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Almost perfect restriction semigroups

Peter R. Jones

Department of Mathematics, Statistics and Computer Science, Marquette University, Milwaukee, WI 53201, USA

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ABSTRACT

We call a restriction semigroup *almost perfect* if it is proper and the least congruence that identifies all its projections is perfect. We show that any such semigroup is isomorphic to a 'W-product' $W(T,Y)$, where T is a monoid, Y is a semilattice and there is a homomorphism from T into the inverse semigroup TI_Y of isomorphisms between ideals of Y. Conversely, all such W-products are almost perfect. Since we also show that every restriction semigroup has an easily computed cover of this type, the combination yields a 'McAlister-type' theorem for all restriction semigroups. It is one of the theses of this work that almost perfection and perfection, the analogue of this definition for restriction monoids, are the appropriate settings for such a theorem. That these theorems do *not* reduce to a general theorem for inverse semigroups illustrates a second thesis of this work: that restriction (and, by extension, Ehresmann) semigroups have a rich theory that does not consist merely of generalizations of inverse semigroup theory. It is then with some ambivalence that we show that all the main results of this work easily generalize to encompass *all* proper restriction semigroups.

The notation $W(T,Y)$ recognizes that it is a far-reaching generalization of a long-known similarly titled construction. As a result, our work generalizes Szendrei's description of almost factorizable semigroups while at the same time

E-mail address: peter.jones@mu.edu.

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including certain classes of free restriction semigroups in its realm.

1. Introduction

The study of the structure of restriction semigroups has in large part been motivated by consideration of structure theorems for inverse semigroups. For instance, the Munn representation of inverse semigroups by isomorphisms between the principal ideals of its semilattice of idempotents generalizes naturally [6] to representations of restriction semigroups by similar mappings of its semilattice of projections, and these generalized representations are at the very heart of our work. The 'inductive groupoid' approach to inverse semigroups has been extended successfully to restriction semigroups by Lawson [16].

Somewhat complementary to the Munn representation is the McAlister theory, whereby every inverse semigroup is an idempotent-separating image of an E-unitary inverse semigroup, and the latter semigroups are described as 'P-semigroups', relative to their semilattices, greatest group images and one further structural parameter. This theory has been extended with success to restriction semigroups, with E-unitariness replaced by 'properness' and pairs of actions replacing a single one.

When moving yet further from inverse semigroups, Branco, Gomes and Gould [1,7] introduced the notion of T-properness of (one-sided) Ehresmann semigroups, with respect to a submonoid T. The main thesis of our work is that (returning to the realm of restriction semigroups) a modification of this idea yields narrower notions of properness that are the appropriate ones in which to prove a 'McAlister-type' theory. That this theory specializes in the case of proper inverse semigroups to a very narrow subclass we take to be a witness to our thesis, rather than the contrary. Our results suggest that the road taken for Ehresmann semigroups in the cited papers is indeed a natural one.

To illustrate that this is not merely a conceit, we state the main result of this paper, to illustrate its simplicity. In fact, we prove a more general theorem, applying to all proper restriction semigroups, about which more will be said below. Recall first that for restriction semigroups, monoids, considered as restriction semigroups with a single projection, play the role that groups play for inverse semigroups and that such a semigroup is *proper* if the least congruence σ that identifies all the projections (loosely, the least monoid congruence) meets each class of the generalized Green relations $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{R}}$ trivially.

We call a restriction semigroup S *almost perfect* if it is proper and σ is perfect (meaning that the product of classes is again a class). The reason for the qualifier 'almost' is that we term a restriction *monoid M perfect* if, further, each σ-class has a greatest element, that is, M is also an *F-restriction monoid*. The connections with T-properness will be made below.

Now suppose T is a monoid, Y is a semilattice and there is a representation of T in the inverse semigroup TI_Y of isomorphisms between ideals of Y. For $t \in T$, denote by Δt and ∇t the domain and range, respectively of its image in TI_Y. The representation may be expressed, in the language of actions, as $(e,t) \mapsto e^t$, for $t \in T$ and $e \in \Delta t$. Let

$$W(T,Y) = \{(t,e^t) \in T \times Y : e \in \Delta t\} \quad (= \{(t,f) \in T \times Y : f \in \nabla t\}).$$

The multiplication in this 'W-product' is defined by $(t,e^t)(u,f^u) = (tu,(e^t f)^u)$; unary operations are defined by $(t,e^t)^+ = (1,e)$ and $(t,f)^* = (1,f)$.

Theorem 1.1. *(See Theorems 5.1 and 6.1, Corollary 4.2.) Every restriction semigroup has an almost perfect (projection-separating) cover. Every semigroup $W(T,Y)$ is almost perfect and, conversely, every almost perfect restriction semigroup S is isomorphic to a semigroup of that form.*

Every restriction monoid has a perfect monoidal cover. If the monoid T acts by isomorphisms between principal *ideals of a semilattice $Y = Y^1$, then $W(T,Y)$ is a perfect monoid and, conversely, every perfect restriction monoid M is isomorphic to such a monoid, where Y is the semilattice P_M of projections of M, $T = M/\sigma$, and the action is induced by the (generalized) Munn representation of M.*

We must emphasize that our construction, although using the same notation, is more general than the original construction [5,8,20], which corresponds precisely to the case that the representation is by *fully defined* isomorphisms between ideals of Y, that is, by injective endomorphisms whose ranges are ideals of Y. See the discussion following Theorem 5.1, and its application in Proposition 7.4.

The connection with the term 'T-proper' mentioned earlier is a central part of this work. We call a proper monoid M *strongly T-proper* if it contains a plain submonoid T such that $M = P_M T$ and T is separated by σ; equivalently, each element of M can be uniquely expressed in the form et, where $e \in P_M, t \in T, e \leq t^+$. In fact, strong T-properness is equivalent to perfection (Proposition 3.2). For a restriction *semigroup*, almost perfection is equivalent to *almost T-properness* (Proposition 3.3), which is the property that the monoid $C(S)$ of permissible sets, in which S embeds, is strongly T-proper (that is, perfect).

In the semigroup case of the theorem above, the representation is obtained as follows: Y is again the semilattice of projections, T is a monoid such that $S^1 = P_{S^1}T$, and the action is induced by the extension of the (generalized) Munn representation of S to the monoid $C(S)$. The key role played by $C(S)$ was suggested by its role in the determination of proper left factorizable and almost factorizable semigroups by Gomes and Szendrei [8,20], which together with their use of the original construction $W(T,Y)$ provided some of the motivation for our techniques.

In Section 7, our construction is applied to both the one- and two-sided versions of factorizable and almost factorizable semigroups, to free restriction semigroups and monoids and to relatively free semigroups in certain varieties of restriction semigroups. The application to freedom provided the second main impetus for the 'T-proper' approach to our work.

In fact, the construction of the semigroups $W(T,Y)$ and the constructions used in the covering theorems – and moreover a description of *all* such covers (Corollary 4.5) – are all instances of a very general construction $S_{T,R}$ (Theorem 4.1) which, despite its somewhat technical hypotheses, has a simple verification since the calculations are performed within a direct product $S \times T$. The virtue of this approach is witnessed in the following section, when all the properties (associativity included) of the W-product follow. Further, the representation theorem by the W-product also then follows from the description of covers mentioned above.

In the penultimate section of the paper, a further level of generality, about which we have a certain ambivalence, is achieved by weakening the requirement that the representation of T in TI_Y, introduced prior to Theorem 1.1, be a homomorphism: we now must allow *subhomomorphisms*. These are mappings α such that only $(a\alpha)(b\alpha) \leq (ab)\alpha$, referring to the natural partial order, is required. By means of this generalization, *all* proper restriction semigroups are described via the W-product construction. In fact, all the general theorems of the paper have straightforward extensions to the proper case by this means. Expressed in rather different language, our construction is an alternative formulation to the semigroups $\mathcal{M}(T,Y)$ used in [2,3] to describe proper restriction semigroups.

Our ambivalence stems from the main thesis of this paper: that the concept of perfection (for monoids) and almost perfection (for semigroups) is in reality the 'optimum' one, rather than properness *per se*, with Theorem 1.1 and the examples in Section 7 our support for this thesis, in practical terms, along with the elegance of homomorphic actions by monoids, rather than subhomomorphic ones.

The final section specializes the general results of the previous section to inverse semigroups: the representation turns out to be essentially that of Petrich and Reilly [19], though this played no role in the development of our work. More importantly, we demonstrate that the covering theorems, in particular, do *not* extend in a meaningful way to inverse semigroups, producing a restriction semigroup even when starting from an inverse semigroup, in general.

Although we believe that this paper incorporates a new approach, there is a already a body of work on the structure of proper restriction semigroups. Lawson [15] obtained a structure theorem for proper 'type A' semigroups, based more on the work of Petrich and Reilly [19], cited above, than on McAlister's theorem *per se*. As mentioned above, Cornock and Gould [3] obtained one for proper restriction semigroups in general, again using as parameters a monoid that acts partially on both sides of a semilattice.

As this paper was nearing submission, the author received a copy of a preprint of the article [14] by G. Kudryavtseva. Although, by and large, her paper is complementary to this one, in goals and approach, being primarily expressed in terms of actions, there is significant overlap. For instance, F-restriction monoids appear under the same name (unsurprisingly, given their genesis in F-inverse monoids). Her 'ultra-F restriction' monoids coincide with our perfect restriction monoids. Similarly her 'ultra proper restriction semigroups' coincide with our almost perfect restriction semigroups. Thus her structural results on these classes of necessity offer alternative viewpoints on ours. At the end of Section 2.2 we outline the connection, basing the connection on our fundamental Corollary 2.8, as suggested by Kudryavtseva in a private communication.

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2. Preliminaries

We first introduce restriction semigroups more formally, along with their basic properties and related definitions. A *left restriction* semigroup is a unary semigroup $(S, \cdot, ^+)$ that satisfies the 'left restriction' identities

$$x^+x = x; \qquad (x^+y)^+ = x^+y^+; \qquad x^+y^+ = y^+x^+; \qquad xy^+ = (xy)^+x.$$

A *right restriction* semigroup is a unary semigroup $(S, \cdot, ^*)$ that satisfies the 'dual' identities, obtained by replacing $^+$ by * and reversing the order of each expression. A *restriction* semigroup is then a biunary semigroup $(S, \cdot, ^+, ^*)$ that satisfies both sets of identities, along with $(x^+)^* = x^+$ and $(x^*)^+ = x^*$.

From these identities it follows that for all $x \in S$, x^+ is idempotent and $(x^+)^+ = x^+$. These idempotents are the *projections* of S; by duality these are also the idempotents x^*, $x \in S$. Denote the set of projections by P_S and the set of idempotents by E_S. Although, by the third identity, P_S is a semilattice, this need by no means be true of E_S. The following consequence of the identities is well known.

