

# Journal Pre-proof

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PII: S0021-8693(20)30543-3

DOI: <https://doi.org/10.1016/j.jalgebra.2020.09.048>

Reference: YJABR 17879

To appear in: *Journal of Algebra*

Received date: 23 January 2020

Please cite this article as: P. Zhou, Grothendieck groups and Auslander-Reiten  $(d + 2)$ -angles, *J. Algebra* (2021), doi: <https://doi.org/10.1016/j.jalgebra.2020.09.048>.

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# Grothendieck groups and Auslander-Reiten $(d + 2)$ -angles<sup>★</sup>

Panyue Zhou

## Abstract

Xiao and Zhu has shown that if  $\mathcal{C}$  is a locally finite triangulated category, then the Auslander-Reiten triangles generate the relations for the Grothendieck group of  $\mathcal{C}$ . The notion of  $(d + 2)$ -angulated categories is a “higher dimensional” analogue of triangulated categories. In this article, we show that a  $(d + 2)$ -angulated category  $\mathcal{C}$  with  $d$  odd is locally finite if and only if the Auslander-Reiten  $(d + 2)$ -angles generate the relations for the Grothendieck group of  $\mathcal{C}$ . This extends the result of Xiao and Zhu, and gives the converse of Xiao and Zhu’s result is also true.

**Key words:**  $(d + 2)$ -angulated categories; Auslander-Reiten  $(d + 2)$ -angles; locally finite; triangulated categories; Grothendieck groups.

**2010 Mathematics Subject Classification:** 16G70; 13D15; 18E30.

## 1 Introduction

Auslander-Reiten theory was introduced by Auslander and Reiten in [AR1, AR2]. Since its introduction, Auslander-Reiten theory has become a fundamental tool for studying the representation theory of Artin algebras. It is well-known that a module category of an Artin algebra has almost split sequences. For an Artin algebra of finite type, Butler [Bu] proved that the relations of its Grothendieck group are generated by all Auslander-Reiten sequences. Soon later Auslander showed that the converse is true in [Au]. The notion of Auslander-Reiten triangles in a triangulated category was introduced by Happel in [Ha]. In contrast to module categories over Artin algebras, not all triangulated categories have Auslander-Reiten triangles [Ha]. It was proved in [Ha] that the bounded derived category of a finite dimensional algebra has Auslander-Reiten triangles if and only if the global dimension of the algebra is finite. Reiten and Van den Bergh [RV] proved that the existence of Auslander-Reiten triangles if and only if the existence of Serre functor in a triangulated category. Recently, Xiao and Zhu [XZ] showed that if  $\mathcal{C}$  is a locally finite triangulated category, then the Auslander-Reiten triangles generate the relations for the Grothendieck group of  $\mathcal{C}$ . Beligiannis [Be] proved the converse of this result holds when  $\mathcal{C}$  is a compactly generated triangulated category. In more recent times, many authors have shown the reverse direction of Xiao and Zhu is true in some special

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<sup>★</sup>This work was supported by the National Natural Science Foundation of China (Grant No. 11901190 and 11671221), and by the Hunan Provincial Natural Science Foundation of China (Grant No. 2018JJ3205), and by the Scientific Research Fund of Hunan Provincial Education Department (Grant No. 19B239).

cases [H1, PPPP]. Extriangulated categories were introduced by Nakaoka and Palu [NP] as a simultaneous generalization of exact categories and triangulated categories. Hence, many results hold on exact categories and triangulated categories can be unified in the same framework. Iyama, Nakaoka and Palu [INP] introduced the notion of almost split extensions and Auslander-Reiten-Serre duality for extriangulated categories, and gave explicit connections between these notions and also with the classical notion of dualizing  $k$ -varieties. Zhu and Zhuang [ZZ] has shown that a locally finite extriangulated category  $\mathcal{C}$  has Auslander-Reiten  $\mathbb{E}$ -triangles and the relations of Grothendieck group are generated by the Auslander-Reiten  $\mathbb{E}$ -triangles. A partial converse result is given when restricting to a triangulated category with a cluster tilting subcategory.

In [GKO], Geiss, Keller and Oppermann introduced  $(d+2)$ -angulated categories. These are a “higher dimensional” analogue of triangulated categories, in the sense that triangles are replaced by  $(d+2)$ -angles, that is, morphism sequences of length  $(d+2)$ . Thus a 3-angulated category is precisely a triangulated category. They appear for example as certain cluster tilting subcategories of triangulated categories. Iyama and Yoshino defined the notion of Auslander-Reiten  $(d+2)$ -angles in special  $(d+2)$ -angulated categories. This notion was generalized to arbitrary  $(d+2)$ -angulated categories by Fedele [Fe2]. Fedele also proved that there are Auslander-Reiten  $(d+2)$ -angles in certain subcategories of  $(d+2)$ -angulated categories. The author [Z1] showed that a  $(d+2)$ -angulated category has Auslander-Reiten  $(d+2)$ -angles if and only if  $\mathcal{C}$  has a Serre functor. Moreover, the author [Z2] also proved that if  $\mathcal{C}$  is a locally finite  $(d+2)$ -angulated category, then  $\mathcal{C}$  has Auslander-Reiten  $(d+2)$ -angles.

Bergh and Thaule [BT2] defined the Grothendieck group of a  $(d+2)$ -angulated category. As in the triangulated case, it is the free abelian group on the set of isomorphism classes of objects, modulo the Euler relations corresponding to the  $(d+2)$ -angles. They showed that when  $d$  is odd, the set of subgroups corresponds bijectively to the complete and dense  $(d+2)$ -angulated subcategories. Fedele [Fe1] showed that under suitable circumstances, the Grothendieck group of a triangulated category can be expressed as a quotient of the split Grothendieck group of a higher cluster tilting subcategory of a triangulated category. Recently, Herschend, Liu and Nakaoka [HLN] defined  $d$ -exangulated categories as a “higher dimensional” analogue of extriangulated categories. Many categories studied in representation theory turn out to be  $d$ -exangulated. In particular,  $d$ -exangulated categories simultaneously generalize  $(d+2)$ -angulated and  $d$ -exact categories [Ja]. Haugland [H2] defined the Grothendieck group of a  $d$ -exangulated category, and classified dense complete subcategories of  $d$ -exangulated categories with a  $d$ -(co)generator in terms of subgroups of the Grothendieck group.

The aim of this article is to discuss the relation between Grothendieck groups of  $(d+2)$ -angulated categories and Auslander-Reiten  $(d+2)$ -angles in  $(d+2)$ -angulated categories. We show that a  $(d+2)$ -angulated category with  $d$  odd is locally finite if and only if the Auslander-Reiten  $(d+2)$ -angles generate the relations for the Grothendieck group of this  $(d+2)$ -angulated category, see Theorem 3.8 and Theorem 3.13. This extends the result of Xiao and Zhu, and gives the converse of Xiao and Zhu’s result holds. We hope that our work would motivate the

further study on  $(d+2)$ -angulated categories.

This article is organised as follows: In Section 2, we review some elementary definitions that we need to use, including  $(d+2)$ -angulated categories and Auslander-Reiten  $(d+2)$  angles. In Section 3, we show our two main results.

## 2 Preliminaries

In this section, we first recall the definition and basic properties of  $(d+2)$ -angulated categories from [GKO]. Let  $\mathcal{C}$  be an additive category with an automorphism  $\Sigma^d : \mathcal{C} \rightarrow \mathcal{C}$ , where  $d$  is an integer greater than or equal to one.

A  $(d+2)$ - $\Sigma^d$ -sequence in  $\mathcal{C}$  is a sequence of objects and morphisms

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{d-1}} A_n \xrightarrow{f_d} A_{d+1} \xrightarrow{f_{d+1}} \Sigma^d A_0.$$

Its *left rotation* is the  $(d+2)$ - $\Sigma^d$ -sequence

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots \xrightarrow{f_d} A_{d+1} \xrightarrow{f_{d+1}} \Sigma^d A_0 \xrightarrow{(-1)^d \Sigma^d f_0} \Sigma^d A_1.$$

A *morphism* of  $(d+2)$ - $\Sigma^d$ -sequences is a sequence of morphisms  $\varphi = (\varphi_0, \varphi_1, \dots, \varphi_{d+1})$  such that the following diagram commutes

$$\begin{array}{ccccccccccc} A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_d} & A_{d+1} & \xrightarrow{f_{d+1}} & \Sigma^d A_0 \\ \downarrow \varphi_0 & & \downarrow \varphi_1 & & \downarrow \varphi_2 & & & & \downarrow \varphi_{d+1} & & \downarrow \Sigma^d \varphi_0 \\ B_0 & \xrightarrow{g_0} & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_d} & B_{d+1} & \xrightarrow{g_{d+1}} & \Sigma^d B_0 \end{array}$$

where each row is a  $(d+2)$ - $\Sigma^d$ -sequence. It is an *isomorphism* if  $\varphi_0, \varphi_1, \varphi_2, \dots, \varphi_{d+1}$  are isomorphisms in  $\mathcal{C}$ .

