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On the asymptotic sequences over an ideal ☆

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ABSTRACT

Let I be an ideal of a Noetherian ring R. Ratliff has shown that very many known results concerning R-sequences and classical grade have a valid analogue for asymptotic sequences and asymptotic grade. As a generalization of the concepts of asymptotic sequences and asymptotic grade, we introduce the concepts of asymptotic I-sequences and asymptotic I-grade. It is shown that they play a role analogous to asymptotic sequences and asymptotic grade.
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1. Introduction

In [16], Rees introduced the important concept of an asymptotic sequence over an ideal I in a Noetherian ring R. Such sequences have been used in [1,15,16] to help proving results concerning the Rees-good basis, I-independent elements and analytic spread.

In [13], Ratliff proved that all asymptotic sequences in R (that is, asymptotic sequences over $(0)R$) which are maximal with respect to being contained in I have the same length and are denoted $\mathrm{agd}(I)$. He showed that asymptotic sequences in R have many basic properties of regular sequences. Later, in [3], Katz showed that any two maximal asymptotic sequences over I in a Noetherian local ring R have the same length and

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are denoted $\operatorname{acogd}(I)$, and in [9,14], Ratliff and McAdam gave another results concerning asymptotic sequences over an ideal and asymptotic cograde.

Let I and J be ideals of a Noetherian ring R. In this paper, we show that all maximal asymptotic sequences over I coming from J (say, *asymptotic I-sequence from J* (see Definition 2.3)) have the same length and are denoted $\operatorname{agd}_I(J)$. By this fact, we introduce the concept of asymptotic I-grade. Using this concept we give more knowledge concerning asymptotic sequences over an ideal.

Section 2 contains several results showing that asymptotic I-sequences from J in a Noetherian ring R behave nicely with respect to passing to certain rings related to R. In Section 3 it is shown that asymptotic I-grade of J is well defined and likewise behaves nicely when passing to the same type of related rings. Also, it is shown that, if $\mathbf{x} = x_1, \dots, x_n$ is a sequence of elements in the Jacobson radical of R, then $\mathbf{x}$ is an asymptotic sequence over I if and only if $\operatorname{agd}_I((\mathbf{x})R) = n$, and some consequences of this result are given. Before proving this result, however, it is first shown that if b be an element in the Jacobson radical of R, then

$$\operatorname{agd}_I(J) \leq \operatorname{agd}_I((J,(b))R) \leq \operatorname{agd}_I(J) + 1.$$

In Section 4 we give four upper bounds and two lower bounds on $\operatorname{agd}_I(J)$. In Section 5 several useful characterizations are given for $\operatorname{agd}_I(J)$ to be equal to height $J(R/I)$ and define *strongly quasi-I-unmixed* rings (see Definition 5.1). Then we give some consequences of these characterizations. For instance, in [9], it was shown that if $x_1, \dots, x_n$ is an asymptotic sequence over I, then $x_1 + I, \dots, x_n + I$ is an asymptotic sequence in R/I. A natural question to ask is when the converse holds. In this section we prove the converse holds if R is locally strongly quasi-I-unmixed. Finally, it is shown that if $\mathfrak{p} \in V(I)$ and $R_\mathfrak{p}$ is strongly quasi-$IR_\mathfrak{p}$-unmixed, then $R_\mathfrak{q}$ is strongly quasi-$IR_\mathfrak{q}$-unmixed for all but finitely many elements of $\{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{p} \subset \mathfrak{q},\ \operatorname{height} \mathfrak{q}/\mathfrak{p} = 1\}$.

All rings in this paper are commutative with a unit $1 \neq 0$. For any R-module M we shall use $\operatorname{mAss}_R M$ to denote the set of minimal elements of $\operatorname{Ass}_R M$. If $(R, \mathfrak{m})$ is a Noetherian local ring, then R^* denotes the completion of R with respect to the $\mathfrak{m}$-adic topology. For any ideal I of R, the radical of I, denoted by $\sqrt{I}$, is defined to be the set $\{x \in R \mid x^n \in I \text{ for some } n \in \mathbb{N}\}$. Moreover, we use $V(I)$ to denote the set of prime ideals of R containing I. Further, the integral closure of I is denoted by $\bar{I}$, so $\bar{I} := \{x \in R \mid x \text{ satisfies an equation of the form } x^n + c_1x^{n-1} + \ldots + c_n = 0, \text{ where } c_i \in I^i \text{ for } i = 1, \dots, n\}$. Finally, we denote by $\mathscr{R}$ the graded Rees ring $R[u, It] := \oplus_{n\in\mathbb{Z}} I^n t^n$ of R with respect to I, where t is an indeterminate and $u = t^{-1}$. For any unexplained notation or terminology we refer the reader to [5,17].

2. Asymptotic *I*-sequences

In this section we give a number of elementary properties of asymptotic I-sequences. We begin with definitions.

Definition 2.1. Let $I \subseteq \mathfrak{p}$ be ideals of a Noetherian ring R such that $\mathfrak{p}$ is prime.

(2.1.1) $\mathfrak{p}$ is called a *quintessential* (resp., *quintasymptotic*) prime ideal of I precisely when there exists $z \in \mathrm{Ass}_{R^*_{\mathfrak{p}}} R^*_{\mathfrak{p}}$ (resp., $z \in \mathrm{mAss}_{R^*_{\mathfrak{p}}} R^*_{\mathfrak{p}}$) such that $IR^*_{\mathfrak{p}}+z$ is $\mathfrak{p}R^*_{\mathfrak{p}}$-primary. The set of *quintessential* (resp., *quintasymptotic*) primes of I is denoted by $Q(I)$ (resp., $\overline{Q^*}(I)$).

(2.1.2) Let $\mathscr{R} = \mathscr{R}(I,R)$ be the Rees ring of R with respect to I, then $\mathfrak{p}$ is *asymptotic* prime of I precisely when there exists $\mathfrak{q} \in \overline{Q^*}(u\mathscr{R})$ such that $\mathfrak{p} = \mathfrak{q} \cap R$. The set of *asymptotic* primes of I is denoted by $\overline{A^*}(I)$.

The following lemma lists the known basic facts concerning $\overline{A^*}(I)$ which will be needed below.

Lemma 2.2. *Let I and J be ideals in a Noetherian ring R. Then the following hold:*

(i) $\overline{A^*}(I) = \bigcup_{n\in\mathbb{N}} \mathrm{Ass}_R(R/\overline{I^n}) = \mathrm{Ass}_R(R/\overline{I^n})$ *for all large n, and it contains all minimal prime divisors of I.*

(ii) *If $I \subseteq \mathfrak{p} \in \mathrm{Spec}\, R$ and S is a multiplicatively closed subset of R which is disjoint from $\mathfrak{p}$, then $\mathfrak{p} \in \overline{A^*}(I)$ if and only if $\mathfrak{p}R_S \in \overline{A^*}(IR_S)$.*

(iii) $\mathfrak{p} \in \overline{A^*}(I)$ *if and only if there is $z \in \mathrm{mAss}_R R$ such that $z \subseteq \mathfrak{p}$ and $\mathfrak{p}/z \in \overline{A^*}(I(R/z))$.*

(iv) *Let the ring T be an integral Noetherian extension of R. If $\mathfrak{p} \in \overline{A^*}(I)$, then there is a $\mathfrak{q} \in \overline{A^*}(IT)$ such that $\mathfrak{q} \cap R = \mathfrak{p}$. Moreover, if every minimal prime in T lies over a minimal prime in R, then the converse holds.*

(v) *Let the ring T be a flat Noetherian extension of R. Let $\mathfrak{q}$ be a prime ideal in T such that $\mathfrak{p} = \mathfrak{q} \cap R$. If $\mathfrak{q} \in \overline{A^*}(IT)$, then $\mathfrak{p} \in \overline{A^*}(I)$. Moreover, if $\mathfrak{p} \in \overline{A^*}(I)$ and $\mathfrak{q}$ is minimal over $\mathfrak{p}T$, then $\mathfrak{q} \in \overline{A^*}(IT)$.*

(vi) $\overline{A^*}((I,X)R[X]) = \{(\mathfrak{p},X)R[X] \mid \mathfrak{p} \in \overline{A^*}(I)\}$.

Proof. (i) follows from [6, Lemmas 0.1]. Since localization commutes with integral closure, then (ii) holds by (i). (iii)–(v) are proved in [7, Propositions 3.18((i) $\Leftrightarrow$ (iv)) and 3.22] and [12, Theorems 6.5 and 6.8]. To prove (vi), let $\mathfrak{p} \in \overline{A^*}(I)$. Then by (ii), (iii) and (v), we may assume that R is a complete local domain with maximal ideal $\mathfrak{p}$. Then, in view of [10, Proposition 7], $\mathscr{R}(I,R)$ and $\mathscr{R}((I,X)R[X],R[X])$ are locally unmixed. Thus $\overline{Q^*}(J) = Q(J)$ for all ideals J in $\mathscr{R}(I,R)$ and $\mathscr{R}((I,X)R[X],R[X])$. Therefore $(\mathfrak{p},X)R[X] \in \overline{A^*}((I,X)R[X])$, by [4, Theorem 2.5(v)]. The other inclusion is similar. $\square$

Definition 2.3. Let I and J be ideals of a Noetherian ring R.

