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Derived categories of surfaces isogenous to a higher product

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ABSTRACT

Let $S = (C \times D)/G$ be a surface isogenous to a higher product of unmixed type with $p_g = q = 0$, $G = (\mathbb{Z}/3)^2$. We construct exceptional sequences of line bundles of maximal length on S. As a consequence we find new examples of quasiphantom categories.

1. Introduction

Recently derived categories of surfaces of general type attracted a lot of attention. Several interesting semiorthogonal decompositions of the derived categories were constructed by Böhning, Graf von Bothmer, and Sosna on the classical Godeaux surface [7]; Alexeev and Orlov on the primary Burniat surfaces [1]; Galkin and Shinder on the Beauville surface [12]; Böhning, Graf von Bothmer, Katzarkov and Sosna on the determinantal Barlow surfaces [6]; Fakhruddin on some fake projective planes [11]; Galkin, Katzarkov, Mellit and Shinder on some different fake projective planes and on a fake cubic surface [13]; Coughlan on some surfaces obtained as abelian coverings of del Pezzo surfaces [8]; Keum

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on some fake projective planes with enough automorphisms [16]. These semiorthogonal decompositions consist of admissible subcategories generated by exceptional sequences of line bundles of maximal lengths and their orthogonal complements. These orthogonal complements have vanishing Hochschild homology groups. An admissible triangulated subcategory of a derived category of a smooth projective variety is called a quasiphantom category if its Hochschild homology group vanishes and its Grothendieck group is finite. When the Grothendieck group of a quasiphantom category also vanishes it is called a phantom category. Gorchinskiy and Orlov in [14] constructed phantom categories using quasiphantom categories constructed in [1,7,12]. Determinantal Barlow surfaces also provide examples of phantom categories [6].

Let S be a surface isogenous to a higher product $(C \times D)/G$ of unmixed type with $p_g = q = 0$. If G is an abelian group, Bauer and Catanese [2] proved that G is one of $(\mathbb{Z}/2)^3, (\mathbb{Z}/2)^4, (\mathbb{Z}/3)^2, (\mathbb{Z}/5)^2$. As mentioned above, Galkin and Shinder [12] constructed exceptional sequences of line bundles of maximal length 4 on the Beauville surface which is a surface isogenous to a higher product with $p_g = q = 0$ and $G = (\mathbb{Z}/5)^2$. Motivated by their work, we study the derived categories of the 2-dimensional family of surfaces isogenous to a higher product with $p_g = q = 0$, $G = (\mathbb{Z}/3)^2$ and prove that there exist similar semiorthogonal decompositions.

Theorem 1.1. *Let $S = (C \times D)/G$ be a surface isogenous to a higher product with $p_g = q = 0$, $G = (\mathbb{Z}/3)^2$. There are exceptional sequences of line bundles of maximal length 4 in $D^b(S)$ and the orthogonal complements of the admissible subcategories generated by these line bundles are quasiphantom categories.*

This result gives new examples of quasiphantom categories having Grothendieck groups $(\mathbb{Z}/3)^5$ and these categories can be used to construct phantom categories by a theorem of Gorchinskiy and Orlov [14]. We also compute Hochschild cohomology groups of quasiphantom categories and prove that for some exceptional sequences we obtained the categories generated by those exceptional sequences are deformation invariant. While adding these results to this paper which was on the arXiv, similar results have been obtained independently by Coughlan in [8] via different method. In his paper [8], Coughlan considers general type surfaces which are obtained as abelian covers of del Pezzo surfaces satisfying some conditions. His method can be applied to surfaces isogenous to a higher product with $G = (\mathbb{Z}/3)^2$, $G = (\mathbb{Z}/5)^2$ and many other surfaces of general type. He constructs many exceptional sequences of maximal lengths on these surfaces and studies deformation invariance and Hochschild cohomology groups.

This paper is organized as follows. In Section 2, we collect some basic facts about the surfaces isogenous to a higher product and compute the Grothendieck groups of these surfaces. In Section 3, we construct exceptional sequences of line bundles on the 2-dimensional family of surfaces isogenous to a higher product with $p_g = q = 0$, $G = (\mathbb{Z}/3)^2$. In Section 4, we discuss quasiphantom and phantom categories. In Section 5, we will consider the cases where $G = (\mathbb{Z}/2)^3$, $G = (\mathbb{Z}/2)^4$.

Notations. We will work over $\mathbb{C}$. A curve will mean a smooth projective curve. A surface will mean a smooth projective surface. Derived category of a variety will mean the bounded derived category of coherent sheaves on that variety. In this paper, G denotes a finite group and $\widehat{G} = Hom(G, \mathbb{C}^*)$ denotes the character group of G. We will use $\sim$ to denote linear equivalence of divisors.

2. Preliminaries

In this section we recall the definition and some basic facts about surfaces isogenous to a higher product. For details, see [2].

Definition 2.1. A surface S is called isogenous to a higher product if $S = (C \times D)/G$ where C, D are curves with genus at least 2 and G is a finite group acting freely on $C \times D$. When G acts diagonally, S is called of unmixed type.

Remark 2.2. (See [2].) Let S be a surface isogenous to a higher product of unmixed type. Then S is a surface of general type. When $p_g = q = 0$, elementary computations show that $K_S^2 = 8$, $C/G \cong D/G \cong \mathbb{P}^1$ and $|G| = (g_C - 1)(g_D - 1)$ where g_C and g_D denote the genus of C and D, respectively.

