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Jacobian Ideals of Trilinear Forms: An Application of 1-Genericity

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1. INTRODUCTION

We analyze the structure of all ideals constructed by taking the first partial derivatives of a trilinear form whose coefficients satisfy a kind of weak genericity property.

Here is the set-up: let K be a field and let R be the polynomial ring over K in the three sets of indeterminates $X_1, \ldots, X_n$, $Y_1, \ldots, Y_m$, $Z_1, \ldots, Z_p$. We will assume throughout that $n \geq m \geq p$. Let

$$A = \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m \\ 1 \leq k \leq p}} a_{ijk} X_i Y_j Z_k$$

be a trilinear form in R, and let J_A denote the ideal of R generated by all the partial derivatives of A.

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A question that arises from the theory of hyperdeterminants (see [GKZ, p. 445]) is the following: What can be said about the ideal J_A? A reason for this question emerges, among other things, from results which show that information on the depth of J_A and, more finely, on the primary decomposition of J_A, is linked to information on the hyperdeterminant of A (see [BW]). The difficulty with hyperdeterminants, whose definition makes sense only when $n \leq m + p - 1$, is that there is no explicit formula for them. However, when $n = m + p - 1$, the hyperdeterminants are better understood. The first author, together with Boffi and Bruns, analyzed in [BBG] the minimal primes of J_A when the entries in A satisfy a specific combinatorial structure; more precisely, A is taken to be a "non-degenerate diagonal trilinear form of boundary type," namely, $n = m + p - 1$ and $a_{ijk} \neq 0$ if and only if $i = j + k - 1$. In that paper the authors also ask if it is possible to relax in any way these assumptions [BBG, Remark 1.17].

We provide an answer to this question in the present work: the structure described in [BBG] holds in a much larger context; see Theorems 4.10 and 4.11. We determine the minimal components and the radical of J_A, and moreover, when $n = m + p - 1$, we give an explicit criterion for when the hyperdeterminant of A vanishes (Proposition 3.13).

The critical idea in this paper which enables these generalizations is the new concept of a trilinear form in *general position*. We develop and analyze the properties of such trilinear forms in Section 3. Whereas the proofs in [BBG] relied on the combinatorial structure of the a_{ijk}, our concept of the generic trilinear form enables us to relax quite a few of the assumptions from [BBG] and still simplify the proofs and yield some extra results. Moreover, our generalizations are in some sense "natural," as, for example, when $n = m + p - 1$, the trilinear forms in general position correspond exactly to those three-dimensional arrays for which the hyperdeterminant is non-zero (see Proposition 3.13).

The organization of the paper is as follows: in Section 2 we introduce the notation and define trilinear forms in *general position* (see Definition 2.2). In Section 3 we show that, when K is algebraically closed, the class of matrices in general position is very large and that it includes those treated in [BBG] (see Corollary 3.12 and Proposition 3.13). We prove, in fact, that there is a Zariski-open subset U of K^{nmp} such that if $(a_{ijk}) \in U$, then the corresponding A is in general position (see Proposition 3.14). The key idea of this part of the paper is that the notion of trilinear form in general position is related to the concept of the 1-generic matrix introduced by Eisenbud in [E2]. More precisely, we give a wider definition of 1-genericity (see Definition 3.1), and we use it to prove some equivalent and simpler formulations of general position (see Theorem 3.11). In this part of the work we exploit the interplay among the three matrices of linear forms obtained by taking appropriate second partial derivatives of A. When the underlying field is algebraically closed, A is in general position if and only if any one (equivalently: each one) of these matrices is 1-generic.

In Section 4 we find the minimal primes of J_A for the trilinear forms in general position. These resuls are analogous to those in [BBG]. However, our proofs use the genericity abstraction rather than the prescribed combinatorial structure of the coefficients of the trilinear form. In Section 5 we go beyond [BBG] and explicitly describe the radical and the minimal components of J_A (see Theorems 5.1 and 5.2). Furthermore, in Section 6 we give explicit primary decompositions in the case that $p = 2$ (see Theorems 6.5 and 6.7), and we discuss some properties of the embedded components in general.

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2. NOTATION

Throughout we use the trilinear form A described in the introduction with $n \geq m \geq p \geq 1$. If one of the $n + m + p$ variables does not appear in A, we may without loss of generality reduce the number of variables, as this makes the problem in principle simpler to solve. Moreover, to prevent degenerate cases we also assume that even after any linear change of variables separately among the three groups of variables, all the variables appear. In particular, this restricts n to be at most mp, as A is a homogeneous linear polynomial in the n variables X_i with coefficients taken from the mp-dimensional vector space of all products $Y_j Z_k$.

Throughout X denotes the p by m matrix whose ijth entry is the second partial derivative $A_{Y_j Z_i}$. Similarly, Y is the p by n matrix whose ijth entry is $A_{X_j Z_i}$, and Z is the m by n matrix whose ijth entry is $A_{X_j Y_i}$. In contrast, $\underline{X}$, $\underline{Y}$, and $\underline{Z}$ denote $(X_1, \ldots, X_n)$, $(Y_1, \ldots, Y_m)$, and $(Z_1, \ldots, Z_p)$, respectively. Depending on the context, these stand for either the ideal or the row vector.

Similarly, A_X stands for either the ideal or the vector $(A_{X_1}, \dots, A_{X_n})$. A_Y and A_Z are defined similarly. Note that A_X, as a vector, is equal to the product of the vector $\underline{Z} = (Z_1, \dots, Z_p)$ with the matrix Y, namely, $A_X = \underline{Z}Y$. Also, $A_X = \underline{Y}Z$. Similarly, $A_Y = \underline{Z}X = \underline{X}Z^T$ and $A_Z = \underline{X}Y^T = \underline{Y}X^T$.

For any matrix M and any integer $q \geq 0$, $I_q(M)$ stands for the ideal generated by the q by q minors of M.

With this notation, the ideal A_X equals $I_1(\underline{Z}Y) = I_1(\underline{Y}Z)$, A_Y equals $I_1(\underline{Z}X) = I_1(\underline{X}Z^T)$, and $A_Z = I_1(\underline{X}Y^T) = I_1(\underline{Y}X^T)$.

LEMMA 2.1. *$\underline{Z}I_p(X) \subseteq A_Y$, $\underline{Z}I_p(Y) \subseteq A_X$ and $\underline{Y}I_m(Z) \subseteq A_X$.*

Proof. Let X' be a $p \times p$ submatrix of X. Then $\underline{Z}I_p(X') = I_1(\underline{Z}X' \operatorname{adj} X') \subseteq I_1(\underline{Z}X') \subseteq I_1(\underline{Z}X)$. As X' was arbitrary, $\underline{Z}I_p(X) \subseteq A_Y$ follows.

