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UNITARY REPRESENTATIONS OF CYCLOTOMIC RATIONAL CHEREDNIK ALGEBRAS

STEPHEN GRIFFETH

ABSTRACT. We classify the irreducible unitary modules in category $\mathcal{O}_c$ for the rational Cherednik algebras of type $G(r, 1, n)$ and give explicit combinatorial formulas for their graded characters. More precisely, we produce a combinatorial algorithm determining, for each r-partition $\lambda^\bullet$ of n, the closed semi-linear set of parameters c for which the contravariant form on the irreducible representation $L_c(\lambda^\bullet)$ is positive definite. We use this algorithm to give a closed form answer for the Cherednik algebra of the symmetric group (recovering a result of Etingof-Stoica and the author) and the Weyl groups of classical type.

1. INTRODUCTION

1.1. The goals of this paper are: first, to obtain the classification of irreducible unitary representations in category $\mathcal{O}_c$ for the rational Cherednik algebras of type $W = G(r, 1, n)$, and second, to give an explicit basis for each of them in terms of Specht-module valued versions of non-symmetric Jack polynomials. The strategy is that of the appendix to [EtSt]. The tactics are somewhat different, requiring the development of tools that were previously unavailable for $r > 1$, and the answer is quite a bit more complicated. The main steps are as follows: first, in Theorem 3.1 we give a presentation of the cyclotomic rational Cherednik algebra compatible with a certain commutative subalgebra discovered by Dunkl and Opdam [DuOp]; second, in the proof of Theorem 1.1, we use this presentation to extend the results of Cherednik [Che1] and Suzuki [Suz] classifying the diagonalizable representations for the type A Cherednik algebra to the cyclotomic case; third, in the proof of Theorem 1.2 we use Theorem 1.1 together with arguments analogous to those in the appendix of [EtSt] to complete the classification of the unitary irreducible objects in category $\mathcal{O}_c$.

The irreducible objects in category $\mathcal{O}_c$ are indexed by the irreducible complex representations of the complex reflection group W. Thus for $W = S_n$ the symmetric group, they are indexed by integer partitions of n, and for $W = G(2, 1, n) = W(B_n)$ the Weyl group of type B_n they are indexed by pairs (λ^0, λ^1) of integer partitions with n total boxes. More generally, for the complex reflection group $G(r, 1, n)$ they are indexed by r-tuples $(\lambda^0, \ldots, \lambda^{r-1})$ of integer partitions with n total boxes. In the case $W = S_n$, the parameter c is a single real number; for the Weyl group of type B it is a pair (c, d) consisting of two real numbers; and for the complex reflection group $G(r, 1, n)$ it is an r-tuple of real numbers. Our main

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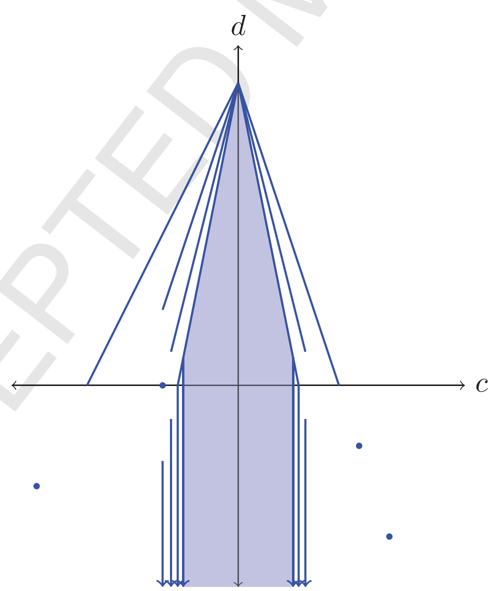
theorem provides an algorithm determing, for each r-partition λ of n, the closed semi-linear set of parameters $U(\lambda^\bullet) \subseteq \mathbf{R}^r$ for which the contravariant form on the irreducible object $L_c(\lambda^\bullet)$ is positive definite. Our algorithm may be made into a closed form answer for $r = 1$ and $r = 2$ (and, presumably, also for any fixed r, given enough patience), and we record this closed form answer in Corollary 8.4 and Corollary 8.5.

We recover the main result of [EtSt] for $r = 1$. Already for $r = 2$ the answer is much more intricate: even to state it in completely explicit fashion requires several pages (see Section 8). To give the reader a rough idea of the form of the answer and a visual demonstration of the varying levels of complexity, here we present miniature drawings, all to the same scale, of the unitary sets for three particular cases: first, for the group $W = S_{30}$ and the partition $\lambda = (6, 6, 6, 4, 4, 4)$. The set of all possible c is one-dimensional, and the unitary set consists of a closed interval, and five isolated points.

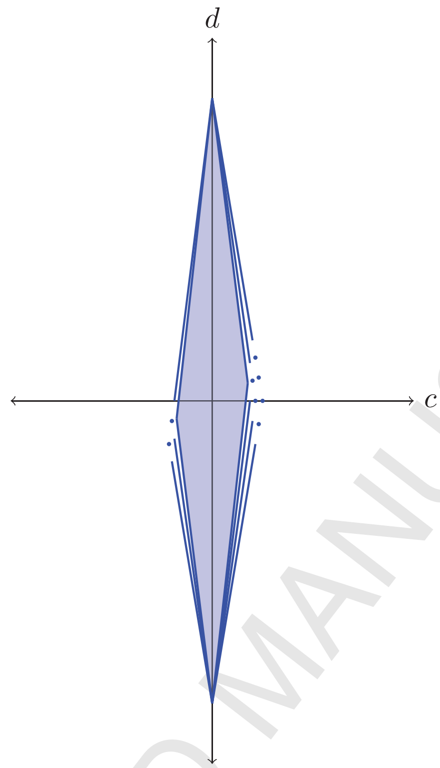
The interior of the closed interval is the set on which the standard module itself is unitary.

Second, for the group $W = G(2, 1, 30)$ and the bipartition $\lambda = ((6, 6, 6, 4, 4, 4), \emptyset)$, the unitary set is the subset of the (c, d) plane shown here:



The shaded region is the set on which the standard module itself is unitary; on the darker lines and points, the standard module is not irreducible, and its proper quotient $L_c(\lambda)$ is unitary.

Finally, for the group $W = G(2, 1, 34)$ and the bipartition $((3, 3, 3, 2, 2), (5, 5, 5, 5, 3, 3))$, the unitary set looks like this (as above, the shaded parallelogram is the set of parameters for which the standard module itself is unitary):



The Cherednik algebra H_c contains a commutative subalgebra $\mathfrak{t}$, the *Dunkl-Opdam subalgebra*, that acts by locally finite, normal, and hence diagonalizable, operators on any unitary representation in category $\mathcal{O}_c$. Therefore a first step towards the classification of irreducible unitary modules in $\mathcal{O}_c$ is the classification of the irreducible modules in $\mathcal{O}_c$ on which $\mathfrak{t}$ acts by diagonalizable operators (A. Ram [Ram] uses the word *calibrated* for the analog of this type of module for the affine Hecke algebra). Once we have achieved this classification (Theorem 1.1), the classification of unitary modules (Theorem 1.2) is obtained by Cherednik's technique of intertwining operators, combined with a detailed study of the combinatorics of certain tableaux. For the groups $W = G(2,1,n)$ we can be completely explicit, but for larger r it seems that a direct description of the unitary irreducibles in $\mathcal{O}_c$ is unavoidably complicated. In particular, the set of c for which $L_c(\lambda)$ is unitary may have components of all dimensions between 0 and r.

In the appendix to [EtSt] we had the advantage that the first part (classification of diagonalizable modules) had been previously carried out by Cherednik and Suzuki for the trigonometric DAHA, and the corresponding classification for the rational DAHA relied on an embedding of the latter into the former. Here we first obtain the classification of diagonalizable modules by working directly with the rational DAHA, making use of a presentation adapted to the technique of intertwining operators (since the first version of this paper appeared on the arxiv, this presentation was rediscovered by Webster [Web], and Braverman-Etingof-Finkelberg [BEF] have constructed a cyclotomic version of the full DAHA). Once this is done there are necessary and sufficient numerical criteria that the eigenvalues of a diagonalizable module must satisfy in order that it be unitary, and the remainder of the paper is devoted to constructing a combinatorial machine for handling these numerics.

We now introduce the notation we will need to state our main results, referring to Section 2 for precise definitions. We identify integer partitions with their Young diagrams. Given a box b of a partition, we will write $\mathrm{ct}(b)$ for its content, equal to $i-j$ if the box is in column i and row j. Given an r-partition $\lambda^\bullet = (\lambda^0, \dots, \lambda^{r-1})$ and a box $b \in \lambda^i$ we will write $\beta(b) = i$. The Cherednik algebra H_c depends on a parameter $c = (c_0, d_0, d_1, \dots, d_{r-1}) \in \mathbf{R}^{r+1}$, where $d_0 + d_1 + \cdots + d_{r-1} = 0$. It contains a certain commutative subalgebra $\mathfrak{t}$ (see 2.12 for the precise definition) which acts by normal operators, and hence diagonalizably, on any unitary representation. Given a parameter c and an r-partition $\lambda^\bullet$, we write $L_c(\lambda^\bullet)$ for the corresponding representation of H_c. We follow the convention that the superscript i in λ^i is always to be taken modulo r, as is the subscript i in d_i (so that λ^i and d_i are defined for all integers $i \in \mathbf{Z}$).

Our first main result is the classification and description of the modules $L_c(\lambda^\bullet)$ that are $\mathfrak{t}$-diagonalizable. For each box $b \in \lambda^\bullet$, define statistics $c(b)$, $k_c(b)$ and $l_c(b)$ as follows: first, $c(b)$ is the *charged content* of b, defined by

$$c(b) = d_{\beta(b)} + r\mathrm{ct}(b)c_0. \tag{1.1}$$

Second, $k_c(b)$ is the smallest positive integer k such that there is a box $b' \in \lambda^{\beta(b)-k}$ with

$$k = c(b) - c(b'),$$

(and $k_c(b) = \infty$ if no such equation holds) and finally, $l_c(b)$ is the smallest positive integer l such that there is an outside addable box b' for $\lambda^{\beta(b)-l}$ with

$$l = c(b) - c(b')$$

(and $l_c(b) = \infty$ if no such equation holds). Recall that a box b is *addable* to a partition λ if $b \notin \lambda$ and adding b to λ produces the diagram of a partition. An *outside* addable box is an addable box b such that $\mathrm{ct}(b) \neq \mathrm{ct}(b')$ for all $b' \in \lambda$.

Given boxes $b, b' \in \lambda^i$ we write $b \leq b'$ if b is (weakly) up and to the left of b' (thus in particular $\beta(b) = \beta(b')$ if $b \leq b'$). We define Γ to be the set of pairs (P, Q) where P and Q are fillings of the boxes of $\lambda^\bullet$ by non-negative integers, P is a bijection from the boxes of $\lambda^\bullet$ to the set $\{1, 2, \dots, n\}$, Q is weakly increasing $Q(b) \leq Q(b')$ if $b \leq b'$, and $P(b) > P(b')$ if $b < b'$ and $Q(b) = Q(b')$.

We define a subset $\Gamma_c \subseteq \Gamma$ as follows: a pair $(P, Q) \in \Gamma$ is in Γ_c if and only if the following conditions hold:

(a) whenever $b \in \lambda^\bullet$ and $k \in \mathbf{Z}_{>0}$ with $k = d_{\beta(b)} - d_{\beta(b)-k} + r\mathrm{ct}(b)c_0$ we have $Q(b) < k$, and

(b) whenever $b_1, b_2 \in \lambda^\bullet$ and $k \in \mathbf{Z}_{>0}$ with $\beta(b_1) - \beta(b_2) = k \bmod r$ and $k = d_{\beta(b_1)} - d_{\beta(b_2)} + r(\mathrm{ct}(b_1) - \mathrm{ct}(b_2) \pm 1)c_0$ we have $Q(b_1) \leq Q(b_2) + k$, with equality implying $P(b_1) > P(b_2)$.

Theorem 1.1. *The module $L_c(\lambda^\bullet)$ is diagonalizable if and only if either*

(a) $c_0 = 0$ or

(b) $c_0 \neq 0$ and for every removable box $b \in \lambda^\bullet$, either $k_c(b) = \infty$ or the inequality $l_c(b) < k_c(b)$ holds.

In case (b), as basis of $L_c(\lambda^\bullet)$ is given by the set $\{f_{(P,Q)} \mid (P,Q) \in \Gamma_c\}$ of non-symmetric Specht-valued Jack polynomials.

Each irreducible representation $L_c(\lambda^\bullet)$ is equipped with a non-degenerate Hermitian contravariant form. We call $L_c(\lambda)$ *unitary* if this form is positive definite. Our second main result is the classification of the modules $L_c(\lambda^\bullet)$ that are unitary. We state the result for $c_0 \geq 0$: $L_c(\lambda^\bullet)$ is unitary if and only if $L_{c'}((\lambda^\bullet)^t)$ is, where c' is the same as c but with c_0 replaced by $-c_0$, and the transpose of an r-partition is the transpose of each of its component λ^j's.

For $0 \leq i, j \leq r-1$ we write m_{ij} for the integer with $1 \leq m_{ij} \leq r$ and $m_{ij} = i - j \mod r$ (thus for $i = j$ we have $m_{ij} = r$). Let (b, j) be a pair consisting of a box $b \in \lambda^i$ and an integer $0 \leq j \leq r-1$. A *blocking sequence* B for (b, j) is a sequence $B = (b_1, b_2, \ldots, b_{2k+1}, \ell)$ of boxes and an integer $0 \leq \ell \leq r-1$ such that

(a) $b \leq b_1$, and for each $1 \leq p \leq k$, $b_{2p} \leq b_{2p+1}$, and
(b) $m_{\beta(b_{2k+1}),\ell} + \sum_{p=1}^{k} m_{\beta(b_{2p-1}),\beta(b_{2p})} \leq m_{ij}$.

Condition (b) may be rephrased and visualized as follows: linearly order the set of integers modulo r so that $i-1 > i-2 > \cdots > i$. Then the sequence $(\beta(b_2), \beta(b_4), \ldots, \ell)$ must be a decreasing subsequence of the interval $i-1 > i-2 > \cdots > j+1 > j$.

Given a blocking sequence $B = (b_1, b_2, \ldots, b_{2k+1}, \ell)$ for (b, j), we define the set L_B of parameters by $c \in L_B$ if and only if

$$d_{\beta(b_{2p-1})} - d_{\beta(b_{2p})} + r(\mathrm{ct}(b_{2p-1}) - \mathrm{ct}(b_{2p}) \pm 1)c_0 = m_{\beta(b_{2p-1}),\beta(b_{2p})} \quad \text{for } 1 \leq p \leq k$$

and

$$d_{\beta(b_{2k+1})} - d_\ell + r\mathrm{ct}(b_{2k+1})c_0 = m_{\beta(b_{2k+1}),\ell}.$$

Given an ordered pair of boxes (b, b') with $b \in \lambda^i$ and $b' \in \lambda^j$, a *blocking sequence* for (b, b') is either a blocking sequence for (b, j), or a sequence $B = (b_1, \ldots, b_{2q})$ of boxes of $\lambda^\bullet$ such that

(a) $b \leq b_1$ and $b_{2q} \leq b'$,
(b) for each $1 \leq k \leq q-1$ we have $b_{2k} \leq b_{2k+1}$, and
(c) we have
$$\sum_{k=1}^{q} m_{\beta(b_{2k-1}),\beta(b_{2k})} \leq m_{ij}.$$

Just as above, condition (c) is equivalent to the requirement that the sequence $(\beta(b_2), \beta(b_4), \ldots, \beta(b_{2q}))$ be an decreasing subsequence of the interval $i-1 > i-2 > \cdots > j+1 > j$.

Given a blocking sequence B for (b, b') of this second type, we define the set L_B of parameters by $c \in L_B$ if and only if

$$d_{\beta(b_{2p-1})} - d_{\beta(b_{2p})} + r(\mathrm{ct}(b_{2p-1}) - \mathrm{ct}(b_{2p}) \pm 1)c_0 = m_{\beta(b_{2p-1}),\beta(b_{2p})} \quad \text{for } 1 \leq p \leq q.$$

Theorem 1.2. *For $c_0 = 0$, the module $L_c(\lambda^\bullet)$ is unitary if and only if for all $0 \leq i \leq r-1$ such that $\lambda^i \neq \emptyset$ we have $d_i - d_j \leq m_{ij}$ for all $0 \leq j \leq r-1$. Suppose $c_0 > 0$. The module $L_c(\lambda^\bullet)$ is unitary if and only if the following conditions are satisfied:*

(a) it is diagonalizable (see Theorem 1.1),

(b) *for every pair $b, b' \in \lambda^\bullet$ of boxes such that*

$$d_{\beta(b)} - d_{\beta(b')} + r(\mathrm{ct}(b) - \mathrm{ct}(b') + 1)c_0 > m_{\beta(b),\beta(b')},$$

either $b \leq b'$ or else there is a blocking sequence B with $c \in L_B$, and

(c) *for every box $b \in \lambda^\bullet$ and $0 \leq j \leq r-1$ such that*

$$d_{\beta(b)} - d_j + r\mathrm{ct}(b)c_0 > m_{\beta(b),j}$$

there is a blocking sequence B with $c \in L_B$.

The theorem exhibits the set of parameters c for which the module $L_c(\lambda^\bullet)$ is unitary as a semi-linear subset of the parameter space. At first sight it might seem to require an analysis of an unmanageable number of inequalities and equalities, but it is a practical computational tool: in Section 8 we will deduce the classification theorem of [EtSt] from it, and use it to explicitly compute the unitary spectrum of each $L_c(\lambda^\bullet)$ for Weyl groups of classical type.

With Theorems 1.1 and 1.2 in hand, the natural next step is the calculation of the Kazhdan-Lusztig polynomials

$$P_{\lambda,\mu}(q) = \sum \dim(\mathrm{Ext}^i(\Delta_c(\lambda), L_c(\mu)))q^i$$

for all unitary representations $L_c(\lambda^\bullet)$. Using Dirac cohomology and the Hodge decomposition theorem from [HuWo] (c.f. [Ciu]) coupled with our character formula in Theorem 1.1 should produce a combinatorial algorithm (much faster than the usual linear algebra using canonical bases) based on jeu de taquin for computing these KL polynomials.

T. Suzuki remarks that in the type A case, the unitary modules correspond to integrable modules and the diagonalizable modules correspond to the *admissible* modules of Kac and Wakimoto via the Arakawa-Suzuki functor. We do not know the analogs of this coincidence for the groups $G(r,p,n)$, though it should be interesting to compare our results with those of Varagnolo and Vasserot, [VaVa]. Finally, we remark that the version of Clifford theory that appears in the last section of [Gri2] allows one to deduce analogous results for the groups $G(r,p,n)$ when $n > 2$.