When $S = (C \times D)/G$ is a surface isogenous to a higher product of unmixed type with $p_g = q = 0$ and G is an abelian group, Bauer and Catanese proved that G should be isomorphic to one of $(\mathbb{Z}/2)^3$, $(\mathbb{Z}/2)^4$, $(\mathbb{Z}/3)^2$ and $(\mathbb{Z}/5)^2$ in [2]. Recently Shabalin [23], Bauer, Catanese and Frapporti [3,4] computed the first homology groups of these surfaces independently.

Theorem 2.3. *(See [3,4,23].) Let S be a surface isogenous to a higher product $(C \times D)/G$ of unmixed type with $p_g = q = 0$ and assume G to be abelian. Then we have the following:*

(1) $H_1(S, \mathbb{Z}) \cong (\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/4)^2$ *for* $G = (\mathbb{Z}/2)^3$;
(2) $H_1(S, \mathbb{Z}) \cong (\mathbb{Z}/4)^4$ *for* $G = (\mathbb{Z}/2)^4$;
(3) $H_1(S, \mathbb{Z}) \cong (\mathbb{Z}/3)^5$ *for* $G = (\mathbb{Z}/3)^2$;
(4) $H_1(S, \mathbb{Z}) \cong (\mathbb{Z}/5)^3$ *for* $G = (\mathbb{Z}/5)^2$.

Remark 2.4. Let S be a surface with $p_g = q = 0$ isogenous to a higher product $(C \times D)/G$ of unmixed type and let G be abelian. From the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

we get

$$Pic(S) \cong H^2(S, \mathbb{Z}).$$

The above theorem and Noether's formula

$$\chi(\mathcal{O}_X) = 1 = \frac{1}{12}(8 + 2b_0 - 2b_1 + b_2) = \frac{1}{12}(K_S^2 + \chi_{top}(S))$$

imply that these surfaces have $b_2 = 2$.

Finally the above remark and the universal coefficient theorem imply the following:

$$Pic(S) \cong H^2(S, \mathbb{Z}) \cong \mathbb{Z}^2 \oplus H_1(S, \mathbb{Z}).$$

From a result of Kimura [17], we know that Bloch's conjecture holds for S (see also [5]). We can compute the Grothendieck group of S by the following lemma.

Lemma 2.5. *(See [7, Proposition 2.1], [12, Lemma 2.7].) Let S be a surface with $p_g = q = 0$ isogenous to a higher product $(C \times D)/G$ of unmixed type and let G be abelian. Then*

$$K(S) \cong \mathbb{Z}^2 \oplus Pic(S).$$

See [2,9] for more details about the geometry of S.

3. Derived categories of surfaces isogenous to a higher product with $G = (\mathbb{Z}/3)^2$

In Section 3 and Section 4, we consider the derived categories of surfaces isogenous to a higher product of unmixed type with $p_g = q = 0$, $G = (\mathbb{Z}/3)^2$. We recall some basic notions to describe the derived category of algebraic variety.

Definition 3.1. (1) An object E in a $\mathbb{C}$-linear triangulated category $\mathcal{D}$ is called exceptional if

$$Hom_{\mathcal{D}}(E, E[i]) = \begin{cases} \mathbb{C} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

(2) A sequence $E_1, \cdots, E_n$ of exceptional objects of $\mathcal{D}$ is called an exceptional sequence if

$$Hom_{\mathcal{D}}(E_i, E_j[k]) = 0, \forall i > j, \forall k.$$

When S is a surface with $p_g = q = 0$, every line bundle on S is an exceptional object in $D^b(S)$. In this paper we want to prove the following theorem.

Theorem 3.2. *Let $S = (C \times D)/G$ be a surface isogenous to a higher product of unmixed type with $p_g = q = 0$, $G = (\mathbb{Z}/3)^2$. There are exceptional sequences of line bundles of maximal length 4 on S. The orthogonal complements of the admissible subcategories in* *the derived category of S are quasiphantom categories whose Grothendieck groups are isomorphic to* $(\mathbb{Z}/3)^5$.

We will construct exceptional sequences of line bundles of maximal length using G-equivariant line bundles on $C \times D$. For this we study the equivariant geometry of C and D.

3.1. Equivariant geometry of C

From [2], we see that C and D are curves with genus 4. The group G acts on C and let $\pi : C \to \mathbb{P}^1$ be the quotient map. There are 4 branch points on $\mathbb{P}^1$ and 4 orbits on C where the G action has nontrivial stabilizers. This is also true for D. See [2, Section 3] for more details. Let E_1, E_2, E_3, E_4 be the set-theoretic orbits of ramification points of G-action on C.

Let X be a smooth projective variety and let G be a finite group acting on X. There is a well-known exact sequence

$$0 \to \widehat{G} \to Pic^G(X) \to Pic(X)^G \to H^2(G, \mathbb{C}^*),$$

and the last homomorphism is surjective when X is a curve (see [10]).

When G is abelian, Galkin and Shinder proved the following lemma.