**Definition 2.1.** [GKO, Definition 2.1] A  $(d+2)$ -angulated category is a triple  $(\mathcal{C}, \Sigma^d, \Theta)$ , where  $\mathcal{C}$  is an additive category,  $\Sigma^d$  is an automorphism of  $\mathcal{C}$  ( $\Sigma^d$  is called the  $d$ -suspension functor), and  $\Theta$  is a class of  $(d+2)$ - $\Sigma^d$ -sequences (whose elements are called  $(d+2)$ -angles), which satisfies the following axioms:

- (N1) (a) The class  $\Theta$  is closed under isomorphisms, direct sums and direct summands.  
 (b) For each object  $A \in \mathcal{C}$ , the trivial sequence

$$A \xrightarrow{1_A} A \rightarrow 0 \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow \Sigma^d A$$

belongs to  $\Theta$ .

- (c) Each morphism  $f_0 : A_0 \rightarrow A_1$  in  $\mathcal{C}$  can be extended to  $(d+2)$ - $\Sigma^d$ -sequence:

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{d-1}} A_d \xrightarrow{f_d} A_{d+1} \xrightarrow{f_{d+1}} \Sigma^d A_0.$$

- (N2) A  $(d+2)$ - $\Sigma^d$ -sequence belongs to  $\Theta$  if and only if its left rotation belongs to  $\Theta$ .

(N3) Each solid commutative diagram

$$\begin{array}{ccccccc}
 A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots \xrightarrow{f_d} A_{d+1} \xrightarrow{f_{d+1}} \Sigma^d A_0 \\
 \downarrow \varphi_0 & & \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_{d+1} \\
 B_0 & \xrightarrow{g_0} & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & \cdots \xrightarrow{g_d} B_{d+1} \xrightarrow{g_{d+1}} \Sigma^d B_0
 \end{array}$$

with rows in  $\Theta$ , the dotted morphisms exist and give a morphism of  $(d+2)$ - $\Sigma^d$ -sequences.

(N4) In the situation of (N3), the morphisms  $\varphi_2, \varphi_3, \dots, \varphi_{d+1}$  can be chosen such that the mapping cone

$$A_1 \oplus B_0 \xrightarrow{\begin{pmatrix} -f_1 & 0 \\ \varphi_1 & g_0 \end{pmatrix}} A_2 \oplus B_1 \xrightarrow{\begin{pmatrix} -f_2 & 0 \\ \varphi_2 & g_1 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} -f_{d+1} & 0 \\ \varphi_{d+1} & g_d \end{pmatrix}} \Sigma^n A_0 \oplus B_{d+1} \xrightarrow{\begin{pmatrix} -\Sigma^d f_0 & 0 \\ \Sigma^d \varphi_1 & g_{d+1} \end{pmatrix}} \Sigma^d A_1 \oplus \Sigma^d B_0$$

belongs to  $\Theta$ .

Later, Bergh and Thaulé [BT1] introduced a higher “octahedral axiom”, and show that it is equivalent to the mapping cone axiom for a  $(d+2)$ -angulated category, see also [ABT].

(N4)\* Given the solid part of the diagram

$$\begin{array}{ccccccccccccccc}
 A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{d-1}} & A_d & \xrightarrow{f_d} & A_{d+1} & \xrightarrow{f_{d+1}} & \Sigma^d A_0 \\
 \parallel & & \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & & & \downarrow \varphi_d & & \downarrow \varphi_{d+1} & & \parallel \\
 A_0 & \xrightarrow{g_0} & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{d-1}} & B_d & \xrightarrow{g_d} & B_{d+1} & \xrightarrow{g_{d+1}} & \Sigma^d A_0 \\
 \downarrow f_0 & & \parallel & & \downarrow \psi_2 & & \downarrow \psi_3 & & & & \downarrow \psi_d & & \downarrow \psi_{d+1} & & \downarrow \Sigma^d f_0 \\
 A_1 & \xrightarrow{\varphi_1} & B_1 & \xrightarrow{h_1} & C_2 & \xrightarrow{h_2} & C_3 & \xrightarrow{h_3} & \cdots & \xrightarrow{h_{d-1}} & C_d & \xrightarrow{h_d} & C_{d+1} & \xrightarrow{h_{d+1}} & \Sigma^d A_1
 \end{array}$$

with commuting squares and with rows in  $\Theta$ , the dotted morphisms exist such that each square commutes, and the  $(d+2)$ - $\Sigma^d$ -sequence

$$\begin{aligned}
 A_2 & \xrightarrow{\begin{pmatrix} f_2 \\ \varphi_2 \end{pmatrix}} A_3 \oplus B_2 \xrightarrow{\begin{pmatrix} -f_3 & 0 \\ \varphi_3 & -g_2 \end{pmatrix}} A_4 \oplus B_3 \oplus C_2 \xrightarrow{\begin{pmatrix} -f_4 & 0 & 0 \\ -\varphi_4 & -g_3 & 0 \\ \phi_4 & \psi_3 & h_2 \end{pmatrix}} A_5 \oplus B_4 \oplus C_3 \\
 & \xrightarrow{\begin{pmatrix} -f_5 & 0 & 0 \\ \varphi_5 & -g_4 & 0 \\ \phi_5 & \psi_4 & h_3 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} -f_d & 0 & 0 \\ (-1)^{d+1} \varphi_d & -g_{d-1} & 0 \\ \phi_d & \psi_{d-1} & h_{d-2} \end{pmatrix}} A_{d+1} \oplus B_d \oplus C_{d-1} \\
 & \xrightarrow{\begin{pmatrix} (-1)^{d+1} \varphi_{d+1} & -g_d & 0 \\ \phi_{d+1} & \psi_d & h_{d-1} \end{pmatrix}} B_{d+1} \oplus C_d \xrightarrow{(\psi_{d+1}, h_d)} C_{d+1} \xrightarrow{\Sigma^d f_1 \circ h_{d+1}} \Sigma^d A_2
 \end{aligned}$$

belongs to  $\Theta$ .

**Theorem 2.2.** [BT1, Theorem 4.4] *If  $\Theta$  is a collection of  $(d+2)$ - $\Sigma^d$ -sequences satisfying axioms (N1), (N2) and (N3), then the following are equivalent:*

- (1)  $\Theta$  satisfies (N4);
- (2)  $\Theta$  satisfies (N4\*).

The following two lemmas are very useful which are needed later on.

**Lemma 2.3.** [Fe2, Lemma 3.13] *Let  $\mathcal{C}$  be a  $(d+2)$ -angulated category, and*

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0. \quad (2.1)$$

*a  $(d+2)$ -angle in  $\mathcal{C}$ . Then the following are equivalent:*

- (1)  $\alpha_0$  is a section (also known as a split monomorphism);
- (2)  $\alpha_d$  is a retraction (also known as a split epimorphism);
- (3)  $\alpha_{d+1} = 0$ .