The other inclusions are proved analogously. ∎

An analysis of the proofs in [BBG] shows that in order to obtain explicitly the minimal components of J_A one needs the following key conditions:

1. $n \geq m \geq p$ and $n \geq m + p - 1$,
2. $I_p(X)$ has height $m - p + 1$ (maximal possible),
3. $I_p(Y)$ has height m (maximal possible),
4. for all $l = 1, \dots, p$, the localization of A_Y at $\{1, Z_l, Z_l^2, Z_l^3, \dots\}$ is a prime ideal of height m,
5. $T_Z^{-1}(A_X) = T_Z^{-1}(\underline{Y})$, where $T_Z = K[Z_1, \dots, Z_p] \setminus \{0\}$.

The above conditions identify a class of trilinear forms. For the sake of clarity we give a name to this class as follows:

Definition 2.2. A trilinear form A and its coefficients a_{ijk} are said to be in *general position* when the five conditions above are satisfied.

Throughout we assume that A is in general position in this sense.

There are two conditions similar to the last one, which are satisfied for every trilinear form in general position. Namely, let $T_X = K[X_1, \dots, X_n] \setminus \{0\}$ and $T_Y = K[Y_1, \dots, Y_m] \setminus \{0\}$. Certainly $T_X^{-1}(A_Y) \subseteq T_X^{-1}(\underline{Z})$. As $I_p(X)$ is a non-zero ideal in $K[X_1, \dots, X_n]$, by Lemma 2.1 then also $T_X^{-1}(\underline{Z}) \subseteq T_X^{-1}(A_Y)$. Thus $T_X^{-1}(A_Y) = T_X^{-1}(\underline{Z})$. Similarly, $T_Y^{-1}(A_X) = T_Y^{-1}(\underline{Z})$.

Of course, whenever $I_m(Z)$ is a nonzero ideal, condition 5 of general position follows from Lemma 2.1.

In the next section we prove some equivalent formulations of general position. In particular, if the underlying field is algebraically closed, we prove that the first and the third conditions imply all the others. We also prove that there are many trilinear forms in general position.

3. TRILINEAR FORMS IN GENERAL POSITION AND 1-GENERIC MATRICES

Definition 3.1 Let $W_1, \ldots, W_s$ be indeterminates over a field K. The term *linear form* in $K[W_1, \ldots, W_s]$ means a homogeneous polynomial of degree 1. Let M be a q by r matrix whose entries are linear forms in $K[W_1, \ldots, W_s]$. We say that M is 1-*generic* if for any invertible row operation on M, the entries of each row generate an ideal of height $\min\{r, s\}$.

Eisenbud [E2, p. 547] defined 1-generic only when $s \geq q + r - 1$, and in that case his definition and ours agree. The simplest example of a matrix which is 1-generic in our sense but not in Eisenbud's is the 1 by r matrix $[\, W_1 \ \cdots \ W_s \ 0 \ \cdots \ 0\,]$, where $s < r$, and more examples are given later in this paper.

It is easy to see that 1-genericity is unaffected by invertible row or column operations, and that when $s \geq q + r - 1$, it is also unaffected by taking transposes.

Many matrices are 1-generic, but here is a large class of matrices which are not:

LEMMA 3.2. *Let K be an algebraically closed field, $W_1, \ldots, W_s$ indeterminates over it, and M a q by s matrix whose entries are linear forms in the W_i. If $q > 1$, M is not* 1-*generic*.

Proof. If the entries of the first row generate a proper subideal of $(W_1, \ldots, W_s)$, we are done, so we may assume instead that $(M_{11}, \ldots, M_{1s}) = (W_1, \ldots, W_s)$, where M_{ij} is, naturally, the ijth entry of M. Thus every entry of the second row can be written as a linear combination of the M_{1i}. Namely, for all $i = 1, \ldots, s$, one has $M_{2i} = \sum_{j=1}^{s} a_{ij} M_{1j}$ for some $a_{ij} \in K$. Let α be an element of K and consider Row $2 + \alpha$Row 1. The entries of this linear combination of the two rows can be written as

$$
\begin{aligned}
&[M_{21} + \alpha M_{11} \cdots M_{2s} + \alpha M_{1s}] \\
&\qquad = [M_{11} \cdots M_{1s}] \begin{bmatrix} a_{11} + \alpha & a_{21} & \cdots & a_{s1} \\ a_{12} & a_{22} + \alpha & \cdots & a_{s2} \\ & & \ddots & \\ a_{1s} & a_{2s} & \cdots & a_{ss} + \alpha \end{bmatrix}.
\end{aligned}
$$

Note that the determinant of the square matrix appearing above is a monic polynomial in α of degree $s \geq 1$. As K is algebraically closed, there exists an $\alpha \in K$ which is a zero of the determinant. This means that for this choice of α, the entries of Row $2 + \alpha$Row 1 do not generate an ideal of height s, so that M is not 1-generic. ∎

We prove in the next two lemmas that when a matrix is 1-generic, the ideal generated by its maximal minors is "large."

LEMMA 3.3. *Assume that K is algebraically closed. If M is a 1-generic q by r matrix in s variables $W_1, \dots, W_s$ and $s \geq q + r - 1$, $q \leq r$, then the height of $I_q(M)$ is $r - q + 1$.*

Proof. As M is 1-generic and $s \geq q + r - 1$, then M is 1-generic also in Eisenbud's sense. Then it follows by [E1, Exercise A2.19, part b, p. 605] or [E2, Proposition 1.3] that the height of the ideal $I_q(M)$ is $r - q + 1$. ∎

Under some conditions the height of $I_q(M)$ is the determining factor of 1-genericity:

LEMMA 3.4. *Assume that K is algebraically closed and that W is a q by r matrix whose entries are linear forms in the variables $W_1, \dots, W_s$. Assume that $s, q \leq r$. Then W is 1-generic if and only if the height of $I_q(W)$ is maximal possible, namely s. Also, W is 1-generic if and only if the radical of $I_q(W)$ is $(W_1, \dots, W_s)$.*

Proof. If $I_q(W)$ has height s, then as $I_q(W)$ is contained in the ideal generated by the entries of any non-trivial linear combination of the rows of W; those entries have to generate an ideal of height at least s. As the entries are all linear forms in $W_1, \dots, W_s$, this proves that W is 1-generic.