Lemma 3.3. *(See [12, Lemma 2.1].) Let G be a finite abelian group. Then the image of $Pic^G(X)$ in $Pic(X)^G$ consists of equivalence classes of G-invariant divisors and there is a short exact sequence*

$$0 \to \widehat{G} \to Pic^G(X) \to Div(X)^G/\sim \to 0,$$

where $\sim$ denotes linear equivalence.

Using the above exact sequences, we analyze the equivariant geometry of C.

Notation 3.4. From now to Section 4, we let $G = (\mathbb{Z}/3)^2$ and $S = (C \times D)/G$ where C and D are curves with genus at least 2 on which G acts such that the diagonal action of G on $C \times D$ is free.

Lemma 3.5.

(1) $Div(C)^G/\sim \cong \mathbb{Z} \oplus \mathbb{Z}/3$.
(2) *There are exactly two G-invariant effective divisors of degree 3 on C which are not linearly equivalent.*

Proof. Let us consider the quotient map $\pi : C \to \mathbb{P}^1$. Let $x_1, \cdots, x_n$ be the branch points of π and $r_1, \cdots, r_n$ be the corresponding ramification indices. Then $Pic^G(C) \cong$

$\mathbb{Z} \oplus \mathbb{Z}/(d_1) \oplus \mathbb{Z}/(d_1/d_2) \oplus \cdots \oplus \mathbb{Z}/(d_{n-1}/d_{n-2})$ where $d_1 = gcd(r_1, \cdots, r_n)$, $d_2 = gcd(r_1r_2, r_1r_3, \cdots, r_{n-1}r_n)$, $\cdots$, $d_{n-1} = gcd(r_1 \cdots r_{n-1}, \cdots, r_2 \cdots r_n)$ (see [10, Equation (2.2)]). Because there are 4 branch points and ramification index is 3 for every branch point, we get $Pic^G(C) \cong \mathbb{Z} \oplus (\mathbb{Z}/3)^3$. Then (1) follows from the short exact sequence in the above lemma. From [2] we may assume that the stabilizer elements of E_1, E_2, E_3, E_4 are $e_1, e_2, -e_1, -e_2$, respectively, where e_1, e_2 are basis of $G = (\mathbb{Z}/3)^2$. Consider $\langle e_1 \rangle$-action on C, and let $\phi : C \to \mathbb{P}^1$ be its quotient map. Then we get $E_2 \sim E_4$ since each of them is a pullback of a point of $\mathbb{P}^1$ via ϕ. Similarly we get $E_1 \sim E_3$. If the four orbits are all linearly equivalent then $Div(C)^G/\!\sim\, \cong \mathbb{Z}$ which contradicts (1). Therefore the four orbits cannot be all linearly equivalent. Since all G-invariant divisors are linear combination of G-orbits, we get (2). □

Lemma 3.6.

$$
\begin{aligned}
h^0(C, \mathcal{O}_C(2E_1 - E_2)) &= 0, \\
h^1(C, \mathcal{O}_C(2E_1 - E_2)) &= 0, \\
h^0(C, \mathcal{O}_C(E_2 - 2E_1)) &= 0, \\
h^1(C, \mathcal{O}_C(E_2 - 2E_1)) &= 6.
\end{aligned}
$$

Proof. From the Riemann–Roch formula we find that

$$h^0(C, \mathcal{O}_C(2E_1 - E_2)) - h^1(C, \mathcal{O}_C(2E_1 - E_2)) = 3 + 1 - 4 = 0.$$

Therefore it suffices to show that $h^0(C, \mathcal{O}_C(2E_1 - E_2)) = 0$. We know that E_1, E_2 are G-invariant divisors on C and hence there is a G-action on $H^0(C, \mathcal{O}_C(2E_1 - E_2))$. If $h^0(C, \mathcal{O}_C(2E_1 - E_2)) \neq 0$, then there is a G-eigensection $f \in H^0(C, \mathcal{O}_C(2E_1 - E_2))$, and $2E_1 - E_2 + (f)$ should be a G-invariant effective divisor of degree 3. Every G-invariant effective divisor of degree 3 on C is linearly equivalent to E_1 or E_2 by the above lemmas. It follows that $2E_1 - E_2 \sim E_1$ or $2E_1 - E_2 \sim E_2$. Then $E_1 - E_2 \sim 0$ or $2E_1 - 2E_2 \sim 0$ which contradicts the assumption that E_1 and E_2 are not linearly equivalent.

Similarly we get

$$h^0(C, \mathcal{O}_C(E_2 - 2E_1)) - h^1(C, \mathcal{O}_C(E_2 - 2E_1)) = -3 + 1 - 4 = -6,$$

and

$$h^0(C, \mathcal{O}_C(E_2 - 2E_1)) = 0$$

because the degree of $\mathcal{O}_C(E_2 - 2E_1)$ is negative. □

Remark 3.7. From [2, Section 3], we see that D is also a curve with genus 4 and there are 4 branch points on $\mathbb{P}^1$ and 4 orbits on D where the G action has nontrivial stabilizers. By the same argument as above, we can find two set-theoretic orbits of ramification points F_1, F_2 on D which are not linearly equivalent. Then we have

$$\begin{aligned} h^0(D, \mathcal{O}_D(2F_1 - F_2)) &= 0, \\ h^1(D, \mathcal{O}_D(2F_1 - F_2)) &= 0, \\ h^0(D, \mathcal{O}_D(F_2 - 2F_1)) &= 0, \\ h^1(D, \mathcal{O}_D(F_2 - 2F_1)) &= 6. \end{aligned}$$

3.2. Exceptional sequences of line bundles on S

Let X be the product of C and D. By abuse of notation, we let $\mathcal{O}_X(E_i)$ (respectively, $\mathcal{O}_X(F_i)$) for $i \in \{1,2\}$ denote the pullback of $\mathcal{O}_C(E_i)$ (respectively, $\mathcal{O}_D(F_i)$). For any character $\chi \in Hom(G, \mathbb{C}^*)$, we can identify equivariant line bundles $\mathcal{O}_X(E_i)(\chi)$ (respectively, $\mathcal{O}_X(F_i)(\chi)$) with line bundles on S.