*If a  $(d+2)$ -angle (2.1) satisfies one of the above equivalent conditions, it is called split.*

**Lemma 2.4.** [BT1, Lemma 4.1] *Suppose that  $\mathcal{C}$  is a  $(d+2)$ -angulated category, and let*

$$\begin{array}{ccccccccccc} A_0 & \xrightarrow{\alpha_1} & A_1 & \xrightarrow{\alpha_2} & A_2 & \xrightarrow{\alpha_3} & \cdots & \longrightarrow & A_{d-1} & \xrightarrow{\alpha_{d-1}} & A_d & \xrightarrow{\alpha_d} & A_{d+1} & \xrightarrow{\alpha_{d+1}} & \Sigma^d A_0 \\ & & & & & & & & & & \downarrow \varphi_d & & \parallel & & \\ B_0 & \xrightarrow{\beta_0} & B_1 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & \cdots & \longrightarrow & B_{d-1} & \xrightarrow{\beta_{d-1}} & B_d & \xrightarrow{\beta_d} & A_{d+1} & \xrightarrow{\beta_{d+1}} & \Sigma^d B_0 \end{array}$$

*be a commutative diagram whose rows are  $(d+2)$ -angles. Then it can be completed the diagram to a morphism*

$$\begin{array}{ccccccccccc} A_0 & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots & \longrightarrow & A_{d-1} & \xrightarrow{\alpha_{d-1}} & A_d & \xrightarrow{\alpha_d} & A_{d+1} & \xrightarrow{\alpha_{d+1}} & \Sigma^d A_0 \\ \downarrow \varphi_0 & & \downarrow \varphi_1 & & \downarrow \varphi_2 & & & & \downarrow \varphi_{d-1} & & \downarrow \varphi_d & & \parallel & & \downarrow \Sigma^d \varphi_0 \\ B_0 & \xrightarrow{\beta_0} & B_1 & \xrightarrow{\beta_1} & B_3 & \xrightarrow{\beta_2} & \cdots & \longrightarrow & B_{d-1} & \xrightarrow{\beta_{d-1}} & B_d & \xrightarrow{\beta_d} & A_{d+1} & \xrightarrow{\beta_{d+1}} & \Sigma^d B_0 \end{array}$$

*of  $(d+2)$ -angles such that*

$$\begin{array}{c} A_0 \xrightarrow{\begin{pmatrix} -\alpha_0 \\ \varphi_0 \end{pmatrix}} A_1 \oplus B_0 \xrightarrow{\begin{pmatrix} \alpha_1 & 0 \\ \varphi_1 & \beta_0 \end{pmatrix}} A_2 \oplus B_1 \xrightarrow{\begin{pmatrix} -\alpha_2 & 0 \\ \varphi_2 & \beta_1 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} (-1)^d \alpha_{d-1} & 0 \\ \varphi_{d-1} & \beta_{d-2} \end{pmatrix}} A_d \oplus B_{d-1} \\ \xrightarrow{\begin{pmatrix} (-1)^{d+1} \varphi_d, \beta_{d-1} \end{pmatrix}} B_d \xrightarrow{\alpha_{d+1} \beta_d} \Sigma^d A_0 \end{array} \text{ is a } (d+2)\text{-angle in } \mathcal{C}.$$

Now we recall an Auslander-Reiten  $(d+2)$ -angle in  $(d+2)$ -angulated categories.

We denote by  $\text{rad}_{\mathcal{C}}$  the Jacobson radical of  $\mathcal{C}$ . Namely,  $\text{rad}_{\mathcal{C}}$  is an ideal of  $\mathcal{C}$  such that  $\text{rad}_{\mathcal{C}}(A, A)$  coincides with the Jacobson radical of the endomorphism ring  $\text{End}(A)$  for any  $A \in \mathcal{C}$ .

**Definition 2.5.** [IY, Definition 3.8] and [Fe2, Definition 5.1] Let  $\mathcal{C}$  be a  $(d+2)$ -angulated category. A  $(d+2)$ -angle

$$A_{\bullet}: A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0$$

in  $\mathcal{C}$  is called an *Auslander-Reiten  $(d+2)$ -angle* if  $\alpha_0$  is left almost split,  $\alpha_d$  is right almost split and when  $d \geq 2$ , also  $\alpha_1, \alpha_2, \dots, \alpha_{d-1}$  are in  $\text{rad}_{\mathcal{C}}$ .

**Remark 2.6.** [Fe2, Remark 5.2] Assume  $A_\bullet$  as in the above definition is an Auslander-Reiten  $(d+2)$ -angle. Since  $\alpha_0$  is left almost split,  $\text{End}(A_0)$  is local and hence  $A_0$  is indecomposable. Similarly, since  $\alpha_d$  is right almost split,  $\text{End}(A_{d+1})$  is local and hence  $A_{d+1}$  is indecomposable. Moreover, when  $d = 1$ , we have  $\alpha_0$  and  $\alpha_d$  in  $\text{rad}_{\mathcal{C}}$ , so that  $\alpha_d$  is right minimal and  $\alpha_0$  is left minimal. When  $d \geq 2$ , since  $\alpha_{d-1} \in \text{rad}_{\mathcal{C}}$ , we have that  $\alpha_d$  is right minimal and similarly  $\alpha_0$  is left minimal.

Let  $k$  be an algebraically closed field and  $\mathcal{C}$  be a  $k$ -linear Hom-finite additive category. Recall that the notion of a Serre functor. We call a  $k$ -linear autoequivalence  $\mathbb{S}: \mathcal{C} \rightarrow \mathcal{C}$  a *Serre functor* of  $\mathcal{C}$  if there exists a functorial isomorphism:

$$\text{Hom}_{\mathcal{C}}(X, Y) \simeq D\text{Hom}_{\mathcal{C}}(Y, \mathbb{S}X)$$

for any pair of objects  $X, Y \in \mathcal{C}$ , where  $D(-) = \text{Hom}_k(-, k)$  is the  $k$ -linear duality functor.

The author gave a relationship between the existences of Auslander-Reiten  $(d+2)$ -angles and a Serre functor in  $(d+2)$ -angulated categories.

**Theorem 2.7.** [Z1, Theorem 4.5] *Let  $\mathcal{C}$  be a  $(d+2)$ -angulated category. Then  $\mathcal{C}$  has Auslander-Reiten  $(d+2)$ -angles if and only if  $\mathcal{C}$  has a Serre functor.*

### 3 Grothendieck groups and Auslander-Reiten $(d+2)$ -angles

In this section, let  $k$  be an algebraically closed field. We always assume that  $\mathcal{C}$  is a  $k$ -linear Hom-finite Krull-Schmidt **connected**  $(d+2)$ -angulated category. We denote by  $\text{ind}(\mathcal{C})$  the set of isomorphism classes of indecomposable objects in  $\mathcal{C}$ . For any  $X \in \text{ind}(\mathcal{C})$ , we denote by  $\text{SuppHom}_{\mathcal{C}}(X, -)$  the subcategory of  $\mathcal{C}$  generated by objects  $Y$  in  $\text{ind}(\mathcal{C})$  with  $\text{Hom}_{\mathcal{C}}(X, Y) \neq 0$ . Similarly,  $\text{SuppHom}_{\mathcal{C}}(-, X)$  denotes the subcategory generated by objects  $Y$  in  $\text{ind}(\mathcal{C})$  with  $\text{Hom}_{\mathcal{C}}(Y, X) \neq 0$ . If  $\text{SuppHom}_{\mathcal{C}}(X, -)$  ( $\text{SuppHom}_{\mathcal{C}}(-, X)$ , respectively) contains only finitely many indecomposable objects, we say that  $|\text{SuppHom}_{\mathcal{C}}(X, -)| < \infty$  ( $|\text{SuppHom}_{\mathcal{C}}(-, X)| < \infty$  respectively).

**Definition 3.1.** [Z1, Definition 3.1] A  $(d+2)$ -angulated category  $\mathcal{C}$  is called *locally finite* if  $|\text{SuppHom}_{\mathcal{C}}(X, -)| < \infty$  and  $|\text{SuppHom}_{\mathcal{C}}(-, X)| < \infty$ , for any object  $X \in \text{ind}(\mathcal{C})$ .

**Theorem 3.2.** [Z1, Theorem 3.8] *Let  $\mathcal{C}$  be a locally finite  $(d+2)$ -angulated category. Then  $\mathcal{C}$  has Auslander-Reiten  $(d+2)$ -angles.*

The following result is essentially already in [L, Lemma 2.1].

