Journal of Algebra 544 (2020) 29–46

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Spinorial representations of symmetric groups

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ARTICLE INFO

Article history:
Received 16 June 2019
Available online 11 October 2019
Communicated by Gunter Malle

MSC:
20C30
57R20

Keywords:
Symmetric groups
Representation theory
Spinoriality
Stiefel-Whitney class
Spin structure
Alternating groups

ABSTRACT

A real representation π of a finite group may be regarded as a homomorphism to an orthogonal group $\mathrm{O}(V)$. For symmetric groups S_n, alternating groups A_n, and products $S_n \times S_{n'}$ of symmetric groups, we give criteria for whether π lifts to the double cover $\mathrm{Pin}(V)$ of $\mathrm{O}(V)$, in terms of character values. From these criteria we compute the second Stiefel-Whitney classes of these representations.

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https://doi.org/10.1016/j.jalgebra.2019.09.025
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1. Introduction

A real representation π of a finite group G can be viewed as a group homomorphism from G to the orthogonal group $\mathrm{O}(V)$ of a Euclidean space V. Recall the double cover $\rho : \mathrm{Pin}(V) \to \mathrm{O}(V)$. We say that π is *spinorial*, provided it lifts to $\mathrm{Pin}(V)$, meaning there is a homomorphism $\hat{\pi} : G \to \mathrm{Pin}(V)$ so that $\rho \circ \hat{\pi} = \pi$.

When the image of π lands in $\mathrm{SO}(V)$, the representation is spinorial precisely when its second Stiefel-Whitney class $w_2(\pi)$ vanishes. Equivalently, when the associated vector bundle over the classifying space BG has a spin structure. (See Section 2.6 of [3], [8], and Theorem II.1.7 in [11].) Determining spinoriality of Galois representations also has applications in number theory: see [14], [6], and [13].

In this paper we give lifting criteria for representations of the symmetric groups S_n, the alternating groups A_n, and a product $S_n \times S_{n'}$ of two symmetric groups. Write $s_i \in S_n$ for the transposition $(i, i+1)$, in cycle notation. A key result of this paper is the following:

Theorem 1.1. Let $n \geq 4$.

(1) A representation π of S_n is spinorial iff $\chi_\pi(1) \equiv \chi_\pi(s_1 s_3) \mod 8$ and $\chi_\pi(1) - \chi_\pi(s_1)$ is congruent to 0 or 6 $\mod 8$.
(2) A representation π of A_n is spinorial iff $\chi_\pi(1) \equiv \chi_\pi(s_1 s_3) \mod 8$.

Combining this with the main result of [9] on character values, one deduces that as $n \to \infty$, "100%" of the irreducible representations of S_n are spinorial. (See Corollary 3.6.)

Next, we leverage this result to compute the second Stiefel-Whitney classes for (real) representations π of S_n:

$$w_2(\pi) = \left[\frac{\chi_\pi(1) - \chi_\pi(s_1)}{4}\right] e_{\mathrm{cup}} + \frac{\chi_\pi(1) - \chi_\pi(s_1 s_3)}{4} w_2(\pi_n), \tag{1.1}$$

where π_n is the standard representation of S_n, and $e_{\mathrm{cup}} \in H^2(G, \mathbb{Z}/2\mathbb{Z})$ is a certain cup product. (See Section 6.3.) Also $[\cdot]$ denotes the greatest integer function.

This formula allows us to compute the second Stiefel-Whitney classes of representations of $S_n \times S_{n'}$ through Künneth theory, and therefore to identify spinorial representations of this product. To state the result, let $\Pi = \pi \boxtimes \pi'$ be the external tensor product of representations π of S_n and π' of $S_{n'}$. Let $g = \frac{1}{2}(\chi_\pi(1) - \chi_\pi(s_1))$, the multiplicity of -1 as an eigenvalue of $\pi(s_1)$, and similarly write g' for the corresponding quantity for π'.

Theorem 1.2. The representation Π of $S_n \times S_{n'}$ is spinorial iff the restrictions of Π to $S_n \times \{1\}$ and $\{1\} \times S_{n'}$ are spinorial, and

$$(\deg \Pi + 1) g g' \equiv 0 \mod 2.$$

We now describe the layout of this paper. Section 2 reviews the group $\mathrm{Pin}(V)$ and other conventions. The S_n case of Theorem 1.1 is proven in Section 3 by means of defining relations for the s_i. Additionally we note corollaries of Theorem 1.1: primarily the aforementioned "100%" result, a connection with skew Young tableau numbers, and the important case of Young permutation modules, meaning the induction of the trivial character from a Young subgroup of S_n. In particular we demonstrate that the regular representation of S_n is spinorial for $n \geq 4$.

Representations of the alternating groups are treated in Section 4, again via generators and relations. The main result is the A_n case of Theorem 1.1. We enumerate the spinorial irreducible representations of A_n in Theorem 4.2. Data for spinoriality of irreducible representations of S_n and A_n for small n is presented in Tables 1 and 2 of Section 5.

In Section 6 we review the axioms of Stiefel-Whitney classes of real representations, and then deduce the Stiefel-Whitney class of a real representation of S_n. In Section 7 we apply Künneth theory to this formula to compute Stiefel-Whitney classes for real representations of $S_n \times S_{n'}$. From this it is straightforward to deduce Theorem 1.2.

Acknowledgments

The authors would like to thank Dipendra Prasad, Rohit Joshi, and Amritanshu Prasad for helpful conversations. The research by the first author for this article was supported by a PhD scholarship from the Indian National Board for Higher Mathematics.

2. Notation and preliminaries

2.1. Representations

All representations are on finite-dimensional vector spaces, which are always real, except in Section 4, where they may be specified as complex. For a representation (π, V) of a group G, write '$\det \pi$' for the composition $\det \circ \pi$; it is a linear character of G. Also write 'χ_π' for the character of π. If $H \leq G$ is a subgroup, write $\pi|_H$ for the restriction of π to H. A real representation $\pi : G \to \mathrm{GL}(V)$ can be conjugated to have image in $\mathrm{O}(V)$, so we will assume that this is the case. When $\det \pi$ is trivial, it maps to $\mathrm{SO}(V)$, and the spinoriality question is whether it lifts to the double cover $\mathrm{Spin}(V)$ (which we review in the next section).

