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A Cacti theoretical interpretation of the axioms of bialgebras and H-module algebras

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ABSTRACT

We establish a dictionary between the Cacti algebra axioms on a Cacti algebra structure with underlying free associative algebra, under suitable good behavior with degrees. Using these ideas, for an associative algebra A and a bialgebra H, we also translate Cacti algebra maps $\Omega(H) \to C^\bullet(A)$ (where $\Omega(H)$ stands for the cobar construction on H and $C^\bullet(A)$ is the Hochschild cohomology complex) with H-module algebra structures on A, and illustrate with examples of applications.

Introduction and preliminaries

In [4], the author defines a Cacti algebra structure on $\Omega(H)$, the cobar construction of a d.g. bialgebra H. Recall that $\Omega(H) = TV$ the tensor algebra on $V = \operatorname{Ker}\epsilon$, with differential of the form $d_i + d_\Delta$. That is, one differential coming from the original differential on the d.g. bialgebra H and a second one coming from its coalgebra structure. In the mentioned article, the author works over $\mathbb{Z}/2\mathbb{Z}$. In [11], signs are introduced for any characteristic. This construction gives examples of Cacti algebras of special type, they are not only graded but naturally bigraded, and operations have extra properties with

E-mail addresses: mfarinat@dm.uba.ar (M.A. Farinati), llombard@dm.uba.ar (L.E. Lombardi).
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0021-8693/

respect to this bigrading. We call these properties *well graded* (see Definition 1.3). We prove a kind of converse for this construction that includes the characterization of the image of the functor $\Omega : d.g.bialg \to Cacti\text{-}alg$. More precisely, we prove that if a Cacti algebra T is well graded and freely generated as associative algebra by elements of (external) degree one, namely $T \cong TV$ as associative algebras (TV = the tensor algebra on a graded vector space), then the Cacti algebra structure on T determines uniquely a d.g. bialgebra structure on $H = V \oplus k1_H$, and hence $T = \Omega(H)$ for a uniquely determined d.g. bialgebra H.

The examples arising from the Kadeishvili construction are not the only well graded ones. The historically most important family of Cacti algebras, namely the Hochschild complex $C^\bullet(A)$ of an associative (eventually d.g.) algebra A is also a well graded Cacti algebra. In Lemma 2.1, we study morphisms between well graded Cacti algebras. A consequence of this result can be seen as a continuation of the dictionary between Cacti algebras and bialgebras. More precisely (Theorem 2.5) we prove that, given an (eventually d.g.) bialgebra H and associative algebra A, the set of morphism between bigraded Cacti algebras $\{\Omega(H) \to C^\bullet(A)\}$ is in 1–1 correspondence with structures of H-module algebra on A.

We end with examples of applications to the Gerstenhaber algebra structure on the Hochschild cohomology of an algebra.

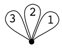
We also mention that an action on TV of a certain PROP that contains the operad of spineless cacti was given in [7] (see Theorem C of [7]). This generalizes an example that appears [3]. This connection exists when V merely has the structure of a vector space (with non-degenerate pairing in some cases). These actions are not d.g. however. Part of our results can be seen as determine what properties V has to have in order to get a d.g. action. We thanks the referee for pointing out this bridge, between the two types of examples in the original work of Gerstenhaber [3] and its Deligne conjecture/string topology generalizations in [7].

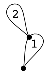
For the purpose of this work, a Cacti algebra is an algebra over the operad $\mathcal{X}_2$ a suboperad of $\mathcal{X}$ defined in [1]. This operad is (up to a sign convention) isomorphic to S_2 (see [8]) and the operad of cellular chains of normalized spineless cacti in [5,6] (where the graphical notation is taken from). An algebra over this operad is a Gerstenhaber–Voronov algebra [10]. We briefly recall the definition: a *Cacti algebra* is a differential graded vector space (T, d) with operations

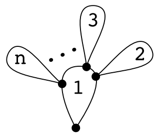
1. $C_2 : T \otimes T \to T$, an associative product: $C_2 \circ_1 C_2 = C_2 \circ_2 C_2 =: C_3$,
2. for any $n \geq 2$, $B_n : T^{\otimes n} \to T$ are brace operations,

satisfying a set of compatibility relations that we list below. In order to write them, it is convenient to use a graphic representation:

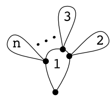
$C_2 =$ 

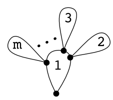
, $C_3 =$ 

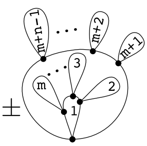
, $B_2 =$ 

, $B_n =$ 

For example, the brace relations can be described pictorially as

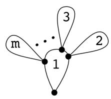
$B_n \circ_1 B_m =$ 

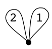
$\circ_1$ 

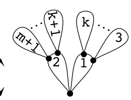
$= \sum_{\text{possibilities}} \pm$ 

where the sign is given by the permutation of the dots belonging to B_n and B_m.

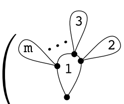
The distributivity law between C_2 and B_m is:

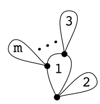


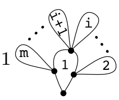
$\circ_1$ 

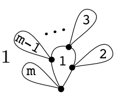
$= \sum_k$ 

And finally, the relation with the differential is $\partial C_2 = 0$ and

$\partial B_m = \partial \Bigg($ 

$\Bigg) =$ 

$+ \sum_{i=2}^{m-1} (-1)^{i+1}$ 

$+ (-1)^{m+1}$ 

where, if $P : T^{\otimes n} \to T$ is an operation, ∂P is by definition the operation given by

$$(\delta P)(t_1 \otimes \cdots \otimes t_n) = d\big(P(t_1 \otimes \cdots \otimes t_n)\big) - \sum_{i=1}^{n} (-1)^{|P|+\sum_{j=1}^{i-1} |t_j|} P\big(t_1 \otimes \cdots \otimes d(t_i) \otimes \cdots \otimes t_n\big)$$

In particular, $\partial C_2 = 0$ means that in any Cacti algebra, the differential is a derivation for the product C_2.

1. $\mathcal{C}acti$-algebra structure in TV

Let V be a graded vector space, then $TV = \bigoplus_{n \geq 0} V^{\otimes n}$ is a free associative algebra, and it is bigraded taking, for $v_k \in V_{i_k}$

$$\operatorname{bideg}(v_1 \otimes \cdots \otimes v_n) = \left(\sum_{i=1}^{n} |v_i|_V, n \right)$$

We call $\sum_{i=1}^{n} |v_i|_V$ the *internal* degree, and n the *external* or tensorial degree. We remark that the total degree

$$|v_1 \otimes \cdots \otimes v_n|_{tot} = \left(\sum_{i=1}^{n} |v_i|_V \right) + n$$

is the same as the usual degree on the tensor algebra of ΣV, the suspension of V. This total degree is most usually considered, but we prefer to keep the information of the bigrading by reasons that will be clear in the rest of this work.

Remark 1.1. Let V be a trivially graded vector space (i.e. $V = V_0$), then the data of a square zero differential in $A = TV$ of total degree one is equivalent to give a (non-necessarily counital) coassociative coalgebra structure in V.