Theorem 3.8. *For any choice of* 4 *characters* $\chi_1, \chi_2, \chi_3, \chi_4$*,*

$$\mathcal{O}_X(\chi_1), \mathcal{O}_X(E_2 - 2E_1)(\chi_2), \mathcal{O}_X(F_2 - 2F_1)(\chi_3), \mathcal{O}_X(E_2 - 2E_1 + F_2 - 2F_1)(\chi_4)$$

form an exceptional sequence of line bundles of maximal length 4 on S.

Proof. Since $p_g = q = 0$, every line bundle on S is exceptional. From the Künneth formula we find that

$$\begin{aligned} h^j(X, \mathcal{O}_X(2E_1 - E_2)) &= 0, \forall j, \\ h^j(X, \mathcal{O}_X(2F_1 - F_2)) &= 0, \forall j, \\ h^j(X, \mathcal{O}_X(2E_1 - E_2 + 2F_1 - F_2)) &= 0, \forall j, \\ h^j(X, \mathcal{O}_X(-2E_1 + E_2 + 2F_1 - F_2)) &= 0, \forall j. \end{aligned}$$

Therefore the G-invariant parts are also trivial. Hence, we find that $\mathcal{O}_X(\chi_1)$, $\mathcal{O}_X(E_2 - 2E_1)(\chi_2)$, $\mathcal{O}_X(F_2 - 2F_1)(\chi_3)$, $\mathcal{O}_X(E_2 - 2E_1 + F_2 - 2F_1)(\chi_4)$ form an exceptional sequence. Since $K(S) \cong \mathbb{Z}^4 \oplus (\mathbb{Z}/3)^5$, the maximal length of exceptional sequences on S is 4. $\square$

3.3. Deformations of categories generated by exceptional sequences

In this subsection we discuss the deformations of categories generated by exceptional sequences. In order to do this we recall definitions and basic facts about A_∞-algebras. For details, see [15].

Definition 3.9. (See [15].) An A_∞-algebra is a $\mathbb{Z}$-graded vector space

$$A = \bigoplus_{p\in\mathbb{Z}} A^p$$

endowed with graded maps

$$m_n : A^{\otimes n} \to A, n \geq 1,$$

of degree $2 - n$ satisfying

$$\sum (-1)^{r+st} m_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0,$$

where the sum runs over all decompositions $n = r + s + t$.

Definition 3.10. (See [15].) An A_∞-algebra A is called strictly unital if it has an element 1 of degree zero such that $m_1(1) = 0$, $m_2(1, a) = m_2(a, 1) = a$ for all $a \in A$ and for $n \geq 3$, $m_n(a_1, \cdots, a_n) = 0$ if one of $a_i \in \{a_1, \cdots, a_n\} \subset A$ is equal to 1.

We want to prove that the A_∞-algebra of endomorphism of the exceptional sequences constructed above is formal. We follows the arguments in [1,7,12].

Proposition 3.11. *Let $T = \mathcal{O}_X(\chi_1) \oplus \mathcal{O}_X(E_2 - 2E_1)(\chi_2) \oplus \mathcal{O}_X(F_2 - 2F_1)(\chi_3) \oplus \mathcal{O}_X(E_2 - 2E_1 + F_2 - 2F_1)(\chi_4)$ be the direct sum of line bundles forming an exceptional collection and let $B = RHom(T, T)$ be the DG-algebra of endomorphisms. Then B is formal, i.e. $H^*(B)$ can be chosen to be a graded algebra.*

Proof. Let $H^*(B)$ be the minimal model of B. By [22, Lemma 2.1], we may assume that the $H^*(B)$ is strictly unital. We want to show that $H^*(B)$ does not have nontrivial m_n for $n \geq 3$. Consider $m_n(b_1, \cdots, b_n)$ for $n \geq 3$. From the Künneth formula we have the following:

$$H^k(X, \mathcal{O}_X(-2E_1 + E_2 + 2F_1 - F_2)) = 0, \quad \forall k \in \mathbb{Z},$$
$$H^k(X, \mathcal{O}_X(2E_1 - E_2 - 2F_1 + F_2)) = 0, \quad \forall k \in \mathbb{Z}.$$

Therefore we see that there is no morphism between $\mathcal{O}_X(E_2 - 2E_1)(\chi_2)$ and $\mathcal{O}_X(F_2 - 2F_1)(\chi_3)$. From this we see that if every b_i is nonzero then at least one b_i should be multiple of 1. Because we assume that $H^*(B)$ is strictly unital, $m_n(b_1, \cdots, b_n) = 0$, for all $n \geq 3$. □

In [1], Alexeev and Orlov asked the following question.