Now assume that W is 1-generic. Since W is a matrix of linear forms, by [E1, Exercise A2.19, part a, p. 605], $\sqrt{I_q(W)}$ is the intersection of a collection of ideals each of which is generated by the entries of a non-trivial linear combination of the rows of W. By assumption on 1-genericity of W, each of these ideals has height $\min\{s, r\} = s$ and is generated by the linear forms in $K[W_1, \dots, W_s]$. Thus each of these ideals equals $(W_1, \dots, W_s)$, and so does their intersection $\sqrt{I_q(W)}$. Thus both $\sqrt{I_q(W)}$ and $I_q(W)$ have height s and $\sqrt{I_q(W)}$ equals $(W_1, \dots, W_s)$.

Finally, if $\sqrt{I_q(W)} = (W_1, \dots, W_s)$, its height is s so that W is 1-generic. ∎

This immediately applies to our matrices Y and Z:

LEMMA 3.5. *Assume that K is algebraically closed, and that $n \geq m + p - 1$. Then Y is* 1*-generic if and only if the height of $I_p(Y)$ is m, and that is true if and only if the radical of $I_p(Y)$ is $(Y_1, \ldots, Y_m)$. Also, Z is* 1*-generic if and only if the height of $I_m(Z)$ is p, and that holds if and only if $\sqrt{I_m(Z)} = (Z_1, \ldots, Z_p)$.* ∎

The field K needs to be algebraically closed. This was already pointed out in [E2, p. 548]. Here is a quick counterexample to the lemma if we omit the assumption that K be algebraically closed: let $K = \mathbb{Q}$, let Y_1, Y_2 be variables over F, and let Y be the 2 by 3 matrix

$$Y = \begin{bmatrix} Y_1 & Y_2 & 0 \\ Y_2 & Y_1 + Y_2 & 0 \end{bmatrix}.$$

Each of the two rows of Y generates (Y_1, Y_2), and for every $b \in \mathbb{Q}$, the entries of (row 1) + b (row 2) generate

$$(Y_1 + bY_2, Y_2 + bY_1 + bY_2) = (Y_1 + bY_2, Y_2(1 - b^2 + b)).$$

As there is no rational number b for which $1 - b^2 + b = 0$, this last ideal also has height 2. Thus every generalized row generates an ideal of height exactly 2; yet $I_2(Y)$ is principal, so it cannot have height 2. Thus this Y is not 1-generic.

The 1-genericity of any one among X, Y, or Z implies the 1-genericity of the others, and even more is true:

PROPOSITION 3.6. *If $n \geq m + p - 1$, the following are equivalent* (*without any assumption on the field K*):

(i) *X is* 1*-generic.*

(ii) *The transpose of X is* 1*-generic.*

(iii) *Y is* 1*-generic.*

(iv) *Z is* 1*-generic.*

Proof. As $n \geq m + p - 1$, X is 1-generic if and only if it is 1-generic in Eisenbud's sense [E1, E2]. But a matrix is 1-generic in Eisenbud's sense if and only if its tranpose is 1-generic in Eisenbud's sense. This proves that the first two statements are equivalent.

The proof that the first and the third statements are equivalent is essentially the same as the proof of the equivalence of statements (ii) and (iv). We explicitly only prove here that if X is 1-generic, so is Y. The converse has a completely analogous proof.

Assume that Y is not 1-generic. First observe that an invertible row operation on Y corresponds naturally to a linear change of variables among $Z_1, \ldots, Z_p$, and thus to an identical invertible row operation on X. Thus without loss of generality we may assume, if Y is not 1-generic, that the entries of the first row of Y generate an ideal L of height strictly smaller than m. Let the entries $i_1, \dots, i_{m-1}$ generate L. Let $i'_1, \dots, i'_{n-m+1}$ be such that $\{i_1, \dots, i_{m-1}, i'_1, \dots, i'_{n-m+1}\}$ is the set $\{1, \dots, n\}$. Then the assumption is that there exist elements $d_{l'l}$ in K with $1 \le l' \le n-m+1$ and $1 \le l \le m-1$ such that

$$\begin{aligned} i_{l'}\text{th entry of the first row of } Y &= \sum_j a_{i_{l'}j1} Y_j \\ &= \sum_{l=1}^{m-1} d_{l'l}(l\text{th entry of the first row of } Y) \\ &= \sum_{l=1}^{m-1} d_{l'l}\left(\sum_j a_{i_l j1} Y_j\right). \end{aligned}$$

Comparing the coefficients of the variable Y_j on both sides we get that, for each index $j = 1, \dots, m$,

$$a_{i_{l'}j1} = \sum_{l=1}^{m-1} d_{l'l} a_{i_l j1}.$$

Now consider the ideal generated by the entries of the first row of X. For every $j = 1, \dots, m$, we have

$$\begin{aligned} \sum_i a_{ij1} X_i &= \sum_{l=1}^{m-1} a_{i_l j1} X_{i_l} + \sum_{l'=1}^{n-m+1} a_{i'_{l'}j1} X_{i'_{l'}} \\ &= \sum_{l=1}^{m-1} a_{i_l j1} X_{i_l} + \sum_{l'=1}^{n-m+1} \left(\sum_{l=1}^{m-1} d_{l'l} a_{i_l j1}\right) X_{i'_{l'}} \\ &= \sum_{l=1}^{m-1} a_{i_l j1}\left(X_{i_l} + \sum_{l'=1}^{n-m+1} d_{l'l} X_{i'_{l'}}\right). \end{aligned}$$

In conclusion

$$\begin{aligned} &\left(\left\{\sum_i a_{ij1} X_i \colon l = 1, \dots, m-1\right\}\right) \\ &\qquad \subseteq \left(\left\{X_{i_l} + \sum_{l'=1}^{n-m+1} d_{l'l} X_{i'_{l'}} \colon l = 1, \dots, m-1\right\}\right) \end{aligned}$$

which is an ideal of height $m-1$. Thus X is not 1-generic. ∎

Remark 3.7 David Eisenbud pointed out another proof of this proposition: X is 1-generic if and only if each generalized row of X gives an injective map from K^m to the space of linear forms in $K[X_1, \ldots, X_n]$, with the jth basis element mapping to the jth entry of this generalized row. Also, Y is 1-generic if and only if each generalized row of Y gives an surjective map from K^n to the space of linear forms in $K[Y_1, \ldots, Y_m]$, with the ith basis element mapping to the ith entry of this generalized row. But the matrices X and Y are adjoints of each other in the sense of Eisenbud and Popescu [EP], with a generalized row of X corresponding to the analogous generalized row of Y, so that by the duality between injectivity and surjectivity between adjoints, X is 1-generic if and only if Y is.