2. Definitions and background

2.1. Let V be a complex vector space of dimension n and let $W \subseteq \mathrm{GL}(V)$ be a finite group of linear transformations of V. The set of *reflections* (sometimes called pseudo-reflections or complex reflections) in W is

$$R = \{r \in W \mid \dim(\mathrm{fix}(r)) = n-1\}.$$

The group W is a *reflection group* if it is generated by R.

2.2. For each reflection $r \in R$ let c_r be a formal variable such that $c_{wrw^{-1}} = c_r$ for all $r \in R$ and $w \in W$, and choose $\alpha_r \in V^*$ such that the zero set of α_r is the fix space of r. Let $A = \mathbf{C}[c_r]_{r \in R}$ be the ring of polynomials generated by these variables (thus A is a polynomial ring in a set of variables corresponding to the conjugacy classes of reflections).

Write $A[V] = A \otimes_{\mathbf{C}} \mathbf{C}[V]$ for the ring of polynomial functions on V with coefficients in A. For each $y \in V$ define a *Dunkl operator* on $A[V]$ by

$$y(f) = \partial_y(f) - \sum_{r \in R} c_r \langle \alpha_r, y \rangle \frac{f - r(f)}{\alpha_r} \quad \text{for } f \in A[V], \tag{2.1}$$

where ∂_y is the partial derivative of f in the direction y and $\langle \cdot, \cdot \rangle$ is the natural pairing between V^* and V. Some authors choose a normalization in which the factor $2(1 - \det(r)^{-1})^{-1}$ appears, but we absorb this factor into the parameter c_r. Each $f \in A[V]$ defines a multiplication operator $h \mapsto fh$ on $A[V]$, and the *rational Cherednik algebra* $H = H(W, V)$ determined by these data is the A-subalgebra of $\mathrm{End}_A(A[V])$ generated by W, $A[V]$, and the Dunkl operators $y \in V$.

A routine computation shows that these operators satisfy the relations

$$wyw^{-1} = w(y) \quad wxw^{-1} = w(x) \quad \text{for } w \in W,\ x \in V^*, \text{ and } y \in V, \tag{2.2}$$

and by induction on the degree of f

$$yf - fy = \partial_y(f) - \sum_{r \in R} c_r \langle \alpha_r, y \rangle \frac{f - r(f)}{\alpha_r} r \quad \text{for } y \in V \text{ and } f \in A[V]. \tag{2.3}$$

Using these relations one then checks

$$[[y_1, y_2], f] = [y_1, [y_2, f]] + [[y_1, f], y_2] = 0$$

which implies

$$y_1 y_2 = y_2 y_1 \quad \text{for all } y_1, y_2 \in V. \tag{2.4}$$

In fact H is generated by $A[V]$, W, and $A[V^*]$ subject to (2.2) and the special case of (2.3) in which $f \in V^*$ is linear, in which case it may be written as

$$yx - xy = \langle x, y \rangle - \sum_{r \in R} c_r \langle \alpha_r, y \rangle \langle x, \alpha_r^\vee \rangle r \tag{2.5}$$

where $x - r(x) = \langle x, \alpha_r^\vee \rangle \alpha_r$ determines $\alpha_r^\vee \in V$.

Multiplication induces an isomorphism

$$A[V] \otimes AW \otimes A[V^*] \to H \tag{2.6}$$

called the *triangular decomposition* of H.

2.3. Let $\{S^\lambda \mid \lambda \in \Lambda\}$ be the set of irreducible representations of $\mathbf{C}W$, where Λ is an index set. To avoid a profusion of subscripts, we abuse notation and write also S^λ for its extension to AW. Write $A[V^*] \rtimes W$ for the subalgebra of H generated by the Dunkl operators $y \in V$ and the group W. The *standard module* corresponding to $\lambda \in \Lambda$ is

$$\Delta(\lambda) = \mathrm{Ind}_{A[V^*] \rtimes W}^H (S^\lambda) \tag{2.7}$$

where the $A[V^*] \rtimes W$-module structure on S^λ is determined by $yS^\lambda = 0$ for all $y \in V$. Thanks to the triangular decomposition (2.6) there is an isomorphism of $A[V] \rtimes W$-modules

$$A[V] \otimes_A S^\lambda \to \Delta(\lambda) \tag{2.8}$$

and via this isomorphism the Dunkl operators act according to the formula

$$y(f\otimes v) = \partial_y(f)\otimes v - \sum_{r\in R} c_r \langle \alpha_r, y\rangle \frac{f-r(f)}{\alpha_r}\otimes r(v) \quad \text{for } y\in V,\ f\in A[V], \text{ and } v\in S^\lambda. \tag{2.9}$$

2.4. There is a reparametrization that simplifies the expression of many numbers arising naturally in the study of the Cherednik algebra (in particular, the eigenvalues of the monodromy of the connections corresponding to the standard modules). Let $\mathcal{A}$ be the set of hyperplanes H in V of the form $H = \mathrm{fix}(r)$ for some $r \in R$. For each $H \in \mathcal{A}$ choose $\alpha_H \in V^*$ with H equal to the zero set of α_H. The subgroup $W_H = \{w \in W \mid w(v) = v \text{ if } v \in H\}$ is a cyclic subgroup, and we write $W_H^\vee$ for its character group. Let $n_H = |W_H|$ be the size of W_H and for $\chi \in W_H^\vee$ let

$$e_{H,\chi} = \frac{1}{n_H}\sum_{w\in W_H} \chi(w^{-1})w \in \mathbf{C}W_H$$

be the corresponding primitive idempotent. Define $c_{H,\chi}$ by

$$c_{H,\chi} n_H = \sum_{r\in W_H - \{1\}} c_r(1-\chi(r)). \tag{2.10}$$

In particular $c_{H,\mathrm{triv}} = 0$, and A may be viewed as a polynomial ring in the variables $c_{H,\chi}$ (modulo the relations $c_{H,\chi} = c_{wH,w\chi w^{-1}}$ for $w \in W$). Using the relation

$$w = \sum_{\chi\in W_H^\vee} \chi(w) e_{H,\chi} \quad \text{for } w \in W_H,$$

the formula for Dunkl operators becomes

$$y(f) = \partial_y(f) - \sum_{H\in\mathcal{A}} \frac{\langle \alpha_H, y\rangle}{\alpha_H} \sum_{\chi\in W_H^\vee - \{1\}} c_{H,\chi} n_H e_{H,\chi},$$

which is, up to a sign, the formula in [DuOp], equation (5).

For each $H \in \mathcal{A}$ fix an eigenvector $\alpha_H^\vee \notin H$ for W_H. In terms of the parameters $c_{H,\chi}$ the relation (2.5) is

$$yx - xy = \langle x, y\rangle - \sum_{H\in\mathcal{A}} \frac{\langle \alpha_H, y\rangle\langle x, \alpha_H^\vee\rangle}{\langle \alpha_H, \alpha_H^\vee\rangle} \sum_{\chi\in W_H^\vee} (c_{H,\chi\otimes\det^{-1}} - c_{H,\chi}) n_H e_{H,\chi} \tag{2.11}$$

2.5. We extend complex conjugation to $A = \mathbf{C}[c_{H,\chi}]$ be declaring $\overline{c_{H,\chi}} = c_{H,\chi}$. Fix a positive definite Hermitian inner product $\langle\cdot,\cdot\rangle$ on S^λ and mutually inverse W-equivariant conjugate linear isomorphisms $V \to V^*$ and $V^* \to V$, written $y \mapsto y^*$ and $x \mapsto x^*$. Then $\langle\cdot,\cdot\rangle$ has a unique extension, also denoted $\langle\cdot,\cdot\rangle$, to $\Delta(\lambda)$ determined by the following rules:

(a) $\langle\cdot,\cdot\rangle$ is bi-additive, A-linear in the second variable, and A-conjugate linear in the first variable with respect to the extension of complex conjugation to A that fixes the variables $c_{H,\chi}$,

(b) $\langle xf, g\rangle = \langle f, x^* g\rangle$ for all $x \in V^*$ and $f, g \in \Delta(\lambda)$.

2.6. We will consider two types of extensions of scalars: first, writing $F = \mathrm{Frac}(A)$ for the fraction field of A, we write $H_F = F \otimes_A H$ for the *generic* Cherednik algebra, and similarly $\Delta_F(\lambda) = F \otimes_R \Delta(\lambda)$ and $\langle \cdot, \cdot \rangle_F$ for the F-conjugate linear extension of $\langle \cdot, \cdot \rangle$ to $\Delta_F(\lambda)$. Second, given a specialization $A \to \mathbf{C}$ of the variables c_r (or $c_{H,\chi}$) to complex numbers, we will write $H_c = \mathbf{C} \otimes_A H$ and $\Delta_c(\lambda) = \mathbf{C} \otimes_A \Delta(\lambda)$ for the corresponding specializations. We think of the symbol c as standing for this specialization, or equivalently, for a set of complex numbers indexed by conjugacy classes of reflections (or conjugacy classes of characters of rank one parabolic subgroups). In case the specialization is such that the variables $c_{H,\chi}$ are all real, the contravariant form also specializes and we write $\langle \cdot, \cdot \rangle_c$ for its specialization.

2.7. Suppose that we have specialized the variables to complex numbers c. *Category* $\mathcal{O}_c$ is the subcategory of the category H_c−mod of finitely generated H_c-modules on which each Dunkl operator $y \in V$ acts locally nilpotently. Thanks to (2.9) each standard module $\Delta_c(\lambda)$ is in $\mathcal{O}_c$. In fact, the quotient $L_c(\lambda)$ of the standard module $\Delta_c(\lambda)$ by its radical is simple and this gives a complete list of inequivalent irreducible objects in $\mathcal{O}_c$. Furthermore, the radical of $\Delta_c(\lambda)$ coincides with the radical of the form $\langle \cdot, \cdot \rangle_c$. Therefore the contravariant form descends to a non-degenerate form on $L_c(\lambda)$, and Cherednik has posed the problem of deciding when this form is positive definite. The study of this problem began with the paper [EtSt].

2.8. From now on, we will assume that W is a monomial group. The precise definitions follow. Fix positive integers r and n. Let $W = G(r, 1, n)$ be the group of n by n matrices such that the entries are either 0 or a power of $e^{2\pi\sqrt{-1}/r}$, and there is exactly one non-zero entry in each row and each column. Then W is a group of matrices acting on $V = \mathbf{C}^n$, and we write $y_1, \dots, y_n$ for the standard basis of V.

There are two W-orbits of reflecting hyperplanes: writing $x_1, \dots, x_n$ for the standard basis of V^*, there those of the form $x_i = 0$ (for which $n_H = r$) and those of the form $x_i = \zeta^l x_j$ (for which $n_H = 2$). Write s_{ij} for the permutation matrix interchanging i and j and fixing all other coordinates, and ζ_i for the matrix that multiplies the ith coordinate by $\zeta = e^{2\pi\sqrt{-1}/r}$ and fixes all other coordinates. We will write $e_{ij} = \frac{1}{r}\sum_{l=0}^{r-1} \zeta^{-lj}\zeta_i^l$, and leave the other idempotents unnamed.

2.9. In the case $W = G(r, 1, n)$ the relations for the rational Cherednik algebra may be written in the following extremely explicit form. Let c_0 and $d_1, \dots, d_{r-1} \in \mathbf{C}$ be variables and let $A = \mathbf{C}[c_0, d_1, \dots, d_{r-1}]$. The Cherednik algebra H for W is generated by the algebras $A[y_1, \dots, y_n]$, $R[x_1, \dots, x_n]$, and AW, subject to the relations $wfw^{-1} = w(f)$ for $f \in A[y_1, \dots, y_n]$ or $f \in A[x_1, \dots, x_n]$ and $w \in W$,

$$y_i x_i = x_i y_i + \kappa - c_0 \sum_{\substack{1 \le j \ne i \le n \\ 0 \le l \le r-1}} \zeta_i^l s_{ij} \zeta_i^{-l} - \sum_{l=0}^{r-1} (d_l - d_{l-1}) e_{il} \tag{2.12}$$

for $1 \le i \le n$, and

$$y_i x_j = x_j y_i + c_0 \sum_{l=0}^{r-1} \zeta^{-l} \zeta_i^l s_{ij} \zeta_i^{-l} \tag{2.13}$$

for $1 \le i \ne j \le n$. Here κ is an additional parameter that the reader interested only in unitary representations may take to be 1. To define d_l for all $l \in \mathbf{Z}$ we specify d_0 by the relation $d_0 + d_1 + \cdots + d_{r-1} = 0$ and impose $d_i = d_j$ if $i = j \mod r$. In terms of the parameters c_r attached to conjugacy classes of reflections, c_0 is the parameter for the conjugacy class of transpositions, and

$$d_j = \sum_{1 \le m \le r-1} \zeta^{jm} c_m,$$

where c_m is the conjugacy class of reflections containing ζ_1^m. Comparing with (2.10), this implies that c_0 and $d_0, d_1, \dots, d_{r-1}$ are all real if and only if the $c_{H,\chi}$'s are all real. Whenever we make use of the specialized contravariant form, we will assume the parameters c_0 and d_l are all real.

2.10. A *partition* $\lambda = \lambda_1 \ge \lambda_2 \ge \cdots$ is a weakly decreasing sequence of integers such that $\lambda_n = 0$ for n large enough. Given a positive integer r, an r-partition is a sequence $\lambda^\bullet = (\lambda^0, \lambda^1, \dots, \lambda^{r-1})$ of r partitions. The *size* of an r-partition $\lambda^\bullet$ is the sum $|\lambda^\bullet| = \sum_{i,j} \lambda^i_j$, and an *$r$-partition of n* is an r-partition $\lambda^\bullet$ with $|\lambda^\bullet| = n$. We picture partitions and r-partitions as *Young diagrams*: collections of boxes stacked in a corner, as in (2.14) (but without the numbers). A *tableau* on an r-partition $\lambda^\bullet$ is a function T from the boxes of $\lambda^\bullet$ to the integers. A *standard tableau* on an r-partition $\lambda^\bullet$ of n is a bijection from the boxes of $\lambda^\bullet$ to $\{1, 2, \dots, n\}$ such that the enties in each λ^i are strictly increasing left to right and top to bottom. An example of a standard tableau on the 2-partition $\lambda^\bullet = ((3,2),(2,2))$ of 9 is

$$\left(\begin{array}{|c|c|c|} \hline 2 & 4 & 6 \\ \hline 3 & 9 \\ \cline{1-2} \end{array} \, , \, \begin{array}{|c|c|} \hline 1 & 5 \\ \hline 7 & 8 \\ \hline \end{array} \right). \tag{2.14}$$

Given a box $b \in \lambda^\bullet$, define $\beta(b) = l$ if $b \in \lambda^l$ and $\mathrm{ct}(b) = j - i$ if b is in the ith row and jth column of λ^l. For the example (2.14) we have $\beta(T^{-1}(5)) = 1$ and $\mathrm{ct}(T^{-1}(5)) = 1$.

2.11. Let $W = G(r,1,n)$. The *Jucys-Murphy elements* of the group algebra $\mathbf{C}W$ are

$$\phi_i = \sum_{\substack{1 \le j < i \\ 0 \le l \le r-1}} \zeta_i^l s_{ij} \zeta_i^{-l} \quad \text{for } 1 \le i \le n. \tag{2.15}$$

Together with the elements ζ_i of W they generate a subalgebra of $\mathbf{C}W$ that acts diagonalizably on every W-module. There is a bijection $\lambda^\bullet \mapsto S^{\lambda^\bullet}$ from the set of r-partitions of n to the set of irreducible W-modules such that $S^{\lambda^\bullet}$ has a basis v_T indexed by standard Young tableaux T on $\lambda^\bullet$, and v_T is determined up to scalars by the equations

$$\phi_i v_T = r\mathrm{ct}(T^{-1}(i)) v_T \quad \text{and} \quad \zeta_i v_T = \zeta^{\beta(T^{-1}(i))} v_T \quad \text{for } 1 \le i \le n. \tag{2.16}$$

We fix a W-invariant positive definite Hermitian form on each $S^{\lambda^\bullet}$ and assume that the norm of v_T with respect to this form is 1. For the groups $G(2,1,n)$, it seems these versions of Jucys-Murphy elements were first written down in [Che2].

2.12. As in [DuOp] Definition 3.7, we put

$$z_i = y_i x_i + c_0 \phi_i \quad \text{for } 1 \leq i \leq n. \tag{2.17}$$

Together with the elements ζ_i they generate a commutative algebra $\mathfrak{t}$ of H_c, and Theorem 5.1 of [Gri2] states that $\mathfrak{t}$ acts on each standard module $\Delta_c(\lambda^\bullet)$ in an upper-triangular fashion.

2.13. We define a partial order on the boxes of $\lambda^\bullet$ by: $b \leq b'$ if $T(b) < T(b')$ for all standard Young tableaux T on $\lambda^\bullet$. Thus $b \leq b'$ if and only if $\beta(b) = \beta(b')$ and b is (weakly) up and to the left of b'.

We write $\Gamma = \Gamma(\lambda^\bullet)$ for the set of pairs (P, Q) of tableaux on $\lambda^\bullet$ such that P is a bijection from the boxes of $\lambda^\bullet$ to the set $\{1, 2, \dots, n\}$, Q is a filling of the boxes of $\lambda^\bullet$ by non-negative integers such that if $b < b'$ then $Q(b) \leq Q(b')$, with $Q(b) = Q(b')$ implying $P(b) > P(b')$. Then Theorem 5.1 of [Gri2] implies that there is a $\mathfrak{t}$-eigenbasis $f_{P,Q}$ of $\Delta(\lambda^\bullet)$ such that

$$\zeta_i f_{P,Q} = \zeta^{\beta(P^{-1}(i)) - Q(P^{-1}(i))} f_{P,Q}$$

and

$$z_i f_{P,Q} = (Q(P^{-1}(i)) + 1 - (d_{\beta(P^{-1}(i))} - d_{\beta(P^{-1}(i)) - Q(P^{-1}(i)) - 1}) - r\mathrm{ct}(P^{-1}(i)) c_0) f_{P,Q}.$$

Then $f_{P,Q}$ is a polynomial function on $\mathbf{C}^n$ with values in $S^{\lambda^\bullet}$, which we will refer to a a (non-symmetric) *Specht-valued Jack polynomial*. The indexing here is related to that in the paper [Gri2] as follows: for a pair (μ, T) consisting of a standard Young tableau T on $\lambda^\bullet$ and $\mu \in \mathbf{Z}_{\geq 0}^n$, we let w_μ be the longest element of S_n such that $w_\mu(\mu)$ is non-decreasing and define $P = w_\mu^{-1} T$ and $Q(b) = \mu_{P(b)}$. Note that we used an unorthodox convention for standard Young tableaux in [Gri2], regarding them as functions from $\{1, 2, \dots, n\}$ to the boxes of $\lambda^\bullet$ (we have now come to our senses). This gives a bijection from the set of pairs (μ, T) as above to Γ.

3. The trigonometric presentation of H

Here we give another presentation of H which is adapted to the application of intertwining operators to the classification of diagonalizable modules in the next section.