Question 3.12. (See [1].) Is true that for any exceptional collection of maximal length on a smooth projective surface S with ample K_S and with $p_g = q = 0$, the DG algebra of endomorphisms of the exceptional collection does not change under small deformations of the complex structure on X?

They constructed exceptional sequences of maximal length for primary Burniat surfaces and proved that the above property holds for their exceptional sequences in [1]. This phenomenon was observed for some exceptional sequences on other surfaces of general type (see [6–8]). We prove that for certain exceptional sequence we constructed the above question is true.

Proposition 3.13. *There is a choice of four characters $\chi_i \in \widehat{G}, i \in \{1,2,3,4\}$ such that the DG algebra of endomorphisms of $T = \mathcal{O}_X(\chi_1) \oplus \mathcal{O}_X(E_2 - 2E_1)(\chi_2) \oplus \mathcal{O}_X(F_2 - 2F_1)(\chi_3) \oplus \mathcal{O}_X(E_2 - 2E_1 + F_2 - 2F_1)(\chi_4)$ does not change under small deformations of the complex structure of S.*

Proof. From the Riemann–Roch theorem for curves and Künneth formula we have the following:

$$H^k(X, \mathcal{O}_X(E_2 - 2E_1)) = \begin{cases} \mathbb{C}^6 & \text{if } k = 1, \\ \mathbb{C}^{24} & \text{if } k = 2, \\ 0 & \text{otherwise.} \end{cases}$$

$$H^k(X, \mathcal{O}_X(F_2 - 2F_1)) = \begin{cases} \mathbb{C}^6 & \text{if } k = 1, \\ \mathbb{C}^{24} & \text{if } k = 2, \\ 0 & \text{otherwise.} \end{cases}$$

$$H^k(X, \mathcal{O}_X(E_2 - 2E_1 + F_2 - 2F_1)) = \begin{cases} \mathbb{C}^{36} & \text{if } k = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Then there are $\chi, \chi' \in \widehat{G}$ such that $H^1(X, \mathcal{O}_X(E_2 - 2E_1)(\chi))^G = 0$ and $H^1(X, \mathcal{O}_X(F_2 - 2F_1)(\chi'))^G = 0$.

From the Riemann–Roch theorem for surfaces we get for any character $\chi'' \in \widehat{G}$,

$$\begin{aligned} &\chi(\mathcal{O}_X(E_2 - 2E_1 + F_2 - 2F_1)(\chi'')) \\ &= \chi(\mathcal{O}_S) + \frac{1}{2}\mathcal{O}_X(E_2 - 2E_1 + F_2 - 2F_1)(\chi'') \\ &\quad \cdot \mathcal{O}_X(E_2 - 2E_1 + F_2 - 2F_1 - K_C - K_D)(\chi'') \\ &= 1 + \frac{1}{18}(3 \cdot 9 + 3 \cdot 9) = 4. \end{aligned}$$

Then we get the following equalities by Künneth formula,

$$H^k(X, \mathcal{O}_X(E_2 - 2E_1 + F_2 - 2F_1)(\chi''))^G = \begin{cases} \mathbb{C}^4 & \text{if } k = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore there is a choice of four characters $\chi_i \in \widehat{G}$, $i \in \{1,2,3,4\}$ such that the minimal model of the DG algebra of endomorphisms of $T = \mathcal{O}_X(\chi_1) \oplus \mathcal{O}_X(E_2 - 2E_1)(\chi_2) \oplus \mathcal{O}_X(F_2 - 2F_1)(\chi_3) \oplus \mathcal{O}_X(E_2 - 2E_1 + F_2 - 2F_1)(\chi_4)$ has only terms in degree 0 and 2. The multiplication of two elements of degree 2 is 0 since there is no Ext^4 between objects. Hence the structure of the DG-algebra is completely determined in this case. Finally we get the desired result by the semicontinuity. □

We do not know whether the DG algebra of endomorphism of $T = \mathcal{O}_X(\chi_1) \oplus \mathcal{O}_X(E_2 - 2E_1)(\chi_2) \oplus \mathcal{O}_X(F_2 - 2F_1)(\chi_3) \oplus \mathcal{O}_X(E_2 - 2E_1 + F_2 - 2F_1)(\chi_4)$ does not change under small deformations of complex structure of S for every choice of four characters $\chi_i \in \widehat{G}$, $i \in \{1,2,3,4\}$.

4. Quasiphantom categories and phantom categories

In this section we consider Hochschild homology groups and cohomology groups of the orthogonal complements of the categories generated by exceptional sequences.

4.1. Hochschild homology and cohomology

We recall the definition and some basic facts about Hochschild homology and cohomology of a smooth projective variety. For details about Hochchild homology and cohomology, see [18].

Definition 4.1. (See [18].) Let S be a smooth projective variety. The Hochschild homology and cohomology of S are defined by

$$HH_*(S) = Hom^*(S \times S, \Delta_*\mathcal{O}_S \otimes \Delta_*\mathcal{O}_S),$$
$$HH^*(S) = Hom^*_{S\times S}(\Delta_*\mathcal{O}_S, \Delta_*\mathcal{O}_S).$$

Hochschild homology and cohomology of a smooth projective variety can be computed using the following theorem.