Lemma 2.1. *Let S be a restriction semigroup. Then S satisfies $x^+ \geq (xy)^+$ and $(xy)^+ = (xy^+)^+$ and their duals, namely $y^* \geq (xy)^*$ and $(xy)^* = (x^*y)^*$.*

Until quite recently, the term 'weakly E-ample' was used for restriction semigroups, providing evidence of a succession of generalizations – by the so-called York school – of Fountain's 'ample semigroups', which we will define below.

A restriction monoid is a restriction semigroup with identity 1. When adjoining an identity element to a restriction semigroup, setting $1^+ = 1^* = 1$ ensures that a restriction monoid is obtained. In the standard terminology, restriction semigroups S with $|P_S| = 1$, necessarily monoids, are termed *reduced*. On the other hand, 'plain' monoids may be regarded as reduced restriction semigroups, setting $a^+ = a^* = 1$ for all a. Since there is considerable room for ambiguity in this article, we shall use either 'plain' or 'reduced' to distinguish such monoids from restriction semigroups that also happen to be restriction monoids.

The relevant generalized Green relations may be defined as follows. In any restriction semigroup we put $\widetilde{\mathcal{R}} = \{(a,b) : a^+ = b^+\}$, $\widetilde{\mathcal{L}} = \{(a,b) : a^* = b^*\}$ and $\widetilde{\mathcal{H}} = \widetilde{\mathcal{L}} \cap \widetilde{\mathcal{R}}$. It follows easily from Lemma 2.1 that each contains the corresponding usual Green relations, that $\widetilde{\mathcal{R}}$ is a left congruence and that $\widetilde{\mathcal{L}}$ is a right congruence. In the standard literature, these relations would have been denoted $\widetilde{\mathcal{R}}_E$, $\widetilde{\mathcal{L}}_E$ and $\widetilde{\mathcal{H}}_E$, respectively, referring to a 'distinguished semilattice of idempotents' E. When restriction semigroups are defined as we have done, E is necessarily P_S and so there is no ambiguity. Due to the potential for conflict, the author has nevertheless used $\mathbb{R}$, $\mathbb{L}$ and $\mathbb{H}$ instead in recent work such as [12], but hopes the notation used here will prove standard in future. A left restriction semigroup is *left ample* if $\widetilde{\mathcal{R}}$ coincides with the relation $\mathcal{R}^*= \{(a,b) : xa = xb \text{ if and only if } xb = yb, \text{ for all } x, y \in S^1\}$; *right ampleness* is defined dually; a restriction semigroup is *ample* if both left and right ample.

The *natural partial order* on a restriction semigroup S is defined by $a \leq b$ if $a = eb$ for some $e \in P_S$; equivalently if $a = a^+b$. It is self-dual, compatible with the operations on S and extends the usual order on P_S. Put $a{\downarrow}= \{b \in S : b \leq a\}$, the principal order ideal generated by a. An *order ideal* of S is then a nonempty subset that is closed under $\downarrow$. The order ideals of a semilattice are just its ideals.

In general, the term 'homomorphism' will be used appropriate to context: that is, it should respect the binary operation for plain semigroups, and either or both unary operations for one-sided or two-sided restriction semigroups. In the case of monoids, we shall use the qualifier 'monoidal' to indicate that it should respect the identity elements. When considering subsemigroups (or submonoids), we shall use qualifiers such as 'biunary' or 'plain' when the situation might not be clear. A plain submonoid of a restriction monoid, for instance, is any subsemigroup having as its identity that of the larger monoid. A biunary subsemigroup of a restriction semigroup S is *full* if it contains all of the projections of S. A homomorphism $S \longrightarrow T$ of restriction semigroups is *P-separating* (or projection-separating) if it is injective on P_S.

Likewise, the term 'congruence' is used appropriate to context. We denote the greatest P-separating congruence by μ and observe that $\mu \subseteq \widetilde{\mathcal{H}}$. In the standard terminology, a congruence ρ on a semigroup is *perfect* if $(a\rho)(b\rho) = (ab)\rho$.

Denote by σ the least congruence on a restriction semigroup S that identifies all projections (easily seen also to be the least 'plain' congruence with this property). Clearly σ is the least congruence on S whose quotient is *reduced*, as defined above. Thus, σ might loosely be called the 'least monoid congruence' by analogy with the term 'least group congruence', to which σ specializes on inverse semigroups. It is well known that

$$\sigma = \{(a,b) \in S : ea = eb \text{ for some } e \in P_S\} = \{(a,b) \in S : af = bf \text{ for some } f \in P_S\}.$$

It is clear that any principal order ideal of S is contained in a σ-class and that any σ-class is an order ideal of S.

A restriction semigroup S is *proper* if $\widetilde{\mathcal{R}} \cap \sigma = \widetilde{\mathcal{L}} \cap \sigma = \iota$ (where ι is the identical relation). From this definition it is immediate that $\sigma \cap \mu = \iota$, that is, S is a subdirect product of S/σ and S/μ.

Given a restriction semigroup U, a *cover for U [over a monoid T]* is a proper restriction semigroup S [such that $S/\sigma \cong T$] having U as a P-separating homomorphic image.

We recall the primary definitions of this paper. A restriction semigroup S is *almost perfect* if it is proper and σ is perfect. A restriction monoid is *perfect* if it is proper, σ is perfect and each σ-class has a greatest element. By extension of the usual term for inverse semigroups, a restriction semigroup is *F-restriction* if it is proper and each σ-class has a greatest element. Since in that case the projections form a σ-class, such a semigroup is of necessity a monoid. In any such monoid, let m_a denote the greatest element of $a\sigma$, so that $a\sigma = m_a{\downarrow}$.

The following elementary lemma plays a key role in this paper (and motivates consideration of subhomomorphisms although, in the body of the paper, full consideration to the latter will only be given in Section 8). A mapping $\alpha : S \longrightarrow T$ of restriction semigroups is a *subhomomorphism* if $(a\alpha)(b\alpha) \leq (ab)\alpha$ for all $a, b \in S$ (and the unary operations are respected, which occurs automatically if S is a monoid, interpreted as either a plain monoid or a restriction monoid, and α is monoidal). The reader should beware that, for inverse semigroups, Lawson [17] calls such maps dual prehomomorphisms, while for Petrich [18] a prehomomorphism is a subhomomorphism that, in addition, respects inversion (also see Section 9).

Lemma 2.2. *Let M be an F-restriction monoid and put $T = M/\sigma$. The relation $m_a m_b \leq m_{ab}$ always holds. Thus the map $\kappa_M : T \longrightarrow M$, $(a\sigma)\kappa_M = m_a$, is a monoidal subhomomorphism.*

Further, M is perfect if and only if $m_a m_b = m_{ab}$ for all $a, b \in M$ and, therefore, if and only if κ_M is a homomorphism.

Proof. Since $(a\sigma)(b\sigma) \subseteq (ab)\sigma$ always holds, the same is true for $m_a m_b \leq m_{ab}$. Thus κ_M is a subhomomorphism, and $(a\sigma)\kappa_M = m_1 = 1$.

Now suppose M is perfect, so that $(a\sigma)(b\sigma) = (ab)\sigma$. Then $m_{ab} = a_1 b_1$ for some $a_1 \in a\sigma$ and $b_1 \in b\sigma$. Since $a_1 \leq m_a$ and $b_1 \leq m_b$, $m_{ab} \leq m_a m_b$, so that equality holds. Conversely, suppose $m_a m_b = m_{ab}$ and let $x \in (ab)\sigma$. Then $x \leq m_a m_b$, so $x = (x^+ m_a)(m_b) \in (a\sigma)(b\sigma)$. $\square$

The motivation for the term 'T-proper' comes from consideration of generators. Suppose a restriction semigroup S is generated, as such, by a subset X. It is well known that every non-projection of S is expressible as the product of a projection and an element of the subsemigroup $\langle X\rangle$ generated by X. Thus for every $s \in S$, either $s \in P_S$ or $s \leq t$ for some $t \in \langle X\rangle$, in which case, $s = s^+ t = ts^*$. Let T denote the plain submonoid of S^1 generated by X. Then $S = P_S T$. In this case we say that S^1 is *P-generated* by T.

Finally, recall that any inverse semigroup $(S, \cdot, ^{-1})$ may be regarded as a restriction semigroup by setting $x^+ = xx^{-1}$ and $x^* = x^{-1}x$ and 'forgetting' the inverse operation. In that case, $P_S = E_S$. As noted above, σ is now the least group congruence and the term 'E-unitary' is more commonly used, rather than 'proper'. Although, in a proper restriction semigroup, P_S is indeed a unitary subset, the converse need not be true.

The inverse semigroups of most significance in this paper are the topic of the next subsection.

2.1. Munn semigroups and the (generalized) Munn representations

If Y is a semilattice, T_Y denotes the *Munn* semigroup on Y: the inverse subsemigroup of the symmetric inverse semigroup $\mathcal{I}_Y$ consisting of the isomorphisms between *principal* ideals of Y. For an exposition of this semigroup, its basic properties, and application to inverse semigroup theory, see [10]. The dual semigroup T_Y^r again consists of the isomorphisms between principal ideals, but with functions instead written on the left and composition reversed. Denote by γ the isomorphism $T_Y \longrightarrow T_Y^r$ that is induced by inversion.

Less familiar is the inverse subsemigroup of $\mathcal{I}_Y$, which we denote TI_Y, consisting of the isomorphisms between *arbitrary* ideals of Y, which plays a central role in this paper. In [18], this semigroup is denoted $\Sigma(Y)$. In Proposition 2.6 we shall provide an alternative representation of TI_Y (and cite another one at the end of the following subsection).

Throughout this paper, the domain and range of a mapping α are denoted $\Delta\alpha$ and $\nabla\alpha$.

Let S be a restriction semigroup and put $Y = P_S$. The *generalized (right) Munn representation* $\theta : S \longrightarrow T_Y$ is defined by $a \mapsto \theta_a$, where $\Delta\theta_a = a^+{\downarrow}$ and, for $e \leq a^+$, $e\theta_a = (ea)^*$. In the case of an inverse semigroup, this reduces to the usual Munn representation. The generalized left Munn representation is the dual map $\psi : S \longrightarrow T_Y^r$, defined by $a \mapsto \psi_a$, where $\Delta\psi_a = a^*{\downarrow}$ and, for $f \leq a^*$, $f\psi_a = (af)^+$. For each a, θ_a and ψ_a are mutually inverse isomorphisms between their respective domains. In the sequel, we shall omit the qualifier 'generalized'.

The bulk of the following result is in [6, Proposition 5.2]. A broad generalization, framed in the language used herein, was found by the author [11].

Result 2.3. *Let S be a restriction semigroup and put $Y = P_S$. The maps θ and ψ are biunary, projection-separating homomorphisms from S onto full subsemigroups of T_Y and T_Y^r, respectively, related by $\psi = \theta\gamma$. Each induces the greatest P-separating congruence μ on S.*

If S is a monoid, then θ and ψ are monoidal.

2.2. The monoids $C(S)$ and their representations

We refer the reader to [20] for more details of the basic results cited here. In that work Szendrei defined $C(S)$ for the case of restriction semigroups in general, closely following [8] (itself based on the one-sided notion of El Qallali [4] and extending the definition in the case of inverse semigroups [18, V.2.2]).