If V is arbitrarily graded, then the data of a square zero differential in TV is equivalent to a differential in V together with an up to homotopy coassociative coalgebra structure in TV, but if the differential in TV is of the form $D = d_i + d_e$ where $\operatorname{bideg} d_i = (1, 0)$ and $\operatorname{bideg} d_e = (0, 1)$ then to give D is equivalent to a (strict) coassociative differential coalgebra structure in V. Take simply $d_V = d_i|_V$, and $\Delta' = d_e|_V$.

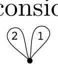
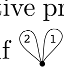
Remark 1.2. The non-necessarily counital coassociative structures in V are in 1–1 correspondence with the unital coassociative structures in $H := V \oplus k1_H$, where 1_H is a new formal element satisfying $\Delta(1_H) = 1_H \otimes 1_H$. The correspondence is given by $\Delta' \leftrightarrow \Delta$ with

$$\Delta : H \to H \otimes H$$
$$\Delta(v) := \Delta'(v) + 1_H \otimes v + v \otimes 1_H = v_1 \otimes v_2 + 1_H \otimes v + v \otimes 1_H$$

for $v \in V$, $\Delta 1_H := 1_H \otimes 1_H$. And given $\Delta : H \to H \otimes H$, let $\pi : H \to V$ be the canonical projection with respect to the direct sum decomposition $H = V \oplus k1_H$, then

$$\Delta'(v) := (\pi \otimes \pi) \circ \Delta$$

The counit in H is given by $\epsilon(v) = 0$ if $v \in V$ and $\epsilon(1_H) = 1$. Working with elements one can easily see that the coassociativity equation for Δ and Δ' is *the same*, so Δ is coassociative iff Δ' is, and letting 1_H having internal degree 0 (but tensorial degree 1), and $d_i(1_H) = 0$ the correspondence works equally well for the graded case.

We will consider Cacti-algebra structures on TV of a certain type. Recall that the cactus $C_2 =$  provides a strict associative product. We will say that the Cacti algebra structure on TV *extends* the one in TV if  $(x, y) = x \otimes y$ (where $x, y \in TV$). Notice that in $\overline{T}V$, this property implies that every element of A can be obtained from V and the action of the cactus C_n, namely if $\mathbf{x} = x_1 \otimes \ldots \otimes x_n$ then $\mathbf{x} = C_n(x_1, \ldots, x_n)$.

The next definition is motivated by the example of the Hochschild complex $C^\bullet(A)$ of an associative algebra A. Recall that in $C^\bullet(A)$, if $f : A^{\otimes n} \to A$, then the brace operation is a formula of type

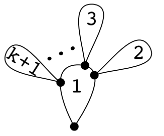
$$f\{g_1, \dots, g_k\} = \sum \pm f\big(\cdots, g_1(-), \cdots, g_2(-), \cdots\big)$$

and this implicitly says that if $n < k$ then

$$f\{g_1, \dots, g_k\} = 0$$

These brace operations correspond to the cactus

$$B_{k+1} =$$



Definition 1.3. Let C be bigraded vector space

$$C = \bigoplus_{p,q} C^{p,q}$$

with a Cacti algebra structure on it with respect to the total degree. We will say that this structure is *well graded* if

$$a \in C^{\bullet,p}, \quad p < n-1 \quad \Longrightarrow \quad B_n(a, \dots) = 0$$

and the differential is compatible with the bigrading in the sense that $d = d_i + d_e$ where

$$\begin{aligned} d_i &: T^{n,\bullet} \to T^{n+1,\bullet} \\ d_e &: T^{\bullet,n} \to T^{\bullet,n+1} \end{aligned}$$

Moreover, we ask C_2 and B_m ($m \geq 2$) to be homogeneous with respect to the internal degree.

Notice that if C is a cactus algebra that is graded (and not bigraded), then it can be considered as trivially bigraded with $C^{0,q} = C^q$ and $C^{p,q} = 0$ for $p \neq 0$, and the definition of well graded makes sense.

Example 1.4. The Hochschild complex of an associative algebra is a well graded Cacti algebra, this example is trivially bigraded. But also if A is a differential graded associative algebra, then $C(A)$ is well graded. In both cases, the bidegree is given by

$$C^{p,q}(A) = \operatorname{Hom}\big(A^{\otimes q}, A\big)_p$$

where $\mathrm{Hom}(-,-)_p$ is the set of homogeneous linear transformations of degree p (between two graded vector spaces).

Example 1.5. If $(H, \cdot, \Delta, d)$ is a differential graded associative bialgebra, then in particular it is a differential graded coalgebra, and the cobar construction makes sense

$$\Omega(H) = (TV, d)$$

where $V = \overline{H} = \mathrm{Ker}\,\epsilon$ and $d = d_H + d_\Delta$. In [4], Kadeishvili exhibits (in characteristic 2) a Cacti-algebra structure on $\Omega(H)$ coming from the bialgebra structure of H. In [11] the author introduce appropriate signs showing that $\Omega(H)$ is a Cacti algebra in any characteristic (e.g. 0). In this construction, the brace structure is given by

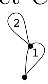
$$B_m(\mathbf{x}, \bar{\mathbf{y}}) := \sum_{1 \le i_1 < \ldots < i_{m-1} \le n} \pm x_1 \otimes \ldots \otimes (x_{i_1} \cdot \mathbf{y}_1) \otimes \ldots \otimes (x_{i_{m-1}} \cdot \mathbf{y}_{m-1}) \otimes \ldots \otimes x_n$$

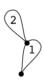
where in each term, the sign is the Koszul-permutation sign of the symbols

$$\cdot \ldots \cdot x_1 \ldots x_n \mathbf{y}_1 \ldots \mathbf{y}_{m-1} \mapsto x_1 \ldots x_{i_1} \cdot \mathbf{y}_1 x_{i_1+1} \ldots x_{i_{m-1}} \cdot \mathbf{y}_{m-1} \ldots x_n$$

and the notation is $\mathbf{x} = x_1 \otimes \cdots \otimes x_n$, and $\bar{\mathbf{y}} = (\mathbf{y}_1 \ldots, \mathbf{y}_{m-1})$. We remark that here, for $x \in H$, its symbol has degree $|x|_{tot} = |x|_H + 1$, and if $\mathbf{y} = y_1 \otimes \cdots \otimes y_n$, then its symbol has degree $n + \sum_{i=1}^{n} |y_i|_H$.

In this formula, it is implicitly assumed that $m - 1 \le n$, otherwise it is zero, so this is also an example of well-graded Cacti algebra.