3.1. The affine Weyl semigroup. Let $W_{\geq 0} = \mathbf{Z}_{\geq 0}^n \rtimes S_n$. It contains the elements $s_1, \dots, s_{n-1}$ and $\Phi = \epsilon_n s_{n-1} \cdots s_2 s_1$, so that $s_1, \dots, s_{n-1}$ satisfy the usual Coxeter relations, and interact with Φ via the relations

$$\Phi s_i = s_{i-1} \Phi \text{ for } 2 \leq i \leq n-1 \text{ and } \Phi^2 s_1 = s_{n-1} \Phi^2. \tag{3.1}$$

In fact, the abstract semigroup with generators $s_1, \dots, s_{n-1}$ and Φ, together with the Coxeter relations and (3.1) is isomorphic to $W_{\geq 0}$, as we now sketch.

Letting G be this semigroup, it follows that there is a map $G \to W_{\geq 0}$ and thus that $s_1, \dots, s_{n-1}$ generate a copy of S_n inside G. Define $\epsilon_n = \Phi s_1 \cdots s_{n-1}$. The relations in (3.1) imply that $s_i \epsilon_n = \epsilon_n s_i$ for $1 \leq i \leq n-2$, and therefore we may unambiguously define $\epsilon_i = w \epsilon_n w^{-1}$ for each $1 \leq i \leq n-1$ and any $w \in S_n$ with $w(n) = i$. It follows from this definition that $w \epsilon_i w^{-1} = \epsilon_{w(i)}$ for all $1 \leq i \leq n$ and $w \in S_n$. Again using (3.1), a direct calculation shows that $\epsilon_n \epsilon_1 = \epsilon_1 \epsilon_n$, and hence for all $1 \leq i \neq j \leq n$ choosing $w \in S_n$ with

$w(1) = i$ and $w(n) = j$ gives $\epsilon_i \epsilon_j = w\epsilon_1\epsilon_n w^{-1} = w\epsilon_n\epsilon_1 w^{-1} = \epsilon_j\epsilon_i$. It follows from this that there is a map $W_{\geq 0} \to G$ inverse to the previous one.

3.2. The Dunkl-Opdam subalgebra. The *Dunkl-Opdam* subalgebra $\mathfrak{t}$ of H is the (commutative, as proved in [DuOp]) subalgebra of H generated by $z_1, \dots, z_n$ and $\zeta_1, \dots, \zeta_n$. By the PBW theorem it is isomorphic to the polynomial ring in the variables $z_1, \dots, z_n$ tensored with the group algebra of $(\mathbf{Z}/r\mathbf{Z})^n$. Define an automorphism ϕ of $\mathfrak{t}$ by

$$\phi(\zeta_i) = \zeta_{i+1} \quad \text{for } 1 \leq i \leq n-1 \text{ and } \quad \phi(\zeta_n) = \zeta^{-1}\zeta_1 \tag{3.2}$$

and

$$\phi(z_i) = z_{i+1} \quad \text{for } 1 \leq i \leq n-1 \text{ and } \quad \phi(z_n) = z_1 + 1 - \sum_{j=0}^{r-1} (d_{j-1} - d_{j-2}) e_{1j}. \tag{3.3}$$

Put $\Phi = x_n s_{n-1} \cdots s_1$ and $\Psi = y_1 s_1 \cdots s_{n-1}$. By Proposition 4.3 and Lemma 5.3 of [Gri1],

$$f\Phi = \Phi\phi(f) \quad \text{and} \quad f\Psi = \Psi\phi^{-1}(f) \tag{3.4}$$

for all $f \in \mathfrak{t}$, and

$$z_i s_j = s_j z_i \text{ if } j \neq i, i+1, \text{ while } \quad z_i s_i = s_i z_{i+1} - c_0 \pi_i \text{ for } 1 \leq i \leq n-1, \tag{3.5}$$

where

$$\pi_i = \sum_{l=0}^{r-1} \zeta_i^l \zeta_{i+1}^{-l}.$$

Thus (as observed in [Dez1] and [Dez2]) the subalgebra H_{gr} of H generated by $\mathfrak{t}$ and $W = G(r,1,n)$ is isomorphic to the generalized graded affine Hecke algebra for $G(r,1,n)$ defined in [RaSh]. The structure of an H_{gr}-module may be put on a vector space by definining an action of $\mathfrak{t}$ together with operators s_i satisfying the Coxeter relations together with (3.5) and $w\zeta_i = \zeta_{w(i)} w$ for all $w \in S_n$ and $1 \leq i \leq n$.

3.3. The trigonometric presentation. H contains the commutative subalgebra $\mathfrak{t}$, elements Φ, Ψ, and $s_1, \dots, s_{n-1}$. These satisfy the following relations: (1) $\mathfrak{t}$ and $s_1, \dots, s_{n-1}$ generate a graded affine Hecke algebra H_{gr} for the group $G(r,1,n)$ inside H, (2) Φ and $s_1, \dots, s_{n-1}$ generate an affine Weyl semigroup inside H, (3) Ψ and $s_1, \dots, s_{n-1}$ generate an affine Weyl semigroup inside H with relations

$$\Psi s_i = s_{i+1}\Psi \text{ for } 1 \leq i \leq n-2 \text{ and } \Psi^2 s_{n-1} = s_1 \Psi^2, \tag{3.6}$$

and the following relations hold:

$$\zeta_i \Phi = \Phi\phi(\zeta_i), \quad \zeta_i \Psi = \Psi\phi^{-1}(\zeta_i), \tag{3.7}$$

(3.8)

$$\Psi\Phi = z_1, \quad \Phi\Psi = z_n - \kappa + \sum_{j=0}^{r-1} (d_j - d_{j-1}) e_{nj}, \quad \text{and} \quad \Psi s_{n-1} \Phi = \Phi s_1 \Psi + c_0 \sum_{0 \leq l \leq r-1} \zeta^{-l} \zeta_1^l \zeta_n^{-l}.$$

In fact, this constitutes a presentation for H, as we will see in the remainder of this section. Constructing an H-module may therefore be done as follows: construct an H_{gr}-module together with operators Φ and Ψ satisfying the relations (3.1), (3.6), (3.7), and (3.8).

Theorem 3.1. *Let A be the algebra generated by H_{gr} together with elements Φ and Ψ satisfying* (3.1), (3.6), (3.7), *and* (3.8). *The natural map $A \to H$ is an isomorphism.*

Proof. Define elements $x_n, y_1 \in A$ by $x_n = \Phi s_1 \cdots s_{n-1}$ and $y_1 = \Psi s_{n-1} \cdots s_1$. Then put $x_i = w x_n w^{-1}$ and $y_i = v y_1 v^{-1}$ where $w, v \in S_n$ are chosen with $w(n) = i = v(1)$. The various x_i's commute with one another by the discussion in 3.1, and by symmetry the y_i's commute. Furthermore, the algebra A is generated by the group W together with $x_1, \dots, x_n$ and $y_1, \dots, y_n$. We will show that it is spanned by the set of all words $x_1^{a_1} \cdots x_n^{a_n} y_1^{b_1} \cdots y_n^{b_n} w$ with $a_i, b_i \in \mathbf{Z}_{\geq 0}$ and $w \in W$. From this together with the PBW theorem for H it will follow that the natural map from A to H is an isomorphism.

It suffices to show that the span of the set of words as above is closed under left multiplication by x_i's, y_i's, and w's. This is clear for x_i's and easy for w's. We will show how to reorder a product $y_i x_j$. First observe that by the definitions of x_1,y_1 and the relation $\Psi\Phi = z_1$,

$$y_1 x_1 = \Psi\Phi = z_1.$$

By using the graded Hecke algebra relations between z_i and s_i it follows by induction on i that

$$z_i = y_i x_i + a_i \quad \text{for some } a_i \in \mathbf{C}W.$$

In particular

$$y_n x_n + a_n = z_n = \Phi\Psi + \kappa - \sum (d_j - d_{j-1}) e_{nj} = x_n y_n + \kappa - \sum (d_j - d_{j-1}) e_{nj}. \tag{3.9}$$

This proves that $y_n x_n = x_n y_n + b_n$ for some $b_n \in \mathbf{C}W$. Conjugating by some $w \in S_n$ with $w(n) = i$ gives $y_i x_i = x_i y_i + b_i$ for some $b_i \in \mathbf{C}W$. Using the last relation in (3.8), the relations (3.1) and (3.6) and the definitions of $y_1 = \Psi s_{n-1} \cdots s_1$ and $x_n = \Phi s_1 \cdots s_{n-1}$ allows one to rewrite $y_1 x_n = x_n y_1 + b_{1n}$ for some $b_{1n} \in \mathbf{C}W$, and conjugating by $w \in S_n$ with $w(1) = i$ and $w(n) = j$ gives $y_i x_j = x_j y_i + b_{ij}$ for some $b_{ij} \in \mathbf{C}W$, finishing the proof. □

4. Specht-valued Jack polynomials

For $\mu, \nu \in \mathbf{Z}^n_{\geq 0}$, write $\mu > \nu$ if either $\mu_+ >_d \nu_+$, where μ_+ and ν_+ are the partition rearrangements of μ and ν and $>_d$ denotes dominance order, or $\mu_+ = \nu_+$ and $w_\mu > w_\nu$ in Bruhat order. Extend this to a partial order on pairs (μ, T) by ignoring T: thus $(\mu, T) \geq (\nu, S)$ exactly if $\mu \geq \nu$. The following is Theorem 5.1 of [Gri2]; the polynomials it constructs are $S^{\lambda^\bullet}$-valued generalizations of non-symmetric Jack polynomials. We use them to construct bases for the irreducible unitary representations in $\mathcal{O}_c$.

Theorem 4.1. *Let λ be an r-partition of n, $\mu \in \mathbf{Z}^n_{\geq 0}$, and let T be a standard tableau on λ. Put $v_T^\mu = w_\mu^{-1}.v_T$ and recall the definitions of β and ct given in 2.10.*

(a) *The action of ζ_i and z_i on $\Delta(\lambda^\bullet)$ are given by*

$$\zeta_i . x^\mu v_T^\mu = \zeta^{\beta(T^{-1}(w_\mu(i))) - \mu_i} x^\mu v_T^\mu$$

and

$$z_i.x^\mu v_T^\mu = \left(\mu_i + 1 - (d_{\beta(T^{-1}(w_\mu(i)))} - d_{\beta(T^{-1}(w_\mu(i)))-\mu_i-1}) - c_0 r\, ct(T^{-1}(w_\mu(i)))\right) x^\mu v_T^\mu + \sum_{(\nu,S)<(\mu,T)} c_{\nu,S} x^\nu v_S^\nu.$$

(b) *Assuming that scalars are extended to $F = \mathbf{C}(c_0, d_1, d_2, \dots, d_{r-1})$, for each $\mu \in \mathbf{Z}^n_{\geq 0}$ and $T \in SYT(\lambda)$ there exists a unique $\mathfrak{t}$ eigenvector $f_{\mu,T} \in \Delta(\lambda^\bullet)$ such that*

$$f_{\mu,T} = x^\mu v_T^\mu + \text{lower terms.}$$

The $\mathfrak{t}$-eigenvalue of $f_{\mu,T}$ is determined by the formulas in part (a).

We will also index these non-symmetric Jack polynomials $f_{P,Q}$, with $(P,Q) \in \Gamma$ as in 2.13.

4.1. Intertwiners. The intertwiners σ_i are defined, for $1 \leq i \leq n-1$, by

$$\sigma_i = s_i + \frac{c_0}{z_i - z_{i+1}} \pi_i. \tag{4.1}$$

Thus σ_i is well-defined on any $\mathfrak{t}$-weight space on which π_i acts by 0 or on which z_i and z_{i+1} have distinct eigenvalues.

For convenience, we reproduce here Lemma 5.3 of [Gri2], which describes how the intertwiners act on the basis $f_{\mu,T}$ of $\Delta(\lambda^\bullet)$. For $\mu \in \mathbf{Z}^n$ define

$$\phi(\mu_1, \dots, \mu_n) = (\mu_2, \mu_3, \dots, \mu_n, \mu_1 + 1) \quad \text{and} \quad \psi(\mu_1, \dots, \mu_n) = \phi^{-1}(\mu_1, \dots, \mu_n). \tag{4.2}$$

Lemma 4.2. *Let $\mu \in \mathbf{Z}^n_{\geq 0}$ and let T be a standard Young tableau on λ.*

(a) *Suppose $\mu_i \neq \mu_{i+1}$. If $\mu_i < \mu_{i+1}$ or $\mu_i - \mu_{i+1} \neq \beta(T^{-1}(w_\mu(i))) - \beta(T^{-1}(w_\mu(i+1)))$ mod r then*

$$\sigma_i.f_{\mu,T} = f_{s_i.\mu,T}.$$

(b) *If $\mu_i > \mu_{i+1}$ and $\mu_i - \mu_{i+1} = \beta(T^{-1}(w_\mu(i))) - \beta(T^{-1}(w_\mu(i+1)))$ mod r then*

$$\sigma_i.f_{\mu,T} = \frac{(\delta - rc_0)(\delta + rc_0)}{\delta^2} f_{s_i\mu,T},$$

where

$$\delta = \kappa(\mu_i - \mu_{i+1}) - (d_{\beta(T^{-1}(w_\mu(i)))} - d_{\beta(T^{-1}(w_\mu(i+1)))}) - c_0 r(ct(T^{-1}(w_\mu(i))) - ct(T^{-1}(w_\mu(i+1)))).$$

(c) *Put $j = w_\mu(i)$. If $\mu_i = \mu_{i+1}$ then*

$$\sigma_i.f_{\mu,T} = \begin{cases} 0 & \text{if } s_{j-1}.T \text{ is not a standard tableau,} \\ f_{\mu,s_{j-1}.T} & \text{if } \zeta^{\beta(T(j))} \neq \zeta^{\beta(T(j-1))}, \\ \left(1 - \left(\frac{1}{ct(T(j-1)) - ct(T(j))}\right)^2\right)^{1/2} f_{\mu,s_{j-1}.T} & \text{else.} \end{cases}$$

(d) *For all $\mu \in \mathbf{Z}^n_{\geq 0}$,*

$$\Phi.f_{\mu,T} = f_{\phi.\mu,T}.$$

(e) *For all* $\mu \in \mathbf{Z}_{\geq 0}^n$,

$$\Psi.f_{\mu,T} = \begin{cases} \left(\kappa\mu_n - (d_{\beta(T^{-1}(w_\mu(n)))} - d_{\beta(T^{-1}(w_\mu(n)))-\mu_n}) - rct(T^{-1}(w_\mu(n)))c_0\right) f_{\psi.\mu,T} & \text{if } \mu_n > 0, \\ 0 & \text{if } \mu_n = 0. \end{cases}$$

We define $\phi(P,Q)$ as follows: ϕ cycles the entries of P, relacing $P(b)$ by $P(b)-1$ for $P(b) > 1$ and replacing $P(b)$ by n if $P(b) = 1$, and adds 1 to the entry of Q in the box that P labels with 1. Thus for example if (P,Q) is the pair

$$\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & 4 \\ \hline \end{array}\;,\;\begin{array}{|c|c|}\hline 0 & 0 \\ \hline 0 & 1 \\ \hline \end{array}$$

then $\phi(P,Q)$ is the pair

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 4 & 3 \\ \hline \end{array}\;,\;\begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 & 1 \\ \hline \end{array}$$

We write $\psi = \phi^{-1}$ for the inverse of ϕ (which does not preserve $\Gamma(\lambda)$.

The lemma, translated into the (P,Q)-indexing and using $c(b) = d_{\beta(b)} + r\mathrm{ct}(b)c_0$ is

Lemma 4.3. *Let* $\mu \in \mathbf{Z}_{\geq 0}^n$ *and let* T *be a standard Young tableau on* λ.

(a) *If* $Q(P^{-1}(i)) < Q(P^{-1}(i+1))$ *or* $Q(P^{-1}(i)) - Q(P^{-1}(i+1)) \neq \beta(P^{-1}(i)) - \beta(P^{-1}(i+1))$ *mod* r *then*

$$\sigma_i.f_{P,Q} = f_{s_iP,Q}.$$

(b) *If* $Q(P^{-1}(i)) > Q(P^{-1}(i+1))$ *and* $Q(P^{-1}(i)) - Q(P^{-1}(i+1)) = \beta(P^{-1}(i)) - \beta(P^{-1}(i+1))$ *mod* r *then*

$$\sigma_i.f_{P,Q} = \frac{(\delta - rc_0)(\delta + rc_0)}{\delta^2} f_{s_iP,Q},$$

where

$$\delta = Q(P^{-1}(i)) - Q(P^{-1}(i+1)) - (c(P^{-1}(i)) - c(P^{-1}(i+1))).$$

(c) *If* $Q(P^{-1}(i))i = Q(P^{-1}(i+1))$ *then*

$$\sigma_i.f_{P,Q} = \begin{cases} 0 & \text{if } (s_iP,Q) \text{ is not an element of } \Gamma(\lambda), \\ f_{s_iP,Q} & \text{if } \beta(P^{-1}(i)) \neq \beta(P^{-1}(i+1)), \\ \left(1 - \left(\frac{1}{ct(P^{-1}(i+1)) - ct(P^{-1}(i))}\right)^2\right)^{1/2} f_{s_iP,Q} & \text{else.} \end{cases}$$

(d) *For all* $(P,Q) \in \Gamma(\lambda)$,

$$\Phi.f_{P,Q} = f_{\phi(P,Q)}.$$

(e) *For all* $(P,Q) \in \Gamma(\lambda)$,

$$\Psi.f_{P,Q} = \begin{cases} \left(Q(P^{-1}(n)) - c(P^{-1}(n)) + d_{\beta(P^{-1}(n)) - Q(P^{-1}(n))}\right) f_{\psi(P,Q)} & \text{if } Q(P^{-1}(n)) > 0, \\ 0 & \text{if } Q(P^{-1}(n)) = 0. \end{cases}$$

5. Diagonalizability

5.1. Weight spaces. Fix an r-partition $\lambda^\bullet$ and let $\Gamma = \mathbf{Z}_{\geq 0} \times \mathrm{SYT}(\lambda^\bullet)$ (via the bijection of 2.13 this is the same Γ as defined there). Given $c = (c_0, d_0, \dots, d_{r-1}) \in \mathbf{C}^{r+1}$, $(\mu, T) \in \Gamma$ and $1 \leq i \leq n$, write $\mathrm{wt}_c(\mu, T)_i$ for the pair

(5.1)
$$\mathrm{wt}_c(\mu,T)_i = (\mu_i + 1 - (d_{\beta(T^{-1}w_\mu(i))} - d_{\beta(T^{-1}w_\mu(i))-\mu_i-1}) - r\mathrm{ct}(T^{-1}w_\mu(i))c_0, \zeta^{\beta(T^{-1}w_\mu(i))-\mu_i})$$