**Lemma 3.3.** *Let  $(\mathcal{C}, \Sigma^d, \Theta)$  be a  $(d+2)$ -angulated category, and*

$$A_\bullet: A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{d-3}} A_{d-2} \xrightarrow{\alpha_{d-2}} A_{d-1} \xrightarrow{\begin{pmatrix} u \\ v \end{pmatrix}} L \oplus M \xrightarrow{(c, d)} N \xrightarrow{h} \Sigma^d A_0$$

*a  $(d+2)$ -angle in  $\mathcal{C}$ . If  $u = 0$ , then the  $(d+2)$ -angle  $A_\bullet$  is isomorphic to the following*

$(d+2)$ -angle:

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{d-3}} A_{d-2} \xrightarrow{\alpha_{d-2}} A_{d-1} \xrightarrow{\begin{pmatrix} 0 \\ v \end{pmatrix}} L \oplus M \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}} L \oplus Q \rightarrow \Sigma^d A_0.$$

Moreover, it is a direct sum of the following two  $(d+2)$ -angles

$$B_\bullet : A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{d-3}} A_{d-2} \xrightarrow{\alpha_{d-2}} A_{d-1} \xrightarrow{v} M \xrightarrow{t} Q \rightarrow \Sigma^d A_0,$$

$$M_\bullet : 0 \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow 0 \rightarrow L \xrightarrow{1_L} L \rightarrow 0.$$

*Proof.* Since  $u = 0$ , by (N3), we have the following commutative diagram

$$\begin{array}{ccccccccccccccc} A_0 & \xrightarrow{\alpha_1} & A_1 & \xrightarrow{\alpha_2} & A_2 & \xrightarrow{\alpha_3} & \cdots & \longrightarrow & A_{d-1} & \xrightarrow{\begin{pmatrix} u \\ v \end{pmatrix}} & L \oplus M & \xrightarrow{(c, d)} & N & \longrightarrow & \Sigma^d A_0 \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & L & \xrightarrow{1} & L & \longrightarrow & 0 \end{array}$$

It follows that  $c'(c, d) = (1, 0)$  and then  $c'c = 1$ . This shows that  $c$  is a section.

Now we can assume that  $A_\bullet$  is the following form

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{d-3}} A_{d-2} \xrightarrow{\alpha_{d-2}} A_{d-1} \xrightarrow{\begin{pmatrix} 0 \\ v \end{pmatrix}} L \oplus M \xrightarrow{\begin{pmatrix} s & w \\ 0 & t \end{pmatrix}} u(M) \oplus Q \xrightarrow{(x, y)} \Sigma^d A_0$$

where  $s$  is an isomorphism. Then  $(x, y) \begin{pmatrix} s & w \\ 0 & t \end{pmatrix} = 0$  implies  $x = 0$  since  $s$  is an isomorphism, and  $\begin{pmatrix} s & w \\ 0 & t \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = 0$  implies  $wv = 0$ . Thus  $(0, w) \begin{pmatrix} 0 \\ v \end{pmatrix} = wv = 0$ . So there exists a morphism  $(a, b) : u(M) \oplus Q \rightarrow u(M)$  such that  $(0, w) = (a, b) \begin{pmatrix} s & w \\ 0 & t \end{pmatrix}$ . In particular,  $as = 0$  and  $aw + bt = w$ . Since  $s$  is an isomorphism, we have  $a = 0$  implies  $w = bt$ . Hence, we have a commutative diagram

$$\begin{array}{ccccccc} A_{d-1} & \xrightarrow{\begin{pmatrix} 0 \\ v \end{pmatrix}} & L \oplus M & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}} & L \oplus Q & \xrightarrow{(0, y)} & \Sigma^d A_0 \\ \parallel & & \parallel & & \downarrow \begin{pmatrix} s & b \\ 0 & 1 \end{pmatrix} & & \parallel \\ A_{d-1} & \xrightarrow{\begin{pmatrix} 0 \\ v \end{pmatrix}} & L \oplus M & \xrightarrow{\begin{pmatrix} s & w \\ 0 & t \end{pmatrix}} & L \oplus Q & \xrightarrow{(0, y)} & \Sigma^d A_0 \end{array}$$

implies that  $A_\bullet \simeq B_\bullet \oplus M_\bullet$  since  $\begin{pmatrix} s & b \\ 0 & 1 \end{pmatrix}$  is an isomorphism. Since  $\Theta$  is closed under direct summands and  $A_\bullet \in \Theta$  we have  $B_\bullet \in \Theta$ , that is, it is a  $(d+2)$ -angle.  $\square$

**Lemma 3.4.** *Let  $\mathcal{C}$  be a locally finite  $(d+2)$ -angulated category. Then for any object  $X$  in  $\mathcal{C}$ , there exists a natural number  $n$  ( $m$ , respectively) such that  $\text{rad}_{\mathcal{C}}^n(-, X) = 0$  ( $\text{rad}_{\mathcal{C}}^m(X, -) = 0$ , respectively).*

*Proof.* This result was proved in [XZ, Lemma 1.2] for the case that  $\mathcal{C}$  is a triangulated category. But their proof can be applied to the context of a  $(d+2)$ -angulated category without any change.  $\square$

Suppose that  $\mathcal{C}$  is an essentially small  $(d+2)$ -angulated category, and let  $F(\mathcal{C})$  be the free



abelian group on the set of isomorphism classes  $\langle A \rangle$  of objects  $A$  in  $\mathcal{C}$ . Given a  $(d+2)$ -angle

$$A_\bullet : A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_d \rightarrow A_{d+1} \rightarrow \Sigma^d A_0$$

in  $\mathcal{C}$ , we denote the corresponding Euler relation in  $F(\mathcal{C})$  by  $\chi(A_\bullet)$ , that is,

$$\chi(A_\bullet) := [A_0] - [A_1] + [A_2] + \cdots + (-1)^{d+1}[A_{d+1}].$$

**Definition 3.5.** [BT2, Definition 2.1] and [Fe1, Definition 2.2] Let  $\mathcal{C}$  be an essentially small  $(d+2)$ -angulated category, and  $F(\mathcal{C})$  the free abelian group on the set of isomorphism classes  $[A]$  of objects  $A$  in  $\mathcal{C}$ . Moreover, let  $R(\mathcal{C})$  be the subgroup of  $F(\mathcal{C})$  generated by the following sets of elements

$$\{\chi(A_\bullet) \mid A_\bullet \text{ is a } (d+2)\text{-angle in } \mathcal{C}\}$$

in  $\mathcal{C}$ . The *Grothendieck group*  $K_0(\mathcal{C})$  of  $\mathcal{C}$  is the quotient group  $F(\mathcal{C})/R(\mathcal{C})$ . Given an object  $A \in \mathcal{C}$ , the residue class  $\langle A \rangle + R(\mathcal{C})$  in  $K_0(\mathcal{C})$  is denoted by  $[A]$ .

**Definition 3.6.** Let  $\mathcal{C}$  be an essentially small  $(d+2)$ -angulated category, and  $F(\mathcal{C})$  the free abelian group on the set of isomorphism classes  $\langle A \rangle$  of objects  $A$  in  $\mathcal{C}$ . Moreover, let  $R'(\mathcal{C})$  be the subgroup of  $F(\mathcal{C})$  generated by the following sets of elements

$$\{\chi(A_\bullet) \mid A_\bullet \text{ is a **split** } (d+2)\text{-angle in } \mathcal{C}\}$$

in  $\mathcal{C}$ . The *split Grothendieck group*  $K_0(\mathcal{C}, 0)$  of  $\mathcal{C}$  is the quotient group  $F(\mathcal{C})/R'(\mathcal{C})$ . Given an object  $A \in \mathcal{C}$ , the residue class  $\langle A \rangle + R'(\mathcal{C})$  in  $K_0(\mathcal{C}, 0)$  is denoted by  $[A]$ . For simplicity, sometimes we also denote by  $[A_\bullet]$  the element in  $K_0(\mathcal{C}, 0)$ .

Note that the definition of the Grothendieck group is the reason why we are only considering essentially small categories: the collection of isomorphism classes in the category must form a set. In addition, there exists a canonical epimorphism  $\phi : K_0(\mathcal{C}, 0) \rightarrow K_0(\mathcal{C})$ .

**Remark 3.7.** [BT2, Proposition 2.2] Let  $\mathcal{C}$  be an essentially small  $(d+2)$ -angulated category, and  $K_0(\mathcal{C}, 0)$  its split Grothendieck group.

- (1) The element  $[0]$  is the zero element in  $K_0(\mathcal{C}, 0)$ .
- (2) If  $A$  and  $B$  are objects in  $\mathcal{C}$ , then  $[A \oplus B] = [A] + [B]$  in  $K_0(\mathcal{C}, 0)$ .

Now we are ready to state and prove our first main result.