(2.3.1) A sequence $\mathbf{x} = x_1,\dots,x_n$ of elements of J is called an *asymptotic I-sequence from J* if, $(I,(\mathbf{x}))R \neq R$ and for all $1 \le i \le n$, we have $x_i \notin \bigcup\{\mathfrak{p} \in \overline{A^*}((I,(x_1,\dots,x_{i-1}))R)\}$.

(2.3.2) An asymptotic I-sequence $\mathbf{x} = x_1,\dots,x_n$ from J is *maximal* if $x_1,\dots,x_n,x_{n+1}$ is not an asymptotic I-sequence from J for any $x_{n+1} \in J$.

(2.3.3) The *asymptotic I-grade* of J, denoted $\mathrm{agd}_I(J)$ is the length of a maximal asymptotic I-sequence from J.

The following proposition shows that the asymptotic I-sequences behave well under passing to localization.

Proposition 2.4. *Let I and J be ideals in a Noetherian ring R and let $\mathbf{x} = x_1, \dots, x_n$ be a sequence of elements of J. Let S be a multiplicatively closed subset of R such that $(I + J)R_S \neq R_S$. Then the following statements hold:*

(i) *If $\mathbf{x}$ is an asymptotic I-sequence from J, then the image of $\mathbf{x}$ in R_S is an asymptotic IR_S-sequence from JR_S. The converse holds if for all $\mathfrak{p} \in \bigcup\{\mathfrak{q} \in \overline{A^*}((I, (x_1, \dots, x_i))R); i = 0, \dots, n-1\}$, we have $\mathfrak{p}R_S \neq R_S$.*

(ii) *If for all $\mathfrak{p} \in \bigcup\{\mathfrak{q} \in \overline{A^*}((I, (x_1, \dots, x_i))R); i = 0, \dots, n\}$, we have $\mathfrak{p}R_S \neq R_S$, then $\mathbf{x}$ is a maximal asymptotic I-sequence from J if and only if the image of $\mathbf{x}$ in R_S is a maximal asymptotic IR_S-sequence from JR_S.*

Proof. It follows readily from Lemma 2.2(ii). $\square$

The following result shows that the asymptotic I-sequences behave well under passing to the factor rings modulo minimal primes of R.

Proposition 2.5. *Let I and J be ideals in a Noetherian ring R and let $\mathbf{x} = x_1, \dots, x_n$ be a sequence of elements of J. Then the following statements hold:*

(i) *$\mathbf{x}$ is an asymptotic I-sequence from J if and only if the image of $\mathbf{x}$ in R/z is an asymptotic $I(R/z)$-sequence from $J(R/z)$ for all $z \in \operatorname{mAss}_R R$.*

(ii) *$\mathbf{x}$ is a maximal asymptotic I-sequence from J if and only if the image of $\mathbf{x}$ in R/z is an asymptotic $I(R/z)$-sequence from $J(R/z)$ for all $z \in \operatorname{mAss}_R R$ and there exists $w \in \operatorname{mAss}_R R$ such that the image of $\mathbf{x}$ in R/w is a maximal asymptotic $I(R/w)$-sequence from $J(R/w)$.*

Proof. It follows readily from Lemma 2.2(iii). $\square$

Corollary 2.6. *Let I and J be ideals of a Noetherian ring R and let $\mathbf{x} = x_1, \dots, x_n$ be a sequence of elements of J. Let K be an ideal of R contained $\operatorname{Rad} R$. Then $\mathbf{x}$ is a/an (resp., maximal) asymptotic I-sequence from J if and only if $\mathbf{x}$ is a/an (resp., maximal) asymptotic $I(R/K)$-sequence from $J(R/K)$.*

Proof. This follows easily from Proposition 2.5, since there is a one-to-one correspondence between the minimal prime ideals in R and in R/K. $\square$

The next result shows that the asymptotic I-sequences behave well under passing to finite integral extension rings of R.

Proposition 2.7. *Let $R \subseteq T$ be an integral extension of Noetherian rings. Let I and J be ideals of R and let $\mathbf{x} = x_1, \dots, x_n$ be a sequence of elements of J. Then the following statements hold:*

(i) *If* $\mathbf{x}$ *is an asymptotic* IT*-sequence from* JT*, then* $\mathbf{x}$ *is an asymptotic* I*-sequence from* J*.*

(ii) *If every minimal prime of* T *lies over a minimal prime in* R*, then* $\mathbf{x}$ *is a/an (resp., maximal) asymptotic* I*-sequence from* J *if and only if* $\mathbf{x}$ *is a/an (resp., maximal) asymptotic* IT*-sequence from* JT*.*

Proof. It follows readily from Lemma 2.2(iv). □

The next result shows that the asymptotic I-sequences behave well under passing to flat Noetherian extension rings of R.

Proposition 2.8. *Let* $R \subseteq T$ *be a flat extension of Noetherian rings. Let* I *and* J *be ideals of* R *and let* $\mathbf{x} = x_1, \ldots, x_n$ *be a sequence of elements of* J*. Then* $\mathbf{x}$ *is a/an (resp., maximal) asymptotic* I*-sequence from* J *if and only if* $\mathbf{x}$ *is a/an (resp., maximal) asymptotic* IT*-sequence from* JT*.*

Proof. It follows readily from Lemma 2.2(v). □

The next proposition concerns the asymptotic I-sequences and asymptotic $IR[X]$-sequences, where X is an indeterminate over R.

Proposition 2.9. *Let* I *and* J *be ideals in a Noetherian ring* R *and let* $\mathbf{x} = x_1, \ldots, x_n$ *be a sequence of elements of* J*. Then the following statements are equivalent:*

(i) *The sequence* $\mathbf{x}$ *is a/an (resp., maximal) asymptotic* I*-sequence from* J*.*

(ii) *The sequence* $x_1, \ldots, x_i, X, x_{i+1}, \ldots, x_n$ *is a/an (resp., maximal) asymptotic* $IR[X]$*-sequence from* $(J, X)R[X]$ *for some* $i = 0, 1, \ldots, n$*.*

(iii) *The sequence* $x_1, \ldots, x_i, X, x_{i+1}, \ldots, x_n$ *is a/an (resp., maximal) asymptotic* $IR[X]$*-sequence from* $(J, X)R[X]$ *for every* $i = 0, 1, \ldots, n$*.*

Proof. In view of Lemma 2.2(v), for $j = 0, 1, \ldots, i$, we have

$$\overline{A^*}((I, (x_1, \ldots, x_j))R[X]) = \{\mathfrak{p}R[X] \mid \mathfrak{p} \in \overline{A^*}((I, (x_1, \ldots, x_j))R)\}$$

(note that, for an ideal J in R, the prime divisors of $JR[X]$ are $\mathfrak{p}R[X]$ such that $\mathfrak{p}$ is a prime divisor of J). Also, it is clear that X is not in any prime divisor of $(I, (x_1, \ldots, x_i))R[X]$. Moreover, for $k = 0, 1, \ldots, n - i$, we have

$$\overline{A^*}((I, (x_1, \ldots, x_i, X, x_{i+1}, \ldots, x_{i+k}))R[X]) = \\ \{(\mathfrak{p}, X)R[X] \mid \mathfrak{p} \in \overline{A^*}((I, (x_1, \ldots, x_{i+k}))R)\},$$

by Lemma 2.2(vi). Now, the result follows. □

Proposition 2.11 concerns asymptotic I-sequences and projectively equivalent ideals. Before this we need the following definition.

Definition 2.10. Let I be an ideal of a Noetherian ring R.

(2.10.1) A sequence $\mathbf{x} = x_1, \ldots, x_n$ of elements of R is called an *asymptotic sequence over* I if, $(I, (\mathbf{x}))R \neq R$ and for all $1 \leq i \leq n$, we have $x_i \notin \bigcup\{\mathfrak{p} \in \overline{A^*}((I, (x_1, \ldots, x_{i-1}))R)\}$.

(2.10.2) An asymptotic sequence over $(0)R$ is simply called an *asymptotic sequence in* R.

(2.10.3) An asymptotic sequence $\mathbf{x} = x_1, \ldots, x_n$ of elements of R over I is *maximal* if $x_1, \ldots, x_n, x_{n+1}$ is not an asymptotic sequence over I for any $x_{n+1} \in R$.

Proposition 2.11. *Let I, I' and J be ideals in a Noetherian ring R and let $\mathbf{x} = x_1, \ldots, x_n$ be a sequence of elements of J. If I and I' are projectively equivalent (i.e., there exist positive integers r and s with $\overline{I^r} = \overline{I'^s}$), then $\mathbf{x}$ is a/an (resp., maximal) asymptotic I-sequence from J if and only if $\mathbf{x}$ is a/an (resp., maximal) asymptotic I'-sequence from J.*

Proof. If $\mathbf{x}$ is an asymptotic I-sequence from J, then $\mathbf{x}$ is an asymptotic sequence over I, by Definitions 2.3 and 2.10. Then $\overline{A^*}((I, (\mathbf{x}))R) = \overline{A^*}((I', (\mathbf{x}))R)$ by the proof of [7, Proposition 6.24], so the conclusion follows from this. □

3. On asymptotic *I*-grade

In this section, we show that any two maximal asymptotic I-sequences from J have the same length and introduce the concept of asymptotic I-grade of an ideal. We begin with a remark. We recall that if I be an ideal of a Noetherian local ring $(R, \mathfrak{m})$, then $\ell(I)$ denotes the analytic spread of the ideal I, so $\ell(I) := \dim \mathscr{R}(I, R)/(\mathfrak{m}, u)\mathscr{R}(I, R)$.