Let $\mathrm{sgn} : S_n \to \{\pm 1\}$ be the usual sign character. For $G = S_n$, we say that π is *chiral* provided $\det \pi = \mathrm{sgn}$ and π is *achiral* provided $\det \pi = 1$. Write $\pi_n : S_n \to \mathrm{GL}_n(\mathbb{R})$ for the standard representation of S_n by permutation matrices.

2.2. Partitions

If λ is a partition of n we write '$\lambda \vdash n$' and '$|\lambda| = n$'. If $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0,$$

write '$\lambda + 1$' for the partition $(\lambda_1, \ldots, \lambda_\ell, 1)$ of $n + 1$.

2.3. The Pin group

We essentially review [4, Chapter 1.6] for defining the groups $\mathrm{Spin}(V)$ and $\mathrm{Pin}(V)$, where V is a Euclidean (i.e., a normed finite-dimensional real vector) space. The Clifford algebra $C(V)$ is the quotient of the tensor algebra $T(V)$ by the two-sided ideal generated by the set

$$\{v \otimes v + |v|^2 : v \in V\}.$$

Write $C(V)^\times$ for its group of units.

We identify V as a subspace of $C(V)$ through the natural injection $i : V \to C(V)$. Write α for the unique involution of the $\mathbb{R}$-algebra $C(V)$ with the property that $\alpha(x) = -x$ for $x \in V$. One has

$$C(V) = C(V)^0 \oplus C(V)^1,$$

where $C(V)^0$ is the 1-eigenspace of α and $C(V)^1$ is the -1-eigenspace.

Write t for the unique anti-involution of $C(V)$ with $t(x) = x$. For $x \in C(V)$, define $\overline{x} = t(\alpha(x))$; it is again an algebra anti-involution. Define

$$N : C(V) \to C(V)$$

by $N(x) = x\overline{x}$. Put

$$\Gamma_V = \{x \in C(V)^\times \mid \alpha(x)Vx^{-1} = V\}.$$

Let $\rho : \Gamma_V \to \mathrm{GL}(V)$ be the homomorphism given by $v \mapsto \alpha(x)vx^{-1}$. We will repeatedly use the fact that if v is a unit vector, then $\rho(v)$ is the reflection determined by v. Write '$\mathrm{Pin}(V)$' for the kernel of the restriction of N to Γ_V. The restriction of ρ to $\mathrm{Pin}(V)$ is a double cover of $\mathrm{O}(V)$ with kernel $\{\pm 1\}$. The preimage of $\mathrm{SO}(V)$ under ρ is denoted '$\mathrm{Spin}(V)$'. Alternately, $\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap C(V)^0$.

3. Symmetric groups

3.1. Lifting criteria

Let $n \geq 2$. The group S_n is generated by the transpositions $s_i = (i, i+1)$ for $1 \leq i \leq n-1$, with the following relations:

(1) ${s_i}^2 = 1, \quad 1 \leq i \leq n-1,$
(2) $s_i s_k = s_k s_i,$ when $\quad |i-k| > 1,$
(3) $(s_i s_{i+1})^3 = 1, \quad 1 \leq i \leq n-2.$

Therefore, defining a homomorphism from S_n to a group G is equivalent to choosing elements $x_1, \ldots, x_{n-1} \in G$ satisfying the same relations. Let us call the relation $x_i^2 = 1$ the "first lifting condition", the relation $x_i x_k = x_k x_i$ the "second lifting condition", and $(x_i x_{i+1})^3 = 1$ the "third lifting condition". Note that this second condition is vacuous for $n < 4$.

Let $\pi : S_n \to \mathrm{O}(V)$ be a representation of degree d. For each $\pi(s_i) \in \mathrm{O}(V)$ there are $\pm c_i \in \mathrm{Pin}(V)$ with $\rho(\pm c_i) = \pi(s_i)$, and the question is whether we may choose signs so that the $x_i = \pm c_i$ satisfy these lifting conditions.

Let $g_\pi = \dfrac{\chi_\pi(1) - \chi_\pi(s_1)}{2}$, as in [2]. This is the multiplicity of the eigenvalue -1 of $\pi(s_1)$, and the eigenvalue 1 occurs with multiplicity $d - g_\pi$. Put $c_i = u_1 \cdots u_{g_\pi} \in \mathrm{Pin}(V)$, where $u_1, \ldots, u_{g_\pi}$ is an orthonormal basis of the -1-eigenspace of $\pi(s_i)$. Since $\pi(s_i)$ is the product of the reflections in each u_j, the elements c_i and $-c_i$ are the lifts of $\pi(s_i)$. One computes that

$$c_i^2 = (-c_i)^2 = (-1)^{\frac{1}{2} g_\pi (g_\pi + 1)},$$

and therefore the first lifting condition is satisfied iff g_π is congruent to 0 or 3 modulo 4. It does not matter for this whether we choose c_i or $-c_i$.

Consider the sequence $(c_1 c_2)^3, (c_2 c_3)^3, \ldots \in \mathrm{Pin}(V)$. Since each $(\pi(s_i)\pi(s_{i+1}))^3 = 1$, this must be a sequence of ± 1's. For the third lifting condition these must each be 1. Thus c_1 may take either sign, but then the signs for $c_2, c_3, \ldots$ are determined. Moreover this does not affect the first lifting condition. Thus:

Proposition 3.1. The first and third lifting conditions hold iff $g_\pi \equiv 0 \text{ or } 3 \mod 4$.

Now let $|i - k| > 1$, and suppose as above that $c_i^2 = 1 = c_k^2 = 1$. Then the second lifting condition holds iff $(c_i c_k)^2 = 1$. By conjugating we may assume that $i = 1$ and $k = 3$. So put $h_\pi = \dfrac{\chi_\pi(1) - \chi_\pi(s_1 s_3)}{2}$; as above the condition is equivalent to $h_\pi \equiv 0, 3$ mod 4. However:

Lemma 3.2. The integer h_π is even.