A *permissible* subset A of a restriction semigroup S is a nonempty order ideal such that $a^+b = b^+a$ and $ab^* = ba^*$ for all $a, b \in A$. (Thus for a semilattice the permissible subsets are simply the nonempty order ideals themselves.) Note that whenever the relation $a^+b = b^+a$ holds in S, it follows that $a \; \sigma \; b$, since $eb = ea$, where $e = a^+b^+ \in P_S$. If S is proper, then the converse also holds: for if $a, b \in S$, the equations $(a^+b)^+ = a^+b^+ = b^+a^+ = (b^+a)^+$ always hold, while if $a \; \sigma \; b$, then also $a^+b \; \sigma \; b^+a$ and equality follows from properness. The dual is true, similarly. We summarize the properties that will be used in the sequel. Originally proved in [8] for the subclass of 'weakly ample' semigroups, the proofs carry over immediately to all restriction semigroups.

Result 2.4. *(See [20, Theorems 3.1, 3.2], [8, Proposition 3.8].) Let S be a restriction semigroup. The set $C(S)$ of all permissible subsets of S is a restriction monoid, under multiplication of subsets, with identity P_S, where if $A \in C(S)$, then $A^+ = \{a^+ : a \in A\}$ and $A^* = \{a^* : a \in A\}$; its natural partial order is inclusion; its semilattice of projections consists of the ideals of P_S, under inclusion (and is thus $C(P_S)$). The map $\tau_S : a \mapsto a{\downarrow}$ embeds S in $C(S)$. Extending τ_S to S^1, if necessary, by setting $1\tau_S = P_S$, embeds S^1 in $C(S)$ monoidally.*

The monoid $C(S)$ is proper if and only if S is proper. In that event, if $A \in C(S)$ then $A \subseteq a\sigma_S$ for any $a \in A$ and $a\sigma_S$ is the greatest element m_A of $A\sigma_{C(S)}$, so that $C(S)$ is an F-restriction monoid. The monoids S/σ and $C(S)/\sigma$ are therefore isomorphic, under the mapping $a\sigma_S \mapsto (a\sigma_S)\sigma_{C(S)}$.

Let $\theta : S \longrightarrow T$ be a homomorphism of restriction semigroups whose image is an order ideal. Then the mapping $\hat{\theta} : C(S) \longrightarrow C(T)$, defined by $A \mapsto A\theta$, is a monoidal homomorphism such that $\tau_S\,\hat{\theta} = \theta\,\tau_T$.

Proposition 2.5. *Let S be a proper restriction semigroup. Then $C(S)$ is perfect if and only S is almost perfect.*

Proof. Since $C(S)$ is an F-restriction monoid, by Lemma 2.2 it is perfect if and only if $m_A m_B = m_{AB}$ for all $A, B \in C(S)$. By the second paragraph of Result 2.4, this is equivalent to $(a\sigma)(b\sigma) = (ab)\sigma$ for all $a, b \in S$, that is, to perfection of σ_S. □

Proposition 2.6. *Let Y be a semilattice. The map $\Sigma : C(T_Y) \longrightarrow TI_Y$, where $A\Sigma$ is the union of the members of A, regarded as relations on Y, is an isomorphism such that $\tau_{T_Y}\Sigma$ is the inclusion map $T_Y \longrightarrow TI_Y$.*

Proof. If $A \in C(T_Y)$, then A is an order ideal that further satisfies $\alpha^+\beta = \beta^+\alpha$ for all $\alpha, \beta \in A$. The latter property says that any two members of A agree on the intersection of their domains, and so $A\Sigma$ is a well-defined order isomorphism between the ideals consisting of the unions of the domains and ranges, respectively, of the members of A.

Conversely, if $\gamma \in TI_Y$, let $\gamma\Psi = \{\gamma_{|\,e\downarrow} : e \in \Delta\gamma\}$. For each $e \in \Delta\gamma$, $\gamma_{|\,e\downarrow} : e\!\downarrow \longrightarrow (e\gamma)\!\downarrow$ and so $\gamma_{|\,e\downarrow} \in T_Y$. If $\alpha \in T_Y$, with $\Delta\alpha = f\!\downarrow$, say, and $\alpha \subseteq \gamma_{|\,e\downarrow}$, then since $f \le e$, $f \in \Delta\gamma$ and so $\alpha = \gamma_{|\,f\downarrow}$. Since the members of $\gamma\Psi$ also obviously agree on their intersections, $\gamma\Psi \in C(T_Y)$.

To show that Σ and Ψ are mutually inverse bijections, first let $\gamma \in TI_Y$. Then $\gamma\Psi\Sigma = \bigcup\{\gamma_{|\,e\downarrow} : e \in \Delta\gamma\} \subseteq \gamma$. But if $(f, g) \in \gamma$, then $f \in \Delta\gamma$ and $(f, g) \in \gamma_{|\,f\downarrow}$. Thus the reverse inclusion also holds. Next let $A \in C(T_Y)$ and put $\delta = A\Sigma = \bigcup_{\alpha\in A} \alpha$. For each $e \in \Delta\delta$, $e \in \Delta\alpha$ for some $\alpha \in A$, so that $\delta_{|\,e\downarrow} = \alpha_{|\,e\downarrow} \in A$ (since A is an order ideal). But if $\alpha \in A$, with $\Delta\alpha = f\!\downarrow$, say, then $f \in \Delta\delta$ and, since δ extends α, $\alpha = \delta_{|\,f\downarrow} \in \delta\Psi$. Therefore $A = A\Sigma\Psi$.

To show Σ is a homomorphism, let $A, B \in C(T_Y)$. By the nature of the definitions, it suffices to show that the domains of $(AB)\Sigma$ and $(A\Sigma)(B\Sigma)$ agree. Put $\gamma = A\Sigma$ and $\delta = B\Sigma$. Then

$$\Delta(\gamma\delta) = (\nabla\gamma \cap \Delta\delta)\gamma^{-1} = ((\bigcup_{\alpha\in A} \nabla\alpha) \cap (\bigcup_{\beta\in B} \Delta\beta))\gamma^{-1} = (\bigcup_{\alpha\in A}\bigcup_{\beta\in B} (\nabla\alpha \cap \Delta\beta))\gamma^{-1}.$$

Now for each $\alpha \in A$ and $\beta \in B$, $\nabla\alpha \cap \Delta\beta \subseteq \nabla\alpha \subseteq \nabla\gamma$, so

$$\Delta(\gamma\delta) = \bigcup_{\alpha\in A}\bigcup_{\beta\in B} (\nabla\alpha \cap \Delta\beta)\alpha^{-1} = \bigcup_{\alpha\in A}\bigcup_{\beta\in B} \Delta(\alpha\beta) = \Delta((AB)\Sigma).$$

Therefore Ψ and Σ are mutually inverse isomorphisms. If $\alpha \in T_Y$, then since $\alpha\!\downarrow$ is a principal order ideal, the union of its members is just α itself, yielding the final statement. $\square$

Proposition 2.7. *For any restriction semigroup S, the Munn representation $\theta : S \longrightarrow T_{P_S}$ induces the homomorphism $\overline{\theta} = \hat{\theta}\Sigma : C(S) \longrightarrow TI_{P_S}$, satisfying $\tau_S\overline{\theta} = \theta$. As usual, if $A \in C(S)$, we denote $A\overline{\theta}$ by $\overline{\theta}_A$. Then $\overline{\theta}_A : A^+ \longrightarrow A^*$ and, if $e \in A^+$, $e\overline{\theta}_A = b^*$, where $b \in A$, $b^+ = e$.*

Proof. For convenience, put $Y = P_S$ and see Fig. 1 below. Since the image of θ in T_Y is full (Result 2.3), it is an order ideal. By Result 2.4, $\hat{\theta} : C(S) \longrightarrow C(T_Y)$ is a well defined homomorphism. Thus $\overline{\theta}$ is a homomorphism. Combining the two previous results, $\tau_S\overline{\theta} = \tau_S\,\hat{\theta}\Sigma = \theta\,\tau_{T_Y}\,\Sigma = \theta$. Finally, if $A \in C(S)$, then $A\overline{\theta} = \bigcup_{a\in A} \theta_a$, from which the formula $e\overline{\theta}_A = b^*$ follows (putting $b = ea$, for any $a \in A$ such that $e \le a^+$). $\square$

We will call $\overline{\theta}$ the *extension* of θ to $C(S)$. It is easily checked that if S is a monoid, then all the homomorphisms above preserve the relevant identity elements.

$$\begin{tikzcd}
T \arrow[r, "\kappa"] & C(S) \arrow[r, "\hat{\theta}"] & C(T_Y) \arrow[r, "\Sigma"] & TI_Y \\
 & S \arrow[u, "\tau_S"] \arrow[r, "\theta"] & T_Y \arrow[u, "\tau_{T_Y}"] &
\end{tikzcd}$$

Fig. 1. The mappings in this section.

Referring back to Result 2.4, in the event that S is proper and we put $T = S/\sigma$, then by a slight abuse of notation we shall denote by κ the injection $T \longrightarrow C(S)$, $a\sigma_S \mapsto a\sigma_S$. Notice that κ is the subhomomorphism induced by $\kappa_{C(S)} : C(S)/\sigma_{C(S)} \longrightarrow C(S)$, according to Lemma 2.2, under the isomorphism $T \cong C(S)/\sigma_{C(S)}$ exhibited in Result 2.4. For if $a \in S$ and we put $A = a\sigma$, then $(A\sigma_{C(S)})\kappa_{C(S)} = m_A = A = A\kappa$. Further, by the same lemma and Proposition 2.5, κ is a homomorphism if and only if S is almost perfect.

Now the combination of Proposition 2.7, Lemma 2.2, applied to $C(S)$, and Proposition 2.5 yields the following corollary, which is at the heart of this paper.

Corollary 2.8. *Let S be a proper restriction semigroup. Put $T = S/\sigma$ and $Y = P_S$ and let κ be the injection of T in $C(S)$ just defined. Then the composition $\kappa\overline{\theta} : T \longrightarrow TI_Y$ is a monoidal subhomomorphism, which is a homomorphism if and only if σ is perfect on S (that is, S is almost perfect).*

Proof. All that remains to show is that if $\kappa\overline{\theta}$ is a homomorphism, then σ is perfect on S. Let $a, b \in S$. In the monoid T, $(ab)\sigma = (a\sigma)(b\sigma)$, so $((ab)\sigma)\kappa\overline{\theta} = (a\sigma)\kappa\overline{\theta}\,(b\sigma)\kappa\overline{\theta} = ((a\sigma)\kappa(b\sigma)\kappa))\overline{\theta}$. Recall that κ injects T into $C(S)$. Regarding the σ-classes now as elements of $C(S)$, we use the notation and details of Proposition 2.7 to write the last equation as $\overline{\theta}_{(ab)\sigma} = \overline{\theta}_{a\sigma b\sigma}$. It follows that the domains of these two mappings agree, that is, again using the details of the cited proposition, $((ab)\sigma)^+ = (a\sigma b\sigma)^+$.