Theorem 4.2 *(Hochschild–Kostant–Rosenberg isomorphisms). (See [18, Theorem 8.3].) Let S be a smooth projective variety of dimension n. Then*

$$HH_t(S) \cong \bigoplus_{p=0}^{n} H^{t+p}(S, \Omega_S^p),$$

$$HH^t(S) \cong \bigoplus_{p=0}^{n} H^{t-p}(S, \wedge^p \mathrm{T}_S).$$

Let S be a smooth projective variety. Kuznetsov furthermore defined Hochschild homology and cohomology for any admissible subcategory $\mathcal{A} \subset D^b(S)$ in [18].

Definition 4.3. (See [18, Definition 4.4].) Let S be a smooth projective variety, and $\mathcal{A} \subset D^b(S)$ be an admissible subcategory. Let $\mathcal{E}_{\mathcal{A}}$ be a strong generator of $\mathcal{A}$ and $\mathcal{C}_{\mathcal{A}} = RHom^*(\mathcal{E}_{\mathcal{A}}, \mathcal{E}_{\mathcal{A}})$. Then the Hochschild homology and cohomology of $\mathcal{A}$ are defined as follows:

$$HH_*(\mathcal{A}) := \mathcal{C}_{\mathcal{A}} \otimes^L_{\mathcal{C}_{\mathcal{A}} \otimes \mathcal{C}_{\mathcal{A}}^{opp}} \mathcal{C}_{\mathcal{A}},$$
$$HH^*(\mathcal{A}) := RHom_{\mathcal{C}_{\mathcal{A}} \otimes \mathcal{C}_{\mathcal{A}}^{opp}}(\mathcal{C}_{\mathcal{A}}, \mathcal{C}_{\mathcal{A}}).$$

Then Kuznetsov proved the additivity of Hochschild homology with respect to the semiorthogonal decomposition in [18] which is a main tool to compute the Hochschild homology of the orthogonal complement of an admissible subcategory.

Theorem 4.4. *(See [18, Corollary 7.5, Corollary 8.4].)*

(1) *For any semiorthogonal decomposition $D^b(S) = \langle \mathcal{A}_1, \cdots, \mathcal{A}_n \rangle$, there is an isomorphism*

$$HH_*(S) \cong HH_*(\mathcal{A}_1) \oplus \cdots \oplus HH_*(\mathcal{A}_n).$$

(2) *If E is an exceptional object in $D^b(S)$, then $HH_*(\langle E \rangle) \cong HH^*(\langle E \rangle) \cong \mathbb{C}$.*

In the rest of this subsection we compute the Hochschild cohomology groups of the orthogonal complements of the exceptional sequences. The method to compute Hochschild cohomology groups was developed by Kuznetsov in [19]. He introduced the notion of the height to understand the restriction morphisms $HH^*(X) \to HH^*(\mathcal{A})$ and he proved that it is easy to determine the height when the exceptional sequence satisfies some conditions.

Definition 4.5. (See [19].) Let $E_1, \cdots, E_n$ be an exceptional sequence on S.

(1) $E_1, \cdots, E_n, E_1 \otimes \omega_S^{-1}, \cdots, E_n \otimes \omega_S^{-1}$ is called Hom-free if $\mathrm{Ext}^k(E_i, E_j)$ for $k \leq 0$ and all $i < j \leq i + n$.
(2) A Hom-free sequence $E_1, \cdots, E_n, E_1 \otimes \omega_S^{-1}, \cdots, E_n \otimes \omega_S^{-1}$ is cyclically Ext^1-connected if there is a chain $1 \leq a_0 < a_1 < \cdots < a_p \leq n$ such that $\mathrm{Ext}^1(E_{a_s}, E_{a_{s+1}}) \neq 0$, for $s = 0, 1, \cdots, p-1$ and $\mathrm{Ext}^1(E_{a_p}, E_{a_0} \otimes \omega^{-1}) \neq 0$.

Now we compute the Hochschild cohomology groups of the orthogonal complements of our exceptional sequences.

Proposition 4.6. *Let $\mathcal{A}$ be the orthogonal complement of the exceptional collection $\mathcal{O}_X(\chi_1), \mathcal{O}_X(E_2 - 2E_1)(\chi_2), \mathcal{O}_X(F_2 - 2F_1)(\chi_3), \mathcal{O}_X(E_2 - 2E_1 + F_2 - 2F_1)(\chi_4)$. Then we have $HH^i(S) = HH^i(\mathcal{A})$, for $i = 0, 1, 2$, and $HH^3(S) \subset HH^3(\mathcal{A})$.*

Proof. It is enough to show that the exceptional sequence is Hom-free and not cyclically Ext^1-connected (see [19] for more details). From the Künneth formula and degree computation we find that $\mathcal{O}_X(\chi_1), \mathcal{O}_X(E_2-2E_1)(\chi_2), \mathcal{O}_X(F_2-2F_1)(\chi_3), \mathcal{O}_X(E_2-2E_1+F_2-2F_1)(\chi_4), \mathcal{O}_X(\chi_1)\otimes\omega_S^{-1}, \mathcal{O}_X(E_2-2E_1)(\chi_2)\otimes\omega_S^{-1}, \mathcal{O}_X(F_2-2F_1)(\chi_3)\otimes\omega_S^{-1}, \mathcal{O}_X(E_2-2E_1+F_2-2F_1)(\chi_4)\otimes\omega_S^{-1}$ is Hom-free.