Proof. Let ζ_4 be a 4-cycle in S_n. Then ζ_4^2 is conjugate to $s_1 s_3$. Let m be the multiplicity of $i = \sqrt{-1}$ as an eigenvalue of $\pi(\zeta_4)$. Then h_π, the multiplicity of -1 as an eigenvalue of $\pi(s_1 s_3)$, is $2m$. □

The above discussion shows that the following statement holds:

Proposition 3.3. The second lifting condition holds iff h_π is a multiple of 4.

We summarize the above as the following:

Theorem 3.4. Let $n \geq 4$, and π a representation of S_n. The following are equivalent:

(1) The representation π is spinorial.
(2) $g_\pi \equiv 0$ or $3 \mod 4$, and $h_\pi \equiv 0 \mod 4$.
(3) $\chi_\pi(1) - \chi_\pi(s_1) \equiv 0$ or $6 \mod 8$, and $\chi_\pi(1) \equiv \chi_\pi(s_1 s_3) \mod 8$.

When π is spinorial it has two lifts.

Proof. The first two statements are equivalent by Propositions 3.1 and 3.3. The last two statements are equivalent by the definitions of g_π and h_π. The two lifts correspond to the choice of sign for c_1 in the argument above. $\square$

When π is spinorial, note that $g_\pi \equiv 0 \mod 4$ iff π is achiral, and $g_\pi \equiv 3 \mod 4$ iff π is chiral.

Remark. The two lifts correspond to the two members of $H^1(S_n, \mathbb{Z}/2\mathbb{Z})$; see Theorem II.1.7 of [11].

Corollary 3.5. A representation π of S_n is spinorial iff its restrictions to the cyclic subgroups $\langle s_1 \rangle$ and $\langle s_1 s_3 \rangle$ are both spinorial.

Proof. This follows from Condition (2) of Theorem 3.4. The property that $g_\pi \equiv 0$ or 3 mod 4 corresponds to the subgroup $\langle s_1 \rangle$, and the property that $h_\pi \equiv 0 \mod 4$ corresponds to $\langle s_1 s_3 \rangle$. $\square$

The irreducible representations of S_n are the Specht modules $(\sigma_\lambda, V_\lambda)$, indexed by partitions of n. (See for instance [10].) Write $f_\lambda = f_{\pi_\lambda}$, and similarly for g_λ, h_λ. Write $p(n)$ for the number of partitions of n.

Corollary 3.6. We have

$$\lim_{n \to \infty} \frac{\#\{\lambda \vdash n \mid \sigma_\lambda \text{ is achiral and spinorial }\}}{p(n)} = 1.$$

In other words, as $n \to \infty$, 100% of irreducible representations of S_n are achiral and spinorial.

Proof. According to [9], as $n \to \infty$, 100% of partitions λ of n have

$$\chi_\lambda(1) \equiv \chi_\lambda(s_1) \equiv \chi_\lambda(s_1 s_3) \equiv 0 \mod 8.$$

The conclusion then follows from Theorem 1.1. $\square$

3.2. Connection with skew Young tableaux

Let μ, λ be partitions for which the Young diagram of λ contains that of μ. The notion of standard Young tableaux generalizes to "skew diagrams" λ/μ. Following Section 7.10 in [15], write $f_{\lambda/\mu}$ for the number of SYT on λ/μ. (If the Young diagram of μ is not contained in that of λ, put $f_{\lambda/\mu} = 0$.)

Proposition 3.7. We have

(1) $g_\lambda = f_{\lambda/(1,1)}$ and
(2) $h_\lambda = 2 \cdot (f_{\lambda/(3,1)} + f_{\lambda/(2,1,1)})$.

Proof. Let $\mu \vdash k$ for some $k \leq n$, and let $\underline{\mu}$ be the partition of n defined by adding $(n-k)$ 1's, i.e. $\underline{\mu} = \mu + \underbrace{1 + \cdots + 1}_{(n-k) \text{ times}}$. Write $w_\mu \in S_n$ be a permutation with cycle type μ. According to [15], Exercise 7.62, we have

$$\chi_\lambda(w_{\underline{\mu}}) = \sum_{\nu \vdash k} \chi_\nu(w_\mu) \cdot f_{\lambda/\nu}.$$

Taking $\mu = (2)$ gives

$$\chi_\lambda(s_1) = f_{\lambda/(2)} - f_{\lambda/(1,1)},$$

and taking $\mu = (1,1)$ gives

$$\chi_\lambda(1) = f_{\lambda/(2)} + f_{\lambda/(1,1)},$$

so that $g_\lambda = f_{\lambda/(1,1)}$.

Similarly, taking $\mu = (2,2)$ and using the character table for S_4, we compute

$$\chi_\lambda(s_1 s_3) = f_{\lambda/(4)} - f_{\lambda/(3,1)} + 2f_{\lambda/(2,2)} - f_{\lambda/(2,1,1)} + f_{\lambda/(1,1,1,1)}. \tag{3.1}$$

Taking $\mu = (1,1,1,1)$ gives

$$\chi_\lambda(1) = f_{\lambda/(4)} + 3f_{\lambda/(3,1)} + 2f_{\lambda/(2,2)} + 3f_{\lambda/(2,1,1)} + f_{\lambda/(1,1,1,1)}. \tag{3.2}$$

Combining (3.1) and (3.2) gives the formula for h_λ. □

3.3. Young permutation modules

Another important class of representations of S_n are the Young permutation modules, which are also indexed by partitions of n. Let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$, and consider the set $\mathcal{P}_\lambda$ of ordered partitions of $\{1,2,\dots,n\}$ with shape λ. Thus a member of $\mathcal{P}_\lambda$ is an ℓ-tuple $(X_1,\dots,X_\ell)$ of disjoint sets with each $|X_i| = \lambda_i$ and $\bigcup_{i=1}^{\ell} X_i = \{1,\dots,n\}$. Note that $\mathcal{P}_\lambda$ has cardinality

$$\binom{n}{\lambda_1,\dots,\lambda_\ell}. \tag{3.3}$$

The group S_n acts on $\mathcal{P}_\lambda$ in the obvious way, and we obtain the Young permutation module $\mathbb{R}[\mathcal{P}_\lambda]$. This representation space is given by formal linear combinations of elements of $\mathcal{P}_\lambda$, so its degree is given by (3.3).