Now let $s \in (ab)\sigma$. Then for some $a_1 \in a\sigma$ and $b_1 \in b\sigma$, $s^+ = (a_1b_1)^+$, that is, $s\widetilde{\mathcal{R}}a_1b_1$. But $s\ \sigma\ ab\ \sigma\ a_1b_1$ so, by properness, $s = a_1b_1 \in a\sigma b\sigma$. Therefore σ is perfect. □

We state without proof, since it is not used in this paper, that for any semilattice Y, TI_Y is also isomorphic to the Munn semigroup of the semilattice $C(Y)$; and the homomorphisms $\hat{\theta} : C(S) \longrightarrow C(T_{P_S})$ and $\overline{\theta} : C(S) \longrightarrow TI_Y$ are equivalent to the Munn representations of $C(S)$.

Implicit in the proof of Corollary 2.8 was a direct description of the map $\kappa\overline{\theta}$. For $t \in T$, identify it with the σ-class $t\kappa \in C(S)$. Then, based again on the details of Proposition 2.7, write $\overline{\theta}_t$ for $(t\kappa)\overline{\theta}$. Here $\Delta\overline{\theta}_t$ and $\nabla\overline{\theta}_t$ are respectively the ideals $t^+ = \{a^+ : a \in t\}$ and $t^* = \{a^* : a \in t\}$; and for $e \in t^+$, $e\overline{\theta}_t = b^*$, where $b \in t$, $b^+ = e$.

Referring back to the article [14] by Kudryavtseva cited in the penultimate paragraph of Section 1, its author has kindly informed me that the interpretation of Corollary 2.8 just stated provides the most transparent linkage between the *ultra proper* restriction semigroups, introduced and studied in that article, and our almost perfect restriction semigroups.

Her definition is in terms of the *partial actions of T on P_S that underlie S* [14, Section 3.1]. While her work is phrased in terms of left actions, it is readily seen that the right partial action that underlies S, exhibited in the dual of equation (3.1), is precisely that induced by the subhomomorphism $\kappa\overline{\theta}$ of Corollary 2.8 (and indeed she offers there the alternative viewpoint of partial actions as being induced by subhomomorphisms – dual prehomomorphisms or dual antiprehomomorphisms, in her language).

She then [14, Section 3.8] calls a restriction semigroup *ultra proper* if its underlying right action is 'a partially defined action', which as she explains there holds if and only if the associated subhomomorphism is in fact a homomorphism. By our Corollary 2.8, this holds for a proper restriction semigroup if and only if the semigroup is almost perfect.

She terms a restriction monoid *ultra F-restriction* [14, Section 3.9] if it is ultra proper and F-restriction, using the latter term just as we do here. Therefore this definition is equivalent to our definition of perfect restriction monoids. We summarize.

Proposition 2.9. *The ultra proper restriction semigroups and ultra F-restriction monoids in [14] are precisely our almost perfect restriction semigroups and perfect restriction monoids, respectively.*

3. Strong T-properness and almost T-properness

Recall from Section 1 that a restriction monoid M, with plain submonoid T, is *strongly T-proper* if $M = P_M T$ and σ separates T; and that a restriction semigroup S is *almost T-proper* if $C(S)$ is strongly T-proper (with respect to some plain submonoid T). The reason for the qualifier 'strongly' is that 'T-properness', as used in [1] in the broader context of Ehresmann semigroups, specializes in proper restriction semigroups to a strictly weaker property. While, technically, we should also use the term 'almost strongly T-proper' instead of 'almost T-proper', the latter term generates no conflict and is less cumbersome.

The equation $M = P_M T$ merely states that M is P-generated by T, as a restriction monoid (in the terminology introduced in Section 2). By abuse of terminology, given any plain monoid T, we may call M strongly T-proper when it is strongly T'-proper with respect to some plain submonoid T' isomorphic to T.

In Propositions 3.2 and 3.3, we show that these definitions are equivalent, respectively, to those of perfection and almost perfection. Each form has its own benefits: the 'T-proper' versions have the virtue of following a path that has been fruitful for Ehresmann semigroups [9], and allow ready identification of examples; the 'perfect' versions have the virtue of independence from T and of straightforward verification once σ has been computed.

We begin with element-wise criteria for the monoidal case, illustrating calculations prevalent in the sequel. The global characterizations immediately follow.

Lemma 3.1. *Let M be a restriction monoid and T a plain submonoid. Then M is strongly T-proper if and only if for all $m \in M$, $m \leq t$ for some unique $t \in T$; equivalently, $m = m^+t$ for some unique $t \in T$; and equivalently if and only if for all $m \in M$, $m = tm^*$ for a unique $t \in T$. In that event, M is necessarily proper and $T \cong M/\sigma$.*

Proof. It was noted in Section 2 that M is P-generated by T if and only if for all $m \in M$, $m \leq t$ for some $t \in T$. Suppose M is strongly T-proper and $m \leq t, u$, where $m \in M$ and $t, u \in T$. Then $t \; \sigma \; m \; \sigma \; u$ and so equality holds. To prove the converse, suppose $t, u \in T$ and $t \; \sigma \; u$. Then for some $e \in P_S$, $et = eu = a$, say. Now $a \leq t, u$ and so, by hypothesis, $t = u$. So M is strongly T-proper. The equivalence of the next two statements follows from the discussion of σ in Section 2.

To prove the last statements, assume M is strongly T-proper for some plain monoid T, which we may take to be a submonoid, and suppose that $(a, b) \in \sigma \cap \widetilde{\mathcal{R}}$. Write $a = a^+t$ and $b = b^+u$, for $t, u \in T$. Then $t \; \sigma \; a \; \sigma \; b \; \sigma \; u$, so that $t = u$; and $a^+ = b^+$; so $a = b$. The dual statement is proved analogously. Now the submonoid T is a transversal of the σ classes, whereby it is isomorphic to the quotient monoid. □

Proposition 3.2. *The following are equivalent for a proper restriction monoid M:*

(i) M is strongly T-proper for some plain monoid T;
(ii) M is strongly M/σ-proper;
(iii) M is perfect.

Proof. The equivalence of (i) and (ii) follows from the last statement of the previous lemma. Now suppose (iii) holds. Then by Lemma 2.2, M is strongly T-proper with respect to $T = \{m_a : a \in M\}$. Under (i), if $a \in M$, then since M is P-generated by T, we have $a \; \sigma \; t$ for some $t \in T$ with $a \leq t$. Now σ separates T so it follows that $t = m_a$. Thus these greatest elements are precisely the members of the submonoid T. That is, $m_a m_b = m_{ab}$ for all $a, b \in M$. By Lemma 2.2, M is perfect. □

Turning now to an arbitrary restriction semigroup, we may of course consider strong T-properness of the monoid $M = S^1$. However, that is not general enough for our purposes, as will be seen in the sequel.

By Lemma 3.1, that a semigroup be almost T-proper is equivalent to the property that every $A \in C(S)$ be contained in a unique member of T. Also by that lemma, in that case $C(S)$ and, therefore, S itself is necessarily proper. The next result is the analogue of Proposition 3.2.

Proposition 3.3. *The following are equivalent for a proper restriction semigroup S:*

(i) S is almost T-proper for some plain monoid T;
(ii) S is almost S/σ-proper;
(iii) S is almost perfect.

Proof. According to Result 2.4, $C(S)/\sigma \cong S/\sigma$. The equivalence of (i) and (ii), and of each with perfection of $C(S)$, then follows from the definition and from applying the corresponding equivalence in Proposition 3.2 to $C(S)$.

Now the equivalence of perfection of $C(S)$ with almost perfection of S itself was shown in Proposition 2.5. □

In the case of a monoid, the relationship between these two definitions needs to be elucidated. We return to the original terminology of 'perfection'.

Proposition 3.4. *(1) Every perfect restriction monoid is almost perfect; the converse need not hold. (2) If S is a proper restriction semigroup without identity and S^1 is almost perfect, then S is almost perfect; the converse holds if S/σ has trivial group of units.*

Proof. The direct statement in (1) is obvious from the definitions. Example 7.1 shows that the converse need not be true.

To prove the direct statement in (2), note that in this case σ on S is simply the restriction of that on S^1. Now by assumption σ is perfect on S^1. But the only product in S^1 that yields 1 is $1 \cdot 1$, so σ is also perfect on S and S is therefore also almost perfect. For the converse, if σ on S is perfect, then so is σ on S^1, under the stated assumption. □

Two natural motivating classes of examples for strong T-properness and almost T-properness are (i) free (and certain relatively free) restriction semigroups and monoids, and (ii) factorizable and almost factorizable restriction semigroups (in one- and two-sided versions). Rather than break the flow at this point, we refer the reader to Section 7 for the precise connection. The constructions in Sections 4 and 5 provide a ready source of further examples.

4. Almost perfect covers

We present in Theorem 4.1 a far-reaching generalization $S_{T,R}$ of a construction first explicitly stated in [13, Theorem 9.1], although special cases have appeared elsewhere in the literature. Its simplicity is somewhat obscured by the generality in which we frame it. The pairs (S, R) of restriction semigroups of most interest satisfy $S \leq R \leq C(S)$ (identifying S with its image in $C(S)$ under τ_S). In particular, the pairs $(S, C(S))$ and (S, S^1) (recalling from Result 2.4 that S^1 embeds in $C(S)$) play distinct major roles in this paper, as demonstrated in Corollaries 4.3 and 4.2 respectively. The slightly more abstract setting that we have chosen simplifies the notation considerably and specializes more straightforwardly to the monoidal setting.

Because the specific monoid T is crucial to this section, we tend to prefer the 'T-proper' terminology instead.

Let R be a restriction monoid and S a restriction subsemigroup such that for each $r \in R$ the following conditions and their duals are satisfied: (i) there exists $e \in P_S$,

$e \leq r^+$, and (ii) for any such e, $er \in S$. Clearly this criterion is satisfied if $R = S^1$. It will be shown after the theorem that this is also the case for the other instances described in the previous paragraph. Let T be a plain monoid and $\alpha : T \longrightarrow R$ a monoidal homomorphism. In Section 8, α will be allowed to be a subhomomorphism. Let

$$S_{T,R} = \{(a,t) \in S \times T : a \leq t\alpha \text{ in } R\}.$$

Write S_T in case $R = S^1$.