Then define (μ, T) to be *c-folded* (or simply *folded* when c is fixed or clear from context) if $\mathrm{wt}_c(\mu,T)_i = \mathrm{wt}_c(\mu,T)_{i+1}$ for some $1 \leq i \leq n-1$. Foldings create non-trivial Jordan blocks:

Lemma 5.1. *Suppose that* $\mathrm{wt}_c(\mu,T)_i = \mathrm{wt}_c(\mu,T)_{i+1} = (\alpha, \beta)$*. Put* $f_1 = f_{\mu,T}$ *and* $f_2 = s_i f_1$*. If* $z_i f_1 = \alpha f_1 = z_{i+1} f_1$*, then* $(z_i - \alpha) f_2 = -rc_0 f_1$ *and* $(z_{i+1} - \alpha) f_2 = rc_0 f_1$*.*

Proof. Apply (3.5). □

As in [Gri2], for a box $b \in \lambda^\bullet$ and a positive integer k define a set

$$\Gamma_{b,k} = \{(\mu,T) \in \Gamma \mid \mu^-_{T(b)} \geq k\}, \tag{5.2}$$

and for an ordered pair of distinct boxes $b_1, b_2 \in \Gamma$ and a positive integer $k \in \mathbf{Z}_{>0}$, define the subset $\Gamma_{b_1,b_2,k}$ of Γ by

$$\begin{aligned} (\mu,T) \in \Gamma_{b_1,b_2,k} \quad \iff \quad & \text{either } \mu^-_{T(b_1)} - \mu^-_{T(b_2)} > k \\ & \text{or } \mu^-_{T(b_1)} - \mu^-_{T(b_2)} = k \text{ and } w_\mu^{-1}(T(b_1)) < w_\mu^{-1}(T(b_2)). \end{aligned} \tag{5.3}$$

Via the bijection with pairs (P,Q) as above, these definitions become somewhat easier on the eyes:

$$\Gamma_{b,k} = \{(P,Q) \mid Q(b) \geq k\} \tag{5.4}$$

and

$$\Gamma_{b_1,b_2,k} = \{(P,Q) \mid Q(b_1) - Q(b_2) > k, \text{ or } Q(b_1) - Q(b_2) = k \text{ and } P(b_1) < P(b_2) \tag{5.5}$$

For a given parameter c, define the set $\Gamma_c \subseteq \Gamma$ by

$$\Gamma_c = \bigcap_{b,k} \Gamma^c_{b,k} \cap \bigcap_{b_1,b_2,k} \Gamma^c_{b_1,b_2,k}, \tag{5.6}$$

where for a subset $X \subseteq \Gamma$ we write X^c for its complement, the first intersection runs over pairs $b \in \lambda^\bullet$ and $k \in \mathbf{Z}_{>0}$ such that

$$k = d_{\beta(b)} - d_{\beta(b)-k} + r\mathrm{ct}(b)c_0$$

and the second intersection runs over triples $b_1, b_2 \in \lambda^\bullet$, $k \in \mathbf{Z}_{>0}$ such that $k = \beta(b_1) - \beta(b_2) \bmod r$ and

$$k = d_{\beta(b_1)} - d_{\beta(b_2)} + r(\mathrm{ct}(b_1) - \mathrm{ct}(b_2) \pm 1)c_0.$$

The motivation for the definition is that the set Γ_c contains exactly those (μ, T) such that $f_{\mu,T}$ may be constructed from some $v_T \in S^{\lambda^\bullet}$ by applying a sequence of *invertible* intertwining operators; this is a consequence of Lemma 7.4 of [Gri2].

The definition of Γ_c may be rephrased in terms of pairs (P,Q) as follows: a pair (P,Q) is in Γ_c if and only if the following conditions hold:

(a) whenever $b \in \lambda^\bullet$ and $k \in \mathbf{Z}_{>0}$ with $k = d_{\beta(b)} - d_{\beta(b)-k} + r\mathrm{ct}(b)c_0$ we have $Q(b) < k$, and
(b) whenever $b_1, b_2 \in \lambda^\bullet$ and $k \in \mathbf{Z}_{>0}$ with $\beta(b_1) - \beta(b_2) = k \mod r$ and $k = d_{\beta(b_1)} - d_{\beta(b_2)} + r(\mathrm{ct}(b_1) - \mathrm{ct}(b_2) \pm 1)c_0$ we have $Q(b_1) \leq Q(b_2) + k$, with equality implying $P(b_1) > P(b_2)$.

The *boundary* of Γ_c is

$$\partial\Gamma_c = \{(\mu, T) \in \Gamma - \Gamma_c \mid (\psi.\mu, T) \in \Gamma_c \text{ or } (s_i.\mu, T) \in \Gamma_c \text{ for some } 1 \leq i \leq n\}. \tag{5.7}$$

Lemma 5.2. *Assume $c_0 \neq 0$.*
(a) Suppose $(\mu, T) \in \Gamma_c$ and $(\nu, S) \in \Gamma$ with $wt_c(\mu, T) = wt_c(\nu, S)$. Then $(\nu, S) = (\mu, T)$.
(b) For all $(\mu, T) \in \Gamma_c$ and $1 \leq i \leq n-1$, we have $\mathrm{wt}_c(\mu, T)_i \neq \mathrm{wt}_c(\mu, T)_{i+1}$.
(c) The non-symmetric generalized Jack polynomials $f_{\mu,T}$ for $(\mu, T) \in \Gamma_c \cup \partial\Gamma_c$ are all well-defined at c.
(d) If $(\mu, T) \in \partial\Gamma_c$ is folded then $L_c(\lambda^\bullet)$ is not $\mathfrak{t}$-diagonalizable.

Proof. The definition of Γ_c and Lemma 7.4 of [Gri2] together imply that the intertwiners connecting different $\mathfrak{t}$-weight spaces indexed by Γ_c are all invertible; it follows that every such weight space has the same dimension. Since the weight spaces in degree 0 (coming from $S^{\lambda^\bullet}$) are all one dimensional (here we use $c_0 \neq 0$), this proves (a). Part (b) follows from (a) together with Lemma 5.1. By part (b) the intertwining operators are well-defined on all weight spaces coming from Γ_c; this allows one to recursively construct all Jack polynomials coming from $\Gamma_c \cup \partial\Gamma_c$ recursively, proving (c).

Now we prove (d). Suppose $(\mu, T) \in \partial\Gamma_c$ is folded and let $f_1 = f_{\mu,T}$. If $\mathrm{wt}_c(\mu, T)_i = \mathrm{wt}_c(\mu, T)_{i+1}$ then part (b) implies $z_i f = \alpha f = z_{i+1} f$ for some $\alpha \in \mathbf{C}$, and by Lemma 5.1 $f_2 = s_i f_1$ witnesses a non-trivial Jordan block for $\mathfrak{t}$: $(z_i - \alpha)f_2 = -rc_0 f_1 = -(z_{i+1} - \alpha)f_2$. We will show that the image of f_2 in $L_c(\lambda^\bullet)$ is non-zero. By (b) we must have either $(s_{i-1}\mu, T) \in \Gamma_c$ or $(s_{i+1}\mu, T) \in \Gamma_c$ or $(\psi\mu, T) \in \Gamma_c$.

Suppose that $(\psi\mu, T) \in \Gamma_c$. It follows that the map Ψ is not an injection on the weight space for (μ, T), and hence by the second equation in (3.8) the z_n-eigenvalue on (μ, T) is given by $\alpha = 1 - (d_{\beta_j} - d_{\beta_j - 1})$ where $\zeta_n f_{\mu,T} = \zeta^{\beta_j} f_{\mu,T}$. Compute using (3.8) and Lemma 5.1

$$\Phi\Psi f_2 = (z_n - 1 + \sum_{j=0}^{r-1} (d_j - d_{j-1})e_{nj})f_2 = (z_n - \alpha)f_2 = rc_0 f_1.$$

This equation implies that $\Psi f_2 = a f_{(\psi\mu,T)}$ for some $a \in \mathbf{C}^\times$ and since the image of $f_{(\psi\mu,T)}$ in $L_c(\lambda^\bullet)$ is non-zero, so is the image of f_2. □

Now we can given our first (not completely explicit) description of the diagonalizable $L_c(\lambda^\bullet)$'s. When $c_0 = 0$ the modules $\Delta_c(\lambda^\bullet)$ are all diagonalizable (but with weight spaces of dimension greater than 1), so the following theorem finishes the classification.

Theorem 5.3. *Suppose $c_0 \neq 0$. The module $L_c(\lambda^\bullet)$ is diagonalizable exactly if no element of $\partial\Gamma_c$ is folded; in this case a basis is given by $\{f_{\mu,T} \mid (\mu, T) \in \Gamma_c\}$.*

Proof. Given Lemma 5.2, it remains to show that if no element of $\partial\Gamma_c$ is folded then $L_c(\lambda^\bullet)$ is diagonalizable with the given basis. Let V be the abstract $\mathbf{C}$-vector space with basis given by $\{f_{\mu,T} \mid (\mu, T) \in \Gamma_c\}$, and define actions of $\mathfrak{t}$, $\sigma_1, \dots, \sigma_{n-1}$, Φ, and Ψ on V as follows: the

$\mathfrak{t}$-action has $f_{\mu,T}$ as eigenfunctions with eigenvalues given by Theorem 4.1, and the action of σ_i, Φ, and Ψ is given by the formulas in Lemma 4.2 with the following exceptions: if $(s_i\mu, T) \notin \Gamma_c$ then we put $\sigma_i f_{\mu,T} = 0$ and if $(\phi\mu, T) \notin \Gamma_c$ then we put $\Phi f_{\mu,T} = 0$. Without using the hypothesis that no element of $\partial\Gamma_c$ is folded, it follows from these definitions that the σ_i's satisfy the braid relations, that

$$\sigma_i^2 = \frac{(z_i - z_{i+1} - c_0\pi_i)(z_i - z_{i+1} + c_0\pi_i)}{(z_i - z_{i+1})^2}, \tag{5.8}$$

that

$$f\sigma_i = \sigma_i s_i(f) \quad \text{for } 1 \leq i \leq n-1 \text{ and } f \in \mathfrak{t}, \tag{5.9}$$

that

$$\Phi\sigma_i = \sigma_{i-1}\Phi \quad \text{for } 2 \leq i \leq n-1 \text{ and } \quad \Phi^2\sigma_1 = \sigma_{n-1}\Phi^2, \tag{5.10}$$

that

$$\Psi\sigma_i = \sigma_{i+1}\Psi \quad \text{for } 1 \leq i \leq n-2 \text{ and } \quad \Psi^2 s_{n-1} = s_1\Psi^2, \tag{5.11}$$

that

$$f\Phi = \Phi\phi(f) \quad \text{and} \quad f\Psi = \Psi\phi^{-1}(f) \quad \text{for } f \in \mathfrak{t} \tag{5.12}$$

and that

$$\Psi\Phi = z_1, \quad \Phi\Psi = z_n - \kappa + \sum_{j=0}^{r-1}(d_j - d_{j-1})e_{nj}, \quad \text{and} \quad \Psi\sigma_{n-1}\Phi = \Phi\sigma_1\Psi. \tag{5.13}$$

Indeed, these formulas hold by Lemma 4.2 when applied to those $f_{\mu,T}$ for which the result stays in Γ_c at each stage; some care must be taken near the boundary, as we indicate next.

We will sketch a check of the relation $\Psi\sigma_{n-1}\Phi = \Phi\sigma_1\Psi$ here; the others involve similar reasoning. Suppose first that $\Phi f_{P,Q} = 0$, that is, $\phi(P,Q) \notin \Gamma_c$. By definition of Γ_c and ϕ, there is a box b with $Q(b) = k-1$, $P(b) = 1$ and such that if $Q'(b) \geq k$ for some $(P',Q') \in \Gamma$ then $(P',Q') \notin \Gamma_c$. Now observe that $\phi s_1 \psi(P,Q) = (P',Q')$ with $Q'(b) = k$ and hence $(P',Q') \notin \Gamma_c$, so both operators act by zero on $f_{P,Q}$.

If $\phi(P,Q) \in \Gamma_c$ but $s_{n-1}\phi(P,Q) \notin \Gamma_c$, then writing $(P',Q') = \phi(P,Q)$, setting $b_1 = P'^{-1}(n)$ and $b_2 = P'^{-1}(n-1)$ we have $Q'(b_1) = Q'(b_2) + k$ for some positive integer k, $k = \beta(b_1) - \beta(b_2) \mod r$, such that if $(P_0,Q_0) \in \Gamma$ with $Q_0(b_1) > Q_0(b_2) + k$, or $Q_0(b_1) = Q_0(b_2) + k$ and $P_0(b_1) < P_0(b_2)$, then $(P_0,Q_0) \notin \Gamma_c$. It follows from this that $s_1\psi(P,Q) \notin \Gamma_c$, so again both operators act by zero on $f_{P,Q}$. The other cases are handled in a similar fashion.

Now define the action of $s_1, \dots, s_{n-1}$ on V by the formula

$$s_i = \sigma_i - \frac{c_0}{z_i - z_{i+1}}\pi_i. \tag{5.14}$$

This makes sense by part (b) of Lemma 5.2. Using Theorem 3.1 we must check that the s_i's and $\mathfrak{t}$ satisfy the graded Hecke relations. The relations (3.5) follow from the definition (5.14) and the relations (5.9). The fact that $s_i^2 = 1$ follows from (5.14) and (5.8). The fact that the braid relations are satisfied will be the first place the hypothesis that $\partial\Gamma_c$ contains no folds is used. Compute:

$$\begin{aligned}
s_i s_{i+1} s_i &= \left(\sigma_i - \frac{c_0}{z_i - z_{i+1}}\pi_i\right)\left(\sigma_{i+1} - \frac{c_0}{z_{i+1} - z_{i+2}}\pi_{i+1}\right)\left(\sigma_i - \frac{c_0}{z_i - z_{i+1}}\pi_i\right) \\
&= \sigma_i\sigma_{i+1}\sigma_i - c_0\sigma_i\sigma_{i+1}\frac{1}{z_i - z_{i+1}}\pi_i - c_0\sigma_i^2\frac{1}{z_i - z_{i+2}}\pi_{i,i+2} - c_0\sigma_{i+1}\sigma_i\frac{1}{z_{i+1} - z_{i+2}}\pi_{i+1} \\
&+ c_0^2\sigma_i\frac{1}{(z_{i+1} - z_{i+2})(z_i - z_{i+1})}\pi_i\pi_{i+1} + c_0^2\sigma_{i+1}\frac{1}{(z_i - z_{i+2})(z_i - z_{i+1})}\pi_{i,i+2}\pi_{i+1} \\
&+ c_0^2\sigma_i\frac{1}{(z_{i+1} - z_i)(z_i - z_{i+2})}\pi_i\pi_{i,i+2} - c_0^3\frac{1}{(z_i - z_{i+1})^2(z_{i+1} - z_{i+2})}\pi_i\pi_{i+1}\pi_i.
\end{aligned}$$

This preceding calculation was formal, but the hypothesis that no element of $\partial\Gamma_c$ is folded is exactly what is needed to ensure that the right-hand side of the above equation is well-defined when applied to $f_{\mu,T}$ for all $(\mu, T) \in \Gamma_c$. Routine arithmetic verifies that it is the same as the corresponding expression for $s_{i+1}s_is_{i+1}$. This verifies that we have the structure of an $\mathbb{H}_{\mathrm{gr}}$-module on V.

Verification of the relations (3.1), (3.6), and the last equation in (3.8) is exactly analogous, again using the hypothesis that no element of $\partial\Gamma_c$ is folded: for instance, one computes

$$\begin{aligned}
\Psi s_{n-1}\Phi &= \Psi\left(\sigma_{n-1} - \frac{c_0}{z_{n-1} - z_n}\pi_{n-1,n}\right)\Phi = \Phi\sigma_1\Psi - \Psi\Phi\phi\left(\frac{c_0}{z_{n-1} - z_n}\pi_{n-1,n}\right) \\
&= \Phi s_1\Psi - \Phi\Psi\phi^{-1}\left(\frac{c_0}{z_1 - z_2}\pi_{1,2}\right) - \Psi\Phi\phi\left(\frac{c_0}{z_{n-1} - z_n}\pi_{n-1,n}\right).
\end{aligned} \tag{5.15}$$

The hypothesis that there are no folded elements of $\partial\Gamma_c$ implies that this last expression makes sense when applied to any $f_{P,Q}$ for $(P, Q) \in \Gamma_c$, and a straightforward calculation shows that it is equivalent to the last relation in (3.8).

We have therefore defined an H_c-module struture on V. It follows from the construction and Lemma 5.2 that the $\mathfrak{t}$-weight spaces on V are all one-dimensional, and from Lemma 7.4 of [Gri2] that any non-zero weight vector generates V as an H_c-module. Thus V is irreducible. It belong to category $\mathcal{O}_c$ since by construction Ψ (and hence each y_i) is locally nilpotent on it. The construction implies that its degree 0 piece is isomorphic to $S^{\lambda^\bullet}$, and hence it is isomorphic to $L_c(\lambda^\bullet)$, which is therefore diagonalizable with basis $f_{P,Q}$ for $(P, Q) \in \Gamma_c$. □

6. Combinatorics of folds

6.1. Near folds. We first obtain some limitations on the types of folds that can occur in $\partial\Gamma_c$. First, we switch from now on to the (P, Q) notation for elements of Γ, and we define a *near fold* to be an element $(P, Q) \in \Gamma_c$ such that $\phi(P, Q)$ or $s_i(P, Q)$ is folded for some $1 \le i \le n-1$ (by Lemma 5.2 this fold is then in the boundary $\partial\Gamma_c$). Here we define $s_i(P, Q) = (s_iP, Q)$, and $\phi(P, Q) = (P', Q')$ with

$$P'(b) = \begin{cases} P(b) - 1 & \text{if } P(b) > 1, \text{ and} \\ n & \text{if } P(b) = 1, \end{cases} \tag{6.1}$$

and

$$Q'(b) = \begin{cases} Q(b) & \text{if } P(b) > 1, \text{ and} \\ Q(b)+1 & \text{if } P(b) = 1. \end{cases} \tag{6.2}$$

These definitions are compatible with the corresponding ones for the (μ, T) notation via the bijection of 2.13.

The *upper rim* of a partition λ is the set of boxes $b \in \lambda$ such that there is no box immediately above b. The *upper rim* of an r-partition $\lambda^\bullet$ is the union of the upper rims of its components λ^l. The *left rim* of a partition (resp. multipartition) is defined analogously as the set of boxes with no box immediately to the left.