Remark 3.1. Let I be an ideal of a Noetherian ring R.

(3.1.1) It follows from [13, Theorem 3.1], that all maximal asymptotic sequences in R with elements coming from I have the same length and are denoted $\operatorname{agd}(I)$. Moreover,

$$\begin{aligned}\operatorname{agd}(I) &= \min\{\operatorname{height}(IR^*_{\mathfrak{p}} + z)/z \mid \mathfrak{p} \in V(I) \text{ and } z \in \operatorname{mAss}_{R^*_{\mathfrak{p}}} R^*_{\mathfrak{p}}\} \\ &= \min\{\dim R^*_{\mathfrak{p}}/z \mid \mathfrak{p} \in V(I) \text{ and } z \in \operatorname{mAss}_{R^*_{\mathfrak{p}}} R^*_{\mathfrak{p}}\}.\end{aligned}$$

Also, if R is local, then

$$\operatorname{agd}(I) = \min\{\operatorname{height}(IR^* + z)/z \mid z \in \operatorname{mAss}_{R^*} R^*\},$$

by [13, Corollary 2.13].

(3.1.2) It is shown in [3] that if R is local, then any two maximal asymptotic sequences over I have the same length and are denoted $\operatorname{acogd}(I)$. Moreover,

$$\operatorname{acogd}(I) = \min\{\dim R^*/z - \ell((IR^* + z)/z) \mid z \in \operatorname{mAss}_{R^*} R^*\}.$$

The next proposition shows that $\mathrm{agd}_I(J)$ is unambiguously defined, for all ideals I and J in a Noetherian ring R.

Proposition 3.2. *Let I and J be ideals in a Noetherian ring R. Then*

$$\mathrm{agd}_I(J) = \min\{\mathrm{acogd}(IR_{\mathfrak{p}}) \mid \mathfrak{p} \in V(I+J)\}.$$

Proof. Let $\mathbf{x} = x_1, \ldots, x_n$ be a maximal asymptotic I-sequence from J and $\mathfrak{p} \in V(I+J)$, then the image of $\mathbf{x}$ in $R_{\mathfrak{p}}$ is an asymptotic sequence over $IR_{\mathfrak{p}}$, by Proposition 2.4(i). So $\mathrm{acogd}(IR_{\mathfrak{p}}) \geq n$. Also, there exists $\mathfrak{q} \in \overline{A^*}((I,(\mathbf{x}))R)$ such that $J \subseteq \mathfrak{q}$. Then $\mathfrak{q} \in V(I+J)$ and the image of $\mathbf{x}$ in $R_{\mathfrak{q}}$ is a maximal asymptotic sequence over $IR_{\mathfrak{q}}$, by Proposition 2.4(i) and Lemma 2.2(ii). Therefore $\mathrm{acogd}(IR_{\mathfrak{q}}) = n$ and the result follows. □

In the next proposition we collect some basic properties of the asymptotic I-grade.

Proposition 3.3. *Let I, I', J and J' be ideals in a Noetherian ring R. Then the following hold:*

(i) *If $J \subseteq J'$, then* $\mathrm{agd}_I(J) \leq \mathrm{agd}_I(J')$.
(ii) *If $J' \subseteq I$, then* $\mathrm{agd}_I(J) = \mathrm{agd}_I(J'+J)$.
(iii) *If $\sqrt{J} = \sqrt{J'}$, then* $\mathrm{agd}_I(J) = \mathrm{agd}_I(J')$.
(iv) *If $\sqrt{J} \subseteq \sqrt{I}$, then* $\mathrm{agd}_I(J) = 0$.
(v) *If I and I' are projectively equivalent, then* $\mathrm{agd}_I(J) = \mathrm{agd}_{I'}(J)$.
(vi) $\mathrm{agd}_I(J) = \min\{\mathrm{agd}_I(\mathfrak{p}) \mid \mathfrak{p}$ *is a minimal prime divisor of* $J\}$.
(vii) $\mathrm{agd}_I(JJ') = \mathrm{agd}_I(J \cap J') = \min\{\mathrm{agd}_I(J), \mathrm{agd}_I(J')\}$.
(viii) *If $\mathbf{x} = x_1, \ldots, x_n$ is an asymptotic sequence over I, then* $\mathrm{agd}_I((\mathbf{x})R) = n$.

Proof. (i)–(iv) and (v) follow easily from Propositions 3.2 and 2.11. For (vi), it is enough to show that there exists a minimal prime divisor $\mathfrak{p}$ of J such that $\mathrm{agd}_I(J) = \mathrm{agd}_I(\mathfrak{p})$, by (i). Let $\mathbf{x} = x_1, \ldots, x_n$ be a maximal asymptotic I-sequence from J. Then there exists a prime ideal $\mathfrak{q} \in \overline{A^*}((I,(\mathbf{x}))R)$ such that $J \subseteq \mathfrak{q}$. Let $\mathfrak{p}$ be minimal prime divisor of J such that $\mathfrak{p} \subseteq \mathfrak{q}$, then $n = \mathrm{agd}_I(J) \leq \mathrm{agd}_I(\mathfrak{p}) \leq \mathrm{agd}_I(\mathfrak{q}) = n$, by (i). For (vii), $\mathrm{agd}_I(JJ') \leq \mathrm{agd}_I(J \cap J') \leq \min\{\mathrm{agd}_I(J), \mathrm{agd}_I(J')\}$, by (i), and $\mathrm{agd}_I(JJ') = \mathrm{agd}_I(\mathfrak{p})$ for some minimal prime divisor $\mathfrak{p}$ of JJ', by (vi). Now either J or J' is contained in $\mathfrak{p}$, say J. Then necessarily $\mathrm{agd}_I(JJ') = \mathrm{agd}_I(J) = \mathrm{agd}_I(\mathfrak{p})$, by (i), so it follows that $\min\{\mathrm{agd}_I(J), \mathrm{agd}_I(J')\} \leq \mathrm{agd}_I(JJ')$. For (viii), since $\mathbf{x}$ is an asymptotic I-sequence from $(\mathbf{x})R$, then $n \leq \mathrm{agd}_I((\mathbf{x})R)$. Let $\mathfrak{p} \in \overline{A^*}((I,(\mathbf{x}))R)$, then the image of $\mathbf{x}$ in $R_{\mathfrak{p}}$ is a maximal asymptotic sequence over $IR_{\mathfrak{p}}$, by Proposition 2.4(i) and Lemma 2.2(ii). Therefore $\mathrm{agd}_I((\mathbf{x})R) \leq \mathrm{acogd}(IR_{\mathfrak{p}}) = n$, by Proposition 3.2 and the conclusion follows. □

The next theorem shows that $\mathrm{agd}_I(J)$ behaves nicely when passing to certain related ideals.

Theorem 3.4. *Let I and J be ideals in a Noetherian ring R. Then the following hold:*

(i) *If S is a multiplicatively closed subset of R disjoint from I and J, then* $\operatorname{agd}_I(J) \leq \operatorname{agd}_{IR_S}(JR_S)$.

(ii) $\operatorname{agd}_I(J) = \min\{\operatorname{agd}_{IR_\mathfrak{p}}(JR_\mathfrak{p}) \mid \mathfrak{p} \in V(I+J)\}$.

(iii) $\operatorname{agd}_I(J) = \min\{\operatorname{agd}_{IR_\mathfrak{m}}(JR_\mathfrak{m}) \mid \mathfrak{m}$ *is a maximal prime ideal in* $V(I+J)\}$.

(iv) $\operatorname{agd}_I(J) = \min\{\operatorname{agd}_{I(R/z)}(J(R/z)) \mid z \in \operatorname{mAss}_R R\}$.

(v) $\operatorname{agd}_I(J) = \operatorname{agd}_{I(R/K)}(J(R/K))$, *for all ideals K contained* $\operatorname{Rad} R$.

(vi) *If T is an integral Noetherian extension ring of R such that every minimal prime of T lies over a minimal prime of R, then* $\operatorname{agd}_I(J) = \operatorname{agd}_{IT}(JT)$.

(vii) *If T is a flat Noetherian extension of R, then* $\operatorname{agd}_I(J) = \operatorname{agd}_{IT}(JT)$.

(viii) $\operatorname{agd}_I(J) = \operatorname{agd}_{(I,X)R[X]}(JR[X]) = \operatorname{agd}_{IR[X]}((J,X)R[X]) - 1$.

Proof. (i) and (iv)–(viii) follow from Propositions 2.4(i), 2.5(ii), 2.7(ii), 2.8, 2.9 and Corollary 2.6.

For (ii), there exists $\mathfrak{p} \in V(I+J)$ such that $\operatorname{agd}_I(J) = \operatorname{acogd}(IR_\mathfrak{p})$, by Proposition 3.2. Then by (i) and Proposition 3.3(i), we have

$$\operatorname{agd}_I(J) \leq \operatorname{agd}_{IR_\mathfrak{p}}(JR_\mathfrak{p}) \leq \operatorname{agd}_{IR_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}) = \operatorname{acogd}(IR_\mathfrak{p}) = \operatorname{agd}_I(J).$$

For (iii), there exists $\mathfrak{p} \in V(I+J)$ such that $\operatorname{agd}_I(J) = \operatorname{agd}_{IR_\mathfrak{p}}(JR_\mathfrak{p})$, by (ii). Then $\operatorname{agd}_I(J) \leq \operatorname{agd}_{IR_\mathfrak{m}}(JR_\mathfrak{m}) \leq \operatorname{agd}_{IR_\mathfrak{p}}(JR_\mathfrak{p}) = \operatorname{agd}_I(J)$, for all maximal prime ideals $\mathfrak{m}$ that contains $\mathfrak{p}$, by (i). $\square$

The following proposition will be useful in Section 5.