Theorem 4.1. *As just defined, $S_{T,R}$ is an almost T-proper restriction subsemigroup of $S \times T$, with $S_{T,R}/\sigma \cong T$, now regarding T as a (reduced) restriction semigroup (that is, a plain monoid), and the first projection is a P-separating homomorphism onto a full subsemigroup of S. Further:*

1. *if $S \subseteq (T\alpha)\!\downarrow$, then $S_{T,R}$ is a subdirect product of S and T and is therefore a cover of S over T; in particular, if $R = S^1$ and R is P-generated by $T\alpha$, then S_T is a cover of S;*
2. *if S is a monoid and α is monoidal, then S_T is strongly T-proper.*

Proof. Let $(a,t), (b,u) \in S_{T,R}$. By compatibility of the natural partial order on R and the homomorphism property of α, $ab \leq (t\alpha)(u\alpha) \leq (tu)\alpha$, so $S_{T,R}$ is closed. Observe that α only need be a subhomomorphism for this to hold. For any $(a,t) \in S_{T,R}$, $(a,t)^+ = (a^+, 1)$ and $(a,t)^* = (a^*, 1)$, where $a^+, a^* \leq 1 = 1\alpha$, so $S_{T,R}$ is a restriction subsemigroup of $S \times T$. The semilattice $P_{S_{T,R}} = \{(e,1) : e \in P_S\} \cong P_S$.

By the assumption on S and R, for any $t \in T$ there exists $e \in P_S$ such that $e \leq (t\alpha)^+$ and $e(t\alpha) \in S$, so that $(e(t\alpha), t) \in S_{T,R}$. That is, the second projection map $S_{T,R} \longrightarrow T$ is surjective. In general, this is not true of the first projection map $S_{T,R} \longrightarrow S$. However since $(e,1) \mapsto e$, it is projection-separating and its image is a full subsemigroup.

Next we compute σ. Let $(a,t), (b,u) \in S_{T,R}$. If $(a,t)\ \sigma\ (b,u)$ then from the triviality of σ on T, $t = u$. Conversely, if $t = u$, then $a, b \leq t\alpha$. In the case $t = 1$, then $(a,1)$, $(b,1)$ are both projections and so are σ-related; otherwise, $(b^+,1)(a,t) = (b^+a^+(t\alpha), t) = (a^+b^+(t\alpha), t) = (a^+,1)(b,t)$, so that $(a,t)\ \sigma\ (b,t)$. Therefore the σ-classes of $S_{T,R}$ are the sets $S_t = \{(a,t) : a \leq t\alpha\}$, $t \in T$, and $S_{T,R}/\sigma \cong T$.

Now suppose $((a,t),(b,u)) \in \widetilde{\mathcal{R}} \cap \sigma$ in $S_{T,R}$. Then $t = u$ and $a = a^+t = b^+u = b$. In combination with the dual argument, this shows that $S_{T,R}$ is proper.

To prove almost T-properness, using Proposition 3.3 it must be shown that $S_tS_u = S_{tu}$. Clearly $S_tS_u \subseteq S_{tu}$; conversely, suppose $(c, tu) \in S_{tu}$, where $c \leq (tu)\alpha = (t\alpha)(u\alpha)$ (using the homomorphism property of α) and $t, u \in T$. Then $c = c^+(t\alpha)(u\alpha)c^*$, where $c^+, c^* \in P_S$, so that $c^+(t\alpha), (u\alpha)c^* \in S$, using the assumption (ii) on S and its dual. Therefore $(c, tu) = (c^+(t\alpha), t)((u\alpha)c^*, u) \in S_tS_u$.

To prove 1, suppose that $S \subseteq (T\alpha)\downarrow$. Then for each $a \in S$, $(a,t) \in S_{T,R}$ for some $t \in T$. Thus $S_{T,R}$ is a subdirect product of S and T and the first projection map, being already projection-separating, is a covering map.

To prove 2, suppose that S is a monoid (and $R = S$). Since S_T contains $(1,1)$, it is a monoid. Now for each $t \in T$, $(t\alpha, t)$ is the greatest element in the σ-class S_t. Thus S_T is F-restriction and therefore perfect (and so strongly T-proper). $\square$

We detail the special case $R = S^1$ since for each restriction semigroup it produces a very simple almost perfect cover. In Corollary 6.2 we will provide a simple alternative representation of the coverings in the next corollary by means of the W-semigroup construction. It will be shown below that, in the monoidal case, all perfect covers may be found as in Corollary 4.2. That when applied to an *inverse* semigroup S, this covering does not in general produce another inverse semigroup is at the heart of the divergence of our work from inverse semigroup theory (see Section 9).

Corollary 4.2. *Let S be a restriction semigroup. If T is a monoid and $\alpha : T \longrightarrow S^1$ is a monoidal homomorphism, the image of which P-generates S^1, then $S_T = \{(s,t) \in S \times T : s \leq t\alpha \text{ in } S^1\}$ is an almost T-proper cover of S that is a subdirect product of S and T. In particular, if S is generated, as a restriction semigroup, by a subset X and T is the plain submonoid of S^1 generated by X, then $S_T = \{(s,t) \in S \times T : s \leq t \text{ in } S^1\}$ is such a cover.*

If S is a monoid to begin with, then the above covers are strongly T-proper and monoidal.

One case of special interest occurs when S is generated, as a restriction semigroup, by a subset X: the cover S_{X^*} associated with the homomorphism $\alpha : X^* \longrightarrow S^1$. In this case the cover is ample.

We now present another useful specialization. In Theorem 4.4 and its corollary, it will be shown that this specialization is in a sense as general as the original theorem. In the following, we shall identify a restriction semigroup S with its image in $C(S)$ under the embedding $\tau_S : a \mapsto a\downarrow$ in $C(S)$, according to the first paragraph of Result 2.4. However we shall at times revert to the explicit use of τ_S when clarification is called for. Recall that the partial order in $C(S)$ is inclusion, whereby the definition of $S_{T,R}$ below is the specialization of that in Theorem 4.1. Note that Corollary 4.2 represents the simplest case of this result, since by Result 2.4 the embedding of S^1 in $C(S)$ is monoidal, so that the element 1 can be identified with the identity of $C(S)$.

Corollary 4.3. *Let S be a restriction semigroup, R a submonoid of $C(S)$ that contains S, T a plain monoid and $\alpha : T \longrightarrow R$ a monoidal homomorphism. The almost T-proper semigroup*

$$S_{T,R} = \{(a,t) \in S \times T : a \in t\alpha\}$$

$$\begin{tikzcd} T \arrow[r, "\kappa"] & C(N) \arrow[r, "\hat{\beta}"] & C(S) \\ & N \arrow[u, "\tau_N"] \arrow[r, "\beta"] & S \arrow[u, "\tau_S"] \end{tikzcd}$$

Fig. 2. The mappings in the proof of Theorem 4.4.

is well defined. In particular, $S_{T,C(S)}$ is well defined.

For any semilattice Y, plain monoid T and monoidal homomorphism $\alpha : T \longrightarrow TI_Y$, the semigroup $(T_Y)_{T,TI_Y}$ is a well defined, almost T-proper restriction semigroup. If Y has an identity element and $\alpha : T \longrightarrow T_Y$, then $(T_Y)_T$ is a strongly T-proper monoid.

Proof. We verify that the criteria (i) and (ii) are met. For the sake of clarity, we set aside the identification of S with $S\tau_S$. Let $A \in R$. Then A is an order ideal of S that satisfies $a^+b = b^+a$ for all $a, b \in A$. Here A^+ is an ideal of P_S and so contains $e{\downarrow} = e\tau_S$, for some $e \in P_S$. For any such e, there exists $b \in A$ such that $b^+ = e$. Then $e{\downarrow} A = b{\downarrow} = b\tau_S$: for if $f \leq e$ and $a \in A$, then $fa = fb^+a = fa^+b \in b{\downarrow}$; and if $c \in b{\downarrow}$, then $c \in A$ and $c = b^+c \in e{\downarrow} A$. The dual statement follows from the self-duality of the pair (S, R).

The statements in the second paragraph follow from the identification of $C(T_Y)$ with TI_Y in Proposition 2.6. In fact, once (i) and (ii) have been established, Theorem 4.1 could itself be applied to the pair (T_Y, TI_Y). $\square$

The following theorem may be regarded as a converse of Theorem 4.1. Its generality allows two distinct important applications. Again we identify S with its image in $C(S)$ under τ_S, except where additional clarity is required. Refer to Section 2.2 for the relevant properties of $C(S)$ and κ.

Theorem 4.4. *Let N and S be restriction semigroups, with $T = N/\sigma$. If N is almost perfect and $\beta : N \longrightarrow S$ is a P-separating homomorphism whose image is full in S, then $N \cong S_{T,C(S)}$, with respect to $\alpha = \kappa\hat{\beta} : T \longrightarrow C(S)$.*

Let N and M be restriction monoids. If N is perfect and $\beta : N \longrightarrow M$ is a P-separating homomorphism whose image is full in M, then $N \cong M_T$, where $\alpha = \kappa\beta : T \longrightarrow M$.

Proof. By Corollary 4.3, $S_{T,C(S)}$ is well defined. Recall that $\kappa : T \longrightarrow C(N)$ and $\hat{\beta} : C(N) \longrightarrow C(S)$ is the monoidal extension of β, according to Result 2.4, noting that since $N\beta$ is full in S, it is an order ideal. See Fig. 2. Since $1\kappa = P_N = 1_{C(N)}$, α is monoidal.

Define $\omega : N \longrightarrow S_{T,C(S)}$ by $n\omega = (n\beta, n\sigma)$. Since $n\beta \in (n\sigma)\beta = (n\sigma)\hat{\beta} = (n\sigma)\alpha$, ω is well defined. Clearly it is a homomorphism. Since β is P-separating, so is the congruence on N that it induces. Since N is proper, $\mu \cap \sigma = \iota$, so ω is injective.

Let $(s, t) \in S_{T,C(S)}$, that is, $s \leq t\alpha$ or, more precisely, $s{\downarrow} \subseteq t\alpha$ in $C(S)$, so that $s \in t\alpha$. Therefore $s = a\beta$ for some a in the σ-class $t = a\sigma$ of N. But then $(s, t) = a\omega$. So ω is surjective.

In the monoidal case, κ maps directly into N and ω may now be regarded as mapping directly into M_T; the proof then proceeds almost identically. $\square$

The first application of this theorem is a description of the almost perfect covers of a restriction semigroup.

Corollary 4.5. *Let N be an almost perfect cover of a restriction semigroup S, via the homomorphism β. Put $T = N/\sigma$. Then $N \cong S_{T,C(S)}$, where $\alpha = \kappa\hat{\beta} : T \longrightarrow C(S)$ and $S \subseteq (T\alpha)\!\downarrow$ is satisfied (cf. 1 in Theorem 4.1).*

If N is a perfect, monoidal cover of the restriction monoid M, again via β, then $N \cong M_T$, where $\alpha = \kappa\beta : T \longrightarrow M$ and M is P-generated by $T\alpha$.

Proof. Here β is, further to the theorem above, surjective. Let $s \in S$, $s = a\beta$, say. Reversing the argument in the proof of surjectivity above, $s \leq t\alpha$, where $t = a\sigma \in T$. The monoidal case proceeds similarly. $\square$

In the case that S itself is almost perfect, regarded as its own cover, that is, β is the identity map, put $T = S/\sigma$. Then $S_{T,C(S)} \cong S$, since if $(s,t) \in S_{T,C(S)}$, then t must be $s\sigma$. Likewise, if M is a perfect monoid, then $M \cong M_T$.

