, N_2 is a graded homomorphic image of M_2 (for a morphism of degree zero) and $H_{N_1} - H_{M_1} = H_{J_{\langle 0\rangle}} - H_{L_{\langle 0\rangle}} + H_{M_2} - H_{N_2}$. Hence we may place ourselves in the context of Lemma 3.2. We have

$$\begin{aligned}
\beta^B(L) - \beta^B(J) &= \beta^A(L_{\langle 0\rangle}) - \beta^A(J_{\langle 0\rangle}) + \beta^A(N_2) - \beta^A(M_2) + \beta^A(N_1) - \beta^A(M_1)\\
&= \beta^A(L_{\langle 0\rangle}) - \beta^A(J_{\langle 0\rangle}) + \beta^A(N_2) - \beta^A(M_2) + (H_{N_1} - H_{M_1})\beta^A(k)\\
&= \big[\beta^A(L_{\langle 0\rangle}) - \beta^A(J_{\langle 0\rangle}) + (H_{J_{\langle 0\rangle}} - H_{L_{\langle 0\rangle}})\beta^A(k)\big]\\
&\quad + \big[\beta^A(N_2) - \beta^A(M_2) + (H_{M_2} - H_{N_2})\beta^A(k)\big].
\end{aligned}$$

Lemma 3.10 below, now, completes the proof of the proposition. □

Lemma 3.10. *Let A be a standard-graded polynomial ring. Let M and N be finitely generated graded A-modules. If N is a graded A-submodule of M or a graded homomorphic image of M, then $\beta^A(M)$ can be obtained from $\beta^A(N)+(H_M-H_N)\beta^A(\Bbbk)$ by consecutive cancellations. Here, for a series $h=\sum_{i\in\mathbb{N}} h_i\mathfrak{z}^i$, we mean, by $h\beta^A(\Bbbk)$, the (infinite) Betti table $\sum_{i\in\mathbb{N}}\beta^A(\Bbbk(-i)^{\oplus h_i})$.*

Proof. We will prove this when N is a graded submodule of M; the other case is similar. From minimal graded A-free resolutions of N and M/N, we can construct a graded A-free resolution of M that is not necessarily minimal. Every non-minimal graded A-free resolution of M is a direct sum of a minimal graded A-free resolution of M with copies of exact complexes of the form $0\longrightarrow A(-j)\xrightarrow{1}A(-j)\longrightarrow 0$ that is concentrated in homological degrees i and $i+1$ [9, Theorem 20.2]; therefore $\beta^A(M)$ can be obtained from $\beta^A(N)+\beta^A(M/N)$ by consecutive cancellations. Since $H_{M/N}=H_M-H_N$, it suffices to show that, after relabelling the modules, for any graded A-module M, $\beta^A(M)$ can be obtained from $H_M\beta^A(\Bbbk)$ by consecutive cancellations.

Let $(K_\bullet,\partial_\bullet)$ be a Koszul complex that is a minimal graded A-free resolution of $\Bbbk$. Note that

$$\begin{aligned}H_{\operatorname{Tor}_i^A(M,\Bbbk)}&=H_{\ker(\partial_i\otimes_A M)}-H_{\operatorname{Im}(\partial_{i+1}\otimes_A M)}\\&=H_{K_i\otimes_R M}-H_{\operatorname{Im}(\partial_i\otimes_A M)}-H_{\operatorname{Im}(\partial_{i+1}\otimes_A M)}.\end{aligned}$$

The series $H_{\operatorname{Tor}_i^A(M,\Bbbk)}, 0\le i\le n$ determine $\beta^A(M)$. Similarly $H_{K_i\otimes_R M}$, $0\le i\le n$, determine $H_M\beta^A(\Bbbk)$. The lemma now follows by noting that for each i, $H_{\operatorname{Im}(\partial_i\otimes_A M)}$ is subtracted from $H_M\beta^A(\Bbbk)$ twice, at i and at $i-1$. □

An Eliahou–Kervaire type resolution for z-stable ideals. When I is a strongly stable A-ideal (see the paragraph above Corollary 3.7 for definition), a minimal graded A-free resolution of I is given by the Eliahou–Kervaire complex (see [10, Theorem 2.1]), which can be constructed as an iterated mapping cone for a specific order on the set of minimal monomial generators of I. Iterated mapping cones can always be used to construct free resolutions, but they need not be minimal in general. When they give minimal resolutions, they can be used to obtain exact expressions for Betti numbers. See, for instance, [28], which uses an iterated mapping cone from [27]. This section does not use explicit results on embeddings, but only the calculations in the proof of Theorem 3.1. We first make an observation:

Observation 3.11. Let $\phi: M\longrightarrow N$ be an injective map of finitely generated graded B-modules and let $F_\bullet$ and $G_\bullet$ be minimal graded free B-resolutions of M and N respectively. Denote the comparison map $F_\bullet\longrightarrow G_\bullet$ by Φ. Then the mapping cone $C_\bullet$ of Φ is a graded free B-resolution of $\operatorname{coker}\phi$. Moreover if $\operatorname{rank}_B C_i=\dim_\Bbbk \operatorname{Tor}_i^B(\operatorname{coker}\phi,\Bbbk)$, then $C_\bullet$ is a minimal resolution.

Suppose that J is a z-stable S-ideal. We give an interpretation of a minimal graded B-free resolution of S/J as an iterated mapping cone. Replace J by its preimage in B. For $i \in \mathfrak{I}$, set $J_{\langle i\rangle} = (J :_B z^i) \cap A$. If t is finite, then, for all $i \geq t$, set $J_{\langle i\rangle} = J_{\langle t-1\rangle}$. Let $J' = \bigoplus_{i\in\mathbb{N}} J_{\langle i\rangle} z^i$. Note that if t is finite then $J = (J' + (z^t))$ while if t is infinite then $J = J'$. Note also that $J'R[z]$ is z-stable.

We first construct a minimal resolution of J'. Let $f_1 z^{j_1}, \ldots, f_r z^{j_r}, f z^j$ be a set of minimal multigraded generators of J' ordered such that $j_1 \leq \cdots \leq j_r \leq j$. We may assume that $j > 0$, for, otherwise, a minimal B-free resolution of J' can be obtained by applying $- \otimes_A B$ to a minimal A-free resolution of $(J' \cap A)$. Further, $J_{\langle i\rangle} = J_{\langle j\rangle}$ for all $i \geq j$. Write $d = \deg f$. Let $J'' = (f_1 z^{j_1}, \ldots, f_r z^{j_r})$. Then for all $0 \leq i < j$, $(J'' :_B z^i) \cap A = J_{\langle i\rangle}$, and for all $i \geq j$, $(J'' :_B z^i) \cap A = (J'' :_B z^j) \cap A$. Moreover, $(J'' :_B f z^j) \supseteq (J_{\langle j-1\rangle} :_A f)B = \mathfrak{m}_A B$, so $(J'' :_B f z^j) = \mathfrak{m}_A B$. Also, $J_{\langle j\rangle}/((J'' :_B z^j) \cap A) \simeq \Bbbk(-d)$. There is a graded exact sequence

$$0 \longrightarrow \frac{B}{\mathfrak{m}_A B}(-d-j) \xrightarrow{f z^j} \frac{B}{J''} \longrightarrow \frac{B}{J'} \longrightarrow 0$$

of B-modules. Arguing as in the proof of Theorem 3.1, we see that $\beta^B(J') = \beta^A(J'/zJ') = \beta^A(J_{\langle 0\rangle}) + \sum_{i\leq j} \mathfrak{z}^i H_{\frac{J_{\langle i\rangle}}{J_{\langle i-1\rangle}}} \beta^A(\Bbbk) = \beta^B(J'') + \beta^A(\Bbbk(-d-j))$. By Observation 3.11, the mapping cone of a comparison morphism from a minimal B-free resolution of $\frac{B}{\mathfrak{m}_A B}(-d-j)$ to that of $\frac{B}{J''}$ gives a minimal B-free resolution of $\frac{B}{J'}$.

If t is infinite, then we have by now constructed a minimal B-free resolution of S/J as an iterated mapping cone.