Proposition 3.5. *Let I, J be ideals in a Noetherian ring R, then there exist an integer $m \geq 0$ and a chain of prime ideals $J \subseteq \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_m$ such that $\mathfrak{p}_m$ is a maximal prime ideal and* $\operatorname{agd}_I(\mathfrak{p}_i) = \operatorname{agd}_I(J) + i$ *for $i = 0, \ldots, m$.*

Proof. Let $\operatorname{agd}_I(J) = n$ and $\mathbf{x} = x_1, \ldots, x_n$ be a maximal asymptotic I-sequence from J. Then there exists a prime ideal $\mathfrak{q}_0 \in \overline{A^*}((I,(\mathbf{x}))R)$ such that $J \subseteq \mathfrak{q}_0$. So $\operatorname{agd}_I(\mathfrak{q}_0) = n$. Let $\mathfrak{m}_0$ be a maximal prime ideal such that $\mathfrak{q}_0 \subseteq \mathfrak{m}_0$.

We have two cases by Proposition 3.3(i):

Case 1: $\operatorname{agd}_I(\mathfrak{q}_0) = \operatorname{agd}_I(\mathfrak{m}_0)$, then we set $m = 0$ and $\mathfrak{p}_0 = \mathfrak{m}_0$.

Case 2: $\operatorname{agd}_I(\mathfrak{q}_0) < \operatorname{agd}_I(\mathfrak{m}_0)$, then we set $\mathfrak{p}_0 = \mathfrak{q}_0$. There exists $x_{n+1} \in \mathfrak{m}_0$ such that x_{n+1} is an asymptotic sequence over $(I,(\mathbf{x}))R$. Let $\mathfrak{q}_1$ be a minimal prime divisor of $(\mathfrak{q}_0,(x_{n+1}))R$, then $\mathfrak{q}_1 \in \overline{A^*}((I,(\mathbf{x}),(x_{n+1}))R)$, by [7, Lemma 6.13]. So $\operatorname{agd}_I(\mathfrak{q}_1) = n+1$. Now, replace $\mathfrak{q}_0$ with $\mathfrak{q}_1$ and continue this process. Note that R is Noetherian ring. $\square$

Proposition 3.6. *Let I and J be ideals in a Noetherian ring R and let b be an element in the Jacobson radical of R. Then*

$$\operatorname{agd}_I(J) \leq \operatorname{agd}_I((J,(b))R) \leq \operatorname{agd}_I(J) + 1.$$

Proof. The first inequality follows from Proposition 3.3(i). Let $\mathrm{agd}_I(J) = n$ and $\mathbf{x} = x_1, \ldots, x_n$ be a maximal asymptotic I-sequence from J. If $(J, (b))R \subseteq \cup\{\mathfrak{p} \mid \mathfrak{p} \in \overline{A^*}((I, (\mathbf{x}))R)\}$, then $\mathbf{x}$ is a maximal asymptotic I-sequence from $(J, (b))R$ and the second inequality holds. Otherwise, there exists $j \in J$ such that $j + b \notin \cup\{\mathfrak{p} \mid \mathfrak{p} \in \overline{A^*}((I, (\mathbf{x}))R)\}$, by [2, Theorem 124]. Let $b_1 = j + b$, then $(J, (b))R = (J, (b_1))R$. There exists a prime ideal $\mathfrak{q} \in \overline{A^*}((I, (\mathbf{x}))R)$ such that $J \subseteq \mathfrak{q}$. Let $\mathfrak{q}_1$ be a minimal prime divisor of $(\mathfrak{q}, (b_1))R$. Since b_1 is an asymptotic sequence over $(I, (\mathbf{x}))R$, then $\mathfrak{q}_1 \in \overline{A^*}((I, (\mathbf{x}), (b_1))R)$, by [7, Lemma 6.13]. We know $(J, (b))R = (J, (b_1))R \subseteq (\mathfrak{q}, (b_1))R \subseteq \mathfrak{q}_1$. Therefore $x_1, \ldots, x_n, b_1$ is a maximal asymptotic I-sequence from $(J, (b))R$ and $\mathrm{agd}_I((J, (b))R) = n + 1$. $\square$

Corollary 3.7. *Let I, J and K be ideals in a Noetherian ring R such that $J \subseteq K$ and K is contained in the Jacobson radical of R. If $\mathrm{agd}_I(K) - \mathrm{agd}_I(J) = n \geq 0$, then there exists a chain of ideals $J = J_0 \subset J_1 \subset \ldots \subset J_n \subseteq K$ such that $\mathrm{agd}_I(J_i) = \mathrm{agd}_I(J) + i$ for $i = 0, \ldots, n$.*

Proof. If $n = 0$, the result is clear. Let $n > 0$ and $x_1, \ldots, x_m$ be a maximal asymptotic I-sequence from J. Since $\mathrm{agd}_I(K) - \mathrm{agd}_I(J) = n > 0$, then there exists $x_{m+1}, \ldots, x_{m+n}$ in K such that $x_1, \ldots, x_{m+n}$ is a maximal asymptotic I-sequence from K. We set $J_i = (J, (x_{m+1}, \ldots, x_{m+i}))R$ for $i = 0, \ldots, n$, then $\mathrm{agd}_I(J_i) \geq m + i$. So $\mathrm{agd}_I(J_i) = m + i$ for $i = 0, \ldots, n$, by Proposition 3.6. $\square$

The following proposition gives a nice characterization for asymptotic sequences over an ideal.

Proposition 3.8. *Let I be an ideal of a Noetherian ring R and $\mathbf{x} = x_1, \ldots, x_n$ be a sequence of elements in the Jacobson radical of R, then $\mathbf{x}$ is an asymptotic sequence over I if and only if $\mathrm{agd}_I((\mathbf{x})R) = n$.*

Proof. Let $\mathrm{agd}_I((\mathbf{x})R) = n$, we prove that $\mathbf{x}$ is an asymptotic sequence over I by induction on n. If $n = 1$, the result is clear. Let $\mathrm{agd}_I((\mathbf{x})R) = n > 1$, then $\mathrm{agd}_I((x_1, \ldots, x_{n-1})R) = n - 1$, by Proposition 3.6 and so $x_1, \ldots, x_{n-1}$ is an asymptotic sequence over I. If $x_n \in \cup\{\mathfrak{p} \mid \mathfrak{p} \in \overline{A^*}((I, (x_1, \ldots, x_{n-1}))R)\}$, then $(\mathbf{x})R \subseteq \cup\{\mathfrak{p} \mid \mathfrak{p} \in \overline{A^*}((I, (x_1, \ldots, x_{n-1}))R)\}$. Hence $\mathrm{agd}_I((\mathbf{x})R) = n - 1$ which is a contradiction. Therefore, $\mathbf{x}$ is an asymptotic sequence over I. The converse follows from Proposition 3.3(viii). $\square$

The next two corollaries were proved by Ratliff in [14, Corollary 6.3 and Theorem 6.4], we reprove them without using the Rees rings.

Corollary 3.9. *Let I be an ideal of a Noetherian ring R and $\mathbf{x} = x_1, \ldots, x_n$ be an asymptotic sequence over I that is contained in Jacobson radical of R, then each permutation of $\mathbf{x}$ is an asymptotic sequence over I.*

Proof. It follows immediately from Proposition 3.8. □

Corollary 3.10. *Let I be an ideal of a Noetherian ring R and $\mathbf{x} = x_1, \ldots, x_n$ be an asymptotic sequence over I that is contained in Jacobson radical of R, then $\mathbf{x}$ is an asymptotic sequence in R.*

Proof. It follows immediately from Proposition 3.8, Theorem 4.6 and Generalized Principal Ideal Theorem. □

Corollary 3.11. *Let $I, J = (x_1, \ldots, x_n)R$ and $J' = (y_1, \ldots, y_n)R$ be ideals of a Noetherian ring R. If J and J' are contained in the Jacobson radical of R and $\sqrt{J} = \sqrt{J'}$, then $x_1, \ldots, x_n$ is an asymptotic sequence over I if and only if $y_1, \ldots, y_n$ are.*

Proof. It follows from Propositions 3.8 and 3.3(iii). □

Corollary 3.12. *Let $I, J = (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)R$ and $J' = (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)R$ be ideals in a Noetherian ring R such that J and J' are contained in the Jacobson radical of R. Then $x_1, \ldots, x_{i-1}, x_i y_i, x_{i+1}, \ldots, x_n$ is an asymptotic sequence over I if and only if $x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n$ and $x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n$ are.*