**Theorem 3.8.** *Let  $\mathcal{C}$  be a locally finite  $(d+2)$ -angulated category. Then  $\text{Ker } \phi$  is generated by the elements  $[A_\bullet]$  in  $K_0(\mathcal{C}, 0)$ , where*

$$A_\bullet : A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0.$$

*runs through all Auslander-Reiten  $(d+2)$ -angles in  $\mathcal{C}$ .*

*Proof.* Let  $A_\bullet : A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0$  be an arbitrary non-split  $(d+2)$ -angle. It suffices to prove that the element  $[A_\bullet]$  in  $K_0(\mathcal{C}, 0)$  can be written as

a sum of the elements in  $K_0(\mathcal{C}, 0)$  corresponding to some Auslander-Reiten  $(d+2)$ -angles. We can assume that  $A_{d+1} \in \text{ind}(\mathcal{C})$ . If it is not the case, without loss of generality, we assume that  $A_{d+1} = U \oplus V$  with  $U, V \in \text{ind}(\mathcal{C})$ . By repeatedly applying (N2), we obtain the  $(d+2)$ -angle

$$A_d \xrightarrow{\alpha_d = \begin{pmatrix} p \\ q \end{pmatrix}} A_{d+1} = U \oplus V \rightarrow \Sigma^d A_0 \rightarrow \Sigma^d A_1 \rightarrow \cdots \rightarrow \Sigma^d A_{d-1} \rightarrow \Sigma^d A_d.$$

Now use (N1)(c) to complete the morphism  $q: A_d \rightarrow V$  to a  $(d+2)$ -angle

$$A_d \xrightarrow{q} V \rightarrow \Sigma^d V_0 \rightarrow \Sigma^d V_1 \rightarrow \cdots \rightarrow \Sigma^d V_{d-1} \rightarrow \Sigma^d A_d.$$

By repeated use of (N2), we also obtain the  $(d+2)$ -angle

$$X_\bullet: V_0 \rightarrow V_1 \rightarrow V_2 \rightarrow \cdots \rightarrow V_{d-1} \rightarrow A_d \xrightarrow{q} V \rightarrow \Sigma^d V_0.$$

By use of (N1)(a) and (N1)(b), we obtain the  $(d+2)$ -angle

$$U \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} U \oplus V \xrightarrow{(0, 1)} V \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow \Sigma^d U.$$

By use of (N2), we also obtain the  $(d+2)$ -angle

$$U \oplus V \xrightarrow{(0, 1)} V \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow \Sigma^d U \rightarrow \Sigma^d U \oplus \Sigma^d V.$$

Thus we have the following solid part of commutative diagram

$$\begin{array}{ccccccccccccccc} A_d & \xrightarrow{\begin{pmatrix} p \\ q \end{pmatrix}} & U \oplus V & \longrightarrow & \Sigma^d A_0 & \longrightarrow & \Sigma^d A_1 & \longrightarrow & \cdots & \longrightarrow & \Sigma^d A_{d-2} & \longrightarrow & \Sigma^d A_{d-1} & \longrightarrow & \Sigma^d A_d \\ \parallel & & \downarrow (0, 1) & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \parallel \\ A_d & \xrightarrow{q} & V & \longrightarrow & \Sigma^d V_0 & \longrightarrow & \Sigma^d V_1 & \longrightarrow & \cdots & \longrightarrow & \Sigma^d V_{d-2} & \longrightarrow & \Sigma^d V_{d-1} & \longrightarrow & \Sigma^d A_d \\ \downarrow \begin{pmatrix} p \\ q \end{pmatrix} & & \parallel & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \Sigma^d \begin{pmatrix} p \\ q \end{pmatrix} \\ U \oplus V & \xrightarrow{(0, 1)} & V & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & \Sigma^d U & \longrightarrow & \Sigma^d U \oplus \Sigma^d V \end{array}$$

whose rows are  $(d+2)$ -angles. By Theorem 2.2, we have that

$$\Sigma^d A_0 \rightarrow \Sigma^d A_1 \oplus \Sigma^d V_0 \rightarrow \Sigma^d A_2 \oplus \Sigma^d V_1 \cdots \rightarrow \Sigma^d A_{d-1} \oplus \Sigma^d V_{d-2} \rightarrow \Sigma^d V_{d-1} \rightarrow \Sigma^d U \rightarrow \Sigma^{2d} A_0$$

is a  $(d+2)$ -angle. By use of (N2), we obtain the  $(d+2)$ -angle

$$Y_\bullet: A_0 \rightarrow A_1 \oplus V_0 \rightarrow A_2 \oplus V_1 \cdots \rightarrow A_{d-1} \oplus V_{d-2} \rightarrow V_{d-1} \rightarrow U \rightarrow \Sigma^d A_0.$$

Hence  $[A_\bullet] = [X_\bullet] + [Y_\bullet]$ .

Suppose  $\alpha_{d+1} \in \text{rad}_{\mathcal{C}}^n(A_{d+1}, \Sigma^d A_0)$ , and

$$B_\bullet: B_0 \xrightarrow{\beta_0} B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{d-1}} B_d \xrightarrow{\beta_d} A_{d+1} \xrightarrow{\beta_{d+1}} \Sigma^d B_0$$

is an Auslander-Reiten  $(d+2)$ -angle ending at  $A_{d+1}$ . Since  $\alpha_{d+1} \neq 0$ , by Lemma 2.3, we know that  $\alpha_d$  is not a retraction. Then there exists a morphism  $\varphi_d: A_d \rightarrow B_d$  such that  $\alpha_d = \beta_d \varphi_d$ .

By Lemma 2.4, there exists a morphism

$$\begin{array}{ccccccccccccccc} A_0 & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots & \longrightarrow & A_{d-1} & \xrightarrow{\alpha_{d-1}} & A_d & \xrightarrow{\alpha_d} & A_{d+1} & \xrightarrow{\alpha_{d+1}} & \Sigma^d A_0 \\ | & & | & & | & & & & | & & | & & \parallel & & | \\ \varphi_0 & & \varphi_1 & & \varphi_2 & & & & \varphi_{d-1} & & \varphi_d & & & & \Sigma^d \varphi_0 \\ \Downarrow & & \Downarrow & & \Downarrow & & & & \Downarrow & & \Downarrow & & & & \Downarrow \\ B_0 & \xrightarrow{\beta_0} & B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & \cdots & \longrightarrow & B_{d-1} & \xrightarrow{\beta_{d-1}} & B_d & \xrightarrow{\beta_d} & A_{d+1} & \xrightarrow{\beta_{d+1}} & \Sigma^d B_0 \end{array}$$

of  $(d+2)$ -angles, moreover,

$$\begin{aligned} C_\bullet : A_0 &\xrightarrow{\begin{pmatrix} -\alpha_1 \\ \varphi_0 \end{pmatrix}} A_1 \oplus B_0 \xrightarrow{\begin{pmatrix} \alpha_1 & 0 \\ \varphi_1 & \beta_0 \end{pmatrix}} A_2 \oplus B_1 \xrightarrow{\begin{pmatrix} -\alpha_2 & 0 \\ \varphi_2 & \beta_1 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} (-1)^d \alpha_{d-1} & 0 \\ \varphi_{d-1} & \beta_{d-2} \end{pmatrix}} A_d \oplus B_{d-1} \\ &\xrightarrow{\begin{pmatrix} (-1)^{d+1} \varphi_d & \beta_{d-1} \end{pmatrix}} B_d \xrightarrow{\alpha_{d+1} \beta_d} \Sigma^d A_0 \end{aligned}$$

is a  $(d+2)$ -angle in  $\mathcal{C}$  with  $\alpha_{d+1} \beta_d \in \text{rad}_{\mathcal{C}}^{n+1}(B_d, \Sigma^d A_0)$ , and that  $[A_\bullet] = [B_\bullet] + [C_\bullet]$ .