Definition 1.6. Let C be an arbitrary Cacti algebra and let us denote $*$ the operation induced by $B_2 =$  in C. More precisely,

$$a * b := (-1)^{|a|}$$  $$(a, b) = (-1)^{|a|} B_2(a, b)$$

This product is always pre-Lie (see the proof of the next lemma), but in well graded Cacti algebras, it is associative when restricted to external degree one:

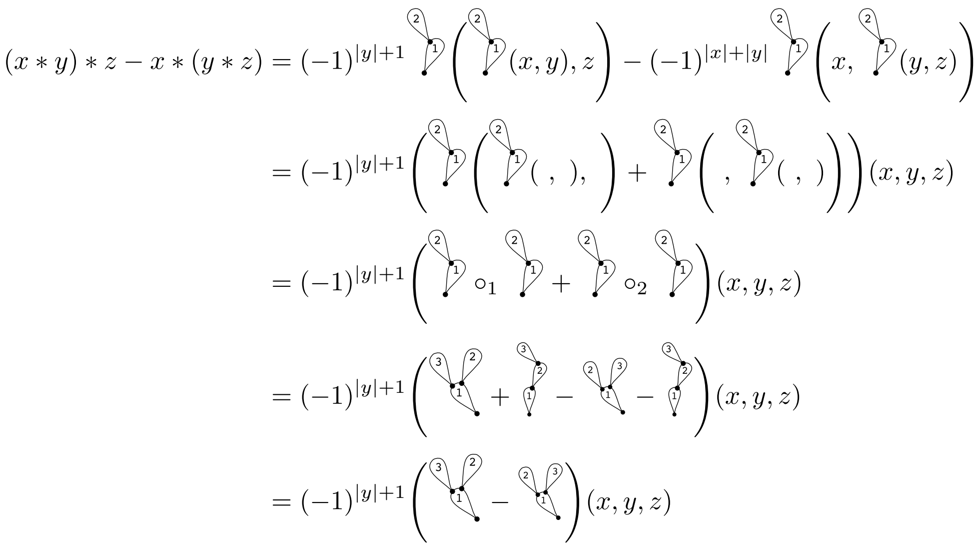
Lemma 1.7. *Let C be a well graded Cacti algebra, set $C^1 = \oplus_p C^{p,1}$ the subspace of elements of external degree one and define $\cdot := *|_{C^1 \times C^1}$, the restriction of $*$ to elements of external degree one. Then $\cdot : C^1 \times C^1 \to C^1$ is associative; moreover, for $y, z \in C$ and $x \in C^1$*

$$(x * y) * z = x * (y * z)$$

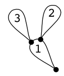
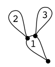
Notice that $C^n * C^m \subseteq C^{n+m-1}$ so, in particular, $(C^1, \cdot)$ is a (non-necessarily unital) associative algebra and C^n is a C^1-module.

Proof. We will compute the associator of B_2 and see that it is governed by B_3, more precisely, it is the 2–3-symmetrization of B_3 (in particular B_2 is pre-Lie). But by hypothesis B_3 acts by zero when the first variable belongs to C^1.

Let $x, y, z \in C$ with $x \in C^1$, we have



(Signs are due to Koszul rule for the total degree of the symbols $x, y, z \in C$ and B_2.)

Because we assume C is well graded, the cactus  and  act trivially when the first variable is in C^1, so the associator vanishes. $\square$

Corollary 1.8. *Let V be a graded vector space and suppose a well graded Cacti algebra structure is given in $\overline{T}V$, then this structure induces by restriction an associative product $\cdot : V \times V \to V$.*

From now on we concentrate in the bigraded associative algebra TV, and we will consider all possible well-graded Cacti algebra structures on it. We recall that the external degree is the tensorial degree, and hence a d-dimensional cactus acts as an operation of (external) degree $-d$, and the differential is of total degree one.

Remark 1.9. A (non-necessarily unitary) operation $\cdot : V \times V \to V$ can be extended to $H := V \oplus k1_H$ declaring 1_H as formal unity for $*$, namely

$$1_H \cdot v := v =: v \cdot 1_H \quad (\forall v \in V) \quad \text{and} \quad 1_H \cdot 1_H := 1_H$$

Notice that $\cdot$ is associative in V if and only if it is associative in H.

Recall that a (well graded) differential in TV induces (by restriction to V) a coassociative and counitary comultiplication in H via

$$\Delta 1_H = 1_H \otimes 1_H$$
$$\Delta v = d_e(v) + v \otimes 1_H + 1_H \otimes v$$

In this way, if TV is given a structure of a well graded Cacti algebra with multiplication equal tensor product, then H is simultaneously a counitary coassociative coalgebra, and a unitary associative algebra. The next theorem shows that H is necessarily a bialgebra. In other words, the coproduct in H is multiplicative, and hence $TV = \Omega(H)$, the Kadeishvili construction.

Theorem 1.10. *Let V be a graded vector space, the following are equivalent*

(i) *To give a well graded $\mathcal{C}acti$ algebra structure on $\overline{T}V$, extending the (free) associative product in $\overline{T}V$ and well graded with respect to the bigrading on $\overline{T}V$.*
(ii) *To give a unitary and counitary differential graded associative bialgebra structure on $H = V \oplus k1_H$.*

More precisely, the correspondence is given in the following way:

From (i) to (ii), the internal differential in $\overline{T}V$, restricted to V gives a differential on V, and the external differential induces the restricted comultiplication in V, that produces the counitary comultiplication in H. The action of B_2 gives the associative product.

From (ii) to (i), we only notice that $(TV, d) = \Omega(H)$, and Kadeishvili construction gives a Cacti algebra structure that is well graded.

Proof. We only need to prove $(i) \Rightarrow (ii)$, and in this part, we only have to check that the comultiplication in H is multiplicative. Namely,

$$\Delta(x \cdot y) = (-1)^{|x_{(2)}|_i |y_{(1)}|_i} (x_{(1)} \cdot y_{(1)}) \otimes (x_{(2)} \cdot y_{(2)})$$

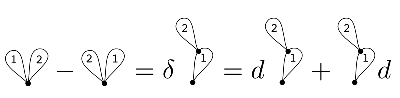
Recall the Sweedler-type notation $\Delta x = x_{(1)} \otimes x_{(2)}$. We observe that, for $a \in TV$,

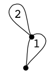
$$da = \Delta(a) - [1_H, a] + d_i$$

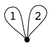
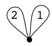
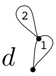
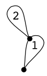
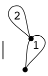
where $[-,-]$ is the super-commutator (using the total degree) in TH, so

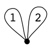
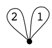
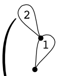
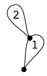
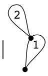
$$[1_H, a] = 1_H \otimes a - (-1)^{|a|} a \otimes 1_H = 1_H \otimes a - (-1)^{|a|_H + 1} a \otimes 1_H$$

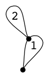
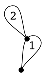
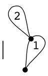
Now, in every $\mathcal{C}acti$-algebra one has

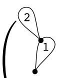
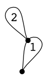
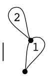


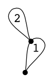
because the first equality comes from computing the boundary of the cactus  and the second is the differential of an operation. When evaluating in elements, using that $d = \Delta - [1,\ ] + d_i$ one gets

 $-$  $(x,y) = d$  $(x,y) +$  $(dx,y) + (-1)^{|x|}$  (x,dy)

 $-$  $(x,y) = \Delta\Big($  $(x,y)\Big) +$  $(\Delta x,y) + (-1)^{|x|}$  $(x,\Delta y)$

$-\Big[1,$  $(x,y)\Big] -$  $([1,x],y) - (-1)^{|x|}$  $(x,[1,y])$

$+\, d_i\Big($  $(x,y)\Big) +$  $(d_ix,y) + (-1)^{|x|}$  (x,d_iy)

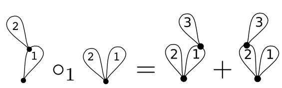
or equivalently, changing notation from  to $\cdot$ or $*$,

$$\begin{aligned}
-[x,y] &= (-1)^{|x|}\Delta(x\cdot y) + (-1)^{|x|+1}\Delta x * y + x * \Delta y \\
&\quad + -(-1)^{|x|}[1,x\cdot y] - (-1)^{|x|+1}[1,x]*y - x*[1,y] \\
&\quad + (-1)^{|x|}d_i(x\cdot y) + (-1)^{|x|+1}d_ix*y + x*d_iy
\end{aligned}$$