Alternatively, S may also be represented in the form $F_{T,R}$ via its Munn representation, as follows. This result is a sort of precursor to the W-semigroup representation Theorem 6.1. See the remarks following that theorem.

Corollary 4.6. *Let S be an almost perfect restriction semigroup, with $T = S/\sigma$ and $Y = P_S$. Then $S \cong F_{T,C(F)}$, where $F \cong S/\mu$ is the image of S in T_Y under the Munn representation.*

5. The general W-product

Let T be a monoid, Y a semilattice and suppose there is a monoidal homomorphism $\alpha : T \longrightarrow TI_Y$, where (see Section 2) TI_Y is the inverse semigroup of isomorphisms between ideals of Y. Adapting the usual language of actions, we say that T acts on Y (on the right) by *isomorphisms between ideals*. If the image lies in T_Y, then we say that the action is by isomorphisms between *principal* ideals. Once more, in Section 8 we will allow *sub*homomorphisms in the construction. The relationship between our construction and the original W-product will be elucidated following Theorem 5.1.

For $t \in T$, write α_t instead of $t\alpha$ and denote by Δt and ∇t, respectively, its domain and range. Expressed in the notation of actions, for $e \in \Delta t$ write e^t instead of $e\alpha_t$.

Consider the set

$$W(T,Y) = \{(t,f) \in T \times Y : f \in \nabla t\} = \{(t,e^t) \in T \times Y : e \in \Delta t\}.$$

The alternative form (t, e^t) for (t, f) results from the bijectivity of α_t. Each form will prove to be the more convenient one at different points.

The product is defined by:

$$(t, e^t)(u, f^u) = (tu, (e^t f)^u).$$

Since Δu is an ideal containing f and $e^t f \leq f$, $(e^t f)^u$ is defined; likewise, $e^t f \in \nabla t$, $e^t f = g^t$, say. Since the action is induced by a homomorphism, $g \in \Delta tu$ and $(e^t f)^u = (g^t)^u = g^{tu} \in \nabla tu$. Thus the operation $W(T, Y)$ is well defined. Note that α only need be a subhomomorphism for this to hold.

Unary operations are defined by:

$$(t, e^t)^+ = (1, e) \quad \text{and} \quad (t, f)^* = (1, f).$$

Theorem 5.1. *Let T be a monoid, Y a semilattice, and $\alpha : T \longrightarrow TI_Y$ a monoidal homomorphism, that is, T acts on Y by isomorphisms between ideals. Then $W = W(T, Y)$ is isomorphic to the semigroup $(T_Y)_{T,TI_Y}$ and is therefore an almost perfect restriction semigroup, with $P_W \cong Y$ and $W/\sigma \cong T$.*

Further if Y is also a monoid and α is a homomorphism into T_Y, that is, T acts on Y by isomorphisms between principal ideals, then W is isomorphic to $(T_Y)_T$ and is therefore a perfect restriction monoid.

Proof. The general conclusions will follow once the isomorphism is proved. Corollary 4.3 shows that $(T_Y)_{T,TI_Y}$ is well defined. Recall that $Y \cong P_{T_Y} = E_{T_Y}$, under the map $e \mapsto e{\downarrow}$.

Define $\Pi : (T_Y)_{T,TI_Y} \longrightarrow W = W(T, Y)$ by

$$(\beta, t)\Pi = (t, e^t), \text{ where } e{\downarrow} = \Delta\beta.$$

By assumption, $\beta \subseteq t\alpha$, so $e \in \Delta t$. Therefore $(\beta, t)\Pi$ is well defined. On the other hand, for any $(t, e^t) \in W$, let β be the restriction of α_t to $e{\downarrow}$. Then $(t, e^t) = (\beta, t)\Pi$, so Π is surjective.

To prove that Π respects the binary operation, let $(\beta, t), (\gamma, u) \in (T_Y)_{T,TI_Y}$, where $\Delta\beta = e{\downarrow}$ and $\Delta\gamma = f{\downarrow}$. Now $(\beta\gamma, tu)\Pi = (tu, g^{tu})$, where $g{\downarrow} = \Delta(\beta\gamma) = (\nabla\beta \cap \Delta\gamma)\beta^{-1}$. Here since $\beta \subseteq t\alpha$, $\nabla\beta = (e\beta){\downarrow} = e\alpha_t{\downarrow} = e^t{\downarrow}$ and so $\nabla\beta \cap \Delta\gamma = e^t{\downarrow} \cap f{\downarrow} = (e^t f){\downarrow}$, whereby $g\beta = g^t = e^t f$. As in the definition of the operation on W, $(e^t f)^u = (g^t)^u = g^{tu}$ and so $(tu, g^{tu}) = (t, e^t)(u, f^u)$.

Finally, $(\beta, t)^+\Pi = (\beta^+, 1)\Pi = (1, e) = (t, e^t)^+$, since β^+ is the identity map on $e{\downarrow}$. Likewise, $(\beta, t)^*\Pi = (\beta^*, 1)\Pi = (1, h)$, where $h{\downarrow} = \nabla\beta = (e\beta){\downarrow} = (e\alpha_t){\downarrow} = e^t{\downarrow}$, that is $(1, h) = (1, e^t) = (t, e^t)^*$. So Π is a biunary isomorphism.

The monoidal case proceeds similarly. □

As noted in Section 1, the original W-semigroup construction [5,8,20] corresponds precisely to the special case whereby $\Delta t = Y$ for all $t \in T$, that is, the action is by *endomorphisms* of Y. In that case, $W(T,Y) = T \times Y$ and so is a 'reverse' semidirect product. Observe that the action is then not simply by endomorphisms, however, since these endomorphisms must be injective and their images must be ideals of Y. The original construction of course includes the case that the action be by *automorphisms* of Y.

In our general situation, the dual to that just considered would be the special case whereby $\nabla t = Y$ for all $t \in T$, that is, the representation is by isomorphisms from ideals of Y *onto* Y itself. See almost right factorizability in Proposition 7.3(c). This stems from the natural self-duality of our construction under the anti-isomorphism $\delta \mapsto \delta^{-1}$ of TI_Y. This dual therefore corresponds to the 'ordinary' semidirect product.

Proposition 7.4 specializes Theorem 6.1 below to the original construction.

The following are consequences of the definitions of the unary operations and the elementary properties of $S_{T,R}$ in Section 4.

Lemma 5.2. *Under the hypotheses of Theorem 5.1, if the pairs $(t,e^t) = (t,g)$ and $(u,f^u) = (u,h)$ belong to $W = W(T,Y)$, then: $(t,e^t)\widetilde{\mathcal{R}}(u,f^u)$ if and only if $e = f$; $(t,g)\widetilde{\mathcal{L}}(u,h)$ if and only if $g = h$; $(t,g) \le (u,h)$ if and only if $t = u$ and $g \le h$; $(t,g)\ \sigma\ (u,h)$ if and only if $t = u$.*

Using Lemma 5.2, the σ-classes of W are precisely the sets $A_t = \{(t,h) : h \in \nabla t\} = \{(t,e^t) : e \in \Delta t\}$, $t \in T$. Translating into this language the corresponding statement from (the proof of) Theorem 4.1, $A_tA_u = A_{tu}$ for all $t,u \in T$. That is, in the original terminology, the strong T-properness of $C(W)$ is witnessed by the submonoid $T\kappa = \{A_t : t \in T\}$.

Likewise, in the case that Y has an identity element and T acts by isomorphisms between principal ideals, then strong T-properness of $W(T,Y)$ is witnessed by the submonoid $T\kappa = \{(t,g^t) : t \in T\}$ of W itself, where for each t, g generates the principal ideal Δt of Y.

Proposition 5.3. *Under the hypotheses of Theorem 5.1, identify P_W with Y under $(e,1) \mapsto e$. Recall the homomorphism $\kappa\overline{\theta}$ introduced in Corollary 2.8: the composition of the injection of T in $C(W)$ with the extension of the Munn representation of W to $C(W)$. This homomorphism is precisely α and so the original action of T on Y is equivalent to that induced by $\kappa\overline{\theta}$.*

In the case where $Y = Y^1$ and the original action is by isomorphisms between principal ideals, the restriction of the Munn representation of W itself to (the image under κ of) T is α and so induces an action of T on Y that is equivalent to the original one.

Proof. In the general case, see Fig. 1 for the mappings involved, with W replacing S. If $t \in T$, then $t\kappa = A_t$, in the notation above. According to Proposition 2.7, $\Delta\overline{\theta}_{A_t} = A_t^+ = \{(1,e) : e \in \Delta t\}$ and, for $(1,e)$ in this domain, $(1,e)\overline{\theta}_{A_t} = b^*$, where $b = (t,f^t) \in A_t$ and $(1,f) = b^+ = (1,e)$. That is, $(1,e)(\kappa\overline{\theta})_t = (t,e^t)^* = (1,e^t) = (1,e\alpha_t)$. Identifying $(1,e)$ with e, α and $\kappa\overline{\theta}$ therefore have the same domains and take the same values.

In the monoidal case, if $t \in T$ and $\Delta t = g\downarrow$, as above, then $\Delta\theta_{(t,g^t)} = (t, g^t)^+\downarrow = (1, g)\downarrow$. For $e \leq g$, $(1, e)\theta_{(t,g^t)} = ((1, e)(t, g^t))^* = (t, e^t)^* = (1, e^t) = (1, e\alpha)$. Again identifying $(1, e)$ with e, and T with its image $T\kappa$ in W, the restriction to T of the Munn representation agrees with α. $\square$

6. Representing proper restriction semigroups as W-products

Proposition 5.3 suggests a straightforward route to the converse of Theorem 5.1, via the homomorphism $\kappa\overline{\theta}$ introduced in Corollary 2.8, using Theorem 4.4.

Theorem 6.1. *Let S be an almost perfect restriction semigroup. Put $T = S/\sigma$ and $Y = P_S$. Then $S \cong W(T, Y)$, where $Y = P_S$ and the action of T on Y, by isomorphisms between ideals, is induced by the homomorphism $\kappa\overline{\theta}$, the composition of the injection of T in $C(S)$ with the extension of the Munn representation of S to $C(S)$.*

If S is a perfect restriction monoid, then $Y = Y^1$ and the action of T on Y is by isomorphisms between principal ideals, induced by the Munn representation of S itself.

Proof. The Munn representation $\theta : S \longrightarrow T_Y$ is P-separating and its image is a full subsemigroup of T_Y, so by Theorem 4.4, $S \cong (T_Y)_{T,C(T_Y)}$, with respect to $\alpha = \kappa\hat{\theta} : T \longrightarrow C(T_Y)$. Now by Proposition 2.6, $(T_Y)_{T,C(T_Y)} \cong (T_Y)_{T,TI_Y}$, with respect to $\alpha\Sigma = \kappa\overline{\theta}$. Alternatively, the identifications in Corollary 4.3 could have been used to combine those steps. Finally, by Theorem 5.1, $(T_Y)_{T,TI_Y} \cong W(T, Y)$, with respect to the same map $\kappa\overline{\theta}$.