The large number of the X_i make it so that X is 1-generic if and only if its transpose is. The analogous statement is false for Y. For example, let

$$Y = \begin{bmatrix} Y_1 & Y_2 & 0 \\ 0 & Y_1 & Y_2 \end{bmatrix}.$$

Then Y is 1-generic but its transpose is not, as, say, the entries of the first column of Y generate an ideal of height strictly smaller than 2.

By the last proposition, we know that for this Y, both X and Z are 1-generic matrices. We calculate them

$$A = [Z_1 Z_2] \begin{bmatrix} Y_1 & Y_2 & 0 \\ 0 & Y_1 & Y_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$
$$= [X_1 Y_1 Z_1 + X_2 Y_1 Z_2 + X_2 Y_2 Z_1 + X_3 Y_2 Z_2],$$

so that

$$X = \begin{bmatrix} X_1 & X_2 \\ X_2 & X_3 \end{bmatrix}, \qquad Z = \begin{bmatrix} Z_1 & Z_2 & 0 \\ 0 & Z_1 & Z_2 \end{bmatrix}.$$

Thus Z is also 1-generic, but its transpose is not.

COROLLARY 3.8. *Let Y be a 1-generic matrix and $T_Z = K[Z_1, \ldots, Z_p] \setminus \{0\}$. Then $T_Z^{-1}(A_X) = T_Z^{-1}(\underline{Y})$.*

Proof. Certainly $T_Z^{-1}(A_X) \subseteq T_Z^{-1}(\underline{Y})$. As Y is 1-generic, so is Z. By Lemma 3.5 then $I_m(Z)$ contains an element of T_Z. Thus by Lemma 2.1, $\underline{Y} I_m(Z) \subset A_X$, so that $T_Z^{-1}(A_X) = T_Z^{-1}(\underline{Y})$. ∎

LEMMA 3.9. *Let X be a 1-generic matrix, $T_Z = K[Z_1, \ldots, Z_p] \setminus \{0\}$, and let $\widehat{T}_Z$ be a multiplicatively closed subset generated by the homogeneous linear polynomials in $K[Z_1, \ldots, Z_p]$. Then $T_Z^{-1}(A_Y)$ and $\widehat{T}_Z^{-1}(A_Y)$ are prime ideals of height m.*

Proof. $T_Z^{-1}(A_Y)$ is generated by m elements each of which is a linear form in $X_1, \ldots, X_n$ with coefficients in the field $T_Z^{-1}K[Z_1, \ldots, Z_p]$. Thus $T_Z^{-1}(A_Y) = T_Z^{-1}(\underline{Z}X)$ is prime ideal which by 1-genericity of X has height m. Let $X_{i_1}, \ldots, X_{i_m}$ be the generators of this ideal. By elementary linear algebra, all the other X_i are expressible as linear combinations of the X_{i_j} with coefficients in $\widehat{T}_Z^{-1}K[\underline{Z}]$, so that $\widehat{T}_Z^{-1}(A_Y) = \widehat{T}_Z^{-1}(X_{i_1}, \ldots, X_{i_m})$. And that is of course a prime ideal of height m. ∎

In fact, $T^{-1}(A_Y)$ is a prime ideal of height m for an even smaller multiplicatively closed subset T of T_Z:

LEMMA 3.10. *Assume that X is* 1*-generic, that K is algebraically closed, and let l be an integer between* 1 *and p. Let T be the multiplicatively closed set $\{1, Z_l, Z_l^2, Z_l^3, \ldots\}$. Then in the localization $T^{-1}R$, $T^{-1}A_Y$ is a prime ideal of height m.*

Proof. We proceed by induction p. First let $p = 1$. Then the ideal $T^{-1}(A_Y)$ is generated by the entries of the 1 by m matrix X. This ideal has height m by 1-genericity of X, and is a prime ideal as it is generated by linear forms.

Now let $p > 1$. Suppose that the height of $T^{-1}(A_Y)$ is strictly less than m or that $T^{-1}(A_Y)$ has two distinct prime ideals minimal over it. As $T \subseteq \widehat{T}_Z$ and $\widehat{T}_Z^{-1}(A_Y)$ is a prime of height m, there exists a prime ideal Q in R, minimal over (A_Y), such that $Z_l \notin Q$ and $\widehat{T}_Z \cap Q$ is non-empty. As Q is a prime ideal and every element of $\widehat{T}_Z$ is a product of linear forms, we may assume that there exists a linear form f_2 in $\widehat{T}_Z \cap Q$. Necessarily f_2 and Z_l are not multiples of each other. Thus there exist linear forms $f_3, \ldots, f_p$ in $K[Z_1, \ldots, Z_p]$ and an invertible p by p matrix M with entries in K such that $\underline{Z} = (Z_l, f_2, \ldots, f_p)M$. Thus

$$A_Y = \underline{Z}X = (Z_l, f_2, \ldots, f_p)MX.$$

Note that MX is still 1-generic. Let X' be the submatrix of MX consisting of all but the second row. X' is 1-generic, so by induction on p, the m entries of $(Z_l, f_3, \ldots, f_p)X'$ generate an ideal of height m in $T^{-1}K[X_1, \ldots, X_n, Y_1, \ldots, Y_m, Z_l, f_3, \ldots, f_p]$. But $Q \supseteq (A_Y) + (f_2) = I_1((Z_l, f_3, \ldots, f_p)X') + (f_2)$, which has height at least $m + 1$. This contradicts the assumption that Q was minimal over an m-generated ideal. Thus the height of $T^{-1}(A_Y)$ is exactly m and its radical is a prime ideal. Thus $T^{-1}(A_Y)$ is generated by a regular sequence, so it has no embedded primes. Hence as a further localization of $T^{-1}(A_Y)$ is a prime, so is $T^{-1}(A_Y)$. ∎

The next result summarizes all the information we have about the interaction between the concepts of general position and 1-genericity. It also underlines the interplay and the properties of the matrices X, Y, and Z.