Let us compute $\mathrm{Ext}^1(\mathcal{O}_X(E_2-2E_1+F_2-2F_1)(\chi_4), \mathcal{O}_X(\chi_1)\otimes\omega_S^{-1})$. By the Serre duality we have the following isomorphisms

$$\begin{aligned}&\mathrm{Ext}^1(\mathcal{O}_X(E_2-2E_1+F_2-2F_1)(\chi_4), \mathcal{O}_X(\chi_1)\otimes\omega_S^{-1})\\ &\quad\cong \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X(E_2-2E_1+F_2-2F_1)(\chi_4-\chi_1)\otimes\omega_S^2)^*\\ &\quad\cong H^1(S, \mathcal{O}_X(E_2-2E_1+F_2-2F_1)(\chi_4-\chi_1)\otimes\omega_S^2)^*.\end{aligned}$$

From Kleiman's criterion we see that $\mathcal{O}_X(E_2-2E_1+F_2-2F_1)(\chi_4-\chi_1)\otimes\omega_S$ is an ample line bundle and from Kodaira vanishing theorem we have

$$H^1(S, \mathcal{O}_X(E_2-2E_1+F_2-2F_1)(\chi_4-\chi_1)\otimes\omega_S^2) = 0.$$

Similarly the above Hom-free sequence cannot be cyclically Ext^1-connected by Serre duality and Kodaira vanishing theorem. Then the height of the exceptional collection $\mathcal{O}_X(\chi_1), \mathcal{O}_X(E_2-2E_1)(\chi_2)$, $\mathcal{O}_X(F_2-2F_1)(\chi_3), \mathcal{O}_X(E_2-2E_1+F_2-2F_1)(\chi_4)$ is 4. Therefore we get the desired result by [19, Theorem 4.5]. □

4.2. Quasiphantom categories and phantom categories

We recall the definitions of quasiphantom and phantom category.

Definition 4.7. (See [14, Definition 1.8].) Let S be a smooth projective variety. Let $\mathcal{A}$ be an admissible triangulated subcategory of $D^b(S)$. Then $\mathcal{A}$ is called a quasiphantom category if the Hochschild homology of $\mathcal{A}$ vanishes, and the Grothendieck group of $\mathcal{A}$ is finite. If the Grothendieck group of $\mathcal{A}$ also vanishes, then $\mathcal{A}$ is called a phantom category.

We now prove the second part of our main theorem.

Proposition 4.8. *Let $\mathcal{A}$ be the left orthogonal complement of the admissible category generated by $\mathcal{O}_X(\chi_1)$, $\mathcal{O}_X(E_2-2E_1)(\chi_2)$, $\mathcal{O}_X(F_2-2F_1)(\chi_3)$, $\mathcal{O}_X(E_2-2E_1+F_2-2F_1)(\chi_4)$. Then we have $K(\mathcal{A}) = (\mathbb{Z}/3)^5$ and $HH_*(\mathcal{A}) = 0$. Therefore $\mathcal{A}$ is a quasiphantom category.*

Proof. This follows from the additivity of Grothendieck groups and Hochschild homology groups with respect to semiorthogonal decomposition. □

Gorchinskiy and Orlov constructed phantom categories using the quasiphantom categories constructed in [1,7,12]. From the same construction, we can provide more examples of phantom categories.

Remark 4.9. Quasiphantom categories of surfaces isogenous to a higher product with $p_g = q = 0$, $G = (\mathbb{Z}/3)^2$ and quasiphantom categories constructed in [1,7,12] can be used to provide more examples of phantom categories by a result of Gorchinskiy and Orlov [14, Theorem 1.12].

5. Discussions

The construction of exceptional sequences of line bundles on $S = (C \times D)/G$ of this paper does not extend to the cases where $G = (\mathbb{Z}/2)^3$, $(\mathbb{Z}/2)^4$, $(\mathbb{Z}/5)^2$. For the $G = (\mathbb{Z}/5)^2$ case, $Div(C)^G/\sim \cong \mathbb{Z}$ [12, Lemma 2.3]. (However there are exceptional sequences of line bundles of maximal length due to the construction of Galkin and Shinder.) For the $G = (\mathbb{Z}/2)^3$, $G = (\mathbb{Z}/2)^4$ cases, the following propositions imply that if there is an exceptional sequence of line bundles of maximal length 4 on S we need another construction to find it.

Proposition 5.1. *Let $G = (\mathbb{Z}/2)^3$, $X := C \times D$ and $S := (C \times D)/G$ be a surface isogenous to a higher product with $p_g = q = 0$ of unmixed type. Then C is a curve of genus 3 and D is a curve of genus 5 (see [2]). Let E_1, E_2 be 2 linear combinations of set-theoretic orbits on C, and F_1, F_2 be 2 linear combinations of set-theoretic orbits on D. By abuse of notation, we let $\mathcal{O}_X(E_i)$ (respectively, $\mathcal{O}_X(F_i)$) denote the pullback of $\mathcal{O}_C(E_i)$ (respectively, $\mathcal{O}_D(F_i)$) for $i \in \{1, 2\}$. For any choice of characters $\chi_1, \chi_2 \in Hom(G, \mathbb{C}^*)$, we can identify the equivariant line bundles $\mathcal{O}_X(E_i + F_i)(\chi_i)$ on X with line bundles on S for $i \in \{1, 2\}$. Then*

$$\mathcal{O}_X, \mathcal{O}_X(E_1 + F_1)(\chi_1), \mathcal{O}_X(E_2 + F_2)(\chi_2)$$

cannot be an exceptional sequence on S.