For example, if $\lambda = (1,\dots,1) \vdash n$, then $\mathbb{R}[\mathcal{P}_\lambda]$ is the regular representation of S_n. If $\lambda = (n-1,1)$, then $\mathbb{R}[\mathcal{P}_\lambda]$ is the standard representation π_n of S_n on $\mathbb{R}^n$. Note that S_n acts transitively on $\mathcal{P}_\lambda$ with a stabilizer equal to the "Young subgroup" $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_\ell}$, so we can also view $\mathbb{R}[\mathcal{P}_\lambda]$ as the induction from S_λ to S_n of the trivial representation.

The characters of the $\mathbb{R}[\mathcal{P}_\lambda]$, though typically reducible, form an important basis of the representation ring of S_n. See, for example, Section 2.2 of [10].

Recall that, for a permutation representation π, the character value $\chi_\pi(g)$ is the number of fixed points of g. Write Θ_λ for the character $\chi_{\mathbb{R}[\mathcal{P}_\lambda]}$.

From this fixed-point principle we compute

$$\Theta_\lambda(s_1) = \sum_{|\lambda_i|\geq 2} \binom{n-2}{\lambda_i,\dots,\lambda_i - 2,\dots,\lambda_\ell},$$

since a partition in $\mathcal{P}_\lambda$ is fixed by s_1 iff 1 and 2 lie in the same part X_i for some i. Similarly, $\Theta_\lambda(s_1 s_3)$ equals

$$\sum_{\substack{1\leq i<j\leq l \\ |\lambda_i|\geq 2, |\lambda_j|\geq 2}} \binom{n-4}{\lambda_1,\dots,\lambda_i-2,\dots,\lambda_j-2,\dots,\lambda_l} + \sum_{|\lambda_k|\geq 4} \binom{n-4}{\lambda_1,\dots,\lambda_i-4,\dots,\lambda_l}.$$

This is because a partition in $\mathcal{P}_\lambda$ is fixed by $s_1 s_3$ iff either the elements $1,2,3,4$ all lie in the same part X_i, or $1,2$ lie in some X_i and $3,4$ lie some other part X_j.

These character values may be used to compute $g_{\mathbb{R}[\mathcal{P}_\lambda]}$ and $h_{\mathbb{R}[\mathcal{P}_\lambda]}$. (Compare Lemma 17 in [2].) For instance if $\lambda = (1,\dots,1) \vdash n$, then $\Theta_\lambda(s_1) = \Theta_\lambda(s_1 s_3) = 0$, so $g_{\mathbb{R}[\mathcal{P}_\lambda]} = h_{\mathbb{R}[\mathcal{P}_\lambda]} = \frac{n!}{2}$. Thus the regular representation of S_n is achiral and spinorial.

The standard representation π_n corresponds to $\lambda = (n-1,1)$. For this λ we have $\Theta_\lambda(s_1) = n-2$, so $g_{\mathbb{R}[\mathcal{P}_\lambda]} = 1$ and it follows that π_n is aspinorial.

For easy reference, we collect here results about common representations of S_n:

Proposition 3.8. For $n \geq 2$ the standard representation π_n is achiral and aspinorial, and the sign representation is achiral and aspinorial. For $n \geq 4$, the regular representation of S_n is achiral and spinorial.

4. Alternating groups

Now we turn to the alternating group A_n, for $n \geq 4$.

4.1. Spinoriality criterion

The group A_n is generated by the permutations

$$u_i = s_1 s_{i+1}, \quad (i = 1, 2, \ldots, n-2)$$

with relations:

$$u_1^3 = u_j^2 = (u_{j-1}u_j)^3 = 1, \quad (2 \leq j \leq n-2),$$
$$(u_i u_j)^2 = 1, \quad (1 \leq i < j-1, j \leq n-2).$$

(See for instance [5].)

Note that u_1 is a 3-cycle and the other u_i are $(2,2)$-cycles.

For a real representation (π, V) of A_n again put $h_\pi = \dfrac{\chi_\pi(1) - \chi_\pi(s_1 s_3)}{2}$. Since this is the multiplicity of the eigenvalue -1 of $\pi(s_1 s_3)$, which has determinant 1, the integer h_π is necessarily even.

Theorem 4.1. A real representation (π, V) of A_n is spinorial if and only if h_π is a multiple of 4. In this case there is a unique lift.

Proof. As in Section 3.1 we must choose c_i with $\rho(c_i) = \pi(u_i)$ satisfying the same relations as the u_i. Let c_1 be a lift of $\pi(u_1)$. Since $\rho(c_1)^3 = \pi(u_1)^3 = 1$, we have $c_1^3 = \pm 1$. This determines the sign of c_1.

The u_j and $u_i u_j$ as above, for $j > 1$, are all conjugate to u_2 in A_n. Therefore all the conditions $c_j^2 = 1$ and $(c_i c_j)^2 = 1$ are equivalent to the condition $c_2^2 = 1$. As before, this is equivalent to h_π being congruent to 0 or 3 mod 4, but since h_π is even, it must be a multiple of 4.

Finally, there is a unique choice of signs normalizing $c_2, \ldots, c_{n-2}$ so that

$$(c_1 c_2)^3 = (c_2 c_3)^3 = \ldots = 1. \quad \square$$

Example. If ρ is the regular representation of A_n (on the group algebra $\mathbb{R}[A_n]$), then $h_\rho = \dfrac{n!}{4}$, so ρ is spinorial iff $n \neq 4, 5$.