Lemma 6.1. *$(P,Q) \in \Gamma_c$ is a near fold if and only if there is a positive integer k and boxes $b_1, b_2 \in \lambda^\bullet$ such that b_1 is a removable box, b_2 is on the upper rim or left rim of $\lambda^\bullet$, $k = \beta(b_1) - \beta(b_2) \mod r$,*

$$k = d_{\beta(b_1)} - d_{\beta(b_2)} + r(\mathrm{ct}(b_1) - \mathrm{ct}(b_2))c_0,$$

and either

(a) $\mathrm{ct}(b_2) = 0$ (so b_2 is the upper left hand corner of $\lambda^{\beta(b_2)}$), $Q(b_1) = k-1$, $Q(b_2) = 0$, $P(b_1) = 1$, and $P(b_2) = n$, or

(b) $\mathrm{ct}(b_2) \neq 0$ and there are $a \in \mathbf{Z}_{\geq 0}$ and a box $b_3 < b_2$ adjacent to b_2 with $\mathrm{ct}(b_3) = \mathrm{ct}(b_2) \pm 1$ with $Q(b_1) = a + k$, $Q(b_2) = a = Q(b_3)$, $P(b_1) = i+1$, $P(b_2) = i-1$, and $P(b_3) = i$ for some $2 \leq i \leq n-1$.

Proof. First, if (a) holds then $\phi(P,Q)$ is folded, and if (b) holds then $s_i(P,Q)$ is folded. This follows from the formula given in 2.13 for the $\mathfrak{t}$-eigenvalue of $f_{P,Q}$, together with the formulas for $\phi(P,Q)$ and $s_i(P,Q)$ given above.

For the converse, assume first that $s_i(P,Q) = (s_iP, Q)$ is folded for some $(P,Q) \in \Gamma_c$ and $1 \leq i \leq n-1$. We will show that in this case we are in situation (b) of the lemma. It follows from our assumption that either

$$Q(P^{-1}(i)) - Q(P^{-1}(i+2)) = d_{\beta(P^{-1}(i))} - d_{\beta(P^{-1}(i+2))} + r(\mathrm{ct}(P^{-1}(i)) - \mathrm{ct}(P^{-1}(i+2)))c_0$$

with $Q(P^{-1}(i)) - Q(P^{-1}(i+2)) = \beta(P^{-1}(i)) - \beta(P^{-1}(i+2)) \mod r$, or that the analogous equations, replacing i and $i+2$ by $i-1$ and $i+1$, hold. In any case there are boxes $b_1, b_2 \in \lambda^\bullet$ and a non-negative integer k with

$$k = d_{\beta(b_1)} - d_{\beta(b_2)} + r(\mathrm{ct}(b_1) - \mathrm{ct}(b_2))c_0, \tag{6.3}$$

$$k = \beta(b_1) - \beta(b_2) \mod r, \quad Q(b_1) - Q(b_2) = k, \quad \text{and} \quad |P(b_1) - P(b_2)| = 2.$$

First observe that $k > 0$ since otherwise $Q(b_1) = Q(b_2)$, $\beta(b_1) = \beta(b_2)$, and $\mathrm{ct}(b_1) = \mathrm{ct}(b_2)$ contradicts $|P(b_1) - P(b_2)| = 2$. Since $(P,Q) \in \Gamma_c$, (6.3) implies $\mathrm{ct}(b_2) \neq 0$. If b_2 is not on the upper or left rim of $\lambda^\bullet$, there are two boxes $b_3, b_4 < b_2$ with $\mathrm{ct}(b_3) = \mathrm{ct}(b_2) \pm 1$ and $\mathrm{ct}(b_4) = \mathrm{ct}(b_2) \pm 1$. Now (6.3) together with $(P,Q) \in \Gamma_c$ implies that $Q(b_3) = Q(b_2) = Q(b_4)$ and that $P(b_1) > P(b_3), P(b_4) > P(b_2)$, contradicting $|P(b_1) - P(b_2)| = 2$. On the other hand, since $\mathrm{ct}(b_2) \neq 0$ there is always at least one box b_3 as above, so we have $P(b_1) = P(b_3) + 1 = P(b_2) + 2$.

If b_1 is not a removable box, then there is a box $b > b_1$ such that $\mathrm{ct}(b) = \mathrm{ct}(b_1) \pm 1$, and (6.3) once more implies $Q(b) = Q(b_1)$ and hence $P(b_1) > P(b) > P(b_2)$, which contradicts $P(b_1) = P(b_3) + 1 = P(b_2) + 2$.

Now assume $\phi(P, Q)$ is folded for some $(P, Q) \in \Gamma_c$. Then

$$Q(P^{-1}(1)) + 1 - Q(P^{-1}(n)) = d_{\beta(P^{-1}(1))} - d_{\beta(P^{-1}(n))} + r(\mathrm{ct}(P^{-1}(1)) - \mathrm{ct}(P^{-1}(n))c_0$$

and

$$Q(P^{-1}(1)) + 1 - Q(P^{-1}(n)) = \beta(P^{-1}(1)) - \beta(P^{-1}(n)) \bmod r.$$

Assume first that $Q(P^{-1}(1))+1-Q(P^{-1}(n)) < 0$ and let $k = Q(P^{-1}(n))-Q(P^{-1}(1))-1$, $b_1 = P^{-1}(n)$ and $b_2 = P^{-1}(1)$. If b_2 is the upper left hand corner of $\lambda^{\beta(b_2)}$ then $\mathrm{ct}(b_2) = 0$ and the equations

$$k = d_{\beta(b_1)} - d_{\beta(b_2)} + r\mathrm{ct}(b_1)c_0 \quad \text{and} \quad k = \beta(b_1) - \beta(b_2) \bmod r$$

together with $(P, Q) \in \Gamma_c$ force $Q(b_1) < k = Q(b_1)-Q(b_2)-1$, contradiction. Thus b_2 is not the upper left-hand corner of $\lambda^{\beta(b_2)}$ and hence there is a box $b_3 < b_2$ with $\mathrm{ct}(b_3) = \mathrm{ct}(b_2) \pm 1$. Now

$$k = d_{\beta(b_1)} - d_{\beta(b_3)} + r(\mathrm{ct}(b_1) - \mathrm{ct}(b_3) \pm 1)c_0 \quad \text{and} \quad k = \beta(b_1) - \beta(b_3) \bmod r$$

together with $(P, Q) \in \Gamma_c$ imply that $Q(b_1) - Q(b_3) \leq k = Q(b_1) - Q(b_2) - 1 \leq Q(b_1) - Q(b_3) - 1$, contradiction. Therefore $Q(P^{-1}(n)) - Q(P^{-1}(1)) - 1 \geq 0$.

Let $k = Q(P^{-1}(1)) + 1 - Q(P^{-1}(n))$, $b_1 = P^{-1}(n)$ and $b_2 = P^{-1}(1)$. Then $k \geq 0$.

If $k = 0$ then $\beta(b_1) = \beta(b_2)$, $Q(b_1) = Q(b_2) + 1$, $\mathrm{ct}(b_1) = \mathrm{ct}(b_2)$ and $Q(b_1) = Q(b_2) - 1$ implying $b_1 < b_2$. But since $P(b_1) = 1$ there can be no box $b > b_1$ with $Q(b) = Q(b_1)$ and likewise since $P(b_2) = n$ there can be no box $b < b_2$ with $Q(b) = Q(b_2)$, contradiction.

Therefore $k > 0$, and the equations

$$k = d_{\beta(b_1)} - d_{\beta(b_2)} + r(\mathrm{ct}(b_1) - \mathrm{ct}(b_2))c_0 \quad \text{and} \quad k = \beta(b_1) - \beta(b_2) \bmod r$$

together with $(P, Q) \in \Gamma_c$, $P(b_1) = 1$ and $P(b_2) = n$ imply that there is no box b with $b > b_1$ or $b < b_2$, and hence also that $Q(b_1) = k - 1$. This proves that we are in case (a) of the lemma. □

6.2. Proof of Theorem 1.1. Now thanks to Theorem 5.3, our first main Theorem 1.1 is a consequence of the following:

Theorem 6.2. *Assume $c_0 \neq 0$. The module $L_c(\lambda^\bullet)$ is diagonalizable if and only if for every removable box $b \in \lambda^\bullet$, either $k_c(b) = \infty$ or $l_c(b) < k_c(b)$.*

Proof. For convenience, we define a statistic $l'_c(b)$ similar to $l_c(b)$, but without restricting to outside addable boxes: more precisely, $l'_c(b)$ is the smallest integer l such that either there exists a box $b' \in \lambda^{\beta(b)-l}$ with

$$l = d_{\beta(b)} - d_{\beta(b)-l} + r(\mathrm{ct}(b) - \mathrm{ct}(b') \pm 1)c_0$$

or

$$l = d_{\beta(b)} - d_{\beta(b)-l} + r\mathrm{ct}(b)c_0.$$

We will prove that $L_c(\lambda^\bullet$ is diagonalizable if and only if for every removable box b of $\lambda^\bullet$, we have $l'_c(b) < k_c(b)$; one checks that $l'_c(b) < k_c(b)$ if and only if $l_c(b) < k_c(b)$, so this will finish the proof.

Suppose first that there is a removable box $b_1 \in \lambda^\bullet$ with $k_c(b_1) < \infty$ and $k_c(b_1) \le l'_c(b_1)$ and write $k = k_c(b_1)$. Then there is b_2 in the upper or left rim of $\lambda^{\beta(b_1)-k}$ with

$$k = d_{\beta(b_1)} - d_{\beta(b_2)} + r(\mathrm{ct}(b_1) - \mathrm{ct}(b_2))c_0.$$

If b_2 is the upper left-hand corner of $\lambda^{\beta(b_1)-k}$ then set $Q(b_1) = k-1$ and $Q(b) = 0$ for all other $b \in \lambda^\bullet$, and put $P(b_1) = 1$, $P(b_2) = n$ and then complete P to a reverse standard Young tableau on $\lambda^\bullet$. It follows from the assumption $k_c(b_1) \le l'_c(b_1)$ that all the inequalities necessary for $(P,Q) \in \Gamma_c$ are satisfied, and therefore (P,Q) is a near fold in Γ_c so $L_c(\lambda^\bullet)$ is not diagonalizable.

If $\mathrm{ct}(b_2) \neq 0$, then we define $Q(b_1) = k$ and $Q(b) = 0$ for all other $b \in \lambda^\bullet$. We now define P for which $(P,Q) \in \Gamma_c$: let b_3 be the box with $b_3 < b_2$ and $\mathrm{ct}(b_3) = \mathrm{ct}(b_2) \pm 1$, let i be maximal so that there exists a reverse standard Young tableau T on $\lambda^\bullet$ with $T(b_2) = i$, and define $P(b_1) = i+1$, $P(b_2) = i-1$, $P(b_3) = i$ and then complete P to a reverse standard Young tableau on $\lambda^\bullet - \{b\}$. It is then straightforward to check that $(P,Q) \in \Gamma_c$ is a near fold so that $L_c(\lambda^\bullet)$ is not diagonalizable.

Conversely, suppose that $l'_c(b) < k_c(b)$ for all corner boxes b such that $k(b) < \infty$. By Theorem 5.3 it suffices to show that there are no near folds in Γ_c. Suppose towards a contradiction that $(P,Q) \in \Gamma_c$ is a near fold. By Lemma 6.1 there is a corner box $b_1 \in \lambda^\bullet$, an integer $k \in \mathbf{Z}_{>0}$, and a box b_2 in the upper or left rim of $\lambda^{\beta(b_1)-k}$ with

$$k = d_{\beta(b_1)} - d_{\beta(b_2)} + r(\mathrm{ct}(b_1) - \mathrm{ct}(b_2))c_0,$$

and one of the following holds:

(a) The box b_2 is the upper left-hand corner of $\lambda^{\beta(b_1)-k}$, and $Q(b_1) = k-1$, $Q(b_2) = 0$, $P(b_1) = 1$, and $P(b_2) = n$, or
(b) there is a box $b_3 < b_2$ with $\mathrm{ct}(b_3) = \mathrm{b}_2 \pm 1$ and so that $Q(b_1) = Q(b_2) + k = Q(b_3) + k$ and $P(b_1) = P(b_3) + 1 = P(b_2) + 2$.

Assume that case (b) holds; (a) is similar.

By hypothesis there is some integer $0 < l < k$ such that either $l = d_{\beta(b_1)} - d_{\beta(b_1)-l} + r\mathrm{ct}(b_1)c_0$ (in which case $(P,Q) \notin \Gamma_c$) or so that there is a box $b_4 \in \lambda^{\beta(b_1)-l}$ with

$$l = d_{\beta(b_1)} - d_{\beta(b_4)} + r(\mathrm{ct}(b_1) - \mathrm{ct}(b_4) \pm 1)c_0$$

in which case also

$$0 < k - l = d_{\beta(b_4)} - d_{\beta(b_2)} + r(\mathrm{ct}(b_4) - \mathrm{ct}(b_2) \pm 1)c_0$$

and $k - l = \beta(b_4) - \beta(b_2) \bmod r$, implying that

(1) $Q(b_1) \le Q(b_4) + l$ with equality implying $P(b_1) > P(b_4)$ and
(2) $Q(b_4) \le Q(b_2) + k - l$ with equality implying $P(b_4) > P(b_2)$.

Observe first that if $b_4 = b_3$ then the requirement $Q(b_1) \le Q(b_4) + l < Q(b_4) + k$ precludes $(P,Q) \in \Gamma_c$. Thus $b_4 \neq b_3$. On the other hand, combining (1) and (2) above shows that $Q(b_1) \le Q(b_2) + k$ with equality implying $P(b_1) > P(b_4) > P(b_2)$, and this contradicts $P(b_1) = P(b_3) + 1 = P(b_2) + 2$. $\square$

The modules $\Delta_c(\lambda^\bullet)$ are graded by polynomial degree, and the modules $L_c(\lambda^\bullet)$ are graded quotients (thanks to the deformed Euler operator from [DuOp]), on which W acts preserving the degree. Writing $L_c^d(\lambda^\bullet)$ for the degree d piece of $L_c(\lambda^\bullet)$, we define the graded character

$$\mathrm{char}(L_c(\lambda^\bullet)^W, q) = \sum_{d=0}^{\infty} \dim_{\mathbf{C}}(L_c^d(\lambda^\bullet)^W) q^d.$$

By using Theorem 2.1 of [DuGr], we obtain the following corollary of Theorem 1.1, where we define a column-strict tableau on $\lambda^\bullet$ to be a filling of its boxes by non-negative integers in such a way that within each component λ^i, the entries are weakly increasing left to right, and strictly increasing top to bottom.

Corollary 6.3. *If $c_0 \neq 0$ and $L_c(\lambda^\bullet)$ is diagonalizable, then the graded character of $L_c(\lambda^\bullet)^W$ is given by the formula*

$$\mathrm{char}(L_c(\lambda^\bullet)^W, q) = \sum_{d=0}^{\infty} a_d q^d,$$

where a_d is the number of column-strict tableaux Q on $\lambda^\bullet$ such that $Q(b) = \beta(b) \bmod r$ for all boxes b of λ, the sum

$$\sum_{b \in \lambda^\bullet} Q(b) = d,$$

and for each positive integer l the following conditions hold:

(a) *we have $Q(b) < l$ whenever $b \in \lambda^i$ and the equation $d_i - d_{i-l} + r\mathrm{ct}(b)c_0 = l$ holds, and*

(b) *we have $Q(b_1) - Q(b_2) \leq l$ whenever $b_1 \in \lambda^i$, $b_2 \in \lambda^{i-l}$, and one of the equations*

$$d_i - d_{i-l} + r(\mathrm{ct}(b_1) - \mathrm{ct}(b_2) \pm 1)c_0 = l$$

holds.

Writing $e = \frac{1}{|W|}\sum_{w \in W} w$ for the symmetrizing idempotent, if the parameter c is not on one of the hyperplanes specified by Theorem 3.4 of [DuGr], then the functor $M \mapsto eM = M^W$ from H_c modules to eH_ce-modules is an equivalence, and the module $L_c(\lambda^\bullet)$ is finite dimensional if and only if $L_c(\lambda^\bullet)^W$ is.

7. Proof of Theorem 1.2

7.1. By Theorem 1.1 $L_c(\lambda^\bullet)$ is unitary exactly if it is diagonalizable and for all $(P,Q) \in \Gamma_c$ we have $\langle f_{P,Q}, f_{P,Q} \rangle \geq 0$. Since every $f_{P,Q}$ for $(P,Q) \in \Gamma_c$ may be obtained from those with $Q = 0$ by applying an invertible sequence of intertwining operators, the following lemma allows for inductive control over the signs of the norm of $f_{P,Q}$ for $(P,Q) \in \Gamma_c$. It is an immediate consequence of the proof of Theorem 6.1 of [Gri2], translated into (P,Q) notation.

Lemma 7.1.

(a) *Suppose $(P,Q) \in \Gamma_c$. Then for $1 \leq i \leq n-1$ we have*

$$\langle \sigma_i f_{P,Q}, \sigma_i f_{P,Q} \rangle = \langle f_{P,Q}, f_{P,Q} \rangle$$

if $Q(P^{-1}(i)) - Q(P^{-1}(i+1)) \neq \beta(P^{-1}(i)) - \beta(P^{-1}(i+1))$ and

$$\langle \sigma_i f_{P,Q}, \sigma_i f_{P,Q} \rangle = \frac{(\delta - rc_0)(\delta + rc_0)}{\delta^2} \langle f_{P,Q}, f_{P,Q} \rangle$$

with

$\delta = Q(P^{-1}(i)) - Q(P^{-1}(i+1)) - (d_{\beta(P^{-1}(i))} - d_{\beta(P^{-1}(i+1))}) - r(\mathrm{ct}(P^{-1}(i)) - \mathrm{ct}(P^{-1}(i+1)))$

else.

(b) *For* $(P,Q) \in \Gamma_c$ *we have*

$\langle \Phi f_{P,Q}, \Phi f_{P,Q} \rangle = \left(Q(P^{-1}(1)) + 1 - (d_{\beta(P^{-1}(1))} - d_{\beta(P^{-1}(1)) - Q(P^{-1}(1)) - 1}) - r\mathrm{ct}(P^{-1}(1))c_0 \right) \langle f_{P,Q}, f_{P,Q} \rangle.$

The lemma implies that for $c_0 \neq 0$, the module $L_c(\lambda^\bullet)$ is unitary if and only if the following two conditions hold: for all $(P,Q) \in \Gamma_c$,

$$Q(P^{-1}(1)) + 1 \geq d_{\beta(P^{-1}(1))} - d_{\beta(P^{-1}(1)) - Q(P^{-1}(1)) - 1} + r\mathrm{ct}(P^{-1}(1))c_0 \tag{7.1}$$

and if $(P,Q) \in \Gamma_c$ and $1 \leq i \leq n-1$ with $Q(P^{-1}(i)) - Q(P^{-1}(i+1)) = \beta(P^{-1}(i)) - \beta(P^{-1}(i+1)) \mod r$ then setting $b_1 = P^{-1}(i)$ and $b_2 = P^{-1}(i+1)$

$$\left(Q(b_1) - Q(b_2) - (d_{\beta(b_1)} - d_{\beta(b_2)}) - r(\mathrm{ct}(b_1) - \mathrm{ct}(b_2))c_0 \right)^2 \geq (rc_0)^2. \tag{7.2}$$

The last condition may be rephrased: the numbers

$$Q(b_1) - Q(b_2) - (d_{\beta(b_1)} - d_{\beta(b_2)}) - r(\mathrm{ct}(b_1) - \mathrm{ct}(b_2) \pm 1)c_0 \tag{7.3}$$

have the same sign (are both weakly positive or both weakly negative). Assuming $c_0 > 0$, the only way this can fail is if

$$Q(b_1) - Q(b_2) - (d_{\beta(b_1)} - d_{\beta(b_2)}) - r(\mathrm{ct}(b_1) - \mathrm{ct}(b_2) - 1)c_0 > 0$$

and

$$Q(b_1) - Q(b_2) - (d_{\beta(b_1)} - d_{\beta(b_2)}) - r(\mathrm{ct}(b_1) - \mathrm{ct}(b_2) + 1)c_0 < 0.$$

7.2. Construction of unitarity-preventing (P,Q)'s. The following lemma is the key step in the proof that the existence of appropriate blocking sequences is a necessary condition for unitarity.