Now suppose that t is finite. Then, as we noted earlier, $J = (J' + (z^t))$. Observe that $j < t$ and hence that $(J' :_B z^t) = J_{\langle t-1\rangle} B$. We have a graded exact sequence

$$0 \longrightarrow \frac{B}{J_{\langle t-1\rangle} B}(-t) \xrightarrow{z^t} \frac{B}{J'} \longrightarrow \frac{B}{J} \longrightarrow 0$$

of B-modules. As we saw above, $\beta^B(J') = \beta^A(J_{\langle 0\rangle}) + \sum_{1\leq i<t} \mathfrak{z}^i H_{\frac{J_{\langle i\rangle}}{J_{\langle i-1\rangle}}} \beta^A(\Bbbk)$. Since $\beta^B(\frac{B}{J_{\langle t-1\rangle} B}(-t)) = \beta^A(\frac{A}{J_{\langle t-1\rangle}}(-t))$, we see from (3.5) that $\beta^B(J) = \beta^B(J') + \beta^B(\frac{B}{J_{\langle t-1\rangle} B}(-t))$. Again, by Observation 3.11, the mapping cone for the comparison map of the minimal B-free resolutions of $\frac{B}{J_{\langle t-1\rangle} B}(-t)$ and $\frac{B}{J'}$ gives a minimal B-free resolution of B/J.

4. The lex-plus-powers conjecture reduces to the Artinian situation

In this section, we discuss some examples of applications of Theorem 3.1. We begin with a recursive application of the result. Then we use the theorem to reduce the lex-plus-powers conjecture to the Artinian situation.

Let $\mathbf{f} = f_1, \ldots, f_c$ be a homogeneous A-regular sequence (where $A = \Bbbk[x_1, \ldots, x_n]$) of degrees $e_1 \leq \cdots \leq e_c$. Let $\mathfrak{a}$ be the ideal generated by $\mathbf{f}$ and let $\mathfrak{b}$ be the ideal generated

by $x_1^{e_1}, \dots, x_c^{e_c}$. We say $\mathbf{f}$ *satisfies the Eisenbud–Green–Harris conjecture* if there exists an inclusion of posets $\mathcal{H}_{A/\mathfrak{a}} \subseteq \mathcal{H}_{A/\mathfrak{b}}$. (Since $\mathcal{H}_{A/\mathfrak{b}}$ admits an embedding induced by the lexicographic order on A by the Clements–Lindström theorem, this definition is equivalent to the seemingly stronger condition that for each $H \in \mathcal{H}_{A/\mathfrak{a}}$ there exists a lex-segment A-ideal L such that $H_{LA/\mathfrak{b}} = H$.) It is known that the Eisenbud–Green–Harris conjecture reduces to the Artinian case. More precisely, after a linear change of coordinates (by replacing $\Bbbk$ by a suitable extension field, if necessary), we may assume that $f_1, \dots, f_c, x_{c+1}, \dots, x_n$ is a maximal regular sequence. Now, write $\bar{\mathbf{f}} = \overline{f}_1, \dots, \overline{f}_c$ for the image of $\mathbf{f}$ after going modulo $x_{c+1}, \dots, x_n$. Then $\bar{\mathbf{f}}$ is a maximal regular sequence in $\Bbbk[x_1, \dots, x_c]$. If $\bar{\mathbf{f}}$ satisfies the Eisenbud–Green–Harris conjecture, then so does $\mathbf{f}$ [7, Proposition 10].

The lex-plus-powers conjecture of Evans (see [11]) asserts that if $\mathbf{f}$ satisfies the Eisenbud–Green–Harris conjecture, I is an A-ideal containing $\mathbf{f}$ and L is the lex-segment A-ideal such that $H_I = H_{LA/\mathfrak{b}}$, then $\beta_{i,j}^A(I) \leq \beta_{i,j}^A(L + \mathfrak{b})$ for all i, j. We show now that, similarly, the lex-plus-powers conjecture reduces to the Artinian case. We keep the notation from the above, and, further denote $L + \mathfrak{b}$ by $\operatorname{lpp}_{\mathbf{f}}(I)$.