Proof. Suppose that $\mathcal{O}_X, \mathcal{O}_X(E_1+F_1)(\chi_1), \mathcal{O}_X(E_2+F_2)(\chi_2)$ is an exceptional sequence of line bundles on S. Then we get $\chi(\mathcal{O}_X(E_i + F_i)(\chi_i), \mathcal{O}_X) = 0$ for $i \in \{1, 2\}$, and $\chi(\mathcal{O}_X(E_2 + F_2)(\chi_2), \mathcal{O}_X(E_1 + F_1)(\chi_1)) = 0$. From the Riemann–Roch formula we get

$$\chi(\mathcal{O}_X(-E_i - F_i)(\chi_i^{-1})) = \chi(\mathcal{O}_S) + \frac{1}{16}\mathcal{O}(-E_i - F_i) \cdot \mathcal{O}(-E_i - F_i + K_C + K_D)$$
$$1 + \frac{1}{16}(-e_i(-f_i + 8) - f_i(-e_i + 4)) = \frac{1}{8}(e_i - 2)(f_i - 4) = 0,$$

where $e_i = \deg E_i$, $f_i = \deg F_i$, $i \in \{1, 2\}$. Because the stabilizer subgroup of a point of a finite group action on a Riemann surface is finite cyclic group, degrees of all set-theoretic orbits on C and D are multiples of 4. Therefore we get $f_1 = f_2 = 4$. However then we have

$$\begin{aligned}
&\chi(\mathcal{O}_X(E_1 + F_1 - E_2 - F_2)(\chi_1 + \chi_2^{-1})) \\
&\quad = \chi(\mathcal{O}_S) + \frac{1}{16}\mathcal{O}(E_1 + F_1 - E_2 - F_2) \cdot \mathcal{O}(E_1 + F_1 - E_2 - F_2 + K_C + K_D) \\
&\quad = 1 + \frac{1}{16}((e_1 - e_2)(f_1 - f_2 + 8) + (f_1 - f_2)(e_1 - e_2 + 4)) \\
&\quad = \frac{1}{8}(e_2 - e_1 - 2)(f_2 - f_1 - 4) \neq 0.
\end{aligned}$$

Therefore $\mathcal{O}_X, \mathcal{O}_X(E_1 + F_1)(\chi_1), \mathcal{O}_X(E_2 + F_2)(\chi_2)$ cannot be an exceptional sequence on S. $\square$

Proposition 5.2. *Let $G = (\mathbb{Z}/2)^4$, $X := C \times D$ and $S := (C \times D)/G$ be a surface isogenous to a higher product with $p_g = q = 0$ of unmixed type. Then C and D are curves of genus 5 (see [2]). Let E be a linear combination of set-theoretic orbits on C, F be a linear combination of set-theoretic orbits on D. By abuse of notation, we let $\mathcal{O}_X(E)$ (respectively, $\mathcal{O}_X(F)$) denote the pullback of $\mathcal{O}_C(E)$ (respectively, $\mathcal{O}_D(F)$). For any character $\chi \in Hom(G, \mathbb{C}^*)$, we can identify the equivariant line bundles $\mathcal{O}_X(E + F)(\chi)$ on X with a line bundles on S. Then*

$$\mathcal{O}_X, \mathcal{O}_X(E + F)(\chi)$$

cannot be an exceptional sequence on S.

Proof. If $\mathcal{O}_X, \mathcal{O}_X(E+F)(\chi)$ is an exceptional sequence of line bundles, then $\chi(\mathcal{O}_X(E+F)(\chi), \mathcal{O}_X) = 0$. From the Riemann–Roch formula we get

$$\chi(\mathcal{O}_X(-E - F)(\chi^{-1})) = \chi(\mathcal{O}_S) + \frac{1}{32}\mathcal{O}(-E - F) \cdot \mathcal{O}(-E - F + K_C + K_D)$$
$$1 + \frac{1}{32}(-e(-f + 8) - f(-e + 8)) = \frac{1}{16}(e - 4)(f - 4),$$

where $e = \deg E$, $f = \deg F$. Because the stabilizer subgroup of a point of a finite group action on a Riemann surface is finite cyclic group, degrees of all set-theoretic orbits on C and D are multiples of 8. Therefore $\chi(\mathcal{O}_X(E + F)(\chi), \mathcal{O}_X) \neq 0$ and $\mathcal{O}_X, \mathcal{O}_X(E + F)(\chi)$ cannot be an exceptional sequence on S. $\square$

We show that there exist exceptional sequences of line bundles of maximal length 4 on surfaces isogenous to a higher product of unmixed type with $p_g = q = 0$, $G = (\mathbb{Z}/2)^3$ or $G = (\mathbb{Z}/2)^4$ or some nonabelian G via different method in [20,21].

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