Lemma 7.2.

(a) *Let* $b \in \lambda^i$ *and* $b' \in \lambda^j$ *and suppose that* $b \not\leq b'$ *and there is no blocking sequence* B *for* (b, b') *with* $c \in L_B$. *Then there exist* $1 \leq a \leq n-1$ *and* $(P,Q) \in \Gamma_c(\lambda^\bullet)$ *with* $P(b) = a+1$, $P(b') = a$, $Q(b) = m_{ij}$ *and* $Q(b') = 0$.

(b) *Let* $b \in \lambda^i$ *and* $0 \leq j \leq r-1$ *and suppose that there is no blocking sequence* B *for* (b, j) *with* $c \in L_B$. *Then there is some* $(P,Q) \in \Gamma_c$ *with* $P(b) = 1$ *and* $Q(b) = m_{ij} - 1$.

Proof. We describe a general procedure for producing an element $(P,Q) \in \Gamma_c$, with different initial steps for parts (a) and (b) of the lemma. For part (a): for each $b'' \geq b$ set $Q(b'') = m_{ij}$, and for each $b''' \leq b'$ set $Q(b''') = 0$. Define $P(b'')$ for all $b'' > b$ and all $b''' < b'$ in such a way that P is decreasing on these posets, and furthermore so that the set of numbers thus defined is equal to the set $\{d, d+1, \dots, n\}$, where $n - d + 1$ is the number of boxes at least b or at most b' (this last condition will force $P(b'') < d$ for the remaining boxes b'' of $\lambda^\bullet$). For part (b): set $Q(b) = m_{ij} - 1$ and for each $b' > b$ set $Q(b') = m_{ij}$. Set $P(b) = 1$ and define $P(b')$ on the set of boxes $b' > b$ in such a way that P is decreasing on this poset and the set of numbers so used is of the form $\{d+1, \dots, n\}$, where $n - d$ is the number of boxes strictly larger than b. The remainder of the construction in both cases (a) and (b) of the lemma is now the same.

Assuming we have defined Q and P on all boxes in $\lambda^{i-1}, \lambda^{i-2}, \dots, \lambda^{i-k+1}$, we define them on λ^{i-k} by induction, choosing the minimal b_1 for which Q and P are not already defined. Choose $P(b_1)$ maximal from among the unused numbers in $\{1, 2, \dots, n\}$. We choose $Q(b_1)$ minimal subject to the conditions:

(a) $Q(b_1) \geq 0$,
(b) $Q(b_1) \geq Q(b_2)$ for all $b_2 \leq b_1$, and
(c) for each box b_2 such that $Q(b_2)$ and $P(b_2)$ have already been defined, and for any positive integer l with $l = \beta(b_2) - \beta(b_1) \bmod r$ and
$$l = d_{\beta(b_2)} - d_{\beta(b_1)} + r(\mathrm{ct}(b_2) - \mathrm{ct}(b_1) \pm 1)c_0,$$
we enforce $Q(b_2) - Q(b_1) \leq l$.

One now checks, using the absence of blocking sequences, that the pair (P, Q) so defined belongs to Γ_c. □

7.3. Proof of Theorem 1.2 in case $c_0 = 0$. If $c_0 = 0$ then $\sigma_i = s_i$ so all $f_{P,Q}$ are well-defined, and by the argument of Theorem 6.1 of [Gri2] we have
$$\langle s_i f_{P,Q}, s_i f_{P,Q} \rangle = \langle f_{P,Q}, f_{P,Q} \rangle$$
and
$$\langle \Phi f_{P,Q}, \Phi f_{P,Q} \rangle = \big(Q(P^{-1}(1)) + 1 - (d_{P^{-1}(1)} - d_{P^{-1}(1) - Q(P^{-1}(1)) - 1}) \big) \langle f_{P,Q}, f_{P,Q} \rangle.$$
It follows that if $d_i - d_j \leq m_{ij}$ for all pairs i, j then the contravariant form is positive semi-definite on $\Delta_c(\lambda^\bullet)$ and hence $L_c(\lambda^\bullet)$ is unitary, and that if $d_i - d_j > m_{ij}$ for some i with $\lambda^i \neq \emptyset$ then there is some $f_{P,Q}$ with negative square norm and hence $L_c(\lambda^\bullet)$ is not unitary.

7.4. From now on we assume $c_0 > 0$. We first prove that if one of the conditions (a), (b) or (c) in the statement of Theorem 1.2 fails to hold then $L_c(\lambda^\bullet)$ is not unitary. If (a) fails, then $L_c(\lambda^\bullet)$ is not diagonalizable, and hence it is not unitary. If (b) fails then there is a box $b \in \lambda^i$ and an integer j for which
$$m_{ij} < d_i - d_j + r\mathrm{ct}(b)c_0$$
and so that there is no blocking sequence B for (b, j) with $c \in L_B$. Then by Lemma 7.2 there is some $(P, Q) \in \Gamma_c$ with $P(b) = 1$ and $Q(b) = m_{ij} - 1$, and using (7.1) shows that $L_c(\lambda^\bullet)$ is not unitary. Finally, suppose that there are boxes $b \in \lambda^i$ and $b' \in \lambda^j$ with
$$m_{ij} < d_i - d_j + r(\mathrm{ct}(b) - \mathrm{ct}(b') + 1)c_0$$
but so that there is no blocking sequence B for (b, b') with $c \in L_B$. Since a blocking sequence for (b, j) would be one for (b, b'), there is no blocking sequence B for (b, j) with $c \in L_B$ and hence by the previous part of the proof we may assume that
$$d_i - d_j + r\mathrm{ct}(b)c_0 \leq m_{ij}.$$
Let $k \in \mathbf{Z}$ be the largest integer so that
$$m_{ij} < d_i - d_j + r(\mathrm{ct}(b) - k + 1)c_0,$$
so that we have $\mathrm{ct}(b') \leq k \leq 0$. It follows that there is some $b'' \leq b'$ with $\mathrm{ct}(b'') = k$. Since a blocking sequence B for (b, b'') is one for (b, b'), there is no blocking sequence B for (b, b'')

with $c \in L_B$. In particular $d_i - d_j + r(\mathrm{ct}(b) - \mathrm{ct}(b'') - 1)c_0 \neq m_{ij}$ (since (b, b'') is a blocking sequence for $(b, b''))$ and it follows that

$$d_i - d_j + r(\mathrm{ct}(b) - \mathrm{ct}(b'') - 1)c_0 < m_{ij} < d_i - d_j + r(\mathrm{ct}(b) - \mathrm{ct}(b'') + 1)c_0.$$

Moreover Lemma 7.2 implies that there is some $(P, Q) \in \Gamma_c$ with $P(b) = i$, $P(b'') = i + 1$, $Q(b) = m_{ij}$, and $Q(b'') = 0$. Now (7.3) shows $L_c(\lambda^\bullet)$ is not unitary.

7.5. For the converse, we must prove that if (a), (b) and (c) in the statement of Theorem 1.2 hold then $L_c(\lambda^\bullet)$ is unitary. We assume that $(P, Q) \in \Gamma$ and $b_1 = b$, $b_2 = b'$ are such that the numbers in (7.3) have different signs, and we will show $(P, Q) \notin \Gamma_c$. First, we may assume that b is not at most b', by interchanging them if necessary. Then the inequality

$$m_{ij} < d_{\beta(b)} - d_{\beta(b')} + r(\mathrm{ct}(b) - \mathrm{ct}(b') + 1)c_0,$$

implies that there is a blocking sequence B for (b, b') with $c \in L_B$. Now Lemma 7.4 below implies that (P, Q) satisfying (7.3) cannot be in Γ_c. Likewise if (7.1) fails for $(P, Q) \in \Gamma_c$ then with $b = P^{-1}(1)$, $i = \beta(b)$, and $j = \beta(b) - Q(b) - 1$ we have

$$m_{ij} < d_i - d_j + r\mathrm{ct}(b)c_0,$$

and hence there exists a blocking sequence B for (b, j) with $c \in L_B$, and Lemma 7.3 below implies that $(P, Q) \notin \Gamma_c$. This completes the proof of Theorem 1.2.

Lemma 7.3. *If $b \in \lambda^i$ and $0 \leq j \leq r - 1$ is a pair for which a blocking sequence B with $c \in L_B$ exists, and there is $(P, Q) \in \Gamma_c$ with $Q(b) \geq m_{ij} - 1$ and $P(b) = 1$ then we must have*

$$d_{\beta(b)} - d_j + r\mathrm{ct}(b)c_0 = m_{ij}.$$

Proof. If $B = (b_1, \ldots, b_{2q+1}, \ell)$ is a blocking sequence for (b, j) with $c \in L_B$ then we obtain

$$Q(b) \leq Q(b_{2q+1}) + \sum_{k=1}^{q} m_{\beta(b_{2k-1}),\beta(b_{2k})} \leq m_{\beta(b_{2q+1}),\ell} - 1 + \sum_{k=1}^{q} m_{\beta(b_{2k-1}),\beta(b_{2k})} \leq m_{ij} - 1,$$

with equality implying $1 = P(b) \geq P(b_{2q+1}) + d - 1$, where d is the number of distinct boxes appearing in the sequence $b, b_1, \ldots, b_{2q+1}$. It follows that $d = 1$, that $q = 0$, that $b = b_1$, and that

$$d_{\beta(b_1)} - d_l + r\mathrm{ct}(b_1)c_0 = m_{il}.$$

But now we have $m_{ij} \leq m_{il}$ since $(P, Q) \in \Gamma_c$, and the opposite inequality follows from the definition of blocking sequence. Thus $m_{ij} = m_{il}$, whence $l = j$ and we are done. □

Lemma 7.4. *If $b \in \lambda^i$, $b' \in \lambda^j$, a blocking sequence for (b, b') exists, and $(P, Q) \in \Gamma_c$ with $Q(b) - Q(b') \geq m_{ij}$ and $P(b) = P(b') + 1$, then $Q(b) = Q(b') + m_{ij}$ and*

$$m_{ij} = d_i - d_j + r(\mathrm{ct}(b) - \mathrm{ct}(b') \pm 1)c_0.$$

Proof. If there is a blocking sequence B for (b, j) then the proof of Lemma 7.3 gives $Q(b) \leq m_{ij} - 1$, contradiction. So any blocking sequence B for (b, b') with $c \in L_B$ must be of the form $B = (b_1, \ldots, b_{2q})$. Since $(P, Q) \in \Gamma$, for $1 \leq k \leq q - 1$ we have $Q(b_{2k}) \leq Q(b_{2k+1})$ with equality implying that $P(b_{2k}) > P(b_{2k+1})$ unless $b_{2k} = b_{2k+1}$. Since $(P, Q) \in \Gamma_c$, for

$1 \leq k \leq q$ we have $Q(b_{2k-1}) \leq Q(b_{2k}) + m_{\beta(b_{2k-1}),\beta(b_{2k})}$ with equality implying $P(b_{2k-1}) > P(b_{2k})$. Combining these inequalities, we have

$$Q(b_0) \leq Q(b') + \sum_{k=1}^{q} m_{\beta(b_{2k-1}),\beta(b_{2k})} \leq Q(b') + m_{ij},$$

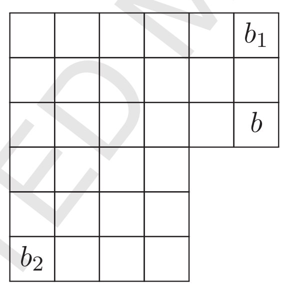
with equality implying $P(b) \geq P(b') + d - 1$ where d is the number of distinct boxes that appear in the sequence $b, b_1, \ldots, b_{2q}, b'$. Since $P(b) = P(b') + 1$ we must have $d = 2$, whence $b_1 = b$, $b_2 = b'$, and

$$d_{\beta(b)} - d_{\beta(b')} + r(\mathrm{ct}(b) - \mathrm{ct}(b') \pm 1)c_0 = m_{ij}$$

so we are done. □

8. Explicit results for classical Weyl groups

8.1. The symmetric group. Here we assume $r = 1$. The parameter is a single number $c = c_0$, which we may assume is non-negative $c \geq 0$. The next corollary recovers the result of [EtSt] classifying the unitary representations of the type A rational Cherednik algebra, and the classification due to Suzuki ([Suz]; see also [SuVa]) of the diagonalizable irreducible representations. The following diagram should help the reader to visualize the notation introduced in the corollary.



Corollary 8.1. *Suppose $c \geq 0$ and λ is a partition of n. If $\lambda = (1^n)$, then $L_c(\lambda)$ is unitary for all $c \geq 0$. Otherwise, let b_1 be the box of λ of maximum content, let b_2 be the box of λ of minimum content, and let b be the removable box of λ of maximum content. Then $L_c(\lambda)$ is diagonalizable unless $c = k/m$ for relatively prime positive integers k and m with $m \leq \mathrm{ct}(b) - \mathrm{ct}(b_2)$, and $L_c(\lambda)$ is unitary if and only if either*

$$c \leq \frac{1}{\mathrm{ct}(b_1) - \mathrm{ct}(b_2) + 1}$$

or $c = 1/m$ for an integer m with

$$m \geq \mathrm{ct}(b) - \mathrm{ct}(b_2) + 1.$$

Proof. We define $b(\lambda) = \mathrm{ct}(b) - \mathrm{ct}(b_2) + 1$. We have $k_c(b) = \infty$ unless $c = k/m$ for relatively prime positive integers k and m with $1 \leq m \leq b(\lambda) - 1$, in which case $k_c(b) = k = l_c(b)$. Moreover, if $k_c(b) = \infty$ then the same is true for all removable boxes of λ. It now follows from Theorem 1.1 that for $c \geq 0$ the module $L_c(\lambda)$ is $\mathfrak{t}$-diagonalizable if and only if c is not a rational number of the form $c = k/m$ with $1 \leq m \leq b(\lambda) - 1$. This recovers the classification from [Suz] (see also [SuVa]).

We now compute the set of $c \geq 0$ for which $L_c(\lambda)$ is unitary. If $\lambda = (1^n)$ then $k_c(b) = \infty$ for the only removable box, so it is diagonalizable for all $c \geq 0$, and since $\mathrm{ct}(b)c \leq 0$ for all boxes b of λ and $b_1 \leq b_2$ for all pairs of boxes with $(\mathrm{ct}(b_1) - \mathrm{ct}(b_2) + 1)c > 0$, Theorem 1.2 shows that $L_c((1^n))$ is unitary for all $c \geq 0$. So from now on we assume $\lambda \neq (1^n)$.

Suppose first that $L_c(\lambda)$ is unitary. If $c > 1/(\mathrm{ct}(b_1) - \mathrm{ct}(b_2) + 1)$, then since $\lambda \neq (1^n)$ we do not have $b_1 \leq b_2$, and hence by Theorem 1.2 there must exist a blocking sequence B for (b_1, b_2) with $c \in L_B$. The definition of L_B and our assumption that $c \geq 0$ now shows that $c = 1/m$ for some positive integer m. Since $L_c(\lambda)$ is diagonalizable, the previous paragraph implies $m \geq b(\lambda)$. Thus we have either $c \leq 1/(\mathrm{ct}(b_1) - \mathrm{ct}(b_2) + 1)$ or $c = 1/m$ for an integer m with $m \geq b(\lambda)$.

Conversely, if $0 \leq c \leq 1/(\mathrm{ct}(b_1) - \mathrm{ct}(b_2) + 1)$ then we have already seen that $L_c(\lambda)$ is diagonalizable. If $b \in \lambda$ then

$$\mathrm{ct}(b)c \leq (\mathrm{ct}(b_1) - \mathrm{ct}(b_2) + 1)c < 1,$$

and likewise we have

$$(\mathrm{ct}(b) - \mathrm{ct}(b') + 1)c \leq (\mathrm{ct}(b_1) - \mathrm{ct}(b_2) + 1)c \leq 1$$

for all pairs b, b' of boxes of λ, so by Theorem 1.2 $L_c(\lambda)$ is unitary.

Finally suppose $c = 1/m$ for some integer $m \geq b(\lambda)$. We have seen above that in this case $L_c(\lambda)$ is diagonalizable. Observe first that we always have $b(\lambda) \geq \mathrm{ct}(b_1)$. Thus for all $b' \in \lambda$, we have

$$\mathrm{ct}(b')c \leq \mathrm{ct}(b_1) \cdot 1/b(\lambda) \leq 1.$$

Hence (c) of Theorem 1.2 holds. If $b', b'' \in \lambda$ are boxes with

$$(\mathrm{ct}(b') - \mathrm{ct}(b'') + 1)c > 1$$

then

$$\mathrm{ct}(b') - \mathrm{ct}(b'') + 1 > m.$$

We show that there is a blocking sequence B for (b', b'') with $c \in L_B$. If $b''' \leq b''$ then a blocking sequence for (b', b''') is one for (b', b'') and if $b''' \geq b'$ then a blocking sequence for (b''', b'') is one for (b', b''). Furthermore, moving b' to the right increases its content and moving b'' to the left decreases its content. So we may assume that there is no box to the right of b' in λ and there is no box to the left of b'' in λ. The inequality

$$\mathrm{ct}(b') - \mathrm{ct}(b'') + 1 > m \geq b(\lambda) \qquad \Longrightarrow \mathrm{ct}(b') > b(\lambda) + \mathrm{ct}(b'') - 1 \geq \mathrm{ct}(b)$$

then implies that $b_1 \leq b' < b$, and hence there is a box b''' with $b' < b''' \leq b$ and

$$\mathrm{ct}(b''') - \mathrm{ct}(b'') + 1 = m,$$

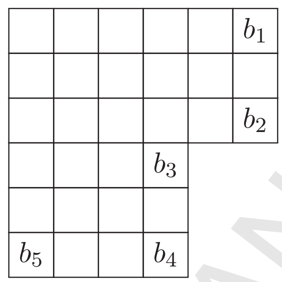
so that $B = (b''', b'')$ is a blocking sequence for (b', b'') with $c \in L_B$. □

8.2. Type B. For $r = 2$ (essentially, for the Weyl groups of type B/C and D) our results can also be made completely explicit. The Weyl group of type B_n is $G(2, 1, n)$, which contains two conjugacy classes of reflections: those conjugate to the transposition (12), and those conjugate to the transformation $(x_1, x_2, \ldots, x_n) \mapsto (-x_1, x_2, \ldots, x_n)$ changing the sign of the first coordinate and leaving the remaining coordinates fixed. We will write c and d

for the parameters c_r attached to these conjugacy classes; in terms of c_0, d_0, and d_1 we then have

$$c = c_0 \quad \text{and} \quad d = d_0 = -d_1.$$

First we will assume $\lambda^\bullet = (\lambda, \emptyset)$ where λ is a partition of n. We remark that this covers the case $(\emptyset, \lambda)$ as well: twisting by a charater of W shows that $L(\emptyset, \lambda)$ at parameter (c, d) is unitary if and only if $L(\lambda, \emptyset)$ at parameter $(c, -d)$ is unitary. For the next two corollaries, we need some notation for certain boxes of λ, and a diagram illustrating the notation. We suppose that $\lambda = (6, 6, 6, 4, 4, 4)$ with Young diagram and certain marked boxes as illustrated here:



The significance of these particular boxes is as follows: b_1 is the box of λ of largest content; b_2 is the removable box of largest content; b_4 is the removable box of second-largest content; b_3 is the highest box in the right rim of λ and directly above b_4; finally, b_5 is the box of smallest content. If λ is a rectangle then the boxes b_3 and b_4 do not exist, and we will not make use of them in this case. It may happen that some of these boxes are the same. The analog of the theorem of Suzuki [Suz] and Cherednik [Che1] in this case is contained in the following corollaries.