Proof. Let $\mathbf{x} = x_1, \ldots, x_{i-1}, x_i y_i, x_{i+1}, \ldots, x_n$. Let $\mathbf{x}$ be an asymptotic sequence over I, then

$$n = \operatorname{agd}_I((\mathbf{x})R) \leq \operatorname{agd}_I(J) \leq \operatorname{height} J(R/I) \leq n,$$

by Propositions 3.8 and 3.3(i), Theorem 4.4 and Generalized Principal Ideal Theorem. So $\operatorname{agd}_I(J) = n$ and $x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n$ is an asymptotic sequence over I, by Proposition 3.8. Also, $x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n$ is an asymptotic sequence over I similarly. For the converse, there exists $\mathfrak{p} \in V((I, (\mathbf{x}))R)$ such that $\operatorname{agd}_I((\mathbf{x})R) = \operatorname{acogd}(IR_\mathfrak{p})$, by Proposition 3.2. Now either $I + J$ or $I + J'$ is contained in $\mathfrak{p}$, say $I + J$. Then $\operatorname{acogd}(IR_\mathfrak{p}) \geq \operatorname{agd}_I(J) \geq \operatorname{agd}_I((\mathbf{x})R)$, by Propositions 3.2 and 3.3(i). Therefore $\mathbf{x}$ is an asymptotic sequence over I, by Proposition 3.8. □

Corollary 3.13. *Let I be an ideal in a Noetherian ring R and let $\mathbf{x} = x_1, \ldots, x_n$ be a sequence of elements of Jacobson radical of R. Then the following statements are equivalent:*

(i) *$x_1, \ldots, x_n$ is an asymptotic sequence over I.*
(ii) *${x_1}^{m_1}, \ldots, {x_n}^{m_n}$ is an asymptotic sequence over I, for some positive integers m_i.*
(iii) *${x_1}^{m_1}, \ldots, {x_n}^{m_n}$ is an asymptotic sequence over I, for all positive integers m_i.*

Proof. It follows from Corollary 3.12. □

In [13, Corollary 3.7], Ratliff proved that if I is an ideal in a Noetherian ring R that can be generated by an asymptotic sequence in R, then $\operatorname{agd}(\mathfrak{p}) = \operatorname{agd}(I)$ for all prime divisors $\mathfrak{p}$ of $\overline{I^n}$. Proposition 3.14 gives a generalization of this result for asymptotic sequences over an ideal.

Proposition 3.14. *Let I and J be ideals in a Noetherian ring R such that J can be generated by an asymptotic sequence over I, then for all $h \geq 1$: $\operatorname{agd}_I(\mathfrak{p}) = \operatorname{agd}_I(J)$ for all prime divisors $\mathfrak{p}$ of $\overline{(I+J)^h}$.*

Proof. Let $\mathbf{x} = x_1, \ldots, x_n$ be an asymptotic sequence over I such that $J = (\mathbf{x})R$, then $\operatorname{agd}_I(J) = n$, by Proposition 3.3(viii). Let $\mathfrak{p}$ be a prime divisor of $\overline{(I+J)^h}$, for some $h \geq 1$, then $\mathfrak{p} \in \overline{A^*}(I+J)$, by Lemma 2.2(i). Therefore $\mathbf{x}$ is a maximal asymptotic I-sequence from $\mathfrak{p}$ and so $\operatorname{agd}_I(\mathfrak{p}) = n$. $\square$

In Proposition 3.16, we give a stronger result than Proposition 3.14. We need a definition.

Definition 3.15. Let I be an ideal in a Noetherian ring R, let $\overline{A^*}(I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$ and $S = R \setminus \bigcup_{i=1}^{k} \mathfrak{p}_i$. Then $(I^h)_{\mathrm{a}} = I^h R_S \cap R$ is the asymptotic component of I^h, for all integers $h > 0$.

Proposition 3.16. *Let I and J be ideals in a Noetherian ring R such that J can be generated by an asymptotic sequence over I, then for all $h \geq 1$: $\operatorname{agd}_I(\mathfrak{p}) = \operatorname{agd}_I(J)$ for all prime divisors $\mathfrak{p}$ of $((I+J)^h)_{\mathrm{a}}$.*

Proof. Let $\mathbf{x} = x_1, \ldots, x_n$ be an asymptotic sequence over I such that $J = (\mathbf{x})R$. Let $\mathfrak{p}$ be a prime divisor of $((I+J)^h)_{\mathrm{a}}$, then there exists $\mathfrak{q} \in \overline{A^*}(I+J)$ such that $\mathfrak{p} \subseteq \mathfrak{q}$. It is clear that $\mathbf{x}$ is a maximal asymptotic I-sequence from $\mathfrak{q}$. Therefore

$$n = \operatorname{agd}_I(J) \leq \operatorname{agd}_I(\mathfrak{p}) \leq \operatorname{agd}_I(\mathfrak{q}) = n$$

by Proposition 3.3((i) and (viii)) and so $\operatorname{agd}_I(\mathfrak{p}) = \operatorname{agd}_I(J)$. $\square$

4. Some bounds for asymptotic *I*-grade

In this section, we give several bounds on asymptotic I-grade. We begin with an upper bound.

Theorem 4.1. *Let I and J be ideals in a Noetherian ring R and $\mathbf{x} = x_1, \ldots, x_h$ be an asymptotic sequence in I. Then there exists a maximal asymptotic I-sequence from $I+J$, say $\mathbf{y} = y_1, \ldots, y_n$ such that $\mathbf{x}, \mathbf{y}$ is an asymptotic sequence in $I+J$. In particular*

$$\operatorname{agd}_I(J) \leq \operatorname{agd}(I+J) - \operatorname{agd}(I).$$

Proof. Let $\operatorname{agd}_I(I+J)=n$. If $n=0$, we are done. If $n>0$, then $I+J\nsubseteq\cup\{\mathfrak{p}\mid\mathfrak{p}\in\overline{A^*}(I)\}$, and so $I+J\nsubseteq\cup\{\mathfrak{p}\mid\mathfrak{p}\in\overline{A^*}((\mathbf{x})R)\}$, by [7, Lemma 5.7]. Pick $y_1\in I+J$ with

$$y_1\notin\cup\{\mathfrak{p}\mid\mathfrak{p}\in\overline{A^*}(I)\}\ \text{and}\ y_1\notin\cup\{\mathfrak{p}\mid\mathfrak{p}\in\overline{A^*}((\mathbf{x})R)\}.$$

Now, y_1 is an asymptotic I-sequence from $I+J$ and $\operatorname{agd}_{(I,(y_1))R}(I+J)=n-1$. Since the choice of y_1 assure that $x_1,\ldots,x_h,y_1$ is an asymptotic sequence in $(I,(y_1))R$, we now may use induction. The inequality follows from this and Proposition 3.3(ii). □

The next result determines when the inequality in Theorem 4.1 becomes equality.

Theorem 4.2. *Let I and J be ideals in a Noetherian ring R. Then the following statements are equivalent:*

(i) $\operatorname{agd}_I(J)+\operatorname{agd}(I)=\operatorname{agd}(I+J)$.

(ii) *There exists $\mathfrak{p}\in V(I+J)$ such that* $\operatorname{agd}_I(J)=\operatorname{acogd}(IR_\mathfrak{p})$, $\operatorname{agd}(I)=\operatorname{agd}(IR_\mathfrak{p})$ *and* $\operatorname{acogd}(IR_\mathfrak{p})+\operatorname{agd}(IR_\mathfrak{p})=\operatorname{agd}(\mathfrak{p}R_\mathfrak{p})$.

(iii) *The equalities in* (ii) *hold for every $\mathfrak{p}\in V(I+J)$ such that* $\operatorname{agd}(I+J)=\operatorname{agd}(\mathfrak{p}R_\mathfrak{p})$.

(iv) *There exist $\mathfrak{p}\in V(I+J)$ and $z\in\operatorname{mAss}_{R_\mathfrak{p}^*}R_\mathfrak{p}^*$ such that*

$$\operatorname{agd}_I(J)=\operatorname{acogd}(IR_\mathfrak{p})=\dim R_\mathfrak{p}^*/(IR_\mathfrak{p}^*+z)$$

and

$$\operatorname{agd}(I)=\operatorname{agd}(IR_\mathfrak{p})=\ell((IR_\mathfrak{p}^*+z)/z)=\operatorname{height}(IR_\mathfrak{p}^*+z)/z.$$

(v) *There exists $\mathfrak{p}\in V(I+J)$ such that the equalities in* (iv) *hold for every $z\in\operatorname{mAss}_{R_\mathfrak{p}^*}R_\mathfrak{p}^*$ with* $\operatorname{agd}(\mathfrak{p}R_\mathfrak{p})=\dim R_\mathfrak{p}^*/z$.

(vi) *For every $\mathfrak{p}\in V(I+J)$ with* $\operatorname{agd}(I+J)=\operatorname{agd}(\mathfrak{p}R_\mathfrak{p})$, *there exists $z\in\operatorname{mAss}_{R_\mathfrak{p}^*}R_\mathfrak{p}^*$ such that the equalities in* (iv) *hold.*

(vii) *The equalities in* (iv) *hold for every $\mathfrak{p}\in V(I+J)$ and for every $z\in\operatorname{mAss}_{R_\mathfrak{p}^*}R_\mathfrak{p}^*$ such that* $\operatorname{agd}(I+J)=\operatorname{agd}(\mathfrak{p}R_\mathfrak{p})=\dim R_\mathfrak{p}^*/z$.