The proof in the monoid case proceeds similarly. $\square$

Note that Corollary 4.6 actually gave a sharper conclusion, based on the precise image F of S in T_Y, one that could be sharpened further by replacing $C(F)$ by the restriction monoid generated by $T\alpha$. The advantage of the $W(T, Y)$ formulation is that its parameters do not require specifying such subsemigroups.

In a different direction, observe that if S is almost perfect, then strong T-properness of $C(S)$ implies that the latter semigroup is isomorphic to $W(T, P_{C(S)})$, via the action induced by its own Munn representation in $T_{P_{C(S)}} = T_{C(P_S)}$. At the end of Subsection 2.2, it was remarked without proof that $T_{C(P_S)} \cong TI_{P_S}$ and that the Munn representation is equivalent to $\overline{\theta}$.

As promised before Corollary 4.2, we interpret the coverings presented there in terms of the W-product. For any restriction semigroup R, with $Y = P_R$, the Munn representation $\theta : R \longrightarrow T_Y$ extends to a monoidal representation $R^1 \longrightarrow TI_Y$, by mapping 1 to the identity of TI_Y. Since this is in essence the Munn representation of R^1 in T_{Y^1}, we again denote it by θ. If R is a monoid, this is just the original representation in T_Y.

Corollary 6.2 *(Alternative version of Corollary 4.2). Let S be a restriction semigroup, T a monoid and $\alpha : T \longrightarrow S^1$ a monoidal homomorphism, the image of which P-generates S. In the notation of Corollary 4.2, $S_T \cong W(T, Y)$, where $Y = P_S$ and the action between ideals of Y is induced by $\alpha\theta$.*

If S is a monoid, then the action is by isomorphisms between principal ideals of Y, induced by the composition of α with the Munn representation of S. In particular, if the plain submonoid T of S itself P-generates S, the action is induced by the restriction of the Munn representation to T.

Proof. Replacing S by S_T in Theorem 6.1, the action specified there is induced by the homomorphism $\kappa\overline{\theta} = \kappa\hat{\theta}\Sigma$ exhibited in Corollary 2.8 and Fig. 1. It only needs to be verified that that action is equivalent to the one in the statement of this corollary. Here $S_T/\sigma \cong T$ and $P_{S_T} = P_S \times \{1\} \cong Y$. Under these identifications, for $t \in T$, the σ-class $t\kappa$ in $C(S_T)$ is $A_t = \{(a,t) : a \leq t\alpha \text{ in } S^1\}$. The action induced by the homomorphism $\overline{\theta}$ is detailed in Proposition 2.7, as follows.

If $t \in T$ then

$$\Delta t = \Delta_{A_t} = A_t^+ = \{(a,t)^+ : a \leq t\alpha\} = \{(e,1) : e \leq (t\alpha)^+\}.$$

The last equation holds because if $a \leq t\alpha$, then $a^+ \leq (t\alpha)^+$; and if $e \leq (t\alpha)^+$, then $(e,1) = (e(t\alpha),t)^+$, where $e(t\alpha) \leq t\alpha$.

Now for $(e,1) \in \Delta t$, $(e,1)\overline{\theta}_{A_t} = (a,t)^*$, where $(a,t)^+ = (e,1)$. Since $a \leq t\alpha$, $a = e(t\alpha)$ and so $(e,1)\overline{\theta}_{A_t} = (e(t\alpha),t)^* = ((e(t\alpha))^*,1)$.

Identifying P_S with P_{S_T}, the action of t therefore has domain P_{S_T}, if $t = 1$, and domain $(t\alpha)^+\!\downarrow$ otherwise, with $e^t = (e(t\alpha))^*$ in either case. The latter action is that induced by the composition of α with the (extension of the) Munn representation θ. $\square$

An explicit isomorphism $S_T \cong W(T,Y)$ is given by $(s,t) \mapsto (t,s^*)$, with inverse $(t,f) \mapsto ((t\alpha)f,t)$.

7. Examples

The W-product provides a simple mechanism for producing specific examples. We begin this section with one promised in Proposition 3.4.

Example 7.1. *An almost perfect restriction monoid need not be perfect.*

Proof. Let $Y = \mathbb{Z}$, under the reverse of the usual order. Let $T = \{x\}^*$, the free monoid on $\{x\}$, and let T act totally on Y, determined by $n^x = n+1$, $n \in \mathbb{Z}$. Then $S = W(T,Y)$ is an almost perfect restriction semigroup. In fact, by Corollary 7.4 below, S is almost factorizable. Note that each of its σ-classes is isomorphic to Y, as a poset. Now the monoid S^1 is again almost perfect, by Proposition 3.4(2), but is not perfect, since the only σ-class with a maximum element is $P_S \cup \{1\}$. $\square$

Given any semilattice Y, a range of examples may be constructed from T_Y itself, for instance by considering TI_Y, or any restriction subsemigroups that contain T_Y, as plain monoids, with α the identical map. For example, $(T_Y)_{TI_Y,TI_Y}$ is such a semigroup.

Examples of strongly T-proper monoids and almost T-proper semigroups were given briefly at the end of Section 3. We shall consider factorizability in the next section.

7.1. Relatively free restriction semigroups

Let FR_X and FRM_X be the free restriction semigroup and monoid respectively, on the set X. It is easily seen (and well known) that $FRM_X = FR_X^1$. The submonoid T of FRM_X generated by X also P-generates FRM_X. Let X^* be the free monoid on X. The natural map $FRM_X \longrightarrow X^*$ restricts to an isomorphism on T and induces σ, cf. [1, Theorem 5.1]. Therefore FRM_X is strongly X^*-proper and so FR_X is almost X^*-proper (and so almost perfect). We omit further reference to FR_X. The semilattice of projections of FRM_X is isomorphic to the semilattice of idempotents of the free *inverse* monoid on X. While this of course follows from the published structure theorems for FRM_X, it can be independently proven (e.g. [13]).

Corollary 7.2. *The free restriction monoid FRM_X on X is isomorphic to $W(X^*, Y)$, where X^* is the free monoid on X, acting on the semilattice of projections Y of FRM_X according to the Munn representation.*

A similar argument applies to the free restriction monoids relative to any variety of restriction semigroups that contains all monoids. Here, however, the semilattice of projections would need to be constructed in order to yield a concrete structure theorem.

7.2. (Almost) factorizable restriction semigroups

In any restriction monoid M, the $\widetilde{\mathcal{R}}$-class $\widetilde{\mathcal{R}}_1$ is a submonoid, since $\widetilde{\mathcal{R}}$ is a left congruence. Such a monoid is *left factorizable* [4,8,20] if $M = P_M\widetilde{\mathcal{R}}_1$. If M is proper, then σ separates $\widetilde{\mathcal{R}}_1$. Therefore the proper left factorizable monoids are precisely the strongly $\widetilde{\mathcal{R}}_1$-proper monoids, in our language. Similarly, the proper right factorizable monoids are the strongly $\widetilde{\mathcal{L}}_1$-proper monoids and the proper factorizable monoids are the $\widetilde{\mathcal{H}}_1$-monoids.

According to [8,20], a restriction *semigroup* S is *almost left factorizable* if every element of S belongs to some member of the $\widetilde{\mathcal{R}}$-class of the identity in $C(S)$. Again there are naturally right and two-sided versions of this definition. They extend to restriction semigroups the definition for inverse semigroups of Lawson [17]. The first equivalence in the following result was no doubt known to the authors of [8,20].

Lemma 7.3. *A proper restriction semigroup S is almost [left, right] factorizable if and only if $C(S)$ is [left, right] factorizable, and thus if and only if S is almost T-proper, where $T = [\widetilde{\mathcal{R}}_1,\ \widetilde{\mathcal{L}}_1]\widetilde{\mathcal{H}}_1$ of $C(S)$.*

Proof. Suppose S is almost left factorizable. Let $B \in C(S)$. By hypothesis, each $a \in B$ belongs to some $A \in \widetilde{\mathcal{R}}_1$. Since both B and A are contained within σ-classes of S, in fact A is the same for each choice of a, that is $B \subseteq A$. By Lemma 3.1, $C(S)$ is strongly $\widetilde{\mathcal{R}}_1$-proper and, by the discussion above, left factorizable. Conversely, if $C(S)$ is strongly $\widetilde{\mathcal{R}}_1$-proper, then for each $a \in S$, $a\!\downarrow\subseteq A$, that is, $a \in A$, for some $A \in \widetilde{\mathcal{R}}_1$. The other cases are similar. $\square$

The semigroups $W(T, Y)$ that arise in the representations of the left and two-sided almost factorizable semigroups in the following result are precisely the 'original' ones of [5,8,20] (see the discussion following Theorem 5.1). Note that the right-hand version falls within the realm of our broader W-product, but not within that of the original. The rather simpler statement of the proposition in the case of factorizability is left to the reader.

Proposition 7.4. *The following are equivalent for a proper restriction semigroup S:*

(a) S is almost left factorizable [almost right factorizable, almost factorizable];
(b) S is almost T-proper, where T is the $\widetilde{\mathcal{R}}$-class [$\widetilde{\mathcal{L}}$-class, $\widetilde{\mathcal{H}}$-class] of the identity in $C(S)$;
(c) the action that is induced by the Munn representation of S, by isomorphisms between ideals of P_S, is by endomorphisms [onto mappings, automorphisms];
(d) $S \cong W(T, Y)$ for such an action of a monoid T upon a semilattice Y (that is, in the left and two-sided cases, the 'original' W-product).

Proof. First consider the left-hand case. The first equivalence was shown in Lemma 7.3. Now suppose $C(S)$ is strongly $\widetilde{\mathcal{R}}_1$-proper. Then the action induced by the Munn representation of S, according to Proposition 2.7 and its corollary, is by fully defined mappings, that is, by endomorphisms. An application of Theorem 6.1 yields (d). That $W(T, Y)$ as in (d) is almost left factorizable can be checked directly, or by using Proposition 5.3 to prove (b).

In the right-hand case, by duality, the action is instead by *onto* partial mappings. In the two-sided case, the requirement becomes that the action be by *automorphisms* of Y. $\square$

That any restriction semigroup with one of the factorizability properties considered above has a proper cover of the same type was shown in [8,20]. This can also be obtained by direct calculation using Corollary 4.2, the simplest form of the coverings in that section.

8. Proper restriction semigroups in general

As noted in Section 1, only minor modifications, based on Lemma 2.2 and the general case of Corollary 2.8, are needed to extend all of the main results of this paper to proper restriction semigroups (and monoids). In this section, we focus on the extensions of each of the cited results to the proper case. We refer the reader to the statement of Theorem 4.1 and the construction that precedes it. We should note here that, in the general situation of that theorem, the connection between S and R may be too tenuous to deduce the converse statement in the next result. Instead, therefore, we phrase it in terms of Corollary 4.3. As noted there, in view of Theorem 4.4 that situation includes the case $R = S^1$ and, moreover, is in a sense general.