Example. For the standard representation π_n of S_n, $h_{\pi_n} = 2$, so the restriction of π_n to A_n is aspinorial.

4.2. Real irreducible representations

Let us review the relationship between real and complex irreducible representations of a finite group G, following [4]. If (π, V) is a complex representation of a group G, write $(\pi_{\mathbb{R}}, V_{\mathbb{R}})$ for the realization of π, meaning that we simply forget the complex structure on V and regard it as a real representation. If moreover (π, V) is an orthogonal complex representation, meaning that it admits a G-invariant symmetric nondegenerate bilinear form, then there is a unique real representation (π_0, V_0), up to isomorphism, so that $\pi \cong \pi_0 \otimes_{\mathbb{R}} \mathbb{C}$.

It is not hard to see that π_0 is self-dual iff π is self-dual, and that an orthogonal π is spinorial, i.e., lifts to $\mathrm{Spin}(V)$, iff π_0 is spinorial, i.e., lifts to $\mathrm{Spin}(V_0)$.

Every real irreducible representation σ of G is either of the form

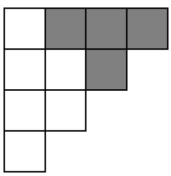
(1) $\sigma = \pi_0$, for an orthogonal irreducible complex representation π of G, or
(2) $\sigma = \pi_{\mathbb{R}}$, for an irreducible complex representation π of G which is not orthogonal.

In the case of $G = S_n$, all complex representations are orthogonal.

4.3. Real irreducible representations of A_n

For a partition λ, write λ' for its conjugate partition. Furthermore write $\epsilon_\lambda = 1$ when the number of cells in the Young diagram of λ above the diagonal is even, and $\epsilon_\lambda = -1$ when this number is odd.

For example, let $\lambda = (4, 3, 2, 1)$. Then $\lambda = \lambda'$, and there are 4 cells above the diagonal, shaded in the Young diagram below, so $\epsilon_\lambda = 1$.



Let σ_λ be the (real) Specht module corresponding to λ, as before. Write $\pi_\lambda = \sigma_\lambda \otimes \mathbb{C}$ for its complexification, i.e., the complex Specht module corresponding to λ.

If $\lambda \neq \lambda'$, then π_λ restricts irreducibly to A_n. When $\lambda = \lambda'$, the restriction of π_λ to A_n decomposes into a direct sum of two nonisomorphic representations π_λ^+ and π_λ^-. Either of $\pi_\lambda^\pm$ is the twist of the other by $\sigma_\lambda(w)$ for any odd permutation w. The set of π_λ with $\lambda \neq \lambda'$, together with the $\pi_\lambda^\pm$ for $\lambda = \lambda'$, is a complete set of irreducible complex representations of A_n.

For $\lambda = \lambda'$ we have

$$\chi_\lambda^+(s_1 s_3) = \chi_\lambda^-(s_1 s_3) = \frac{1}{2}\chi_\lambda(s_1 s_3). \tag{4.1}$$

If moreover $\epsilon_\lambda = 1$, then the representations π_λ^+ and π_λ^- are orthogonal. We may then define real irreducible representations of A_n by $\sigma_\lambda^\pm = (\pi_\lambda^\pm)_0$. However when $\epsilon_\lambda = -1$, the representations $\pi_\lambda^\pm$ are not orthogonal, and therefore the realizations $(\pi_\lambda^\pm)_{\mathbb{R}}$ are irreducible.

Then

$$\begin{aligned}\pi_\lambda^+ \oplus \pi_\lambda^- &\cong \pi_\lambda|_{A_n}\\ &\cong \sigma|_{A_n} \otimes \mathbb{C},\end{aligned}$$

so that

$$\begin{aligned}(\pi_\lambda^+)_{\mathbb{R}} \oplus (\pi_\lambda^-)_{\mathbb{R}} &\cong (\sigma_\lambda|_{A_n} \otimes \mathbb{C})_{\mathbb{R}}\\ &\cong \sigma_\lambda|_{A_n} \oplus \sigma_\lambda|_{A_n}.\end{aligned}$$

Thus we have isomorphisms of real A_n-representations:

$$(\pi_\lambda^+)_{\mathbb{R}} \cong (\pi_\lambda^-)_{\mathbb{R}} \cong \sigma_\lambda|_{A_n}.$$

From these considerations and Theorem 4.1 we conclude:

Theorem 4.2. A complete list of real irreducible representations of A_n is given by

(1) $\sigma_\lambda|_{A_n}$, where either $\lambda \neq \lambda'$, or $\lambda = \lambda'$ and $\epsilon_\lambda = -1$, and
(2) $\sigma_\lambda^\pm$, where $\lambda = \lambda'$ and $\epsilon_\lambda = 1$.

In the first case, $\sigma_\lambda|_{A_n}$ is spinorial iff $\chi_\lambda(s_1s_3) \equiv \chi_\lambda(1) \mod 8$. In the second case, σ_λ^+ is spinorial iff σ_λ^- is spinorial iff $\chi_\lambda(s_1s_3) \equiv \chi_\lambda(1) \mod 16$.

Remark. When $\lambda = \lambda'$ the restriction $\sigma_\lambda|_{A_n}$ is necessarily spinorial by (4.1), since all h_π are even.

5. Tables

We illustrate the theory of this paper by means of two tables. Table 1 contains the following information for $2 \le n \le 6$:

(1) Whether the Specht Module σ_λ is chiral, i.e., whether g_λ is odd.
(2) Whether σ_λ is spinorial, by Theorem 3.4.
(3) Whether the restriction of σ_λ to A_n is spinorial, by Theorem 4.2.

Table 2 lists for self-conjugate λ with $\epsilon_\lambda = 1$, whether the constituents σ_λ^+ and σ_λ^- are spinorial, following Theorem 4.2. This is done for all such λ with $3 \le |\lambda| \le 15$.

Table 1
Spinoriality/chirality of σ_λ with $2 \leq |\lambda| \leq 6$.