Proof. For (i) ⇒ (iii), let $\mathfrak{p}\in V(I+J)$ such that $\operatorname{agd}(I+J)=\operatorname{agd}(\mathfrak{p}R_\mathfrak{p})$, then

$$\begin{aligned}\operatorname{agd}(I+J)&=\operatorname{agd}(\mathfrak{p}R_\mathfrak{p})\\&\geq\operatorname{agd}(IR_\mathfrak{p})+\operatorname{acogd}(IR_\mathfrak{p})\\&\geq\operatorname{agd}(I)+\operatorname{agd}_I(J)=\operatorname{agd}(I+J),\end{aligned}$$

by [7, Proposition 6.9], Proposition 3.2 and Theorem 3.4(i). So the equalities in (ii) hold.

The implication (iii) ⇒ (ii) is obvious, since there exists $\mathfrak{p}\in V(I+J)$ such that $\operatorname{agd}(I+J)=\operatorname{agd}(\mathfrak{p}R_\mathfrak{p})$, by Proposition 3.2.

For (ii) ⇒ (i), since $\operatorname{agd}(I+J)\leq\operatorname{agd}(\mathfrak{p}R_\mathfrak{p})$ for all $\mathfrak{p}\in V(I+J)$, by Proposition 3.2, so the result follows from Theorem 4.1.

Finally, (ii) ⇔ (iv) ⇔ (v) and (iii) ⇔ (vi) ⇔ (vii) follow from [7, Proposition 6.11]. □

Theorem 4.3. *Let I and J be ideals in a Noetherian ring R, then*

$$\mathrm{agd}_I(J) \leq \mathrm{agd}(J(R/I)).$$

Proof. There exists a prime ideal $\mathfrak{p} \in V(I+J)$ such that $\mathrm{agd}(J(R/I)) = \mathrm{agd}(\mathfrak{p}R_\mathfrak{p}/IR_\mathfrak{p})$, by Proposition 3.2. Therefore the conclusion follows from Proposition 3.2 and [7, Proposition 6.6]. □

The next theorem follows from Theorem 4.3 and [7, Lemma 5.13], but we give another proof.

Theorem 4.4. *Let I and J be ideals in a Noetherian ring R, then*

$$\mathrm{agd}_I(J) \leq \mathrm{height}\, J(R/I).$$

Proof. There exists a prime ideal $\mathfrak{p} \in V(I+J)$ such that $\mathrm{height}\, J(R/I) = \mathrm{height}\, \mathfrak{p}/I$, then $\mathrm{agd}_I(J) \leq \mathrm{acogd}(IR_\mathfrak{p}) \leq \dim R^*_\mathfrak{p}/z - \ell((IR^*_\mathfrak{p} + z)/z)$, by Proposition 3.2 and Remark 3.1(ii). Since height does not exceed analytic spread and $R^*_\mathfrak{p}/z$ is catenary, then $\mathrm{agd}_I(J) \leq \dim R^*_\mathfrak{p}/(IR^*_\mathfrak{p} + z)$. Now $\mathrm{agd}_I(J) \leq \mathrm{height}\, \mathfrak{p}/I$. □

Corollary 4.5. *Let I and J be ideals in a Noetherian ring R such that J is generated by $\mathrm{agd}_I(J)$ elements, then $\mathrm{agd}_I(J) = \mathrm{height}\, J(R/I)$.*

Proof. It follows from Theorem 4.4 and Generalized Principal Ideal Theorem. □

Theorem 4.6. *Let I and J be ideals in a Noetherian ring R, then*

$$\mathrm{agd}_I(J) \leq \mathrm{agd}(J) \leq \mathrm{height}\, J.$$

Proof. There exists a prime ideal $\mathfrak{p} \in V(J)$ such that $\mathrm{agd}(J) = \mathrm{agd}(\mathfrak{p}R_\mathfrak{p})$, by Proposition 3.2. We have $\mathrm{agd}_I(J) \leq \mathrm{agd}_I(\mathfrak{p}) \leq \mathrm{agd}_{IR_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p})$, by Proposition 3.3(i) and Theorem 3.4(i). Since $\mathrm{agd}_{IR_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}) \leq \mathrm{agd}(\mathfrak{p}R_\mathfrak{p})$, by Theorem 4.1, then $\mathrm{agd}_I(J) \leq \mathrm{agd}(J)$. The second inequality follows from Theorem 4.4. □

Corollary 4.7. *Let I and J be ideals in a Noetherian ring R such that J is generated by $\mathrm{agd}_I(J)$ elements, then $\mathrm{agd}_I(J) = \mathrm{height}\, J$.*

Proof. It follows from Theorem 4.6 and Generalized Principal Ideal Theorem. □

Now we give two lower bounds on $\mathrm{agd}_I(J)$.

Theorem 4.8. *Let I and J be ideals of a locally quasi-unmixed ring R, then $\mathrm{agd}_I(J) \geq \mathrm{grade}(J(R/I^n))$ for all large n.*

Proof. There exists a prime ideal $\mathfrak{p} \in V(I+J)$ such that $\mathrm{agd}_I(J) = \mathrm{acogd}(IR_\mathfrak{p})$, by Proposition 3.2. Also, for all large n, $\mathrm{acogd}(IR_\mathfrak{p}) \geq \mathrm{grade}(\mathfrak{p}R_\mathfrak{p}/I^n R_\mathfrak{p})$, by [9, Theorem 7.4]. So the conclusion follows from the standard facts on classical grade. $\square$

Theorem 4.9. *Let I and J be ideals in a Noetherian local ring R. Then,*

$$\mathrm{agd}_I(J) \geq \mathrm{agd}(I+J) - \ell(I).$$

Proof. There exists a prime ideal $\mathfrak{p} \in V(I+J)$ such that $\mathrm{agd}_I(J) = \mathrm{acogd}(IR_\mathfrak{p})$, by Proposition 3.2, and $\mathrm{acogd}(IR_\mathfrak{p}) \geq \mathrm{agd}(\mathfrak{p}R_\mathfrak{p}) - \ell(IR_\mathfrak{p})$, by [9, Theorem 7.6]. Also, $\ell(IR_\mathfrak{p}) \leq \ell(I)$, by [9, Lemma 2.5], and $\mathrm{agd}(\mathfrak{p}R_\mathfrak{p}) \geq \mathrm{agd}(\mathfrak{p}) \geq \mathrm{agd}(I+J)$, by Proposition 3.3(i) and Theorem 3.4(i). Now the desired result follows. $\square$

5. Some characterizations for $\mathrm{agd}_I(J) = \mathrm{height}\, J(R/I)$

It was shown in Theorem 4.4 that $\mathrm{agd}_I(J) \leq \mathrm{height}\, J(R/I)$ always holds. The main result in this section, Theorem 5.2 gives several characterizations of when the equality holds. We begin with a definition.

Definition 5.1. Let I be an ideal of a Noetherian local ring R. We say R is *strongly quasi-I-unmixed* ring, if $\dim R^*/(IR^* + z) = \dim R/I$ and $\ell((IR^* + z)/z) = \mathrm{height}(IR^* + z)/z$ for all $z \in \mathrm{mAss}_{R^*} R^*$. More generally, if R is not necessarily local, R is *locally strongly quasi-I-unmixed* ring if for every $\mathfrak{p} \in V(I)$, $R_\mathfrak{p}$ is strongly quasi-$IR_\mathfrak{p}$-unmixed.

Theorem 5.2. *Let I be an ideal of a Noetherian ring R. The following are equivalent:*

(i) $\mathrm{agd}_{(I,(\mathbf{x}))R}(J) = \mathrm{height}\, J(R/(I,(\mathbf{x}))R)$ *for all ideals J and all asymptotic I-sequences $\mathbf{x} = x_1, \ldots, x_n$ from J.*

(ii) $\mathrm{agd}_I(J) = \mathrm{height}\, J(R/I)$ *for all ideals J.*

(iii) $\mathrm{agd}_I(\mathfrak{p}) = \mathrm{height}\, \mathfrak{p}/I$ *for all prime ideals $\mathfrak{p} \in V(I)$.*

(iv) $\mathrm{agd}_I(\mathfrak{m}) = \mathrm{height}\, \mathfrak{m}/I$ *for all maximal prime ideals $\mathfrak{m} \in V(I)$.*

(v) *R is locally strongly quasi-I-unmixed ring.*

(vi) *If $\mathbf{x} = x_1, \ldots, x_n$ is an asymptotic sequence over I and $\mathfrak{p} \in \overline{A^*}((I,(\mathbf{x}))R)$, then* $\mathrm{height}\, \mathfrak{p}/I = n$.