In order to prove what we want, we will use some identities:

$$\begin{aligned}
d_i(x\cdot y) &= d_ix*y + (-1)^{|x|_i}x*d_iy \\
[x,y] &= (-1)^{|x|}[1,x\cdot y] - (-1)^{|x|}[1,x]*y \\
x*[1,y] &= (-1)^{|x|+1}\Delta x * y \\
x*\Delta y &= (-1)^{|x|+1+|x_{(2)}|_i|y_{(1)}|_i}\big(x_{(1)}\cdot y_{(1)}\big)\otimes\big(x_{(2)}\cdot y_{(2)}\big)
\end{aligned}$$

The first one is simply that the internal differential is a derivation for the product. The second comes from the identity in Cacti

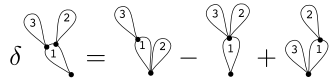


because, if one evaluates this in elements, we get that $*$ verifies a left distributive law with respect to tensor product:

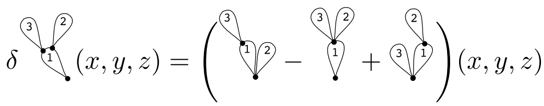
$$\begin{aligned}(a \otimes b) * c &= a \otimes (b * c) + (-1)^{|b|(|c|+1)}(a * c) \otimes b \\ &= (a * 1) \otimes (b * c) + (-1)^{|b|(|c|+1)}(a * c) \otimes (b * 1)\end{aligned}$$

and this implies immediately the equation (considering $a = 1_H$, $b = x$ and $c = y$).

The last two equations have terms of the form $a * (b \otimes c)$ (on their left side). The central idea is that, in any $\mathcal{C}acti$-algebra, even thought $*$ is not distributive on the right with $\otimes$, the failure of this is given by the boundary of B_3. The hypothesis of well graded allows us to control it. In this way, we obtain that $a * (b \otimes c)$ has to be the diagonal action. In order to see this, we calculate δB_3:



and when we evaluate in elements $x, y, z \in V$ we have



$$\begin{aligned}&= (-1)^{|x||y|+|x|+|y|} y \otimes (x \cdot z) \\ &\quad - (-1)^{|x|} x * (y \otimes z) \\ &\quad + (-1)^{|x|}(x \cdot y) \otimes z\end{aligned}$$

But also $\delta B_3 = dB_3 - B_3 d$ in $\overline{T}V$, so

$$\begin{aligned}(\delta B_3)(x, y, z) = d\big(\underbrace{B_3(x, y, z)}_{=0}\big) &- B_3(dx, y, z) \\ &+ (-1)^{|x|} \underbrace{B_3(x, dy, z)}_{=0} - (-1)^{|x|+|y|} \underbrace{B_3(x, y, dz)}_{=0}\end{aligned}$$

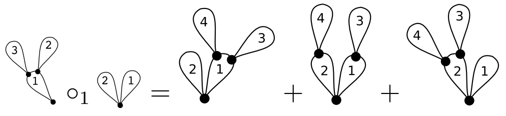
(the vanishing terms are due to the well grading hypothesis). So,

$$-B_3(dx, y, z) = (-1)^{|x||y|+|x|+|y|} y \otimes (x \cdot z) - (-1)^{|x|} x * (y \otimes z) + (-1)^{|x|}(x \cdot y) \otimes z$$

Now, for elements in tensorial degree two $\mathbf{x} = x_1 \otimes x_2$, the cactus B_3 acts by

$$B_3(\mathbf{x}, y, z) = B_3(x_1 \otimes x_2, y, z) = (-1)^{|x_2|+|y|+|x_2||y|}(x_1 \cdot y) \otimes (x_2 \cdot z)$$

because in $\mathcal{C}acti$ we have



where only the second term acts non-trivially in $V^{\otimes 4}$.

Using this identity for $\mathbf{x} = dx$ and recalling

$$dx = \Delta(x) - [1_H, x] = x_{(1)} \otimes x_{(2)} - 1_H \otimes x + (-1)^{|x|} x \otimes 1_H$$

one has

$$\begin{aligned} B_3(dx, y, z) &= B_3\big(x_{(1)} \otimes x_{(2)} - 1_H \otimes x + (-1)^{|x|} x \otimes 1_H, y, z\big) \\ &= (-1)^{|x_{(2)}|+|y|+|x_{(2)}||y|} (x_{(1)} \cdot y) \otimes (x_{(2)} \cdot z) \\ &\quad - (-1)^{|x|+|y|+|x||y|} y \otimes (x \cdot z) \\ &\quad + (-1)^{|x|+1+|y|+1|y|} (x \cdot y) \otimes z \\ &= (-1)^{|x_{(2)}|+|y|+|x_{(2)}||y|} (x_{(1)} \cdot y) \otimes (x_{(2)} \cdot z) \\ &\quad - (-1)^{|x|+|y|+|x||y|} y \otimes (x \cdot z) \\ &\quad - (-1)^{|x|} (x \cdot y) \otimes z \end{aligned}$$

from which one gets the equation

$$x * (y \otimes z) = (-1)^{|x|+|x_{(2)}|+|y|+|x_{(2)}||y|} (x_{(1)} \cdot y) \otimes (x_{(2)} \cdot z)$$

namely, the diagonal action.

From this general equation, using $[1_H \otimes y]$ instead of $y \otimes z$, we deduce

$$x * [1, y] = (-1)^{|x|+1} \Delta x * y$$

And replacing again $y \otimes z$ by $\Delta y = y_{(1)} \otimes y_{(2)}$ (and of course taking into account the signs, noticing that if $v \in V$ then $|v|_{tot} = |v|_i + 1$):

$$x * (\Delta y) = (-1)^{|x|+1+|x_{(2)}|_i |y_{(1)}|_i} (x_{(1)} \cdot y_{(1)}) \otimes (x_{(2)} \cdot y_{(2)})$$

that is precisely the last thing we needed to verify. $\square$

Example 1.11. Let $\mathfrak{g}$ be a Lie algebra and consider $H = U(\mathfrak{g})$, and as always $V = \overline{U}(\mathfrak{g}) = \operatorname{Ker}(\epsilon : U(\mathfrak{g}) \to k)$, then the cohomology of $(\overline{T}V, d)$ is

$$H^\bullet(\overline{T}V) \simeq \Lambda^\bullet \mathfrak{g}$$

(where here $\Lambda\mathfrak{g}$ is the non-unital exterior algebra in $\mathfrak{g}$). Even more, in degree one, the Lie bracket in $H^1(\overline{T}V, d)$ is the commutator of the primitive elements in $U\mathfrak{g}$, namely, the Lie bracket in $\mathfrak{g}$. Since $\Lambda^\bullet\mathfrak{g}$ is generated (as associative algebra) in degree one, the Gerstenhaber structure is determined by the bracket in this degree. So we get the standard Gerstenhaber algebra structure in $\Lambda^\bullet\mathfrak{g}$ from the $\mathcal{C}acti$-algebra in $\overline{T}V$. In other words, the Gerstenhaber algebra structure in $\Lambda^\bullet\mathfrak{g}$ lifts to a well graded $\mathcal{C}acti$-algebra structure in $\overline{T}V = \overline{T}(\overline{U}(\mathfrak{g}))$.