Corollary 8.2. *Suppose λ is a rectangle or $b_4 = b_5$, and let $\lambda^\bullet = (\lambda, \emptyset)$. If $\lambda = (1^n)$, then $L_c(\lambda^\bullet)$ is diagonalizable for all $c \geq 0$. Otherwise, for $c > 0$ the module $L_c(\lambda^\bullet)$ is diagonalizable if and only if*

(a) c is not a rational number of denominator at most $\mathrm{ct}(b_2) - \mathrm{ct}(b_5)$, or

(b) $c = k/\ell$ with k and ℓ positive coprime integers with $\ell \leq \mathrm{ct}(b_2) - \mathrm{ct}(b_5)$ and an equation of the form

$$d + \mathrm{ct}(b_2)c = m/2$$

holds for some positive odd integer $m < 2k$.

Corollary 8.3. *Suppose λ is not a rectangle and $b_4 \neq b_5$. With notation as above, and assuming $c \geq 0$ and $\lambda^\bullet = (\lambda, \emptyset)$, then $L_c(\lambda^\bullet)$ is diagonalizable if and only if*

(a) c is not a rational number of denominator at most $\mathrm{ct}(b_2) - \mathrm{ct}(b_5)$, or

(b) $c = k/l$ for coprime positive integers k and ℓ such that $\mathrm{ct}(b_4) - \mathrm{ct}(b_5) + 1 \leq \ell \leq \mathrm{ct}(b_2) - \mathrm{ct}(b_5)$ and an equation of the form

$$d + \mathrm{ct}(b_2)c = m/2$$

holds for some positive odd integer $m < 2k$.

One proves these corollaries by the same technique as the type A case. Using these corollaries we will deduce the next result on unitarity. We again remark that $L(\emptyset, \lambda)$ at parameter (c, d) is unitary if and only if $L(\lambda, \emptyset)$ is unitary for $(c, -d)$.

Corollary 8.4. *If $\lambda = (1^n)$ and $c \geq 0$, then $L_c(\lambda^\bullet)$ is unitary if and only if either $d \leq 1/2$ or $d + \ell c = 1/2$ for some integer ℓ with $-(n-1) \leq \ell \leq -1$. Assuming $c \geq 0$ and $\lambda \neq (1^n)$, the representation $L_c(\lambda, \emptyset)$ is unitary if and only if (c,d) belongs to at least one of the following sets:*

(a) The set of parameters (c,d) satisfying the inequalities

$$c \leq \frac{1}{\mathrm{ct}(b_1) - \mathrm{ct}(b_5) + 1} \quad \textit{and} \quad d + \mathrm{ct}(b_1)c \leq \frac{1}{2},$$

(b) for each positive integer ℓ with $\mathrm{ct}(b_2) - \mathrm{ct}(b_5) + 1 \leq \ell \leq \mathrm{ct}(b_1) - \mathrm{ct}(b_5) + 1$, the ray consisting of points (c,d) such that $c = 1/\ell$ and

$$d + \mathrm{ct}(b_1)c \leq \frac{1}{2},$$

(c) for each positive integer ℓ with $\mathrm{ct}(b_2) < \ell \leq \mathrm{ct}(b_1)$, the segment consisting of points (c,d) with

$$d + \ell c = \frac{1}{2} \textit{ and } c \leq \frac{1}{\ell - \mathrm{ct}(b_5)},$$

(d) the points (c,d) such that $d + \mathrm{ct}(b_2)c = \frac{1}{2}$, and, if (i) λ is not a rectangle and (ii) $b_4 \neq b_5$, such that either

$$c \leq \frac{1}{\mathrm{ct}(b_3) - \mathrm{ct}(b_5) + 1} \quad \textit{or} \quad c = \frac{1}{\ell - \mathrm{ct}(b_5) + 1}$$

for an integer ℓ satisfying $\mathrm{ct}(b_4) \leq \ell < \mathrm{ct}(b_3)$.

(e) For each pair (ℓ, m) of integers with $\mathrm{ct}(b_2)+1 \leq \ell \leq \mathrm{ct}(b_1)-1$ and $\mathrm{ct}(b_2)-\mathrm{ct}(b_5)+1 \leq m \leq \ell - \mathrm{ct}(b_5)$ the point $P_{\ell,m}$ satisfying

$$d + \ell c = 1/2 \quad \textit{and} \quad c = 1/m.$$

Proof. Theorem 1.2 implies that if $c = 0$ then $L_c(\lambda^\bullet)$ is unitary if and only if $d \leq 1/2$. Suppose first that (c,d) is such that $L_c(\lambda^\bullet)$ is unitary. If $\lambda = (1^n)$ and $d > 1/2$, then by Theorem 1.2 there must exist a blocking sequence B for $(b_1, 1)$ with $c \in L_B$. Thus an equation of the form $d + \ell c = 1/2$ holds for some $-(n-1) \leq \ell \leq -1$. So we may suppose $\lambda \neq (1^n)$ and $c > 0$.

We will prove that (c,d) is of one of the types (a)-(e) in the statement of the theorem above. We may assume that at least one of the inequalities in (a) fails.

Case 1: if $d + \mathrm{ct}(b_1)c \leq 1/2$ then we must have

$$c > \frac{1}{\mathrm{ct}(b_1) - \mathrm{ct}(b_5) + 1}.$$

Suppose first that $d + \mathrm{ct}(b_1)c < 1/2$. It follows that $d + \mathrm{ct}(b)c < 1/2$ for all boxes $b \in \lambda$, and hence $c \notin L_B$ for all blocking sequences of the form $(b, 1)$. Since $\lambda \neq (1^n)$ we do not have $b_1 \leq b_5$ and by Theorem 1.2 there must exist a blocking sequence $B = (b, b')$ for (b_1, b_2) with $c \in L_B$. This implies $c = 1/\ell$ for some positive integer ℓ. By Corollary 8.3 we must have $\ell \geq \mathrm{ct}(b_2) - \mathrm{ct}(b_5) + 1$ and we are in case (b).

Suppose next that $d + \mathrm{ct}(b_1)c = 1/2$ (thus $(b_1, 1)$ is a blocking sequence for (b_1, b_5) with $c \in L_B$ so we cannot quite conclude as we just did). Either $b_1 = b_2$ or there is a box $b > b_1$ with $\mathrm{ct}(b) = \mathrm{ct}(b_1) - 1$. We will treat these two subcases next.

Subcase (i): $b_1 = b_2$. If λ is a rectangle (necessarily $\lambda = (n)$), if $b_4 = b_5$, or if

$$c \leq \frac{1}{\mathrm{ct}(b_3) - \mathrm{ct}(b_5) + 1}$$

then we are in case (d). So we suppose λ is not a rectangle, $b_4 \neq b_5$, and we have

$$c > \frac{1}{\mathrm{ct}(b_3) - \mathrm{ct}(b_5) + 1}.$$

Since $b_4 \neq b_5$ we do not have $b_3 \leq b_5$, and (b) of Theorem 1.2 implies that there is a blocking sequence B for (b_3, b_5) with $c \in L_B$. Since $d + \mathrm{ct}(b_1)c = 1/2$ there is no blocking sequence $B = (b, 1)$ for (b_3, b_5) with $c \in L_B$. Thus there is a blocking sequence (b, b') for (b_3, b_5) with $c \in L_B$, implying $c = 1/\ell$ for some positive integer ℓ. By Corollary 8.3 we have $\ell \geq \mathrm{ct}(b_4) - \mathrm{ct}(b_5) + 1$, implying that we are in case (d).

Subcase (ii): If there is a box $b > b_1$ with $\mathrm{ct}(b) = \mathrm{ct}(b_1) - 1$ then

$$c > \frac{1}{\mathrm{ct}(b) - \mathrm{ct}(b_5) + 1},$$

and hence a blocking sequence B for (b, b_5) must exist with $c \in L_B$. We cannot have $B = (b', 1)$ since $c > 0$ implies that only one equation of the form $d + \ell c = 1/2$ can hold and we are assuming $d + \mathrm{ct}(b_1)c = 1/2$ already. So $B = (b', b'')$ and $c = 1/\ell$ for some positive integer ℓ. By Corollaries 8.2 and 8.3 we have $\ell \geq \mathrm{ct}(b_2) - \mathrm{ct}(b_5) + 1$, and we are in case (b).

Case 2: if $d + \mathrm{ct}(b_1)c > 1/2$ then there must be a blocking sequence B for $(b_1, 1)$ with $c \in L_B$. This blocking sequence must be of the form $(b, 1)$ for some $b > b_1$. Thus an equation $d + \ell c = 1/2$ holds with $\mathrm{ct}(b_2) \leq \ell = \mathrm{ct}(b) < \mathrm{ct}(b_1)$. Suppose first that $\ell > \mathrm{ct}(b_2)$. If $c \leq \frac{1}{\ell - \mathrm{ct}(b_5)}$ then we are in case (c) above. Thus we may assume

$$c > \frac{1}{\ell - \mathrm{ct}(b_5)} = \frac{1}{\mathrm{ct}(b') - \mathrm{ct}(b_5) + 1}$$

where b' is the box directly below b (there is such a box since $\ell > \mathrm{ct}(b_2)$). Hence there must exist a blocking sequence B' for (b', b_5) with $c \in L_{B'}$. Since $c > 0$ the only equation of the form $d + \ell' c = 1/2$ that holds is $d + \ell c = 1/2$, and hence this blocking sequence is necessarily of the form $B' = (b'', b''')$ for some $b'' \geq b'$ and $b''' \leq b_5$. Thus $c = 1/m$ for some integer m with

$$m = \mathrm{ct}(b'') - \mathrm{ct}(b''') + 1 \leq \mathrm{ct}(b') - \mathrm{ct}(b_5) + 1 \leq \ell - \mathrm{ct}(b_5).$$

Since $L_c(\lambda^\bullet)$ is diagonalizable, we have $m \geq \mathrm{ct}(b_2) - \mathrm{ct}(b_5) + 1$ by Corollaries 8.2 and 8.3. Thus we are in case (e).

Finally we suppose that $\ell = \mathrm{ct}(b_2)$, or in other words $b = b_2$. We now repeat the argument of Subcase (i) of Case 1 above to conclude that we are in case (d). This completes the proof of necessity of at least one of the conditions (a)-(e).

For the converse, we first suppose $\lambda = (1^n)$. By Corollary 8.2 $L_c(\lambda^\bullet)$ is diagonalizable for all $c \geq 0$. The condition (b) in Theorem 1.2 always holds so we need only check condition (c) there to verify unitarity. If $d \leq 1/2$ then we have $d + \mathrm{ct}(b)c \leq d \leq 1/2$ for all boxes b and hence (c) holds. On the other hand, if $d + \mathrm{ct}(b)c = 1/2$ for some box $b \in \lambda$, then

$B = (b, 1)$ is a blocking sequence for all $(b', 1)$ and $(b', 2)$ with $b' \leq b$. Moreover for any b' with $b' \not\leq b$ we have

$$d + \mathrm{ct}(b')c < d + \mathrm{ct}(b)c = 1/2,$$

so condition (c) holds.

Now we suppose $\lambda \neq (1^n)$ and treat cases (a)-(e) in the statement one at a time. First we verify diagonalizability in each case. In cases (a), (b), (c), and (e) we have

$$c \leq 1/(\mathrm{ct}(b_2) - \mathrm{ct}(b_5) + 1)$$

and hence $L_c(\lambda^\bullet)$ is diagonalizable by Corollaries 8.2 and 8.3. In case (d), if λ is a rectangle then Corollary 8.2 shows that $L_c(\lambda^\bullet)$ is diagonalizable, and if λ is not a rectangle then Corollary 8.3 shows that $L_c(\lambda^\bullet)$ is diagonalizable.

It remains to verify that conditions (b) and (c) from Theorem 1.2 hold provided that one of (a)-(e) of the present corollary does. In case (a) of this corollary conditions (b) and (c) of Theorem 1.2 automatically hold. In case (b) of this corollary the inequality $d+\mathrm{ct}(b_1)c \leq 1/2$ ensures that condition (c) of Theorem 1.2 holds, and arguing as in the last part of the proof of Corollary 8.1 shows that condition (b) of Theorem 1.2 holds.

Suppose now that we are in case (c) of this corollary, with $d + \ell c = 1/2$ and

$$c \leq \frac{1}{\ell - \mathrm{ct}(b_5)}$$

for some $\mathrm{ct}(b_2) < \ell \leq \mathrm{ct}(b_1)$. Let $b \in \lambda$ be the box with $\mathrm{ct}(b) = \ell$ and $b_1 \leq b \leq b_2$. Thus $B = (b, 1)$ is a blocking sequence with $c \in L_B$ for all $(b', 1)$ and all (b', b'') with $b' \leq b$. If

$$d + \mathrm{ct}(b')c > 1/2$$

then we must have $\ell < \mathrm{ct}(b')$. We prove that $B = (b, 1)$ is a blocking sequence for $(b', 1)$. We may assume that there is no box to the right of b', and hence $b' \leq b$. A similar argument shows that if $\mathrm{ct}(b')c > 1$ then $b' \leq b$. Hence (c) of Theorem 1.2 holds. If b', b'' are boxes with

$$(\mathrm{ct}(b') - \mathrm{ct}(b'') + 1)c > 1$$

then

$$1 < (\mathrm{ct}(b') - \mathrm{ct}(b'') + 1)c \leq \frac{\mathrm{ct}(b') - \mathrm{ct}(b'') + 1}{\ell - \mathrm{ct}(b_5)} \leq \frac{\mathrm{ct}(b') - \mathrm{ct}(b_5) + 1}{\ell - \mathrm{ct}(b_5)}$$

and hence $\mathrm{ct}(b') \geq \ell = \mathrm{ct}(b)$. We now conclude as before that $(b, 1)$ is a blocking sequence for (b', b''), and hence (b) of Theorem 1.2 holds.

We now assume we are in case (d) of this corollary. Since $d+\mathrm{ct}(b_2)c = 1/2$ then $B = (b_2, 1)$ is a blocking sequence with $c \in L_B$ for all $(b', 1)$, $(b', 2)$, and (b', b'') with $b' \leq b_2$. If $b' \not\leq b_2$ then we have $\mathrm{ct}(b') \leq \mathrm{ct}(b_2)$ and hence

$$d + \mathrm{ct}(b')c \leq 1/2.$$

If λ is a rectangle there are no $b' \not\leq b_2$ and if $b_4 = b_5$ then any $b' \not\leq b_2$ has $\mathrm{ct}(b') \leq 0$ so that $\mathrm{ct}(b')c \leq 0 \leq 1$. It follows that (c) of Theorem 1.2 holds in these two cases. Otherwise either

$$c \leq \frac{1}{\mathrm{ct}(b_3) - \mathrm{ct}(b_5) + 1} \quad \text{or} \quad c = \frac{1}{\ell - \mathrm{ct}(b_5) + 1}$$

for an integer ℓ satisfying $\mathrm{ct}(b_4) \leq \ell < \mathrm{ct}(b_3)$. Note that for all $b' \not\leq b_2$ we have $\mathrm{ct}(b') \leq \mathrm{ct}(b_4) - \mathrm{ct}(b_5) + 1$ (c.f. the proof of Corollary 8.1), implying $\mathrm{ct}(b')c \leq 1$ for all $b' \not\leq b_2$. This establishes (c) of Theorem 1.2 in this case. To establish (b) we may assume $(\mathrm{ct}(b') - \mathrm{ct}(b'') + 1)c > 1$ for some $b' \not\leq b_2$ and $b'' \in \lambda$. If

$$c \leq \frac{1}{\mathrm{ct}(b_3) - \mathrm{ct}(b_5) + 1}$$

this is impossible, so we may assume

$$c = \frac{1}{\ell - \mathrm{ct}(b_5) + 1}$$

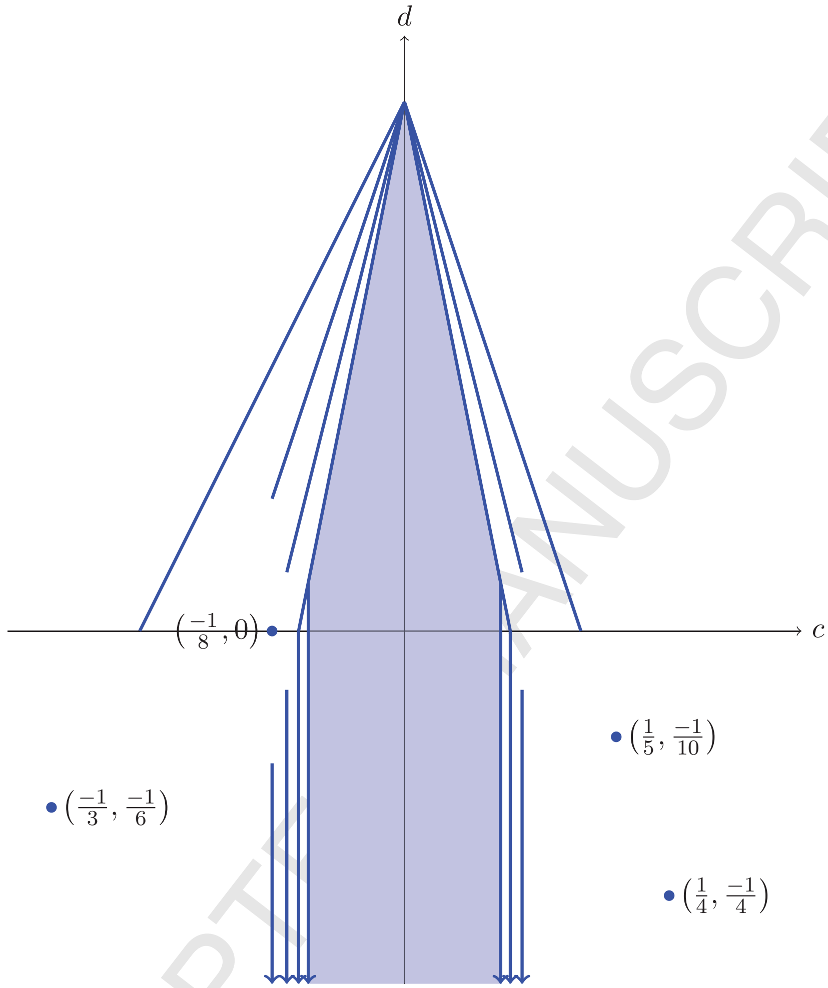
for an integer ℓ satisfying $\mathrm{ct}(b_4) \leq \ell < \mathrm{ct}(b_3)$. But now the same argument as in the proof of Corollary 8.1 establishes (b) of Theorem 1.2.