(vii) *If $(I,(x_1, \ldots, x_n))R \neq R$ and if* $\mathrm{height}(x_1, \ldots, x_i)(R/I) = i$ *for $1 \leq i \leq n$, then $x_1, \ldots, x_n$ is an asymptotic sequence over I.*

Proof. It is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). Let (iii) hold and let $\mathfrak{p} \in V(I)$, then

$$\mathrm{height}\, \mathfrak{p}/I = \mathrm{agd}_I(\mathfrak{p}) \leq \mathrm{agd}_{IR_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}) \leq \mathrm{height}\, \mathfrak{p}R_\mathfrak{p}/IR_\mathfrak{p} = \mathrm{height}\, \mathfrak{p}/I,$$

by Theorems 4.4 and 3.4(i). So $\mathrm{agd}_{IR_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}) = \mathrm{height}\, \mathfrak{p}R_\mathfrak{p}/IR_\mathfrak{p}$. Let $z \in \mathrm{mAss}_{R_\mathfrak{p}^*} R_\mathfrak{p}^*$, then

$$\begin{aligned}\dim R_{\mathfrak{p}}/IR_{\mathfrak{p}} &= \operatorname{height} \mathfrak{p}R_{\mathfrak{p}}/IR_{\mathfrak{p}} \\ &= \operatorname{agd}_{IR_{\mathfrak{p}}}(\mathfrak{p}R_{\mathfrak{p}}) \\ &\leq \dim R^*_{\mathfrak{p}}/z - \ell((IR^*_{\mathfrak{p}} + z)/z) \\ &\leq \dim R^*_{\mathfrak{p}}/z - \operatorname{height}(IR^*_{\mathfrak{p}} + z)/z \\ &= \dim R^*_{\mathfrak{p}}/(IR^*_{\mathfrak{p}} + z) \\ &\leq \dim R^*_{\mathfrak{p}}/IR^*_{\mathfrak{p}} = \dim R_{\mathfrak{p}}/IR_{\mathfrak{p}},\end{aligned}$$

by Remark 3.1(ii). Therefore $\dim R^*_{\mathfrak{p}}/(IR^*_{\mathfrak{p}} + z) = \dim R_{\mathfrak{p}}/IR_{\mathfrak{p}}$ and $\ell((IR^*_{\mathfrak{p}} + z)/z) = \operatorname{height}(IR^*_{\mathfrak{p}} + z)/z$ and so (v) holds.

Let (v) hold. Assume $\mathbf{x} = x_1, \ldots, x_n$ is an asymptotic sequence over I and $\mathfrak{p} \in \overline{A^*}((I, (\mathbf{x}))R)$, then $\mathfrak{p}R_{\mathfrak{p}} \in \overline{A^*}((I, (\mathbf{x}))R_{\mathfrak{p}})$, by Lemma 2.2(ii) and the image of $\mathbf{x}$ in $R_{\mathfrak{p}}$ is a maximal asymptotic sequence over $IR_{\mathfrak{p}}$, by Proposition 2.4(i). Therefore, there exists $z \in \operatorname{mAss}_{R^*_{\mathfrak{p}}} R^*_{\mathfrak{p}}$ such that $\dim R^*_{\mathfrak{p}}/z - \ell((IR^*_{\mathfrak{p}} + z)/z) = n$, by Remark 3.1(ii). Now $\operatorname{height} \mathfrak{p}/I = n$, by supposition and catenariness of $R^*_{\mathfrak{p}}/z$ and so (vi) holds.

Assume that (vi) holds and assume inductively that $x_1, \ldots, x_{i-1}$ have already been shown to be an asymptotic sequence over I (the case $i = 1$ works equally well). If $\mathfrak{p} \in \overline{A^*}((I, (x_1, \ldots, x_{i-1}))R)$, then $\operatorname{height} \mathfrak{p}/I = i - 1$, and since $\operatorname{height}(x_1, \ldots, x_i)(R/I) = i$ we have $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \overline{A^*}((I, (x_1, \ldots, x_{i-1}))R)$. Thus $x_1, \ldots, x_i$ is an asymptotic sequence over I, so by induction (vii) holds.

Let (vii) hold. Let $\mathbf{x} = x_1, \ldots, x_n$ be an asymptotic I-sequence from J such that $\operatorname{height} J(R/(I, (\mathbf{x}))R) = m$. Then there exists $x_{n+1}, \ldots, x_{n+m}$ in J such that $\operatorname{height}(x_{n+1}, \ldots, x_{n+i})(R/(I, (\mathbf{x}))R) = i$, for $1 \leq i \leq m$, by [11, Theorem 8, p. 61]. If $1 \leq i \leq n$, then $\operatorname{height}(x_1, \ldots, x_i)(R/I) = i$, by Corollary 4.5 and Proposition 3.3(viii). If $n + 1 \leq i \leq n + m$, we have

$$\operatorname{height}(\mathbf{x}, x_{n+1}, \ldots, x_i)(R/I) \geqslant$$
$$\operatorname{height}(x_{n+1}, \ldots, x_i)(R/(I, (\mathbf{x}))R) + \operatorname{height}(\mathbf{x})(R/I) = i,$$

and so $\operatorname{height}(\mathbf{x}, x_{n+1}, \ldots, x_i)(R/I) = i$ by the Generalized Principal Ideal Theorem. Therefore $\operatorname{height}(x_1, \ldots, x_i)(R/I) = i$ for $1 \leq i \leq n + m$. Thus $x_1, \ldots, x_{n+m}$ is an asymptotic sequence over I. Since $x_{n+1}, \ldots, x_{n+m}$ is an asymptotic $(I, (\mathbf{x}))R$-sequence from J, then $\operatorname{agd}_{(I,(\mathbf{x}))R}(J) \geqslant m$. Therefore $\operatorname{agd}_{(I,(\mathbf{x}))R}(J) = m$, by Theorem 4.4 and so (i) holds.

For (iv) $\Rightarrow$ (ii), there exist an integer $m \geq 0$ and a chain of prime ideals $I + J \subseteq \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_m$ such that $\mathfrak{p}_m$ is a maximal prime ideal and $\operatorname{agd}_I(\mathfrak{p}_i) = \operatorname{agd}_I(J) + i$ for $i = 0, \ldots, m$, by Propositions 3.5 and 3.3(ii). Since $\mathfrak{p}_m$ is a maximal prime ideal, then

$$\operatorname{agd}_I(\mathfrak{p}_{m-1}) = \operatorname{agd}_I(\mathfrak{p}_m) - 1 = \operatorname{height} \mathfrak{p}_m/I - 1 \geq \operatorname{height} \mathfrak{p}_{m-1}/I \geq \operatorname{agd}_I(\mathfrak{p}_{m-1}),$$

by Theorem 4.4 and so height $\mathfrak{p}_{m-1}/I = \mathrm{agd}_I(\mathfrak{p}_{m-1})$. By continuation of this process we have height $\mathfrak{p}_0/I = \mathrm{agd}_I(\mathfrak{p}_0)$. Therefore

$$\mathrm{agd}_I(J) = \mathrm{agd}_I(\mathfrak{p}_0) = \text{height}\,\mathfrak{p}_0/I \geq \text{height}\,J(R/I) \geq \mathrm{agd}_I(J),$$

by Theorem 4.4 and so $\mathrm{agd}_I(J) = \text{height}\,J(R/I)$. □

Corollary 5.3. *Let I be an ideal of a Noetherian local ring $(R, \mathfrak{m})$ such that R is a strongly quasi-I-unmixed, then R is locally strongly quasi-I-unmixed.*

Proof. If R is a strongly quasi-I-unmixed, then $\mathrm{agd}_I(\mathfrak{m}) = \dim R/I$, by Definition 5.1 and Remark 3.1(ii). Now R is locally strongly quasi-I-unmixed, by Theorem 5.2. □

Corollary 5.4. *Let I be an ideal of a Noetherian local ring $(R, \mathfrak{m})$, then the following are equivalent:*

(i) *R is strongly quasi-I-unmixed ring.*

(ii) *There exists a system of parameters in R/I, $x_1+I, \dots, x_n+I$, such that $x_1, \dots, x_n$ is an asymptotic sequence over I.*

(iii) *If $x_1+I, \dots, x_i+I$ is a subset of a system of parameters in R/I, then $x_1, \dots, x_i$ is an asymptotic sequence over I.*

Proof. For (i) ⇒ (iii), let $x_1 + I, \dots, x_i + I$ be a subset of a system of parameters in R/I, then $\dim R/(I, (x_1, \dots, x_j))R = \dim R/I - j$ for $j = 1, \dots, i$. Since R is strongly quasi-I-unmixed, then $\mathrm{agd}(J(R/I)) = \text{height}\,J(R/I)$ for all ideals J of R, by Theorems 4.3 and 4.4, and Corollary 5.3. Therefore R/I is quasi-unmixed by [7, Corollary 5.8] and so is catenary. Therefore $\text{height}(x_1, \dots, x_j)(R/I) = j$ for $j = 1, \dots, i$ and so $x_1, \dots, x_i$ is an asymptotic sequence over I, by Theorem 5.2 and Corollary 5.3.

It is clear that (iii) ⇒ (ii).

For (ii) ⇒ (i), let $x_1 + I, \dots, x_n + I$ be a system of parameters in R/I, such that $x_1, \dots, x_n$ is an asymptotic sequence over I. Then

$$\dim R/I = n = \mathrm{agd}_I((I, (x_1, \dots, x_n))R) = \mathrm{agd}_I(\mathfrak{m})$$

by Proposition 3.3((ii), (iii) and (viii)). Therefore by Theorem 5.2 and Corollary 5.3 it follows that R is strongly quasi-I-unmixed ring. □

Corollary 5.5. *Let I and J be ideals of a Noetherian ring R such that* $\mathrm{agd}_I(J) = \text{height}\,J(R/I)$, *then $R_\mathfrak{p}$ is strongly quasi-$IR_\mathfrak{p}$-unmixed ring for all minimal prime divisors $\mathfrak{p}$ of $I + J$ such that* $\text{height}\,\mathfrak{p}/I = \text{height}\,J(R/I)$.