As a subexample, if W is any vector space and $\mathfrak{g} = \mathrm{Lie}(W)$ is the free Lie algebra on W, then $\Lambda^\bullet\mathfrak{g} = \Lambda^\bullet\mathrm{Lie}(W)$ is the free Gerstenhaber algebra in W. Again this structure lifts to a (well graded) $\mathcal{C}acti$-algebra structure in $\overline{T}V = \overline{T}(\overline{U(\mathrm{Lie}(W))}) = \overline{T}\overline{TW}$, in the sense that its $\mathcal{C}acti$-algebra structure induces the Gerstenhaber algebra structure on its homology.

Remark 1.12. There is another way of giving algebraic structure to $\overline{T}V$ that is relevant to Cacti, but different from the one we described before. If A is a Frobenius algebra, then one has a particular way of identifying $A^* \cong A$ and so one have isomorphisms of vector spaces

$$\bigoplus_{n\geq 0}\mathrm{Hom}\big(A^{\otimes n}, A\big) \cong \bigoplus_{n\geq 0}\big(A^*\big)^{\otimes n} \otimes A \cong \bigoplus_{n\geq 0}(A)^{\otimes n+1} = \overline{T}A$$

This is considered in [6] in relation with the cyclic version of Deligne's conjecture.

2. Morphisms and well gradings

Recall the notation, for a bigraded algebra $T = \bigoplus_{p,q} T^{p,q}$

$$T^n := \bigoplus_{q\in\mathbb{Z}} T^{n,q}$$

The next lemma is relatively simple to proof, but is the key point of our main result. It formalizes the fact that in TV, all the $\mathcal{C}acti$-algebra structure depends on B_2, the differential, and the associative product.

Lemma 2.1. *Let T and C be two well graded Cacti algebras, and $f: T \to C$ a linear transformation, that is homogeneous with respect to the bigrading. If we assume that*

- *T is generated by T^1 as associative algebra (in particular $T = \bigoplus_{n\geq 1}(T^1)^n$),*
- *f is a morphism of associative algebras,*
- *$f(dt) = df(t)$ for all $t \in T^1$,*
- *$f|_{T^1} : (T^1, \cdot) \to (C^1, \cdot)$ is a morphism of associative algebras,*

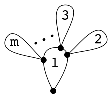
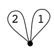
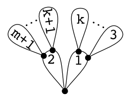
then f is a morphism of Cacti-algebras.

Proof. Let us denote by $\cup$ the associative product given by C_2 (in T and in C). In an analogous way to Theorem 1.10, the signs are given by the Koszul rule, but in this proof it is not necessary to make them explicit, so we will omit then for clarity.

The proof consists of the following reductions:

1. If $f(B_2(\mathbf{x},\mathbf{y})) = B_2(f\mathbf{x}, f\mathbf{y})$, then $f(M,\mathbf{x}_1,\dots,\mathbf{x}_n) = f(M, f\mathbf{x}_1,\dots,f\mathbf{x}_n)$ for every cactus M.

   Proof. Since $\mathcal{C}acti$ is generated by C_2 and B_m $(m \geq 2)$, it is enough to see that f commutes with this operations. Notice that f is a morphism of associative algebras by assumption. In order to reduce from B_m to B_2, we proceed by induction in the external degree. Recall the identity

    $\circ_1$  $= \sum_k$ 

   If we want to compute $B_m(\mathbf{x},\mathbf{y}_1,\dots,\mathbf{y}_{m-1})$, with $\mathbf{x} \in T^{p,\bullet}$, the well grading implies that the non-trivial terms are only with $p \geq m$. Considering elements $\mathbf{x} = x_1 \cup \mathbf{x}'$ with $x \in T^1$ and $\mathbf{x}' \in T^{m-1}$ (this is possible because we assume T is generated by T^1) we have

$$B_m\big((x_1 \cup \mathbf{x}'), \mathbf{y}_1,\dots,\mathbf{y}_{m-1}\big)$$
$$= \sum_{k=1}^{m} \pm B_k\big(x_1,,\mathbf{y}^1,\dots,\mathbf{y}^{k-1}\big) \cup B_{m-k+1}\big(\mathbf{x}',\mathbf{y}^k,\dots,\mathbf{y}^{m-1}\big)$$

   and because of the well-grading ($|x_1|_e = 1$, so every term is zero except two of them)

$$B_m\big(C_2(x_1,\mathbf{x}'),\mathbf{y}_1,\dots,\mathbf{y}_m\big) = \pm C_2\big(x_1, B_m\big(\mathbf{x}',\mathbf{y}^1,\dots,\mathbf{y}^{m-1}\big)\big)$$
$$\pm\, C_2\big(B_2(x_1,y_1), B_{m-1}\big(\mathbf{x}',\mathbf{y}^2,\dots,\mathbf{y}^{m-1}\big)\big)$$

   where the first term has $|x'|_e < |x|_e$, and the second is written using B_2 and B_{m-1}. Hence, f commutes with all B_m if it does with B_2. $\square$

2. If $fB_2(x,\mathbf{y}) = B_2(fx, f\mathbf{y})$ for all $x \in T^1$, $\mathbf{y} \in T$, then $fB_2(\mathbf{x},\mathbf{y}) = B_2(f\mathbf{x}, f\mathbf{y})$ for all $\mathbf{x},\mathbf{y} \in T$.

   Proof. If $\mathbf{x} = x_1 \cup \ldots \cup x_r$, since B_2 distribute the $\cup$-product in the first variable, we have

$$B_2(\mathbf{x},\mathbf{y}) = \sum_{k=1}^{r} \pm x_1 \cup \ldots B_2(x_k,\mathbf{y}) \ldots \cup x_r$$

   so the claim follows. $\square$

3. If $fB_2(x,y) = B_2(fx,fy)$ for all $x,y \in T^1$ (which is true by assumption), then $fB_2(x,\mathbf{y}) = B_2(fx,f\mathbf{y})$ for all $x \in T^1$, $\mathbf{y} \in T$.

Proof. Let $\mathbf{y} = \mathbf{y}' \cup \mathbf{y}' \in T$, notice that the external degree of $\mathbf{y}'$ and $\mathbf{y}''$ are both strict less than the degree of $\mathbf{y}$. For $x \in T^1$, we compute

$$\begin{aligned} B_2(x,\mathbf{y}) &= B_2\big(x,\mathbf{y}' \cup \mathbf{y}''\big) = B_2 \circ_2 C_2\big(x,y',y''\big) \\ &= \pm B_2\big(x,\mathbf{y}'\big) \cup \mathbf{y}'' \pm \mathbf{y}' B_2\big(x,\mathbf{y}''\big) + (\delta B_3)\big(x,\mathbf{y}',\mathbf{y}''\big) \end{aligned}$$

Notice that (due to the well grading and the fact that $x \in T^1$):

$$\begin{aligned} (\delta B_3)\big(x,\mathbf{y}',\mathbf{y}''\big) &= d\big(B_3\big(x,\mathbf{y}',\mathbf{y}''\big)\big) + B_3\big(dx,\mathbf{y}',\mathbf{y}''\big) \\ &\quad \pm B_3\big(x,d\mathbf{y}',\mathbf{y}''\big) \pm B_3\big(x,\mathbf{y}',d\mathbf{y}''\big) \\ &= B_3\big(dx,\mathbf{y}',\mathbf{y}''\big) \end{aligned}$$