λ	Chirality of σ_λ	Spinoriality of σ_λ	Spinoriality of $\sigma_\lambda\|_{A_n}$
$\|\lambda\| = 2$			
(2)	achiral	spinorial	spinorial
(1^2)	chiral	aspinorial	spinorial
$\|\lambda\| = 3$			
(3)	achiral	spinorial	spinorial
$(2, 1)$	chiral	aspinorial	spinorial
(1^3)	chiral	aspinorial	spinorial
$\|\lambda\| = 4$			
(4)	achiral	spinorial	spinorial
$(3, 1)$	chiral	aspinorial	aspinorial
$(2, 2)$	chiral	aspinorial	spinorial
$(2, 1^2)$	achiral	aspinorial	aspinorial
(1^4)	chiral	aspinorial	spinorial
$\|\lambda\| = 5$			
(5)	achiral	spinorial	spinorial
$(4, 1)$	chiral	aspinorial	aspinorial
$(3, 2)$	achiral	aspinorial	aspinorial
$(3, 1^2)$	chiral	spinorial	spinorial
$(2^2, 1)$	chiral	aspinorial	aspinorial
$(2, 1^3)$	chiral	aspinorial	aspinorial
(1^5)	chiral	aspinorial	spinorial
$\|\lambda\| = 6$			
(6)	achiral	spinorial	spinorial
$(5, 1)$	chiral	aspinorial	aspinorial
$(4, 2)$	chiral	aspinorial	spinorial
$(4, 1^2)$	achiral	aspinorial	aspinorial
(3^2)	achiral	aspinorial	aspinorial
$(3, 2, 1)$	achiral	spinorial	spinorial
$(3, 1^3)$	achiral	aspinorial	aspinorial
(2^3)	chiral	aspinorial	aspinorial
$(2^2, 1^2)$	achiral	aspinorial	spinorial
$(2, 1^4)$	achiral	aspinorial	aspinorial
(1^6)	chiral	aspinorial	spinorial

Table 2
Spinoriality of $\sigma_\lambda^\pm$ with $\lambda = \lambda'$, $\epsilon_\lambda = 1$, and $3 \leq |\lambda| \leq 15$.

λ	$\|\lambda\|$	$\sigma_\lambda^\pm$
$(3, 1, 1)$	5	aspinorial
$(3, 2, 1)$	6	spinorial
$(5, 1^4)$	9	spinorial
$(5, 2, 1^3)$	10	spinorial
$(4, 3, 2, 1)$	10	spinorial
$(4, 3, 3, 1)$	11	aspinorial
$(7, 1^6)$	13	spinorial
$(7, 2, 1^5)$	14	spinorial
$(6, 3^2, 1^3)$	15	spinorial
$(5, 4, 3, 2, 1)$	15	spinorial
$(4^3, 3)$	15	spinorial

6. Stiefel-Whitney classes

6.1. Basic properties

Let G be a finite group and π a real representation of G. Stiefel-Whitney classes $w_i(\pi)$ are defined for $0 \leq i \leq \deg \pi$ as members of the cohomology groups $H^i(G) = H^i(G, \mathbb{Z}/2\mathbb{Z})$. Here $\mathbb{Z}/2\mathbb{Z}$ is trivial as a G-module. One considers the total Stiefel-Whitney class in the $\mathbb{Z}/2\mathbb{Z}$-cohomology ring:

$$w(\pi) = w_0(\pi) + w_1(\pi) + \cdots + w_d(\pi) \in H^*(G) = \bigoplus_{i=0}^{\infty} H^i(G),$$

where $d = \deg \pi$.

According to, for example [8], these characteristic classes satisfy the following properties:

(1) $w_0(\pi) = 1$.
(2) $w_1(\pi) = \det \pi$, regarded as a linear character in $H^1(G) \cong \mathrm{Hom}(G, \{\pm 1\})$.
(3) If π' is another real representation, then $w(\pi \oplus \pi') = w(\pi) \cup w(\pi')$.
(4) If $f : G' \to G$ is a group homomorphism, then $w(\pi \circ f) = f^*(w(\pi))$, where f^* is the induced map on cohomology.
(5) Suppose $\det \pi = 1$. Then $w_2(\pi) = 0$ iff π is spinorial.

Note in particular that $w(\chi) = 1 + \chi$, if χ is a linear character of G.

The last property generalizes as follows:

Proposition 6.1. A real representation π is spinorial iff $w_2(\pi) = w_1(\pi) \cup w_1(\pi)$.

We will deduce this proposition from the following lemma.

Lemma 6.2. Let $\pi' = \pi \oplus \det \pi$. Then π is spinorial iff π' is spinorial.

Proof. Let V' be the representation space of π'; say $V' = V \oplus \mathbb{R}v'$ for some unit vector v' perpendicular to V. Write $\iota : C(V) \to C(V')$ for the canonical injection. Note that if $x \in C(V)^0$, then $\iota(x)v' = v' \cdot \iota(x)$, and if $x \in C(V)^1$, then $\iota(x)v' = -v' \cdot \iota(x)$. Write $\rho' : \mathrm{Pin}(V') \to O(V')$ for the usual double cover.

Define $\varphi : \mathrm{O}(V) \to \mathrm{SO}(V')$ by

$$\varphi(g) = g \oplus \det(g),$$

so that $\pi' = \varphi \circ \pi$. Write $\Phi_V < \mathrm{SO}(V')$ for the image of φ. The essential problem is to construct a lift of $\varphi \circ \rho$. Since $\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap C(V)^0$, the map $\tilde{\varphi} : \mathrm{Pin}(V) \to \mathrm{Spin}(V')$ defined by

$$\tilde{\varphi}(x) = \begin{cases} \iota(x), & \text{if } x \in \mathrm{Spin}(V) \\ \iota(x)v', & \text{if } x \notin \mathrm{Spin}(V) \end{cases}$$

is a group homomorphism. Note that

$$\rho' \circ \tilde{\varphi} = \varphi \circ \rho. \tag{6.1}$$

Let us see that $\tilde{\varphi}$ is injective; suppose $\tilde{\varphi}(x_1) = \tilde{\varphi}(x_2)$. Clearly if x_1 and x_2 are both in $\mathrm{Spin}(V)$, or both not in $\mathrm{Spin}(V)$, then $x_1 = x_2$. Suppose $x_1 \in \mathrm{Spin}(V)$ but $x_2 \notin \mathrm{Spin}(V)$. Then $v' = \iota(x_2^{-1}x_1)$ and in particular $v' \in \iota(C(V))$. But this is impossible by Corollary 6.7 in Chapter I of [4]. We conclude that $\tilde{\varphi}$ is injective.