THEOREM 3.11. *Let $n \geq m + p - 1$ and $n \geq m \geq p$. Let K be an algebraically closed field. Then the following are equivalent*:

(i) *X is a 1-generic matrix.*
(ii) *The transpose of X is a 1-generic matrix.*
(iii) *Y is a 1-generic matrix.*
(iv) *$I_p(Y)$ has height m.*
(v) *The radical of $I_p(Y)$ is $(Y_1, \ldots, Y_m)$.*
(vi) *Z is a 1-generic matrix.*
(vii) *$I_m(Z)$ has height p.*
(viii) *The radical of $I_m(Z)$ is $(Z_1, \ldots, Z_p)$.*
(ix) *A is a trilinear form in general position.*

Proof. Proposition 3.6 proves that (i), (ii), (iii), and (vi) are equivalent. Lemma 3.5 proves that (iii), (iv), and (v) are equivalent and also that (vi), (vii), and (viii) are equivalent. Thus the first eight statements are equivalent.

By the third condition of general position, (ix) implies (iv). Finally, Lemmas 3.3, 3.5, 3.8, and 3.10 prove that the first eight statements imply the last one. ∎

This theorem shows that perhaps one should define general position by a simpler formulation such as statement (iv). However, for the proofs in the following it is more convenient if we keep referring to the conditions of general position in its original definition, Definition 2. Moreover, the equivalences in the theorem only hold when K is algebraically closed, but we do not use an algebraically closed field throughout the paper.

In the rest of this section we prove that there are many trilinear forms in general position. First of all, all the examples in [BBG] are in general position:

COROLLARY 3.12. *Assume that K is algebraically closed, that $n = m + p - 1$, and that $a_{ijk} \neq 0$ if and only if $i = j + k - 1$. Then the a_{ijk} are in general position.*

Proof. Remark 1.3 in [BBG] says that $I_p(Y) = (Y_1, \ldots, Y_m)^p$. Thus the height of $I_p(Y)$ is m and the conclusion follows from the previous proposition. ∎

The trilinear forms analyzed in [BBG] describe a particular class of three-dimensional arrays with non-zero hyperdeterminant. Much more is true for the trilinear forms in general position:

PROPOSITION 3.13. *Let K be an algebraically closed field. When $n = m + p - 1$, A is in general position if and only if the three-dimensional array identified by its coefficients has non-zero hyperdeterminant.*

Proof. In [GKZ, Theorem 3.1, p. 458] it is shown that the hyperdeterminant of the three-dimensional array identified by the coefficients of a trilinear form A, with $n = m + p - 1$, is zero if and only if the system of multilinear equations

$$A_{X_1}(\underline{Y}, \underline{Z}) = \cdots = A_{X_n}(\underline{Y}, \underline{Z}) = 0$$

has a non-trivial solution.

We show that A is in general position if and only if $A_{X_1}(\underline{a}, \underline{b}) = \cdots = A_{X_n}(\underline{a}, \underline{b}) = 0$ if and only if $\underline{a}$ and $\underline{b}$ are both 0, (here $\underline{a} \in K^m$ and $\underline{b} \in K^p$). By Theorem 3.11, A is in general position if and only if the corresponding matrix X is 1-generic. Since $n = m + p - 1$ this happens if and only if X is 1-generic in Eisenbud's sense (see [E1, p. 604, E2, p. 547]). In other words taking any two non-zero vectors in K^m and K^p, say $\underline{a}$ and $\underline{b}$,

$$\underline{b} X \underline{a}^t = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} \sum_{k=1}^{p} a_{ijk} a_j b_k \right) X_i$$

is different from zero. Naturally this is equivalent to saying that $\sum_{i=1}^{n} A_{X_i}(\underline{a}, \underline{b}) X_i \neq 0$, and we conclude that A is in general position if and only if given any two non-zero vectors $\underline{a}$ and $\underline{b}$, there is an index i for which $A_{X_i}(\underline{a}, \underline{b})$ is different from zero, as desired. ∎

Clearly this means that when $n = m + p - 1$, the coefficients of the trilinear forms in general position vary in a Zariski-open subset U of K^{nmp}. As shown below, this statement remains true in the case $n > m + p - 1$:

PROPOSITION 3.14. *Let K be an algebraically closed field. There exists a non-empty Zariski-open subset U of K^{nmp} such that if $(a_{ijk}) \in U$, then the corresponding A is in general position.*

Proof. We will prove that whenever $(a_{ijk}) \in U$, then $I_p(Y) = (Y_1, \ldots, Y_m)^p$.

Let A_{ijk}, $i = 1, \ldots, n$, $j = 1, \ldots, m$, and $k = 1, \ldots, p$ be indeterminates over $K[Y_1, \ldots, Y_m]$. Let $\widehat{Y}$ be the "generalized" version of Y; namely, let it be a p by n matrix whose kith entry is $\sum_j A_{ijk} Y_j$. Let $M_1, \ldots, M_{\binom{n}{p}}$ be all the p by p submatrices of $\widehat{Y}$ and let $F_1, \ldots, F_{\binom{m+p-1}{p}}$ be a generating set for $\underline{Y}^p$. Note that for all l, $\det M_l \in \underline{Y}^p K[A_{ijk}, Y_j]$ and that there exist $s_{ij} \in K[A_{ijk}]$ such that

$$\det M_i = \sum_j s_{ij} F_j.$$

Let S be the $\binom{n}{p}$ by $\binom{m+p-1}{p}$ matrix whose ijth entry is s_{ij}. By the assumption that $n \geq m + p - 1$ it follows that $\binom{n}{p} \geq \binom{m+p-1}{p}$.

Now, after some specialization $A_{ijk} \mapsto a_{ijk} \in K$, $I_p(Y) = (Y_1, \dots, Y_m)^p$ if and only if some $\binom{m+p-1}{p}$ by $\binom{m+p-1}{p}$ minor of S is non-zero (after the same specialization). Thus it suffices to determine that the ideal I in $K[A_{ijk}]$ generated by the maximal minors of S is non-zero. Then U is the non-empty set of all points on which I does not vanish. This ideal I is non-zero if and only if there exist examples of a_{ijk} for which $I_p(Y)$ equals $(Y_1, \dots, Y_m)^p$. If $n = m + p - 1$, all cases in [BBG] (see Remark 1.3 in [BBG]) satisfy the condition. If, however, $n > m + p - 1$, we make up examples as follows: into the first $m + p - 1$ columns of Y we place an example from [BBG], and place zeros in the rest of the columns. ∎

In conclusion, the trilinear forms in general position represent a much wider class than that described in [BBG]: they include the catalecticant, generic, generic symmetric, and a lot more kinds of matrices.

4. THE MINIMAL PRIMES OF J_A

We determine explicitly all the minimal primes of J_A for A in general position. Several proofs of this section employ ideas of [BBG]. However, our results are more general, and proofs often simpler.