Proof. Let $\mathfrak{p}$ be a minimal prime divisor of $I + J$ such that $\text{height}\,\mathfrak{p}/I = \text{height}\,J(R/I)$. Then

$$\begin{aligned}\operatorname{height} J(R/I) = \operatorname{agd}_I(J) \leq \operatorname{agd}_{IR_\mathfrak{p}}(JR_\mathfrak{p}) \\ \leq \operatorname{agd}_{IR_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}) \leq \operatorname{height} \mathfrak{p}/I = \operatorname{height} J(R/I),\end{aligned}$$

by Proposition 3.3(i) and Theorems 3.4(i) and 4.4. So $\operatorname{agd}_{IR_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}) = \operatorname{height} \mathfrak{p}/I$ and so $R_\mathfrak{p}$ is strongly quasi-$IR_\mathfrak{p}$-unmixed ring by Theorem 5.2 and Corollary 5.3. □

Corollary 5.6. *Let I be an ideal of a Noetherian ring R and $\mathbf{x} = x_1, \ldots, x_n$ be an asymptotic sequence over I. Then $R_\mathfrak{p}$ is strongly quasi-$IR_\mathfrak{p}$-unmixed ring for all minimal prime divisors $\mathfrak{p}$ of $(I, (\mathbf{x}))R$.*

Proof. We have $\operatorname{agd}_I((I, (\mathbf{x}))R) = \operatorname{height}(\mathbf{x})(R/I) = \operatorname{height} \mathfrak{p}/I$, for all minimal prime divisors $\mathfrak{p}$ of $(I, (\mathbf{x}))R$, by Proposition 3.3((ii) and (viii)), Corollary 4.5 and Generalized Principal Ideal Theorem, and so the result follows from Corollary 5.5. □

Corollary 5.7. *Let I be an ideal of a Noetherian ring R such that R is locally strongly quasi-I-unmixed ring. Let $x_1, \ldots, x_n$ be elements of R. Then $x_1 + I, \ldots, x_n + I$ is an asymptotic sequence in R/I if and only if $x_1, \ldots, x_n$ is an asymptotic sequence over I.*

Proof. Let $x_1 + I, \ldots, x_n + I$ be an asymptotic sequence in R/I, then $x_1, \ldots, x_n$ is an asymptotic sequence over I by [13, Remark 3.3(iii)] and Theorem 5.2. The converse follows from [9, Proposition 4.1]. □

In [13, Proposition 4.6], Ratliff proved that if I is an ideal of the principal class in a locally quasi-unmixed ring R and $\mathfrak{p}$ is a prime divisor of $\overline{I^n}$, for some $n \geq 1$, then $\operatorname{height} \mathfrak{p} = \operatorname{height} I$. We give a generalization of this result for asymptotic sequences over an ideal. Before this we need a lemma.

Lemma 5.8. *Let I and J be ideals of a Noetherian ring R and $x_1, \ldots, x_n$ be elements of J such that $J = (x_1, \ldots, x_n)$. If $\operatorname{agd}_I(J) = n$, then there exists an asymptotic sequence over I, $y_1, \ldots, y_n$, such that $J = (y_1, \ldots, y_n)$.*

Proof. If $n = 0$ the result is clear. Let $n > 0$, then $J \nsubseteq \bigcup\{\mathfrak{p} \mid \mathfrak{p} \in \overline{A^*}(I)\}$. There exists $b_1 \in (x_2, \ldots, x_n)R$ such that $x_1 + b_1 \notin \bigcup\{\mathfrak{p} \mid \mathfrak{p} \in \overline{A^*}(I)\}$ by [2, Theorem 125]. We set $y_1 = x_1 + b_1$. It is clear that $J = (y_1, x_2, \ldots, x_n)R$ and $(x_2, \ldots, x_n)R \nsubseteq \bigcup\{\mathfrak{p} \mid \mathfrak{p} \in \overline{A^*}((I, (y_1))R)\}$. There exists $b_2 \in (x_3, \ldots, x_n)R$ such that $x_2 + b_2 \notin \bigcup\{\mathfrak{p} \mid \mathfrak{p} \in \overline{A^*}((I, (y_1))R)\}$ by [2, Theorem 125]. We set $y_2 = x_2 + b_2$. It is clear that $J = (y_1, y_2, x_3, \ldots, x_n)R$. We continue this process and replace x_i with y_i for $i = 1, \ldots, n$. Therefore $J = (y_1, \ldots, y_n)R$ such that $y_1, \ldots, y_n$ is an asymptotic sequence over I. □

Proposition 5.9. *Let I and J be ideals of a Noetherian ring R such that R is locally strongly quasi-I-unmixed. If $J(R/I)$ is an ideal of principal class, then $\operatorname{height} \mathfrak{p}/I = \operatorname{height} J(R/I)$, for all prime divisors $\mathfrak{p}$ of $\overline{(I + J)^h}$ and $h \geq 1$.*

Proof. There exists a sequence of elements of R, $\mathbf{x} = x_1, \ldots, x_n$, such that $I + J = (I, (\mathbf{x}))R$, then $\operatorname{agd}_I((\mathbf{x})R) = n$, by Proposition 3.3(ii) and Theorem 5.2. Then there exists an asymptotic sequence over I, $y_1, \ldots, y_n$, such that $(\mathbf{x})R = (y_1, \ldots, y_n)R$, by Lemma 5.8. So the result follows from Proposition 3.14 and Theorem 5.2. □

In Proposition 5.10, we give a stronger result than Proposition 5.9.

Proposition 5.10. *Let I and J be ideals of a Noetherian ring R such that R is locally strongly quasi-I-unmixed. If $J(R/I)$ is an ideal of principal class, then* $\operatorname{height} \mathfrak{p}/I = \operatorname{height} J(R/I)$*, for all prime divisors $\mathfrak{p}$ of $((I+J)^h)_a$ and $h \geq 1$.*

Proof. The proof is the same as that given to prove Proposition 5.9, but use Proposition 3.16 in place of Proposition 3.14. □

Theorem 5.11. *Let $I \subseteq \mathfrak{p}$ be ideals of a Noetherian ring R such that $\mathfrak{p}$ is a prime ideal and $R_\mathfrak{p}$ is strongly quasi-$IR_\mathfrak{p}$-unmixed ring. Let* $\mathscr{P} = \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{p} \subset \mathfrak{q}, \operatorname{height} \mathfrak{q}/\mathfrak{p} = 1$ and $R_\mathfrak{q}$ is not strongly quasi-$IR_\mathfrak{q}$-unmixed ring$\}$*, then $\mathscr{P}$ is finite.*

Proof. Let $\mathscr{P}$ be infinite. Let

$$\mathscr{P}' = \{\mathfrak{q} \in \mathscr{P} \mid \operatorname{height} \mathfrak{q}/I = \operatorname{height} \mathfrak{p}/I + 1\},$$

then $\mathscr{P}'$ is infinite by [8, Theorem 1]. Let $\operatorname{height} \mathfrak{p}/I = n$, then $\operatorname{agd}_{IR_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}) = n$ by Theorem 5.2 and Corollary 5.3, and so there exists a sequence, $\mathbf{x} = x_1, \ldots, x_n$, in R such that its image in $R_\mathfrak{p}$ is a maximal asymptotic sequence over $IR_\mathfrak{p}$. Let $\mathscr{Q} = \bigcup_{i=1}^n \{\mathfrak{P} \in \overline{A^*}((I, (x_1, \ldots, x_{i-1}))R) \mid x_i \in \mathfrak{P}\}$ and

$$\mathscr{P}'' = \{\mathfrak{q} \in \mathscr{P}' \mid \mathfrak{P} \nsubseteq \mathfrak{q} \text{ for all } \mathfrak{P} \in \mathscr{Q}\}.$$

Since $\mathscr{Q}$ is finite and each $\mathfrak{P} \in \mathscr{Q}$ can be contained in at most finitely many $\mathfrak{q} \in \mathscr{P}'$ it follows $\mathscr{P}''$ is infinite. Also, the image of $\mathbf{x}$ in $R_\mathfrak{q}$ is an asymptotic sequence over $IR_\mathfrak{q}$ for all $\mathfrak{q} \in \mathscr{P}''$. Let

$$\mathscr{P}''' = \{\mathfrak{q} \in \mathscr{P}'' \mid \mathfrak{q} \notin \overline{A^*}((I, (\mathbf{x}))R)\},$$

then $\mathscr{P}'''$ is infinite. It is clear that the image of $\mathbf{x}$ in $R_\mathfrak{q}$ is not a maximal asymptotic sequence over $IR_\mathfrak{q}$ for all $\mathfrak{q} \in \mathscr{P}'''$ and so $\operatorname{agd}_{IR_\mathfrak{q}}(\mathfrak{q}R_\mathfrak{q}) > n$. On the other hand $\operatorname{agd}_{IR_\mathfrak{q}}(\mathfrak{q}R_\mathfrak{q}) \leq \operatorname{height} \mathfrak{q}R_\mathfrak{q}/IR_\mathfrak{q} = \operatorname{height} \mathfrak{q}/I = n+1$, by Theorem 4.4. Therefore $\operatorname{agd}_{IR_\mathfrak{q}}(\mathfrak{q}R_\mathfrak{q}) = \operatorname{height} \mathfrak{q}R_\mathfrak{q}/IR_\mathfrak{q}$ and so $R_\mathfrak{q}$ is strongly quasi-$IR_\mathfrak{q}$-unmixed ring for all $\mathfrak{q} \in \mathscr{P}'''$, by Theorem 5.2 and Corollary 5.3. This is a contradiction. □

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