Theorem 8.1. *(Cf. Theorem 4.1.) If $\alpha : T \longrightarrow R$ is a subhomomorphism, then $S_{T,R}$ is a proper restriction semigroup.*

If $S \leq R \leq C(S)$ (cf. Corollary 4.3), then $S_{T,R}$ is almost perfect if and only if α is a homomorphism. In particular, this is true for S_T.

If S is a monoid and α is monoidal, then S_T is an F-restriction monoid; thus S_T is strongly T-proper (that is, perfect) if and only if α is also a homomorphism.

Outline of proof. It was mentioned during the proof of the cited theorem that only the subhomomorphism property of α was required for closure; the homomorphism property was not used in the proof of properness. In the monoidal case, the stated property of σ-classes was also proved there.

Next we show that in the case $S \leq R \leq C(S)$, if σ is perfect on $S_{T,R}$ then α must be a homomorphism. For clarity's sake, we shall explicitly use the embedding τ_S, where $a\tau_S = a{\downarrow}$. In the notation of the cited theorem, the σ-classes of $S_{T,R}$ are the sets S_t, $t \in T$, where $S_t = \{(a{\downarrow}, t) : a{\downarrow} \leq t\alpha\}$ in $C(S)$. Note that if $A \in C(S)$ and $a \in S$, then $a{\downarrow} \leq A$ if and only if $a \in A$.

Let $t, u \in T$. Each of $t\alpha$, $u\alpha$ and $(tu)\alpha$ is an order ideal of S. It must be shown that $(tu)\alpha \subseteq (t\alpha)(u\alpha)$ in $C(S)$. Let $a \in (tu)\alpha$. So $a{\downarrow} \leq (tu)\alpha$ and therefore $(a\tau_S, tu) \in S_{tu}$. By perfection of σ, $(a\tau_S, tu) = (b\tau_S, t)(c\tau_S, u)$ for some $b \in t\alpha$, $c \in u\alpha$. Since τ_S is injective, $a = bc \in (t\alpha)(u\alpha)$, as required.

In the monoidal case, either the proof above may be modified or one can directly use the assumption that the elements $(t\alpha, t)$, $t \in T$ form a subsemigroup to show that α is a homomorphism. □

With 'almost perfect' replaced by 'proper' in the semigroup case, and 'perfect' replaced by 'F-restriction' in the monoid case (so that the map $\kappa : T \longrightarrow N$ is defined), Theorem 4.4 and Corollary 4.5 now characterize the respective covers in terms of the broader construction in the theorem just stated.

In the context of this section, if there is a subhomomorphism $\alpha : T \longrightarrow TI_Y$, then T is said to *sub-act* on the semilattice Y. We otherwise retain the notation of Section 5. The construction $W(T, Y)$ is as in that section; as noted at that point, the argument for closure only required a sub-action.

Theorem 8.2. *(Cf. Theorem 5.1.) Let T be a monoid, Y a semilattice, and $\alpha : T \longrightarrow TI_Y$ a monoidal subhomomorphism, that is, T sub-acts on Y by isomorphisms between* *ideals. Then $W = W(T, Y)$ is isomorphic to the semigroup $(T_Y)_{T,TI_Y}$ and so is a proper restriction semigroup, with $P_W \cong Y$ and $W/\sigma \cong T$; it is almost T-proper if and only if α is a homomorphism.*

Further if Y is a monoid and α is a subhomomorphism into T_Y, that is, T sub-acts on Y by isomorphisms between principal ideals, then W is isomorphic to the monoid $(T_Y)_T$ and so is an F-restriction monoid; it is strongly T-proper if and only if α is a homomorphism.

Proof. This follows from Theorem 8.1 and the cited theorem. □

Finally, the converse again follows from application of the above and the cited theorem.

Theorem 8.3. *(Cf. Theorem 6.1.) Let S be a proper restriction semigroup. Put $T = S/\sigma$ and $Y = P_S$. Then $S \cong W(T, Y)$, where the sub-action of T on Y, by isomorphisms between ideals, is induced by the subhomomorphism $\kappa\overline{\theta}$, the composition of the injection of T in $C(S)$ with the extension of the Munn representation of S to $C(S)$.*

If S is an F-restriction monoid then $Y = Y^1$ and the sub-action of T on Y is by isomorphisms between principal ideals, induced by the Munn representation of S itself.

Cornock and Gould [3] provided a structure theorem for proper restriction semigroups in general, based on *pairs* of partial actions of a monoid T on a semilattice Y. For such a pair, they define a semigroup $\mathcal{M}(T, Y)$ and show that this construction describes proper restriction semigroups. Clearly, there must be a correspondence between their construction and that in this section, but we have not pursued this explicitly since the general case is not the one of main interest in our work.

9. Specialization to inverse semigroups

Recall from Section 2 that proper inverse semigroups are usually termed E-unitary. The specializations of the general parts of Theorems 8.2 and 8.3 to inverse semigroups are easily obtained. We use the terminology of [18], that a *prehomomorphism* of inverse semigroups is a subhomomorphism that respects inverses. We leave the statement in the case of monoids to the reader.

Corollary 9.1. *(Cf. [19].) Let T be a group, Y a semilattice, and $\alpha : T \longrightarrow TI_Y$ a monoidal prehomomorphism. Then $W = W(T, Y)$ is an E-unitary inverse semigroup, isomorphic to $(T_Y)_{T,TI_Y}$, with $E_W \cong Y$ and $W/\sigma \cong T$. Conversely, for any E-unitary inverse semigroup S, let T be the group S/σ and $Y = E_S$. Then $S \cong W(T, Y)$, where the sub-action of T on Y, by isomorphisms between ideals, is induced by the prehomomorphism $\kappa\overline{\theta}$.*

Proof. In the direct case, the conclusion is clearer if we consider the semigroups $S_{T,R}$ of Theorem 5.1, with R and S inverse semigroups, under the same assumptions on T and α, for we may then quote the second isomorphism in the theorem. If $(s, t) \in S_{T,R}$, then $s \leq t\alpha$; by the assumption on α and compatibility of the natural partial with inverses in inverse semigroups, $s^{-1} \leq t^{-1}\alpha$, so (s^{-1}, t^{-1}) is an inverse for (s, t).

In the converse case, T is necessarily a group and $P_S = E_S$. Since $(a\sigma)^{-1} = a^{-1}\sigma$ in any inverse semigroup, κ, and therefore $\kappa\overline{\theta}$, is a prehomomorphism. $\square$

The theory of E-unitary inverse semigroups is very well mined and it is hardly to be expected that the general theory, such as this corollary, would reduce to anything but well-trodden ground (albeit in somewhat different language), even though that ground played no part in the development of our theory. As cited incidentally above, the main body of Corollary 9.1 above is essentially the description of E-unitary inverse semigroups found by Petrich and Reilly [19] (see also [18, Theorem VI.8.12], where the inverse semigroup of isomorphisms between ideals of a semilattice Y is denoted $\Sigma(Y)$).

Since we are more interested in where our theory does not simply extend inverse semigroup theory, we refer the reader to Chapter VII of the monograph [18] by Petrich for comprehensive coverage. Of course, McAlister's P-theory is the gold standard, especially when treated in concert with the various alternative descriptions of E-unitary covers discovered in subsequent years ([18, Section VII.4]), where the semigroup $C(S)$ not surprisingly plays a significant role. Chapter 7 of the monograph [17] by Lawson presents the P-theorem from a somewhat different perspective.

The point that we wish to emphasize here is that the specialization of almost perfection and perfection to inverse semigroups does not yield a general theory as it does for restriction semigroups. They are well-studied classes:

Proposition 9.2. *The almost perfect inverse semigroups are the semidirect products of semilattices and groups. The perfect inverse monoids are the monoidal such products.*

Proof. This is simply the combination of Propositions VII.5.11, VII.5.14, and VII.5.24 of [18]. $\square$

The key point of divergence in our work is then the covering result Theorem 4.1 and, more particularly, the simpler result Corollary 4.2. The latter produces an almost perfect cover S_T for S which, as we have just seen, is a semidirect product of a semilattice and a group. By [17, Theorem 7.10] only the almost factorizable inverse semigroups are quotients of such semidirect products. In the monoidal case, only the factorizable inverse semigroups are quotients.

References

[1] M. Branco, G. Gomes, V. Gould, Left adequate and left Ehresmann monoids, Internat. J. Algebra Comput. 21 (2011) 1259–1284.

[2] C. Cornock, Restriction Semigroups: Structure, Varieties and Presentations, Ph.D. Thesis, York, England, 2011.

[3] C. Cornock, V. Gould, Proper two-sided restriction semigroups and partial actions, J. Pure Appl. Algebra 216 (2012) 935–949.
[4] A. El Qallali, Factorisable right adequate semigroups, Proc. Edinb. Math. Soc. 24 (1981) 171–178.
[5] J. Fountain, G. Gomes, Proper left type-A monoids revisited, Glasg. Math. J. 35 (1993) 293–306.
[6] J. Fountain, G. Gomes, V. Gould, A Munn type representation for a class of E-semiadequate semigroups, J. Algebra 218 (1999) 69–714.
[7] G. Gomes, V. Gould, Fundamental Ehresmann semigroups, Semigroup Forum 63 (2001) 11–33.
[8] G. Gomes, M. Szendrei, Almost factorizable weakly ample semigroups, Comm. Algebra 35 (2007) 3503–3523.
[9] G. Gomes, V. Gould, Left adequate and left Ehresmann monoids II, J. Algebra 348 (2011) 171–195.
[10] John M. Howie, Fundamentals of Semigroup Theory, Clarendon Press, Oxford, 1995.
[11] Peter R. Jones, A common framework for restriction semigroups and regular $*$-semigroups, J. Pure Appl. Algebra 216 (2012) 618–632.
[12] Peter R. Jones, On lattices of varieties of restriction semigroups, Semigroup Forum 86 (2013) 337–361.
[13] Peter R. Jones, Varieties of P-restriction semigroups, Comm. Algebra 42 (2014) 1811–1834.
[14] G. Kudryavtseva, Partial monoid actions and a class of restriction semigroups, J. Algebra 429 (2015) 342–370.
[15] M. Lawson, The structure of type A semigroups, Quart. J. Math. Oxford (2) 37 (1986) 279–298.
[16] M. Lawson, Semigroups and ordered categories I: the reduced case, J. Algebra 141 (1991) 422–462.
[17] M. Lawson, Inverse Semigroups: The Theory of Partial Symmetries, World Scientific, Singapore, 1998.
[18] M. Petrich, Inverse Semigroups, Wiley, New York, 1984.
[19] M. Petrich, N. Reilly, A representation of E-unitary inverse semigroups, Quart. J. Math. Oxford 30 (1979) 339–350.
[20] M. Szendrei, Embedding into almost left factorizable restriction semigroups, Comm. Algebra 41 (2013) 1458–1483.