In this section the underlying field does not need to be algebraically closed.

PROPOSITION 4.1. *Let A be a trilinear form such that the height of $I_p(Y)$ is m. If Q is a prime ideal containing J_A, then Q contains either the ideal $(Z_1, \dots, Z_p, A_Z)$, or the ideal $(Y_1, \dots, Y_m, A_Y) + I_p(X)$.*

Proof. If $(Z_1, \dots, Z_p) \subseteq Q$, certainly $(Z_1, \dots, Z_p, A_Z) \subseteq Q$.

Now suppose that not all the Z_i lie in Q. By Lemma 2.1 we conclude that $I_p(Y)$ and $I_p(X)$ are contained in Q. By Lemma 3.5, $(Y_1, \dots, Y_m) \subseteq \sqrt{I_p(Y)}$, so that $(Y_1, \dots, Y_m) \subseteq Q$. Thus Q contains A_Y (by definition), all the Y_i, and $I_p(X)$. ∎

Thus by the definition of general position:

COROLLARY 4.2. *Let A be a trilinear form in general position. If Q is a prime ideal containing J_A, then Q contains either the ideal $(Z_1, \dots, Z_p, A_Z)$, or the ideal $(Y_1, \dots, Y_m, A_Y) + I_p(X)$.* ∎

To find the minimal primes of J_A one needs, as in [BBG], to use some techniques from the theory of symmetric algebras. We recall that if M is a free module over R of rank g, then the symmetric algebra $S(M)$ is just the polynomial ring in g indeterminates over R: $S(M) \cong R[T_1, \dots, T_g]$. If M has a presentation $F \xrightarrow{(c_{ij})} G \longrightarrow M \longrightarrow 0$ with F and G free of ranks f

and g, respectively, then $S(M)$ is isomorphic to $R[T_1, \ldots, T_g]/I$, where I is generated by the f elements $\sum_{j=1}^{g} c_{ji}T_j$, $1 \leq i \leq f$.

PROPOSITION 4.3. *If the height of $I_p(Y)$ is m and $m > p$, then $(Z_1, \ldots, Z_p, A_Z)$ is a minimal prime ideal of J_A of height $2p$.*

Proof. By Proposition 4.1 it suffices to prove that $(Z_1, \ldots, Z_p, A_Z)$ is a prime ideal of height $2p$. For that it suffices to prove that A_Z is a prime ideal of height p.

Let S be the ring $K[Y_1, \ldots, Y_m]$. Consider the map from S^p to S^n given by the transpose Y^T of Y. Then as $I_p(Y)$ has height and grade $m \geq 1$, by the Buchsbaum–Eisenbud criterion for exactness [BE], Y^T is injective. Let N be the cokernel. Then $0 \longrightarrow S^p \xrightarrow{Y^T} S^n \longrightarrow N \longrightarrow 0$ is exact, so that the symmetric algebra $S(N)$ of N can be represented as

$$S(N) = \frac{K[X_1, \ldots, X_n, Y_1, \ldots, Y_m]}{(A_{Z_1}, \ldots, A_{Z_p})}.$$

For all t between 1 and p, grade $(I_t(Y^T)) \geq$ grade $(I_p(Y))$, which by assumption is $m \geq p + 1$. Thus one may use [H, Theorem 1.1] to conclude that $S(N)$ is a Cohen–Macaulay domain of dimension $m + n - p$. Hence $(A_Z)S$ is a prime ideal of height p, which proves the proposition. ∎

Our next step is to show that under some assumptions, the ideal $(Y_1, \ldots, Y_m) + A_Y + I_p(X)$ is perfect. Of course, it is enough to show that $A_Y + I_p(X)$ is perfect.

LEMMA 4.4. *Assume that A is in general position, or equivalently, that X is 1-generic. Then the height of $I_1(\underline{Z}X)\colon \underline{Z}$ is at least m. Also, the height of $I_p(X) + I_1(\underline{Z}X)$ is at least m.*

Proof. As $I_p(X) + I_1(\underline{Z}X) \subseteq (I_1(\underline{Z}X)\colon \underline{Z})$, it suffices to prove that the height of $I_p(X) + I_1(\underline{Z}X)$ is at least m.

Let Q be a prime ideal in $K[X_1, \ldots, X_n, Z_1, \ldots, Z_p]$ containing $I_p(X) + I_1(\underline{Z}X)$. If Q contains all the Z_k, then $(Z_1, \ldots, Z_p) + I_p(X) \subseteq Q$. Since A is in general position we have ht $I_p(X) = m - p + 1$ and we deduce that ht $Q \geq$ ht $((Z_1, \ldots, Z_p) + I_p(X)) \geq p + m - p + 1 = m + 1$.

If Q does not contain all the Z_k, then again ht $Q \geq m$ because A is in general position and satisfies condition 4 of Definition 2.2. ∎

Remark 4.5 There always exists a minimal prime ideal Q of $I_p(X) + I_1(\underline{Z}X)$ which does not contain all the Z_k. This is so for otherwise $(Z_1, \ldots, Z_p) \subseteq \sqrt{I_p(X) + I_1(\underline{Z}X)} \subseteq (X_1, \ldots, X_n)$, which is a contradiction. So let Q be a minimal prime not containing some Z_k. Then after localization at Z_k, the ideals $I_p(X) + I_1(\underline{Z}X)$, $I_1(\underline{Z}X)\colon (Z_1, \ldots, Z_p)$, and $I_1(\underline{Z}X)$ are all equal. As $I_1(\underline{Z}X)$ is generated by m elements, then after localization at Z_k the ideal $I_p(X)+I_1(\underline{Z}X)$ has height at most m. Thus with hypotheses in the lemma, the height of $I_p(X)+I_1(\underline{Z}X)$ is exactly m.

PROPOSITION 4.6. *Let X be a 1-generic matrix, or equivalently, let A be a trilinear form in general position. Then $I_p(X)+I_1(\underline{Z}X)$ is a perfect ideal of height m.*

Proof. Let U be a p by n matrix of indeterminates U_{ij}, and S the polynomial ring generated over K by all the U_{ij} and all the Z_i. Let $I = (Z_1, \ldots, Z_p)S$, $A = I_1(\underline{Z}U) \subseteq I$ and $J = I_1(\underline{Z}U):_S(Z_1, \ldots, Z_p)$.