Write $\tilde{\Phi}_V < \mathrm{Spin}(V')$ for the image of $\tilde{\varphi}$; then $\tilde{\varphi}$ is an isomorphism from $\mathrm{Pin}(V)$ onto $\tilde{\Phi}_V$ and

$$(\rho')^{-1}\Phi_V = \tilde{\Phi}_V.$$

If $\hat{\pi}$ is a lift of π, then $\tilde{\varphi} \circ \hat{\pi}$ is a lift of π'. Conversely, suppose $\hat{\pi}'$ is a lift of π'. Then its image lies in $\tilde{\Phi}_V$, and therefore $\hat{\pi}' = \tilde{\varphi} \circ \hat{\pi}$ for some homomorphism $\hat{\pi} : G \to \mathrm{Pin}(V)$. Since

$$\rho' \circ \hat{\pi}' = \varphi \circ \pi,$$

it follows that

$$\varphi \circ \rho \circ \hat{\pi} = \varphi \circ \pi.$$

Thus $\hat{\pi}$ is a lift of π. □

Proof of Proposition 6.1. Note that $\det \pi' = 1$, so π is spinorial iff $w_2(\pi') = 0$. But

$$\begin{aligned} w(\pi') &= w(\pi) \cup w(\det \pi) \\ &= (1 + \det \pi + w_2(\pi) + \cdots) \cup (1 + \det \pi) \\ &= 1 + w_2(\pi) + w_1(\pi) \cup w_1(\pi) + \cdots , \end{aligned}$$

whence the theorem. □

6.2. The group of order 2

Let C be a cyclic group of order 2, and write 'sgn' for its nontrivial linear character. Then $H^2(C) \cong \mathbb{Z}/2\mathbb{Z}$; the nonzero element is '$\mathrm{sgn} \cup \mathrm{sgn}$'. Let π be the sum of m copies of the trivial representation with n copies of sgn. Then

$$
\begin{aligned}
w(\pi) &= w(\mathrm{sgn}) \cup \cdots \cup w(\mathrm{sgn}) \\
&= 1 + n \cdot \mathrm{sgn} + \binom{n}{2} \cdot \mathrm{sgn} \cup \mathrm{sgn} + \cdots .
\end{aligned}
$$

In particular, $w_2(\pi) = \binom{n}{2} \cdot \mathrm{sgn} \cup \mathrm{sgn}$. By Proposition 6.1, π is spinorial iff $n^2 \equiv \binom{n}{2} \mod 2$; equivalently, $n \equiv 0$ or $3 \mod 4$.

6.3. *Calculation for S_n*

Write

$$e_{\mathrm{cup}} = w_1(\mathrm{sgn}) \cup w_1(\mathrm{sgn}) = w_2(\mathrm{sgn} \oplus \mathrm{sgn}) \in H^2(S_n).$$

Again write π_n for the standard representation of S_n on $\mathbb{R}^n$. From [14, Section 1.5] we know that e_{cup} and $w_2(\pi_n)$ comprise a basis for the $\mathbb{Z}/2\mathbb{Z}$-vector space $H^2(S_n)$.

Proposition 6.3. The map

$$\Phi : H^2(S_n) \to H^2(\langle s_1 \rangle) \oplus H^2(\langle s_1 s_3 \rangle),$$

given by the two restrictions, is an isomorphism for $n \geq 4$.

Proof. Since Φ is a linear map between 2-dimensional $\mathbb{Z}/2\mathbb{Z}$-vector spaces, it suffices to prove that its rank is 2. Let b_1 be the generator of $H^2(\langle s_1 \rangle)$, and b_2 be the generator of $H^2(\langle s_1 s_3 \rangle)$.

The restriction of π_n to $\langle s_1 \rangle$ decomposes into a trivial $(n-1)$-dimensional representation plus one copy of sgn. The restriction to $\langle s_1 s_3 \rangle$ contains two copies of sgn. Therefore $\Phi(w_2(\pi_n)) = (0, b_2)$. Similarly $\Phi(w_2(\mathrm{sgn} \oplus \mathrm{sgn})) = \Phi(e_{\mathrm{cup}}) = (b_1, 0)$. Thus Φ has rank 2, as required. $\square$

Thus the second $\mathbb{Z}/2\mathbb{Z}$-cohomology of S_n is “detected” by these cyclic subgroups; compare Corollary 3.5 above and Theorem VI.1.2 in [1].

Theorem 6.4. For π a real representation of S_n, with $n \geq 4$, we have

$$
\begin{aligned}
w_2(\pi) &= \left[\frac{g_\pi}{2}\right] e_{\mathrm{cup}} + \frac{h_\pi}{2} w_2(\pi_n) \\
&= \left[\frac{\chi_V(1) - \chi_V(s_1)}{4}\right] e_{\mathrm{cup}} + \frac{\chi_V(1) - \chi_V(s_1 s_3)}{4} w_2(\pi_n).
\end{aligned}
$$

Here $[\cdot]$ denotes the greatest integer function.