Theorem 4.1. *If $\bar{\mathbf{f}}$ satisfies the Eisenbud–Green–Harris and the lex-plus-powers conjectures, then so does $\mathbf{f}$.*

Proof. By [7, Proposition 10], $\mathbf{f}$ satisfies the Eisenbud–Green–Harris conjecture. Hence we need to show that if J is an A-ideal containing $\mathbf{f}$, then $H_{\operatorname{Tor}_i^A(J,\Bbbk)} \preccurlyeq H_{\operatorname{Tor}_i^A(\operatorname{lpp}_{\mathbf{f}}(J),\Bbbk)}$. We may assume that $c < n$. Let $\tilde{\mathbf{f}} = \tilde{f}_1, \dots, \tilde{f}_c$ the images of $\mathbf{f}$ in $\tilde{A} := \Bbbk[x_1, \dots, x_{n-1}]$. Inductively we can assume that for all $\tilde{A}$-ideals I containing $\tilde{\mathbf{f}}$, $H_{\operatorname{Tor}_i^{\tilde{A}}(I,\Bbbk)} \preccurlyeq H_{\operatorname{Tor}_i^{\tilde{A}}(\operatorname{lpp}_{\tilde{\mathbf{f}}}(I),\Bbbk)}$. Let J be an A-ideal containing $\mathbf{f}$. By taking the initial ideal with respect to a weight w, $w(x_1) = \cdots = w(x_{n-1}) = 1$ and $w(x_n) = 0$, we may assume that J contains $\tilde{\mathbf{f}}$ and that it has a decomposition (as an $\tilde{A}$-submodule of A) $J = \bigoplus_i J_{\langle i \rangle} x_n^i$ where the $J_{\langle i \rangle}$ are $\tilde{A}$-ideals containing $\tilde{\mathbf{f}}$. By applying [5, Lemma 4.1] we may assume that $J(A/\mathfrak{a})$ is x_n-stable. The proof of this lemma involves taking initial ideals and applying distraction. The values of $\beta_{i,j}^A(-)$ do not decrease while taking initial ideals (which can be argued, e.g., the same way as in [9, Theorem 15.17]), and remain unchanged while applying distraction (since distraction can be interpreted as polarization followed by going modulo a regular element [8, Section 1.3]). Let $J' = \bigoplus_i \operatorname{lpp}_{\tilde{\mathbf{f}}}(J_{\langle i \rangle}) x_n^i$, where, for each i, $\operatorname{lpp}_{\tilde{\mathbf{f}}}(J_{\langle i \rangle})$ is the lex-plus-powers ideal of $\tilde{A}$ for the sequence $\tilde{\mathbf{f}}$. Note that we can obtain the ideals $\operatorname{lpp}_{\tilde{\mathbf{f}}}(J_{\langle i \rangle})\tilde{A}/\mathfrak{b}$ by embedding their Hilbert functions, so $J'A/\mathfrak{b}$ is x_n-stable. We now claim that for all i, $H_{\operatorname{Tor}_i^A(J',\Bbbk)} \succcurlyeq H_{\operatorname{Tor}_i^A(J,\Bbbk)}$. Assume the claim. Now $\operatorname{lpp}_{\mathbf{f}}(J)$ is the preimage of $\epsilon_{A/\mathfrak{b}}(H_{J'})$, so by Theorem 3.1, $H_{\operatorname{Tor}_i^A(\operatorname{lpp}_{\mathbf{f}}(J),\Bbbk)} \succcurlyeq H_{\operatorname{Tor}_i^A(J,\Bbbk)}$.

Now to prove the claim, we follow the strategy of the proof of Theorem 3.1. Note that since J' and J are x_n-stable,

$$J'/x_nJ' = \operatorname{lpp}_{\tilde{\mathbf{f}}}(J_{\langle 0 \rangle}) \oplus M \quad \text{and} \quad J/x_nJ = J_{\langle 0 \rangle} \oplus N$$

for some $\tilde{A}$-modules M and N that are annihilated by $(x_1, \dots, x_{n-1})$. Since $H_{\operatorname{lpp}_{\tilde{\mathbf{f}}}(J_{\langle 0\rangle})} = H_{J_{\langle 0\rangle}}$, we see that $H_M = H_N$. By the induction hypothesis, $H_{\operatorname{Tor}_i^{\tilde{A}}(\operatorname{lpp}_{\tilde{\mathbf{f}}}(J_{\langle 0\rangle}),\Bbbk)} \succcurlyeq H_{\operatorname{Tor}_i^{\tilde{A}}(J_{\langle 0\rangle},\Bbbk)}$. Hence $H_{\operatorname{Tor}_i^{A}(J',\Bbbk)} \succcurlyeq H_{\operatorname{Tor}_i^{A}(J,\Bbbk)}$. □

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