By the initial assumption that all the variables appear even after a linear change of variables, we get that $I_1(X) = (X_1, \ldots, X_n)$. As X is a p by m matrix, there are exactly $mp-n$ linearly independent linear relations $\widetilde{f}_1, \ldots, \widetilde{f}_{mp-n}$ among the entries of X. The $\widetilde{f}_l$ are linear forms in $K[X_1, \ldots, X_n]$. For each $l = 1, \ldots, mp-n$, let f_l be the linear form obtained from $\widetilde{f}_l$ by replacing each ijth entry of X by U_{ij}. Then $f_1, \ldots, f_{mp-n}$ is a regular sequence on S and S/I. Also, $S/(f_1, \ldots, f_{mp-n}) \cong K[X_1, \ldots, X_n, Z_1, \ldots, Z_p]$, and the image of U modulo $(f_1, \ldots, f_{mp-n})$ is X.

Let $'$ denote images modulo $(f_1, \ldots, f_{mp-n})$.

By Lemma 4.4, ht $(A' : I') = $ ht $(I_1(\underline{Z}X) : \underline{Z}) \geq m$.

By a result of Bruns et al. [BKM, Proposition 4.2], the ideal J has height m, and S/J is a Cohen–Macaulay ring. If we knew that $I'_P = A'_P$ for every prime ideal P containing I' with ht $P \leq m$, we could conclude by using a result of Huneke and Ulrich [HU, Proposition 4.2, ii)]. So we now verify $I'_P = A'_P$.

Since $I' = (Z_1, \ldots, Z_p)S$ and $A' = I_1(\underline{Z}X)$, it is enough to show that $(Z_1, \ldots, Z_p)_P \subseteq I_1(\underline{Z}X)_P$ for every prime ideal P containing $(Z_1, \ldots, Z_p)$ and of height $\leq m$. Clearly $I_p(X)$ is not contained in P; otherwise P would contain the ideal $(Z_1, \ldots, Z_p) + I_p(X)$ which by the generic assumption has height $\geq m+1$. Thus $I_p(X) \not\subseteq P$. Then $\underline{Z}I_p(X) \subseteq I_1(\underline{Z}X)$ implies that $(Z_1, \ldots, Z_p)_P \subseteq I_1(\underline{Z}X)_P$. Hence we can indeed apply the Huneke–Ulrich result to finish the proof. ∎

PROPOSITION 4.7. *Assume that A is in general position. Then the ideal*

$$(Y_1, \ldots, Y_m, A_Y) + I_p(X)$$

is a perfect prime of height $2m$, hence a minimal prime ideal of J_A.

Proof. By Corollary 4.2 it suffices to prove that $(Y_1, \ldots, Y_m, A_Y) + I_p(X)$ is a perfect prime of height $2m$. For that it suffices to prove that $A_Y + I_p(X) = I_p(X) + I_1(\underline{Z}X)$ is a perfect prime of height m. As perfection and the height were already proved in Proposition 4.6, it suffices to prove that $I_p(X) + I_1(\underline{Z}X)$ is a prime.

First we prove that Z_1 is a regular element modulo $I_p(X) + I_1(\underline{Z}X)$. By perfection it suffices to prove that the height of $I_p(X) + I_1(\underline{Z}X) + (Z_1)$ is at least $m+1$. Set $\widetilde{Z} = (Z_2, \ldots, Z_p)$ and let $\widetilde{X}$ be the submatrix of X without the first row. Then

$$I_p(X) + I_1(\underline{Z}X) + (Z_1) = I_1(\widetilde{Z}\widetilde{X}) + I_p(X) + (Z_1).$$

Let Q be a prime ideal minimal over this ideal. If Q contains $(Z_2, \ldots, Z_p)$, then Q contains $Z_1, \ldots, Z_p$ and $I_p(X)$. Then by genericity, the height of Q is at least $m+1$. If instead Q does not contain $(Z_2, \ldots, Z_p)$, then as $\widetilde{Z}I_{p-1}(\widetilde{X}) \subseteq I_1(\widetilde{Z}\widetilde{X}) \subseteq Q$, we get that $I_{p-1}(\widetilde{X}) \subseteq Q$, so that Q contains $I_{p-1}(\widetilde{X}) + I_1(\widetilde{Z}\widetilde{X}) + (Z_1)$. By Proposition 4.6, $I_{p-1}(\widetilde{X}) + I_1(\widetilde{Z}\widetilde{X}) + (Z_1)$ has height at least $m+1$, so that $\operatorname{ht} Q \geq m+1$.

This proves that Z_1 is a regular element modulo $I_p(X) + I_1(\underline{Z}X)$. By Lemma 2.1, in the localization at $\{1, Z_1, Z_1^2, Z_1^3, \ldots\}$, the ideals $I_p(X) + I_1(\underline{Z}X)$, $I_1(\underline{Z}X)$, and A_Y are all the same ideal, and by genericity this ideal is a prime of height m. But Z_1 is a regular element modulo $I_p(X) + I_1(\underline{Z}X)$, so that even before localization, $I_p(X) + I_1(\underline{Z}X)$ is a prime ideal of height m. ∎

COROLLARY 4.8. *If A is in general position, then $I_p(X)$ is a prime ideal in $F[\underline{X}]$. Its height is $m - p + 1$.*

Proof. As $I_p(X) = \big((Y_1, \ldots, Y_m, A_Y) + I_p(X)\big) \cap F[\underline{X}]$, the first part follows from the proposition above. The height part follows by the definition of general position. ∎

When K is algebraically closed, this amounts to saying that if X is 1-generic, then $I_p(X)$ is a prime ideal of height $m - p + 1$, as was already proved in [Ke, E2, p. 542].

THEOREM 4.9. *Assume that A is in general position, and that $m > p$. Then the minimal primes of J_A are $(Z_1, \ldots, Z_p, A_Z)$ and $(Y_1, \ldots, Y_m, A_Y) + I_p(X)$.*

Proof. Use Propositions 4.1, 4.3, and 4.7. ∎

If $p = m$, it follows by symmetry from Proposition 4.7 that

$$(Y_1, \ldots, Y_m, A_Y) + I_p(X) \qquad \text{and} \qquad (Z_1, \ldots, Z_p, A_Z) + I_p(X)$$

are both minimal prime ideals of J_A. Here $I_p(X) = (\det(X))$. Note that $\underline{Y} \cdot \det(X) \subseteq I_1(\underline{Y}X) = A_Z$ but neither $\underline{Y}$ nor $\det(X)$ lies in A_Z. Thus neither A_Z nor $(Z_1, \ldots, Z_p, A_Z)$ are prime ideals.