Put  $C_i := A_i \oplus B_{i-1}$ ,  $i = 1, 2, \dots, d$ . At this time  $C_\bullet$  is of the form

$$A_0 \rightarrow C_1 \rightarrow C_2 \rightarrow \cdots \rightarrow C_d \rightarrow B_d \xrightarrow{\alpha_{d+1} \beta_d} \Sigma^d A_0.$$

We decompose  $B_d$  as a direct sum of indecomposable objects:  $B_d = M_1 \oplus M_2 \oplus \cdots \oplus M_k$ . Without loss of generality, we can assume  $k = 2$ . Then the  $(d+2)$ -angle  $C_\bullet$  can be written as

$$A_0 \rightarrow C_1 \rightarrow C_2 \rightarrow \cdots \rightarrow C_{d-1} \rightarrow C_d \xrightarrow{\begin{pmatrix} c \\ e \end{pmatrix}} M_1 \oplus M_2 \xrightarrow{(x, y)} \Sigma^d A_0$$

with  $(x, y) = \alpha_{d+1} \beta_d \in \text{rad}_{\mathcal{C}}^{n+1}(B_d, \Sigma^d A_0)$ . If  $c$  is a retraction, we can assume that

$$C_\bullet : A_0 \rightarrow C_1 \rightarrow C_2 \rightarrow \cdots \rightarrow C_{d-1} \xrightarrow{\begin{pmatrix} u \\ v \end{pmatrix}} M_1 \oplus M' \xrightarrow{\begin{pmatrix} 1 & 0 \\ * & * \end{pmatrix}} M_1 \oplus M_2 \xrightarrow{(x, y)} \Sigma^d A_0$$

It follows that  $\begin{pmatrix} 1 & 0 \\ * & * \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0$  and then  $u = 0$ . By Lemma 3.3, the  $(d+2)$ -angle  $C_\bullet$  is isomorphic to the  $(d+2)$ -angle

$$D_\bullet : A_0 \rightarrow C_1 \rightarrow C_2 \rightarrow \cdots \rightarrow C_{d-1} \xrightarrow{\begin{pmatrix} 0 \\ v \end{pmatrix}} M_1 \oplus M' \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}} M_1 \oplus Q \xrightarrow{(0, z)} \Sigma^d A_0,$$

and  $D_\bullet$  is a direct sum of the following  $(d+2)$ -angles:

$$\begin{aligned} N_\bullet : A_0 &\rightarrow C_1 \rightarrow \cdots \rightarrow C_{d-2} \rightarrow C_{d-1} \xrightarrow{v} M' \xrightarrow{t} Q \xrightarrow{z} \Sigma^d A_0, \\ 0 &\rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow 0 \rightarrow M_1 \xrightarrow{1_{M_1}} M_1 \rightarrow 0. \end{aligned}$$

Hence  $z \in \text{rad}_{\mathcal{C}}^{n+1}(Q, \Sigma^d A_0)$  since  $(0, z) \in \text{rad}_{\mathcal{C}}^{n+1}(Q, \Sigma^d A_0)$  and  $[C_\bullet] = [N_\bullet]$ . Thus we can continue this process with  $[N_\bullet]$  instead of  $[C_\bullet]$ . By Lemma 3.4, we know that this process must stop at a finite steps, that is, up to some finite step, we can get a splitting  $(d+2)$ -angle. Then there are finitely many Auslander-Reiten  $(d+2)$ -angles  $Q_\bullet^1, Q_\bullet^2, \dots, Q_\bullet^q$ , such that  $[C_\bullet] = [N_\bullet] = [Q_\bullet^1] + [Q_\bullet^2] + \cdots + [Q_\bullet^q]$ . Hence we have proved the assertion in this case.

If  $e$  is a retraction, the proof of this result is similar to the case of  $c$  is a retraction.

Now we return to the  $(d+2)$ -angle  $C_\bullet$  and assume that  $c$  and  $d$  are not retraction in the

following. Assume that the Auslander-Reiten  $(d+2)$ -angle endings at  $M_1, M_2$  are respectively  $U_\bullet, V_\bullet$ :

$$\begin{aligned} U_\bullet : U_0 &\xrightarrow{u_0} U_1 \xrightarrow{u_1} U_2 \xrightarrow{u_2} \cdots \xrightarrow{u_{d-1}} U_d \xrightarrow{u_d} M_1 \xrightarrow{u_{d+1}} \Sigma^d U_0, \\ V_\bullet : V_0 &\xrightarrow{v_1} V_1 \xrightarrow{v_2} V_2 \xrightarrow{v_3} \cdots \xrightarrow{v_{d-1}} V_d \xrightarrow{v_d} M_2 \xrightarrow{v_{d+1}} \Sigma^d V_0. \end{aligned}$$

We form the direct sum of them:  $U_\bullet \oplus V_\bullet$ .

$$U_0 \oplus V_0 \xrightarrow{\delta_0} U_1 \oplus V_1 \xrightarrow{\delta_1} U_2 \oplus V_2 \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_{d-1}} U_d \oplus V_d \xrightarrow{\delta_d} M_1 \oplus M_2 \xrightarrow{\delta_{d+1}} \Sigma^d U_0 \oplus \Sigma^d V_0,$$

where  $\delta_i = \begin{pmatrix} u_i & 0 \\ 0 & v_i \end{pmatrix}$ ,  $i = 1, 2, \dots, d+1$ . Since  $c$  is not retraction and  $U_\bullet$  is an Auslander-Reiten  $(d+2)$ -angle, there exists a morphism  $w_d: C_d \rightarrow U_d$  such that  $c = u_d w_d$ . Since  $e$  is not retraction and  $V_\bullet$  is an Auslander-Reiten  $(d+2)$ -angle, there exists a morphism  $w'_d: C_d \rightarrow U_d$  such that  $e = v_d w'_d$ . It follows that  $\begin{pmatrix} u_d & 0 \\ 0 & v_d \end{pmatrix} \begin{pmatrix} w_d \\ w'_d \end{pmatrix} = \begin{pmatrix} c \\ e \end{pmatrix}$ . By Lemma 2.4, we have the following commutative diagram

$$\begin{array}{ccccccccccc} A_0 & \longrightarrow & C_1 & \longrightarrow & \cdots & \longrightarrow & C_{d-1} & \longrightarrow & C_d & \xrightarrow{\begin{pmatrix} c \\ e \end{pmatrix}} & M_1 \oplus M_2 & \xrightarrow{(x, y)} & \Sigma^d A_0 \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \begin{pmatrix} u_d \\ u'_d \end{pmatrix} & & \parallel & & \downarrow \\ U_0 \oplus V_0 & \xrightarrow{\delta_0} & U_1 \oplus V_1 & \xrightarrow{\delta_1} & \cdots & \longrightarrow & U_{d-1} \oplus V_{d-1} & \xrightarrow{\delta_{d-1}} & U_d \oplus V_d & \xrightarrow{\delta_d} & M_1 \oplus M_2 & \xrightarrow{\delta_{d+1}} & \Sigma^d U_0 \oplus \Sigma^d V_0 \end{array}$$

of  $(d+2)$ -angles such that

$$L_\bullet : A_0 \rightarrow C_1 \oplus U_0 \oplus V_0 \rightarrow \cdots \rightarrow C_d \oplus U_{d-1} \oplus V_{d-1} \rightarrow U_d \oplus V_d \xrightarrow{(x, y)\delta_d} \Sigma^d A_0$$

is a  $(d+2)$ -angle. Since  $(x, y) \in \text{rad}_C^{n+1}(B_d, \Sigma^d A_0)$ , we have  $(x, y)\delta_d \in \text{rad}_C^{n+2}(U_d \oplus V_d, \Sigma^d A_0)$ . Thus we can continue this process with  $[L]_\bullet$  instead of  $[C]_\bullet$ . By Lemma 3.4, we know that this process must stop at a finite steps, that is, up to some finite step, we can get a splitting  $(d+2)$ -angle. Then there are finitely many Auslander-Reiten  $(d+2)$ -angles  $P_\bullet^1, P_\bullet^2, \dots, P_\bullet^p$ , such that  $[C]_\bullet = [P_\bullet^1] + [P_\bullet^2] + \cdots + [P_\bullet^p]$ . Hence we have proved the assertion in this case.

This completes the proof.  $\square$

**Remark 3.9.** As a special case of Theorem 3.8 when  $d = 1$ , it is just the Theorem 2.1 of Xiao and Zhu in [XZ].

**Example 3.10.** Let  $\mathcal{C}$  be a triangulated category with a  $d$ -cluster tilting subcategory  $\mathcal{X}$  which is closed under the  $d$ -th power of the shift functor. By [GKO, Theorem 1], we know that  $\mathcal{X}$  is a  $(d+2)$ -angulated category. If  $\mathcal{X}$  is locally finite, by Theorem 3.8, then the relations of its Grothendieck group are generated by all Auslander-Reiten  $(d+2)$ -angulated category, see also [Fe1, Remark 5.5].