Finally we assume we are in case (e) of the present corollary. Let b be the box of the right rim of λ with $\ell = \mathrm{ct}(b)$. The equality $d + \ell c = 1/2$ ensures that $B = (b, 1)$ is a blocking sequence with $c \in L_B$ for all $(b', 1), (b', 2)$, and (b', b'') with $b' \leq b$. Moreover if $b' \not\leq b$ then we have $\mathrm{ct}(b') < \mathrm{ct}(b)$ and hence

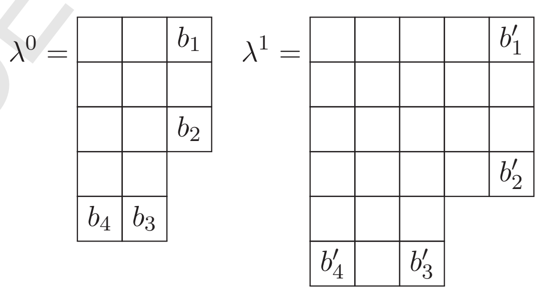
$$d + \mathrm{ct}(b')c < d + \mathrm{ct}(b)c = 1/2.$$

Now the equality $c = 1/m$ for some integer m with $\mathrm{ct}(b_2) - \mathrm{ct}(b_5) + 1 \leq m \leq \ell - \mathrm{ct}(b_5)$ implies as in the proof of Corollary 8.1 that $\mathrm{ct}(b')c \leq 1$ for all $b' \in \lambda$, and that if $(\mathrm{ct}(b') - \mathrm{ct}(b'') + 1)c > 1$ then there is a blocking sequence B for (b', b'') with $c \in L_B$. We have completed the proof. $\square$

The unitary spectrum for $\lambda^\bullet = ((6, 6, 6, 4, 4, 4), \emptyset)$ is drawn next. Again, the shaded area (which should be understood to extend infinitely downwards) is the region in which the standard module itself is unitary.



We next treat the case $\lambda^\bullet = (\lambda^0, \lambda^1)$. We mark certain boxes as follows:



Here b_1 is the box of λ^0 of largest content, b_2 is the removable box of largest content, b_3 is the removable box of smallest content, and b_4 is the box of smallest content. We similarly define the boxes b'_i of λ^1. We do not state the classification of c so that $L_c(\lambda^\bullet)$ is diagonalizable here; it is quite complicated. As it turns out, the classification of c for which $L_c(\lambda^\bullet)$ is unitary does does not require it and is actually simpler to state.

Corollary 8.5. *With notation as above the unitary spectrum $U(\lambda^\bullet)$ is the union of the following sets:*

(a) The parallelogram

$$d + (\mathrm{ct}(b_1) - \mathrm{ct}(b_4') + 1)c \leq \frac{1}{2} \quad d + (\mathrm{ct}(b_4) - \mathrm{ct}(b_1') - 1)c \leq \frac{1}{2}$$
$$-d + (\mathrm{ct}(b_1') - \mathrm{ct}(b_4) + 1)c \leq \frac{1}{2} \quad -d + (\mathrm{ct}(b_4') - \mathrm{ct}(b_1) - 1)c \leq \frac{1}{2}$$

(b) For each integer l with $\mathrm{ct}(b_2) - \mathrm{ct}(b_4') + 1 \leq l \leq \mathrm{ct}(b_1) - \mathrm{ct}(b_4') + 1$, the line segment S_l consisting of points (c, d) with

$$d + lc = \frac{1}{2}, \quad c \geq 0, \quad \textit{and} \quad -d + (\mathrm{ct}(b_1') - \mathrm{ct}(b_4) + 1)c \leq \frac{1}{2},$$

(c) For each integer l with $\mathrm{ct}(b_2') - \mathrm{ct}(b_4) + 1 \leq l \leq \mathrm{ct}(b_1') - \mathrm{ct}(b_4) + 1$ a line segment S_l' consisting of the points (c, d) with

$$-d + lc = \frac{1}{2}, \quad c \geq 0, \quad \textit{and} \quad d + (\mathrm{ct}(b_1) - \mathrm{ct}(b_4') + 1)c \leq \frac{1}{2},$$

(d) For each integer l with $\mathrm{ct}(b_4) - \mathrm{ct}(b_1') - 1 \leq l \leq \mathrm{ct}(b_3) - \mathrm{ct}(b_1') - 1$, a line segment T_l consisting of points (c, d) with

$$d + lc = \frac{1}{2}, \quad c \leq 0, \quad \textit{and} \quad -d + (\mathrm{ct}(b_4') - \mathrm{ct}(b_1) - 1)c \leq \frac{1}{2},$$

(e) For each integer l with $\mathrm{ct}(b_4') - \mathrm{ct}(b_1) - 1 \leq l \leq \mathrm{ct}(b_3') - \mathrm{ct}(b_1) - 1$, a line segment T_l' consisting of points (c, d) with

$$-d + lc = \frac{1}{2}, \quad c \leq 0, \quad \textit{and} \quad d + (\mathrm{ct}(b_4) - \mathrm{ct}(b_1') - 1)c \leq \frac{1}{2},$$

(f) For each pair (l, m) of integers with $\mathrm{ct}(b_2) - \mathrm{ct}(b_4') + 1 \leq l \leq \mathrm{ct}(b_1) - \mathrm{ct}(b_4') + 1$ and $\mathrm{ct}(b_2') - \mathrm{ct}(b_4) + 1 \leq m \leq \mathrm{ct}(b_1') - \mathrm{ct}(b_4) + 1$, the point $P_{l,m}$ solving the equations

$$d + lc = \frac{1}{2} \quad \textit{and} \quad -d + mc = \frac{1}{2},$$

(g) For each pair (l, m) of integers with $\mathrm{ct}(b_4) - \mathrm{ct}(b_1') - 1 \leq l \leq \mathrm{ct}(b_3) - \mathrm{ct}(b_1') - 1$ and $\mathrm{ct}(b_4') - \mathrm{ct}(b_1) - 1 \leq m \leq \mathrm{ct}(b_3') - \mathrm{ct}(b_1) - 1$, the point $Q_{l,m}$ solving the equations

$$d + lc = \frac{1}{2} \quad \textit{and} \quad -d + mc = \frac{1}{2}.$$

Proof. If $c = 0$ then by Theorem 1.2 $L_c(\lambda^\bullet)$ is unitary if and only if $-1/2 \leq d \leq 1/2$. So we may assume $c > 0$. Assume first that if $L_c(\lambda^\bullet)$ is unitary. If

$$d + (\mathrm{ct}(b_1) - \mathrm{ct}(b_4') + 1)c > 1/2$$

then there must exist a blocking sequence B for (b_1, b_4) with $c \in L_B$. This B must be of the form (b, b') with $b \geq b_1$ and $b' \leq b_4'$, and hence

$$d + \ell c = 1/2, \quad \text{where} \quad \ell = \mathrm{ct}(b) - \mathrm{ct}(b') + 1.$$

We observe that

$$\ell \leq \mathrm{ct}(b_1) - \mathrm{ct}(b_4') + 1$$

since $\mathrm{ct}(b_1) \geq \mathrm{ct}(b)$ and $\mathrm{ct}(b_4') \leq \mathrm{ct}(b')$. Moreover we must have $\mathrm{ct}(b_2) - \mathrm{ct}(b_4') + 1 \leq \ell$ since otherwise

$$\mathrm{ct}(b) - \mathrm{ct}(b') < \mathrm{ct}(b_2) - \mathrm{ct}(b_4') \quad \Longrightarrow \quad \mathrm{ct}(b) - \mathrm{ct}(b_2) < \mathrm{ct}(b') - \mathrm{ct}(b_4'),$$

so there is a box $b'' \in \lambda^1$ with $\ell = \mathrm{ct}(b_2) - \mathrm{ct}(b'')$. But this implies $k_c(b_2) = 1$ and hence $L_c(\lambda^\bullet)$ is not diagonalizable, contradiction.

We have proved that if

$$d + (\mathrm{ct}(b_1) - \mathrm{ct}(b_4') + 1)c > 1/2$$

then there is ℓ with

$$\mathrm{ct}(b_2) - \mathrm{ct}(b_4') + 1 \leq \ell \leq \mathrm{ct}(b_1) - \mathrm{ct}(b_4') + 1$$

and

$$d + \ell c = 1/2.$$

By symmetry, if $-d + (\mathrm{ct}(b_1') - \mathrm{ct}(b_4) + 1)c > 1/2$ then there is some ℓ with

$$\mathrm{ct}(b_2') - \mathrm{ct}(b_4) + 1 \leq \ell \leq \mathrm{ct}(b_1') - \mathrm{ct}(b_4) + 1$$

and

$$-d + \ell c = 1/2.$$

This shows that, for $c > 0$, if we are not in case (a) of the present corollary then we are in one of cases (b), (c), or (f) (replacing c by $-c$ and transposing $\lambda^\bullet$ gives the others cases).

Conversely, assume we are in one of cases (a), (b), (c), or (f) (the others may be treated by replacing c by $-c$ as above). We will show first that $L_c(\lambda^\bullet)$ is diagonalizable. In each case we have

$$d + (\mathrm{ct}(b_2) - \mathrm{ct}(b_4') + 1)c \leq 1/2 \quad \text{and} \quad -d + (\mathrm{ct}(b_2') - \mathrm{ct}(b_4) + 1)c \leq 1/2$$

which implies

$$(\mathrm{ct}(b_2) - \mathrm{ct}(b_4') + 1 + \mathrm{ct}(b_2') - \mathrm{ct}(b_4) + 1)c \leq 1.$$

Thus

$$c \leq \frac{1}{\mathrm{ct}(b_2) - \mathrm{ct}(b_4) + 1} \quad \text{and} \quad c \leq \frac{1}{\mathrm{ct}(b_2') - \mathrm{ct}(b_4') + 1}.$$

Together these inequalities imply $k_c(b) = \infty$ for all removable boxes b of $\lambda^\bullet$, and hence by Theorem 1.1 $L_c(\lambda^\bullet)$ is diagonalizable.

In case (a) of the present corollary, the conditions (b) and (c) of Theorem 1.1 automatically hold. Suppose we are in case (b). We have $d + \ell c = 1/2$, for some $\mathrm{ct}(b_2) - \mathrm{ct}(b_4') + 1 \leq \ell \leq \mathrm{ct}(b_1) - \mathrm{ct}(b_4') + 1$, $c > 0$, and $-d + (\mathrm{ct}(b_1') - \mathrm{ct}(b_4) + 1)c \leq 1/2$. Let b be the box with $b_1 \leq b \leq b_2$ and $\mathrm{ct}(b) - \mathrm{ct}(b_4') + 1 = \ell$. Thus $B = (b, 1)$ is a blocking sequence with $c \in L_B$ for all (b', j) and all (b', b'') with $b' \leq b$. Suppose now that $b', b'' \in \lambda^0$ with

$$(\mathrm{ct}(b') - \mathrm{ct}(b'') + 1)c > 1.$$

Since

$$(\mathrm{ct}(b) - \mathrm{ct}(b_4') + 1 + \mathrm{ct}(b_1') - \mathrm{ct}(b_4) + 1)c \leq 1,$$

if $\mathrm{ct}(b') \leq \mathrm{ct}(b)$ we obtain

$$1 < (\mathrm{ct}(b') - \mathrm{ct}(b'') + 1)c \leq (\mathrm{ct}(b) - \mathrm{ct}(b_4') + 1 + \mathrm{ct}(b_1') - \mathrm{ct}(b_4) + 1)c \leq 1.$$

Thus $\mathrm{ct}(b') > \mathrm{ct}(b)$ and hence $b' \leq b$, so $B = (b, 1)$ is a blocking sequence for (b', b'') with $c \in L_B$. Likewise if $b', b'' \in \lambda^1$ then since $\mathrm{ct}(b') \leq \mathrm{ct}(b'_1)$

$$(\mathrm{ct}(b') - \mathrm{ct}(b'') + 1)c \leq (\mathrm{ct}(b) - \mathrm{ct}(b'_4) + 1 + \mathrm{ct}(b'_1) - \mathrm{ct}(b_4) + 1)c \leq 1.$$

Suppose now that $b' \in \lambda^1$ and $b'' \in \lambda^0$. Then

$$-d + (\mathrm{ct}(b') - \mathrm{ct}(b'') + 1)c \leq -d + (\mathrm{ct}(b'_1) - \mathrm{ct}(b_4) + 1)c \leq 1/2$$

and

$$-d + \mathrm{ct}(b')c \leq -d + (\mathrm{ct}(b'_1) - \mathrm{ct}(b_4) + 1)c \leq 1/2.$$

Finally, for $b' \in \lambda^0$ and $b'' \in \lambda^1$ with $\mathrm{ct}(b') > \mathrm{ct}(b)$ we have $b' \leq b$ and hence $B = (b, 1)$ is a blocking sequence for $(b', 1)$ and (b', b'') with $c \in L_B$, while if $\mathrm{ct}(b') \leq \mathrm{ct}(b)$ then

$$d + (\mathrm{ct}(b') - \mathrm{ct}(b'') + 1)c \leq d + (\mathrm{ct}(b) - \mathrm{ct}(b'_4) + 1)c = 1/2$$

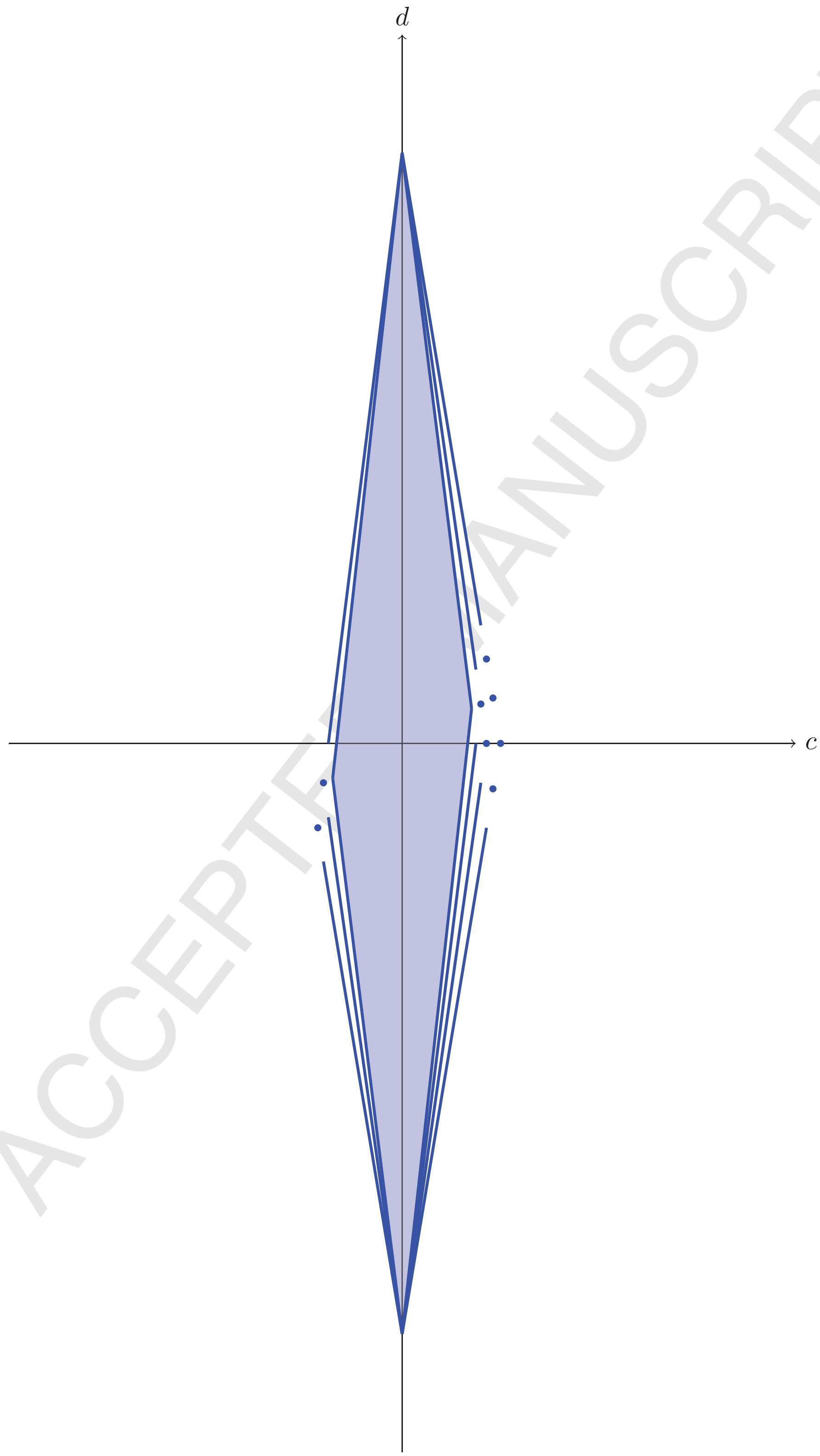
and

$$d + \mathrm{ct}(b')c \leq d + (\mathrm{ct}(b) - \mathrm{ct}(b'_4) + 1)c = 1/2.$$

It follows that conditions (b) and (c) from Theorem 1.2 hold.

Case (c) of the present corollary implies unitarity by symmetry (interchanging d and $-d$, and λ^0 and λ^1), and case (f) is completed by analogous arguments. $\square$

A picture of the unitary spectrum for $\lambda^\bullet = ((3, 3, 3, 2), (5, 5, 5, 5, 3, 3))$ follows. The area shaded light blue is the region in which the standard module itself is unitary.



d
c

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