Proof. Suppose first that π is achiral. Since e_{cup} and $w_2(\pi_n)$ form a basis of $H^2(S_n)$ we must have

$$w_2(\pi) = c_1 e_{\mathrm{cup}} + c_2 w_2(\pi_n),$$

for some $c_1, c_2 \in \mathbb{Z}/2\mathbb{Z}$. Thus $\Phi(w_2(\pi)) = c_1 b_1 + c_2 b_2$. By the Stiefel-Whitney class properties (4) and (5),

$$c_1 = 0 \Leftrightarrow \pi|_{\langle s_1 \rangle} \text{ is spinorial } \Leftrightarrow 4|g_\pi$$

and

$$c_2 = 0 \Leftrightarrow \pi|_{\langle s_1 s_3 \rangle} \text{ is spinorial } \Leftrightarrow 4|h_\pi.$$

Thus $c_1 \equiv \frac{g_\pi}{2} \mod 2$ and $c_2 \equiv \frac{h_\pi}{2} \mod 2$.

If π is chiral, then $\pi' = \pi \oplus \mathrm{sgn}$ is achiral. From the identity $w_2(\pi) = w_2(\pi') + e_{\mathrm{cup}}$, we deduce that

$$w_2(\pi) = \frac{g_\pi - 1}{2} e_{\mathrm{cup}} + \frac{h_\pi}{2} w_2(\pi_n). \quad \square$$

Remark. For $n = 2, 3$, similar reasoning gives $H^2(S_n) \cong H^2(\langle s_1 \rangle)$ and

$$w_2(\pi) = \left[\frac{g_\pi}{2}\right] e_{\mathrm{cup}} = \left[\frac{\chi_V(1) - \chi_V(s_1)}{4}\right] e_{\mathrm{cup}}.$$

Remark. Since the groups $H^2(A_n, \mathbb{Z}/2\mathbb{Z})$ have order 1 or 2, computing the Stiefel-Whitney class of a real representation of A_n is equivalent to determining its spinoriality, which we have already done.

7. Products

Spinoriality for representations of $S_n \times S_{n'}$ can also be determined by means of generators and relations. (See Theorem 5.4.1 in [7].) However we will instead obtain a satisfactory criterion by simply feeding our calculation of $w_2(\pi)$ into the machinery of Stiefel-Whitney classes.

7.1. External tensor products

Let G, G' be finite groups, let (π, V) be a real representation of G, and let (π', V') be a real representation of G'. Write $\pi \boxtimes \pi'$ for the external tensor product representation of $G \times G'$ on $V \otimes V'$. One computes

$$\det(\pi \boxtimes \pi') = \det(\pi)^{\deg \pi'} \cdot \det(\pi')^{\deg \pi},$$

and hence

$$w_1(\pi \boxtimes \pi') = \deg \pi' \cdot w_1(\pi) + \deg \pi \cdot w_1(\pi'),$$

which is an element of

$$H^1(G \times G') \cong H^1(G) \oplus H^1(G').$$

The famous "splitting principle" (e.g., proceeding as in Problem 7-C of [12]) similarly gives

$$\begin{aligned} w_2(\pi \boxtimes \pi') = \deg \pi' \cdot w_2(\pi) + \binom{\deg \pi'}{2} w_1(\pi) \cup w_1(\pi) \\ + (\deg \pi \deg \pi' - 1) w_1(\pi) \otimes w_1(\pi') \\ + \binom{\deg \pi}{2} w_1(\pi') \cup w_1(\pi') + \deg \pi \cdot w_2(\pi'), \end{aligned}$$

as an element of

$$H^2(G \times G') \cong H^2(G) \oplus \left(H^1(G) \otimes H^1(G')\right) \oplus H^2(G').$$

Finally, $w_2(\pi \boxtimes \pi') + w_1(\pi \boxtimes \pi') \cup w_1(\pi \boxtimes \pi')$ comes out to be

$$\begin{aligned} \deg \pi' \cdot w_2(\pi) + \binom{\deg \pi' + 1}{2} w_1(\pi) \cup w_1(\pi) + (\deg \pi \deg \pi' + 1) w_1(\pi) \otimes w_1(\pi') \\ + \binom{\dim \pi + 1}{2} w_1(\pi') \cup w_1(\pi') + \deg \pi \cdot w_2(\pi'). \end{aligned}$$

Thus $\pi \boxtimes \pi'$ is spinorial (by Proposition 6.1) iff all of the following vanish:

(1) $\deg \pi' \cdot w_2(\pi) + \binom{\deg \pi' + 1}{2} w_1(\pi) \cup w_1(\pi)$,
(2) $(\deg \pi \deg \pi' + 1) w_1(\pi) \otimes w_1(\pi')$, and
(3) $\binom{\deg \pi + 1}{2} w_1(\pi') \cup w_1(\pi') + \deg \pi \cdot w_2(\pi')$.

7.2. Products of symmetric groups

We now prove Theorem 1.2. Let π, π' be representations of S_n and $S_{n'}$. Write $f = f_\pi$, $f' = f_{\pi'}$ and similarly for g, h, g' and h'. Let $\Pi = \pi \boxtimes \pi'$; all representations of $S_n \times S_{n'}$ are sums of such representations.

Proof. By Proposition 6.1, Π is spinorial iff

$$w_2(\Pi) = w_1(\Pi) \cup w_1(\Pi).$$

From Theorem 6.4 and (1)-(3) of Section 7.1 we deduce that Π is spinorial iff all of the following are even:

(1) $f' \cdot \frac{h}{2}$,
(2) $f' \left[\frac{g}{2}\right] + \binom{f'+1}{2} g$,
(3) $(ff'+1)gg'$,
(4) $f \cdot \frac{h'}{2}$, and
(5) $f \left[\frac{g'}{2}\right] + \binom{f+1}{2} g'$.

Note that if Π is spinorial, then its restriction to $S_n \times \{1\}$, which amounts to f' copies of π, is spinorial. From before, this implies that $f' \cdot \frac{h}{2}$ is even, and $f'g$ is congruent to 0 or 3 mod 4. One can verify this $f'g$ condition is equivalent to (2) being even. Thus (1), (2), (4), and (5) above are all even iff the restrictions of Π to $S_n \times \{1\}$ and $\{1\} \times S_{n'}$ are spinorial. Theorem 1.2 follows from this, since $ff' = \deg \Pi$. □

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