Now we will show that the converse of Theorem 3.8 is also true when  $d$  is an odd. For convenience, we will use the notation  $[A, B] := \dim_k \text{Hom}_{\mathcal{C}}(A, B)$ .

**Lemma 3.11.** Let  $A_\bullet : A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0$  be an Auslander-Reiten  $(d+2)$ -angle in  $\mathcal{C}$  and  $U$  an object in  $\mathcal{C}$ . Then the following statements hold:

- (1) The morphism  $\text{Hom}_{\mathcal{C}}(U, \alpha_d): \text{Hom}_{\mathcal{C}}(U, A_d) \rightarrow \text{Hom}_{\mathcal{C}}(U, A_{d+1})$  is an epimorphism if and only if  $A_{d+1}$  is not a direct summand in  $U$ .
- (2) If  $U \in \text{ind}(\mathcal{C})$ , then the morphism  $\text{Hom}_{\mathcal{C}}(U, \alpha_0): \text{Hom}_{\mathcal{C}}(U, A_0) \rightarrow \text{Hom}_{\mathcal{C}}(U, A_1)$  is a monomorphism if and only if  $U \not\simeq \Sigma^{-d}A_{d+1}$ .
- (3) If  $U \in \text{ind}(\mathcal{C})$ , we have  $[U, A_0] - [U, A_1] + [U, A_2] + \cdots + (-1)^{d+1}[U, A_{d+1}] \neq 0$  if and only if  $U \simeq A_{d+1}$  or  $U \simeq \Sigma^{-d}A_{d+1}$ .

*Proof.* (1) We know that  $A_{d+1}$  is a direct summand in  $U$  if and only if there exists a retraction  $U \rightarrow A_{d+1}$ . By the definition of an Auslander-Reiten  $(d+2)$ -angle, this is equivalent to  $\text{Hom}_{\mathcal{C}}(U, \alpha_d): \text{Hom}_{\mathcal{C}}(U, A_d) \rightarrow \text{Hom}_{\mathcal{C}}(U, A_{d+1})$  not being epimorphism, which proves (1).

(2) Applying the functor  $\text{Hom}_{\mathcal{C}}(U, -)$  to the  $(d+2)$ -angle  $A_{\bullet}$ , we have the following exact sequence:

$$\cdots \rightarrow \text{Hom}_{\mathcal{C}}(U, \Sigma^{-d}A_d) \xrightarrow{\beta} \text{Hom}_{\mathcal{C}}(U, \Sigma^{-d}A_{d+1}) \rightarrow \text{Hom}_{\mathcal{C}}(U, A_0) \xrightarrow{\gamma} \text{Hom}_{\mathcal{C}}(U, A_1) \rightarrow \cdots$$

where  $\beta = \text{Hom}_{\mathcal{C}}(U, \Sigma^{-d}\alpha_{d+1})$  and  $\gamma = \text{Hom}_{\mathcal{C}}(U, \alpha_0)$ . It follows that the morphism  $\text{Hom}_{\mathcal{C}}(U, \alpha_0)$  is monomorphism if and only if the morphism  $\text{Hom}_{\mathcal{C}}(U, \Sigma^{-d}\alpha_{d+1})$  is epimorphism. By applying (1) to the object  $\Sigma^{-d}U$ , we obtain that  $\text{Hom}_{\mathcal{C}}(U, \Sigma^{-d}\alpha_{d+1})$  is an epimorphism if and only if  $A_{d+1}$  is not a direct summand in  $\Sigma^d U$ . Since  $U \in \text{ind}\mathcal{C}$ , we have that  $\Sigma^d U$  is also indecomposable, this is equivalent to  $U \not\simeq \Sigma^{-d}A_{d+1}$ . Thus (2) holds.

(3) Applying the functor  $\text{Hom}_{\mathcal{C}}(U, -)$  to the  $(d+2)$ -angle  $A_{\bullet}$ , we have the following exact sequence:

$$\cdots \rightarrow \text{Hom}_{\mathcal{C}}(U, A_0) \xrightarrow{\mu} \text{Hom}_{\mathcal{C}}(U, A_1) \rightarrow \cdots \rightarrow \text{Hom}_{\mathcal{C}}(U, A_{d-1}) \xrightarrow{\nu} \text{Hom}_{\mathcal{C}}(U, A_{d+1}) \rightarrow \cdots$$

where  $u = \text{Hom}_{\mathcal{C}}(U, \alpha_0)$  and  $\nu = \text{Hom}_{\mathcal{C}}(U, \alpha_{d+1})$ . We also get the following exact sequence:

$$0 \rightarrow K \rightarrow \text{Hom}_{\mathcal{C}}(U, A_0) \xrightarrow{\mu} \text{Hom}_{\mathcal{C}}(U, A_1) \rightarrow \cdots \rightarrow \text{Hom}_{\mathcal{C}}(U, A_{d-1}) \xrightarrow{\nu} \text{Hom}_{\mathcal{C}}(U, A_{d+1}) \rightarrow M \rightarrow 0$$

where  $K = \text{Ker}\mu$  and  $M = \text{Coker}\nu$ . Splitting into short exact sequences and using our finiteness assumption, we get that the equation

$$[U, A_0] - [U, A_1] + [U, A_2] + \cdots + (-1)^{d+1}[U, A_{d+1}] = \dim_k K + \dim_k M$$

where  $d$  is odd and

$$[U, A_0] - [U, A_1] + [U, A_2] + \cdots + (-1)^{d+1}[U, A_{d+1}] = \dim_k K - \dim_k M$$

where  $d$  is even. Hence  $[U, A_0] - [U, A_1] + [U, A_2] + \cdots + (-1)^{d+1}[U, A_{d+1}] \neq 0$  if and only if the righthand sides of the two equations are also non-zero. This means that either  $K$  or  $M$  (or both) must be non-zero. The object  $K$  is non-zero if and only if  $\text{Hom}_{\mathcal{C}}(U, \alpha_0)$  is not monomorphism. By (2), we know that  $\text{Hom}_{\mathcal{C}}(U, \alpha_0)$  is not monomorphism if and only if  $U \simeq \Sigma^{-d}A_{d+1}$ . Similarly,  $M$  is non-zero if and only if  $\text{Hom}_{\mathcal{C}}(U, \alpha_{d+1})$  is not epimorphism. By

(1), we know that  $\text{Hom}_{\mathcal{C}}(U, \alpha_{d+1})$  is not epimorphism if and only if  $A_{d+1}$  is a direct summand in  $U$ . Since  $U$  is indecomposable, we have  $U \simeq A_{d+1}$ . This completes the proof.  $\square$

**Lemma 3.12.** *Assume that  $b_1[C_1] + b_2[C_2] + \cdots + b_m[C_m] = 0$  in  $K_0(\mathcal{C}, 0)$  for integers  $b_i$  and objects  $C_1, C_2, \dots, C_m$  in  $\mathcal{C}$ . Then we have  $b_1[X, C_1] + b_2[X, C_2] + \cdots + b_m[X, C_m] = 0$  in  $\mathbb{Z}$  for any object  $X$  in  $\mathcal{C}$ .*

*Proof.* Suppose  $b_1[C_1] + b_2[C_2] + \cdots + b_m[C_m] = 0$  in  $K_0(\mathcal{C}, 0)$ . By moving negative terms to the right-hand side of the equality, without loss of generality, we can assume  $b_i \geq 0$  for each  $i = 1, 2, \dots, m$ . Using the defining Euler relations for  $K_0(\mathcal{C}, 0)$ , we obtain

$$b_1[C_1] + b_2[C_2] + \cdots + b_m[C_m] = [b_1C_1 \oplus b_2C_2 \oplus \cdots \oplus b_mC_m] = 0$$

where  $b_iC_i$  is the coproduct of the object  $C_i$  with itself  $b_i$  times. Thus the object  $b_1C_1 \oplus b_2C_2 \oplus \cdots \oplus b_mC_m$  is zero in  $\mathcal{C}$ . Applying  $[X, -]$  and using additivity, we get that the equation  $b_1[X, C_1] + b_2[X, C_2] + \cdots + b_m[X, C_m] = 0$ .  $\square$

Our second main result is the following.