Now, since $dx \in T^1 \oplus T^2$ and T is generated by T^1 as associative algebra, we can write

$$dx = d_i x + \sum x_1 \cup x_2$$

and so

$$\begin{aligned} B_3\big(dx,\mathbf{y}',\mathbf{y}''\big) &= B_3\big(d_i x,\mathbf{y}',\mathbf{y}''\big) + B_3\big(x_1 \cup x_2,\mathbf{y}',\mathbf{y}''\big) \\ &= B_3\big(x_1 \cup x_2,\mathbf{y}',\mathbf{y}''\big) \\ &= (B_3 \circ_1 C_2)\big(x_1,x_2,\mathbf{y}',\mathbf{y}''\big) \\ &= \pm B_3\big(x_1,\mathbf{y}',\mathbf{y}''\big) \cup x_2 \\ &\quad \pm B_2\big(x_1,\mathbf{y}'\big) \cup B_2\big(x_2,\mathbf{y}''\big) \\ &\quad \pm x_1 \cup B_3\big(x_2,\mathbf{y}',\mathbf{y}''\big) \\ &= \pm B_2\big(x_1,\mathbf{y}'\big) \cup B_2\big(x_2,\mathbf{y}''\big) \end{aligned}$$

(we have used again the well grading hypothesis and the fact that $d_i x$, x_1 and x_2 belong to T^1).
We conclude

$$B_2(x,\mathbf{y}) = \pm B_2(x,\mathbf{y}) \cup \mathbf{y}'' \pm \mathbf{y}' \cup B_2\big(x,\mathbf{y}''\big) \pm B_2\big(x_1,\mathbf{y}'\big) \cup B_2\big(x_2,\mathbf{y}''\big)$$

With this in mind, we compute

$$f\big(B_2(x,\mathbf{y})\big) = f\big(\pm B_2\big(x,\mathbf{y}'\big) \cup \mathbf{y}'' \pm \mathbf{y}' B_2\big(x,\mathbf{y}''\big) \pm B_2\big(x_1,\mathbf{y}'\big) \cup B_2\big(x_2,\mathbf{y}''\big)\big)$$

and since f commutes with $\cup$

$$= \pm f B_2\big(x, \mathbf{y}'\big) \cup f\mathbf{y}'' \pm f\mathbf{y}' \cup f B_2\big(x, \mathbf{y}''\big) \pm f B_2\big(x_1, \mathbf{y}'\big) \cup f B_2\big(x_2, \mathbf{y}''\big)$$

Because $\mathbf{y}'$ and $\mathbf{y}''$ have strict less external degree than $\mathbf{y}$, we may assume inductively that f preserves the operation $B_2(x, -)$ in those degrees, and so the above formula is equal to

$$= \pm B_2\big(fx, f\mathbf{y}'\big) \cup f\mathbf{y}'' \pm f\mathbf{y}' \cup B_2\big(fx, f\mathbf{y}''\big) \pm B_2\big(fx_1, f\mathbf{y}'\big) \cup B_2\big(fx_2, f\mathbf{y}''\big)$$

and since f preserves degrees, and also C is well graded, the arguments used to eliminate the terms of type $B_3(x, -)$ can also be used for $B_3(fx, -)$, and we conclude

$$\begin{aligned} &= \pm B_2\big(fx, f\mathbf{y}'\big) \cup f\mathbf{y}'' \pm f\mathbf{y}' \cup B_2\big(fx, f\mathbf{y}''\big) \pm B_2\big(fx_1 \cup fx_2, f\mathbf{y}' \cup f\mathbf{y}''\big) \\ &= \pm B_2\big(fx, f\mathbf{y}'\big) \cup f\mathbf{y}'' \pm f\mathbf{y}' \cup B_2\big(fx, f\mathbf{y}''\big) \pm B_2\big(fdx, f\big(\mathbf{y}' \cup \mathbf{y}''\big)\big) \end{aligned}$$

Finally because f commutes with the differential in T^1, we have that $fdx = d(fx)$ and so

$$= B_2\big(fx, f\mathbf{y}' \cup f\mathbf{y}''\big) = B_2(fx, f\mathbf{y}) \qquad \square$$

Since the requirement of the last reduction holds by assumption of the lemma, we have finished the proof. $\square$

As an immediate corollary, we see that Theorem 1.10 actually gives an equivalence of categories between d.g. bialgebras and Cacti algebras that are well graded and freely generated in external degree one:

Corollary 2.2. *Let H and H' be two (d.g.) unitaries and counitaries bialgebras, and endow $\Omega H = \overline{T}\overline{H}$ and $\Omega H' = \overline{T}\overline{H}'$ with its natural Cacti algebra structure, then*

$$\mathrm{Hom}_{\mathcal{C}acti}\big(\Omega H, \Omega H'\big) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{d.g.bialg}}\big(H, H'\big)$$

Proof. We only remark that both ΩH and $\Omega H'$ are well graded and generated in external degree one, so the lemma above applies. $\square$

Remark 2.3. The Cacti algebra structure on ΩH is unique if one requires well grading, and that the operation B_2 restricted to H agree with the product of H. This is true because if $\widetilde{\Omega H}$ is equal to $\Omega(H)$ as d.g. algebras, but with eventually different Cacti algebra structure with this properties, then the identity map $\Omega H \to \widetilde{\Omega H}$ verifies the hypothesis of the above lemma, and hence it must be a Cacti-algebra isomorphism.

The next theorem is a continuation of the dictionary between Cacti axioms and bialgebra axioms. Before presenting it, we recall a standard definition of a module-algebra.

Definition 2.4. Let A be a unital associative algebra and H a unitary and counitary bialgebra. We say that $\rho : H \otimes A \to A$ is an *H-module algebra structure* on A if it makes A into an H-modulo but also satisfying the property that the multiplication map

$$m_A : A \otimes A \to A$$

is H-linear (with the diagonal action on $A \otimes A$).

In case A is a d.g. algebra and H a d.g. bialgebra, the H-module algebra structure is called *differential* if

$$d\big(h(a)\big) = d_H(h)(a) + (-1)^{|h|} h\big(d_A(a)\big)$$

or equivalently if the map

$$\rho : H \otimes A \to A$$

is a morphism of complexes.

Theorem 2.5. *Let A be a d.g. unital associative algebra and H a d.g. unital and counital bialgebra. Then there exists a 1–1 correspondence between Cacti algebra morphism $\Omega(H) \to C^\bullet(A)$ and differential H-module algebra structures on A. The correspondence is given by restriction:*

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}acti}\big(\Omega H, C^\bullet(A)\big) &\to \mathrm{Hom}_{d.g.alg}\big(\overline{H}, \mathrm{End}(A)\big) \\ &\cong \mathrm{Hom}_{d.g.alg_1}\big(H, \mathrm{End}(A)\big) \cong \mathrm{Hom}(H \otimes A, A) \end{aligned}$$

and in the other direction, if $\rho : H \to \mathrm{End}(A)$, $x \mapsto \rho_x$, the map $\Omega(H) \to C^\bullet(A)$ is given by

$$TV \ni x_1 \otimes \cdots \otimes x_n \mapsto \big(a_1 \otimes \cdots \otimes a_n \mapsto \rho_{x_1}(a_1) \cdots \rho_{x_n}(a_n)\big)$$

In this theorem, $d.g.alg$ means non-necessarily unital differential graded associative algebras, and $d.g.alg_1$ are the $d.g.alg$ maps that also preserve the unit.

Proof. Since ΩH and $C(A)$ are both well graded Cacti algebras, we can use Lemma 2.1. Then, a morphism $f : \Omega H \to C(A)$ is the same as a d.g. algebra morphism such that its restriction on elements of external degree one (i.e. to elements of H) is multiplicative with respect to the operation $*$. This produces a morphism

$$\rho := f|_V : V \to \mathrm{End}(A)$$

where $V = \overline{H} = \mathrm{Ker}(\epsilon)$. This shows that morphisms whose restriction are $*$-multiplicative are the same as (non-unital) V-modulo structures on A, that is the same as unital H-module structures on A.

Notice that given an H-module structure $\rho : H \to \mathrm{End}(A)$, the restriction to V produces a multiplicative map $V \to \mathrm{End}(A)$. Then, the universal property of the tensor algebra gives a multiplicative map $\widehat{\rho} : (TV, \otimes) \to (C(A), \cup)$. The theorem follows if we show that "$\widehat{\rho}$ commutes with the differential if and only if the H-module structure is a (differential) H-module *algebra* structure".

Let us denote, for $h \in H$ and $a \in A$,

$$h(a) := \big(\rho(h)\big)(a)$$

When computing the Hochschild boundary of $\rho(h)$ we get

$$\big(d_e\rho(h)\big)(a \otimes b) = -ah(b) + h(ab) - h(a)b$$

On the other hand, the internal differential is

$$\big(d_i\rho(h)\big)(a) = d\big(h(a)\big) - (-1)^{|h|}h\big(d(a)\big)$$

Because $d = d_e + d_i$ and their bidegrees are different, the equality

$$d\rho(h) = \widehat{\rho}dh$$

is equivalent to two equations

$$d_e\rho(h) = \widehat{\rho}d_eh, \qquad d_i\rho(h) = \rho d_ih$$

The equation with d_e tells us that A is an H-module algebra, because

$$\begin{aligned}(\widehat{\rho}d_eh)(a \otimes b) &= \big(\widehat{\rho}(\Delta h - 1_H \otimes h - h \otimes 1_H)\big)(a \otimes b)\\ &= \big(\widehat{\rho}(h_1 \otimes h_2 - 1_H \otimes h - h \otimes 1_H)\big)(a \otimes b)\\ &= h_1(a)h_2(b) - h(a)b - ah(b)\end{aligned}$$

and hence

$$\partial\rho(h) = \widehat{\rho}d_eh \quad \Longleftrightarrow \quad h(ab) = h_1(a)h_2(b)$$

And the equation with d_i says

$$\big(d_i\rho(h)\big)(a) = d_A\big(h(a)\big) - (-1)^{|h|}h\big(d_A(a)\big) = d_H(h)(a)$$

namely, the d.g. condition for ρ. □

An immediate consequence is the following

Corollary 2.6. *Let H be a bialgebra and A an H-module algebra with structure map $H \otimes A \xrightarrow{\rho} A$. Then ρ induces a Gerstenhaber algebra map*

$$H^\bullet(\Omega H, d) \to HH^\bullet(A)$$

Examples

Let $\mathfrak{g}$ be a Lie algebra and $H = U(\mathfrak{g})$. If A is an associative algebra, then an H-module algebra map is the same as an action of $\mathfrak{g}$ by derivations. If one takes $\mathfrak{g} = Der(A)$, then the morphism $\Omega H \to C(A)$ induces a map on homology

$$\Lambda^\bullet Der(A) \to HH^\bullet(A)$$

whose image is the associative subalgebra of $HH^\bullet(A)$ generated by derivations. This example shows that the map from Theorem 2.5 is in general non-trivial. But it can happen that a bialgebra H has no primitive elements but non-trivial cohomology. This will produce maps with no derivations in its image, but giving elements of higher (cohomological) degree. We present a minimal example of this situation.

Let $H = k1 \oplus kx \oplus kg \oplus kgx$ be the Sweedler or Taft algebra of dimension 4, that may be described in terms of generators and relations as the k-algebra generated by x and g with relations

$$x^2 = 0, \qquad g^2 = 1, \qquad xg = -gx$$

(we assume characteristic different from 2). This algebra is a bialgebra with comultiplication determined by

$$\Delta(g) = g \otimes g, \qquad \Delta x = x \otimes 1 + g \otimes x$$

This algebra has no primitive elements, so $H^1(\Omega H) = 0$, but a direct computation shows that the (class of the) element $xg\otimes x$ generates $H^2(\Omega H)$ over k. A less direct computation shows that $H^\bullet(\Omega H)$ is a polynomial ring on one variable, with generator in degree two (given by this element). Next, we include the verification of this fact, that follows from the following three items:

- $H \cong H^*$ as Hopf algebras, for instance, taking the elements in H^* defined by

$$\widehat{g} : \begin{cases} 1 \mapsto 1 \\ g \mapsto -1 \\ x \mapsto 0 \\ xg \mapsto 0 \end{cases} \qquad \widehat{x} : \begin{cases} 1 \mapsto 0 \\ g \mapsto 0 \\ x \mapsto 1 \\ xg \mapsto 1 \end{cases}$$

one can easily verify that $\widehat{g}^2 = \epsilon$, $\widehat{g}\widehat{x} = -\widehat{x}\widehat{g}$, $\widehat{x}^2 = 0$. For that reason, we have an isomorphism

$$H^\bullet(\Omega H) = \operatorname{Ext}^\bullet_{H^*}(k,k) \cong \operatorname{Ext}^\bullet_H(k,k)$$

- Also, $H = (k[x]/x^2)\# k\mathbb{Z}_2$, so one can compute Ext with the formula

$$\operatorname{Ext}^\bullet_H(k,k) = \operatorname{Ext}^\bullet_{k[x]/x^2}(k,k)^{\mathbb{Z}_2}$$

  (see for instance [9]).
- $\operatorname{Ext}^\bullet_{k[x]/x^2}(k,k)$ is a polynomial ring in one variable, call it D, of degree one (this is the easiest example of classical quadratic Koszul algebra). There are two possibilities: the action of the generator of $\mathbb{Z}_2$ is trivial in D, or it acts by $D \mapsto -D$. In the first case it should be $\operatorname{Ext}_{k[x]/x^2}(k,k)^{\mathbb{Z}_2} = k[D]$, while in the second it should be $\operatorname{Ext}_{k[x]/x^2}(k,k)^{\mathbb{Z}_2} = k[D^2]$. But in H there are no primitive elements, so $H^1(\Omega H) = 0$ and only the second possibility can be true.

A consequence of this commutation is that the Gerstenhaber bracket (in cohomology) of the generator with itself is trivial, just by degree considerations. This implies that in any H-module algebra A, the bilinear map given by

$$\begin{aligned} \Psi : A^{\otimes 2} &\to A \\ a\otimes b &\mapsto xg(a)x(b) \end{aligned}$$

is an integrable 2-cocycle in the sense that $[\Psi,\Psi] = 0$.