**Theorem 3.13.** *Let  $d$  be an odd. Suppose that  $\text{Ker}\phi$  is generated by the elements  $[A_\bullet]$  in  $K_0(\mathcal{C}, 0)$ , where*

$$A_\bullet : A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0.$$

*runs through all Auslander-Reiten  $(d+2)$ -angles in  $\mathcal{C}$ . Then  $\mathcal{C}$  is locally finite.*

*Proof.* For any  $X \in \text{ind}(\mathcal{C})$ , we suppose  $U \in \text{Supp Hom}_{\mathcal{C}}(-, X)$ . Note that the non-split  $(d+2)$ -angle

$$O_\bullet : \Sigma^{-d}X \rightarrow 0 \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow X \xrightarrow{1_X} X,$$

we have  $[O_\bullet] = [\Sigma^{-d}X] + (-1)^{d+1}[X]$  belongs to  $\text{Ker}\phi$ . Thus there are Auslander-Reiten  $(d+2)$ -angles  $B_\bullet^i$  such that  $[O_\bullet] = \sum_{i=1}^r a_i[B_\bullet^i]$ , where

$$B_\bullet^i : B_0^i \xrightarrow{\beta_0^i} B_1^i \xrightarrow{\beta_1^i} B_2^i \xrightarrow{\beta_2^i} \cdots \xrightarrow{\beta_{d-1}^i} B_d^i \xrightarrow{\beta_d^i} B_{d+1}^i \xrightarrow{\beta_{d+1}^i} \Sigma^d B_0^i.$$

Since  $d$  is an odd, we obtain  $[O_\bullet] = [\Sigma^{-d}X] + [X] = \sum_{i=1}^r a_i[B_\bullet^i]$ . By Lemma 3.12, we obtain that the equality

$$[U, O_\bullet] = [U, \Sigma^{-d}X] + [U, X] = \sum_{i=1}^r a_i[U, B_\bullet^i].$$

Since  $\text{Hom}_{\mathcal{C}}(U, X) \neq 0$ , we have

$$[U, O_\bullet] = [U, \Sigma^{-d}X] + [U, X] = \dim_k \text{Hom}_{\mathcal{C}}(U, \Sigma^{-d}X) + \dim_k \text{Hom}_{\mathcal{C}}(U, X) \neq 0.$$

Hence there exists an integer  $i \in \{1, 2, \dots, r\}$  such that  $[U, B_\bullet^i] \neq 0$ , that is,

$$[U, B_0^i] - [U, B_1^i] + [U, B_2^i] + \cdots + (-1)^{d+1}[U, B_{d+1}^i] \neq 0.$$

By Lemma 3.11, this means that the indecomposable  $U$  is isomorphic to an object in the finite set  $\{B_{d+1}^i, \Sigma^d B_{d+1}^i\}_{i=1}^r$ . This shows that  $|\mathrm{SuppHom}_{\mathcal{C}}(-, X)| < \infty$ .

Since  $\mathcal{C}$  has Auslander-Reiten  $(d+2)$ -angles, by Theorem 2.7, we know that  $\mathcal{C}$  has a Serre functor  $\mathbb{S}$ . It follows that there exists an isomorphism  $\mathrm{Hom}_{\mathcal{C}}(X, \mathbb{S}U) \simeq D\mathrm{Hom}_{\mathcal{C}}(U, X) \neq 0$  implies that  $|\mathrm{SuppHom}_{\mathcal{C}}(X, -)| < \infty$ . Therefore  $\mathcal{C}$  is locally finite.  $\square$

**Remark 3.14.** As a special case of Theorem 3.13 when  $d = 1$ , we get that the converse of Theorem 2.1 of Xiao and Zhu in [XZ] is also true.

**Remark 3.15.** When  $d$  is an even, we don't know whether Theorem 3.13 holds.

## Acknowledgements

The author is grateful to Osamu Iyama to point out a shortcoming in a previous version of this article, and give me helpful advice. I also would like to thank Francesca Fedele for many useful comments. Finally, the author wishes to express his sincere thanks to the anonymous referee for her/his detailed comments on a previous version of this article.

## References

- [Au] M. Auslander. Relations for Grothendieck groups of Artin algebras. Proc. Amer. Math. Soc. 91(3): 336-340, 1984.
- [AR1] M. Auslander, I. Reiten. Representation theory of Artin algebras. III. Almost split sequences. Comm. Algebra 3: 239-294, 1975.
- [AR2] M. Auslander, I. Reiten. Representation theory of Artin algebras. IV. Invariants given by almost split sequences. Comm. Algebra 5(5): 443-518, 1977.
- [ABT] E. Arentz-Hansen, P. Bergh, M. Thäule. The morphism axiom for  $n$ -angulated categories. Theory Appl. Categ. 31(18): 477-483, 2016.
- [Be] A. Beligiannis. Auslander-Reiten triangles, Ziegler spectra and Gorenstein rings. K-Theory 32(1): 1-82, 2004.
- [Bu] M. C. Butler. Grothendieck groups and almost split sequences. Integral representations and applications, pp. 357-368, Lecture Notes in Math., 882, Springer, Berlin-New York, 1981.
- [BT1] P. Bergh and M. Thäule. The axioms for  $n$ -angulated categories. Algebr. Geom. Topol. 13(4): 2405-2428, 2013.
- [BT2] P. Bergh and M. Thäule. The Grothendieck group of an  $n$ -angulated category. J. Pure Appl. Algebra, 218(2): 354-366, 2014.
- [Fe1] F. Fedele. Grothendieck groups of triangulated categories. arXiv:1812.08493, 2018.

- [Fe2] F. Fedele. Auslander-Reiten  $(d + 2)$ -angles in subcategories and a  $(d + 2)$ -angulated generalisation of a theorem by Brüning. *J. Pure Appl. Algebra*, 223(8): 3554-3580, 2019.
- [GKO] C. Geiss, B. Keller and S. Oppermann.  $n$ -angulated categories. *J. Reine Angew. Math.* 675: 101-120, 2013.
- [Ha] D. Happel. *Triangulated categories in the representation theory of finite-dimensional algebras*. London Mathematical Society Lecture Note Series, 119. Cambridge University Press, Cambridge, 1988.
- [H1] J. Haugland. Auslander-Reiten triangles and Grothendieck groups of triangulated categories. *arXiv:1904.02506*, 2019.
- [H2] J. Haugland. The Grothendieck group of an  $n$ -exangulated category. *arXiv:1912.04328*, 2019.
- [HLN] M. Herschend, Y. Liu, H. Nakaoka.  $n$ -exangulated categories. *arXiv:1709.06689*, 2017.
- [INP] O. Iyama, H. Nakaoka, Y. Palu. Auslander-Reiten theory in extriangulated categories. *arXiv: 1805.03776*, 2018.
- [IY] O. Iyama, Y. Yoshino. Mutations in triangulated categories and rigid Cohen-Macaulay modules. *Invent. Math.* 172(1): 117-168, 2008.
- [Ja] G. Jasso.  $n$ -abelian and  $n$ -exact categories. *Math. Z.* 283(3-4): 703-759, 2016.
- [L] Z. Lin. Idempotent completion of  $n$ -angulated categories. *arXiv:1701.04223*, 2017.
- [NP] H. Nakaoka, Y. Palu. Extriangulated categories, Hovey twin cotorsion pairs and model structures. *Cah. Topol. Géom. Différ. Catég.* 60(2): 117-193, 2019.
- [OT] S. Oppermann, H. Thomas. Higher-dimensional cluster combinatorics and representation theory. *J. Eur. Math. Soc.* 14(6): 1679-1737, 2012.
- [PPPP] A. Padrol, Y. Palu, V. Pilaud, P. Plamondon. Associahedra for finite type cluster algebras and minimal relations between  $g$ -vectors. *arXiv:1906.06861*, 2019.
- [RV] I. Reiten, M. Van den Bergh. Noetherian hereditary abelian categories satisfying Serre duality. *J. Amer. Math. Soc.* 15(2): 295-366, 2012.
- [XZ] J. Xiao, B. Zhu. Relations for the Grothendieck groups of triangulated categories. *J. Algebra* 257(1): 37-50, 2002.
- [Z1] P. Zhou. On the relation between Auslander-Reiten  $(d+2)$ -angles and Serre duality. *arXiv: 1910.01454*, 2019.
- [Z2] P. Zhou. On the existence of Auslander-Reiten  $(d + 2)$ -angles in  $(d + 2)$ -angulated categories. *arXiv:1912.07983*, 2019.



[ZZ] B. Zhu, X. Zhuang. Grothendieck groups in extriangulated categories. arXiv: 1912.00621, 2019.

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