We finally recall that the data of an H-module algebra structure on A is the same as a $\mathbb{Z}_2$-grading (given by the eigenvectors of eigenvalues ± 1 of g, we assume char $\neq 2$) and a square zero super-derivation (with respect to that grading), because the general formula

$$h(ab) = h_1(a)h_2(b)$$

for $h = x$ says (if a is homogeneous):

$$x(ab) = x(a)b + g(a)x(b) = x(a)b + (-1)^{|a|}ax(b)$$

In that way, every square zero super-derivation x in A gives an unobstructed formal deformation of A.

We finish by collecting some general information on Hopf algebras and its cohomology:

1. If H is finite dimensional bialgebra, then

$$H^\bullet\big(\Omega(H)\big) = \operatorname{Ext}^\bullet_{H^*}(k,k) = H^\bullet\big(H^*,k\big) \subset HH^\bullet\big(H^*\big)$$

These equalities are immediate from the definition if one uses the standard complex for solving H^* as H^*-bimodule when computing Hochschild cohomology. The last inclusion was proved (to be a split inclusion) in [2], by giving a specific map at the level of complexes, that reserves the cup product and i-th compositions. Now this map can be interpreted from the fact that any finite dimensional bialgebra is an H^*-module algebra. The finite dimensional hypothesis is only needed for H^* to be a bialgebra as well.

2. If H is any bialgebra and H' is a bialgebra in duality with H, namely there is a pairing $(-,-) : H \otimes H' \to k$ satisfying

$$\big(\Delta a, x' \otimes y'\big) = \big(a, x' \otimes y'\big)$$

and

$$(a \otimes b, \Delta x) = \big(ab, x'\big)$$

then H is an H'-module algebra.

3. If A is an H-comodule algebra, that is, it is given a comodule structure map

$$A \to A \otimes H$$

such that the multiplication $m_A : A \otimes A \to A$ is H-colinear, and H' is in duality with H, then A is an H'-module algebra. Geometrical examples come in this way: if X and G are affine algebraic varieties and G is an algebraic group, to have an algebraic action of G on X is the same as a comodule algebra structure $\mathcal{O}_X \to \mathcal{O}_X \otimes \mathcal{O}_G$. If $\mathfrak{g}$ is the Lie algebra associated to the algebraic group G, then $U\mathfrak{g}$ is in duality with $\mathcal{O}_G$, and hence A is a $U\mathfrak{g}$-module algebra.

4. If G is a discrete group and $H = k[G]$, an H-module algebra structure on A is the same as a G-grading, but this gives nothing interesting because $k[G]$ is cosemisimple.

5. The general Taft algebra $H = T_m$: This algebra is generated by x, g with relations

$$g^p = 1; \qquad x^m = 0; \qquad gx = \xi xg$$

where ξ is a primitive m-th root of unity. The comultiplication is given by $\Delta g = g \otimes g$, $\Delta x = x \otimes g + 1 \otimes x$. In order to compute the cohomology, one can see $H \cong (k[x]/x^m)\#k[G]$ with $G = \langle g : g^m = 1\rangle$, and so H^* is also of the form $H^* \cong A\#k^G$. The same result in [9] gives the formula

$$H^\bullet\big(A\#k^G, k\big) = H^0\big(k^G, H^\bullet(A, k)\big) = H^\bullet(A, k)_0$$

where $H^\bullet(A, k)_0$ is the homogeneous component of degree zero with respect to the G-grading of $H^\bullet(A, k)$.

6. Let g be a group-like element in some bialgebra H, and denote $u_g = g - 1_H$ (notice $g \notin \operatorname{Ker} \epsilon$, but $u_g \in \operatorname{Ker} \epsilon$). Then $d(u_g) = \Delta' u_g = u_g \otimes u_g$. If h is another grouplike element and x is g–h-primitive, namely $\Delta x = g \otimes x + x \otimes h$, then $dx = u_g \otimes x + x \otimes u_h$.

Proof.

$$\begin{aligned} d(u_g) &= d(g - 1_H) - 1_H \otimes u_g - u_g \otimes 1_H = g \otimes g - 1_H \otimes 1_H - 1_H \otimes u_g - u_g \otimes 1_H \\ &= g \otimes g - 1_H \otimes 1_H - 1_H \otimes g + 1_H \otimes 1_H - g \otimes 1_H + 1_H \otimes 1_H \\ &= (g - 1_H) \otimes (g - 1_H) = u_g \otimes u_g \end{aligned}$$

The formula $dx = u_g \otimes x + x \otimes u_h$ is proved in an analogous way, we omit it. □

This computation allows us to generalize the example of the H-module algebra action of the Sweedler algebra in the following way:
Let $d_1, \dots, d_n: A \to A$ be skew-derivation of an associative algebra A. That means there exist automorphisms g_i and h_i of algebras of A such that

$$d_i(ab) = g_i(a) d_i(b) + d_i(a) h_i(b) \quad \forall a, b \in A$$

Let $f : A^{\otimes n} \to A$ be defined as

$$f(a_1, \dots, a_n) = d_1(a_1) \cdots d_n(a_n)$$

If, in addition, $g_0 = g_{n+1} = \mathrm{Id}$ and $h_i = g_{i+1}$ for all $i = 1, \dots, n-1$, then f is a Hochschild n-cocycle, coming from $\Omega(H)$ for some bialgebra H.

Proof. Let us consider the free algebra generated by $x_i : i = 1, \dots, n$ and $G_i : i = 0, \dots, n+1$, with comultiplication determined by

$$\Delta G_i = G_i \otimes G_i; \qquad \Delta x_i = G_i \otimes x_i + x_i \otimes G_{i+1}$$

and define the H-module structure on A by

$$x_i(a) = d_i(a), \qquad G_i(a) = g_i(a)$$

where, by notation, $g_{n+1} = h_n$. Then A is an H-module algebra. We need to check that $\omega := x_1 \otimes \cdots \otimes x_n \in \Omega(H)$ satisfies $d\omega = 0$. But this is easy because

$$\begin{aligned} d\omega = d(x_1 \otimes \cdots \otimes x_n) &= \sum_{i=1}^{n} (-1)^{i+1} x_1 \otimes \cdots \otimes d(x_i) \otimes \cdots \otimes x_n \\ &= \sum_{i=1}^{n} (-1)^{i+1} (x_1 \otimes \cdots \otimes u_{G_i} \otimes x_i \otimes \cdots \otimes x_n + x_1 \otimes \cdots \otimes x_i \otimes u_{G_{i+1}} \otimes \cdots \otimes x_n) \end{aligned}$$

and all terms cancel telescopically except the first and the last:

$$= u_{G_0} \otimes x_1 \otimes \cdots \otimes x_n + (-1)^{n-1} x_1 \otimes \cdots \otimes x_n \otimes u_{G_{n+1}}$$

But $u_{G_0} = u_{G_{n+1}} = u_{id} = 0$, so $d\omega = 0$, and hence $\partial f = 0$ in $C^\bullet(A)$. $\square$

We remark that also the other Cacti operations that one may do with f in $C^\bullet(A)$ may also be done in $\Omega(H)$.

It would be interesting to know, given an associative algebra A, whether or not any class in $HH^\bullet(A)$ comes from an element in $H^\bullet(\Omega(H))$, for some bialgebra H acting on A, making A into an H-module algebra.

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