THEOREM 4.10. *Assume that A is in general position and that $m = p$. Then the minimal primes of J_A are*

$$(Z_1, \ldots, Z_p, A_Z) + I_p(X), \qquad (Y_1, \ldots, Y_m, A_Y) + I_p(X), \quad \textit{and}$$

$$(Y_1, \ldots, Y_m, Z_1, \ldots, Z_p).$$

Proof. There are no inclusion relations among the listed three ideals. By the observation above, the first two ideals are minimal primes. If Q is any other minimal prime, it follows from Proposition 4.1 and symmetry that Q must contain both $(Y_1, \ldots, Y_m, A_Y)$ and $(Z_1, \ldots, Z_p, A_Z)$. Hence Q must contain $(Y_1, \ldots, Y_m, Z_1, \ldots, Z_p)$. As the latter ideal is prime and contains J_A, it is minimal over J_A. ∎

PROPOSITION 4.11. *Assume that A is in general position and that $n \geq m - p + 1$. Then* ht $J_A = 2p$. *If $m = p$, all the minimal primes have the same height.*

Proof. First let $m > p$. By Proposition 4.3, ht $(\underline{Z}, A_Z) = 2p$, and by Proposition 4.7, the height of the other minimal prime ideal, namely, the ideal $(\underline{Y}, A_Y) + I_p(X)$, is $2m > 2p$.

If $m = p$, then by Remark 4, ht $(\underline{Y}, A_Y, I_p(X)) = 2m$ and ht $(\underline{Z}, A_Z, I_p(X)) = 2p$. Hence ht $(\underline{Y}, A_Y, I_p(X)) =$ ht $(\underline{Z}, A_Z, I_p(X)) = m + p =$ ht $(\underline{Y}, \underline{Z})$. ∎

Note that in this section we only used the first four conditions of Definition 2.2.

5. MINIMAL COMPONENTS AND THE RADICAL OF J_A

In this section again the underlying field does not need to be algebraically closed. The minimal components and the radical of J_A are straightforward to compute when A is in general position.

THEOREM 5.1. *Let A be in general position and let P be a prime ideal minimal over J_A. Then the P-primary component of J_A is P.*

Proof. First assume that $P = (Z_1, \ldots, Z_p, A_Z)$. Let $T_Y = K[Y_1, \ldots, Y_m] \setminus \{0\}$. By the remark after Definition 2.2, $T_Y^{-1}(A_X) = T_Y^{-1}(\underline{Z})$. As T_Y has no elements in common with P, then also $(A_X)_P = (\underline{Z})_P$. Thus the P-primary component contains $\underline{Z}$; hence it is equal to P.

Now assume that $P = (Y_1, \ldots, Y_m, A_Y) + I_p(X)$. Since A is in general position, $T_Z^{-1}(A_X) = T_Z^{-1}(\underline{Y})$, where $T_Z = K[Z_1, \ldots, Z_p] \setminus \{0\}$. As T_Z has no elements in common with P, then also $(J_A)_P$ contains $\underline{Y}$. Moreover, by

Lemma 2.1, $(J_A)_P$ also contains $I_p(X)$. Thus again the P-primary component equals to P.

Finally, let $P = (Y_1, \ldots, Y_m, Z_1, \ldots, Z_p)$. Since $T_X^{-1}(A_Y) = T_X^{-1}(\underline{Z})$, where $T_X = K[X_1, \ldots, X_n] \setminus \{0\}$, then $\underline{Z}$ lies in the P-primary component. But in this case $m = p$, so by symmetry also $\underline{Y}$ lies in the P-primary component. ∎

THEOREM 5.2. *If A is a trilinear form in general position,*

$$\sqrt{J_A} = J_A + \underline{Y}\,\underline{Z}.$$

Proof. First assume that $m > p$. Then

$$\begin{aligned}
\sqrt{J_A} &= (\underline{Z}, A_Z) \cap \big((\underline{Y}, A_Y) + I_p(X)\big) \\
&= A_Z + \underline{Z} \cap \big((\underline{Y}, A_Y) + I_p(X)\big) \\
&= A_Z + A_Y + \underline{Z} \cap \big(\underline{Y} + I_p(X)\big) \\
&= A_Z + A_Y + \underline{Z}(\underline{Y} + I_p(X)) \quad \text{(by multi-homogeneity)} \\
&= A_Z + A_Y + \underline{Y}\,\underline{Z} \quad \text{(by Lemma 2.1)} \\
&= J_A + \underline{Y}\,\underline{Z}.
\end{aligned}$$

Similarly, if $m = p$,

$$\begin{aligned}
\sqrt{J_A} &= \big((\underline{Y}, A_Y) + I_p(X)\big) \cap \big((\underline{Z}, A_Z) + I_p(X)\big) \cap (\underline{Y}, \underline{Z}) \\
&= \Big(\big((\underline{Y}, A_Y) + I_p(X)\big) \cap \big((\underline{Z}, A_Z) + I_p(X)\big)\Big) \cap (\underline{Y}, \underline{Z}) \\
&= \Big(I_p(X) + A_Y + \underline{Y} \cap \big((\underline{Z}, A_Z) + I_p(X)\big)\Big) \cap (\underline{Y}, \underline{Z}) \\
&= \Big(I_p(X) + A_Y + A_Z + \underline{Y} \cap \big(\underline{Z} + I_p(X)\big)\Big) \cap (\underline{Y}, \underline{Z}) \\
&= (\underline{Y}, \underline{Z}) \cdot I_p(X) + A_Y + A_Z + \underline{Y} \cdot \big(\underline{Z} + I_p(X)\big).
\end{aligned}$$

By Lemma 2.1, $\underline{Z} \cdot I_p(X)$ is contained in A_Y, and as $m = p$, by symmetry also $\underline{Y} \cdot I_p(X)$ is contained in A_Z. Thus this radical also simplifies to $J_A + \underline{Y}\,\underline{Z}$. ∎

If $\underline{Y} \cdot \underline{Z} \subseteq J_A$, then of course we have found a primary decomposition of J_A. Note that if $mp > n$, then $\underline{Y} \cdot \underline{Z} \not\subseteq A_X$ and $\underline{Y} \cdot \underline{Z} \not\subseteq J_A$, so there exist embedded primes.

6. ABOUT THE EMBEDDED COMPONENTS OF J_A

We find the embedded components in the case that $p = 2$ and K is algebraically closed. Not all the embedded components are equal—for example, they depend on n and m.

We also discuss the embedded components in cases when $p > 2$, and raise some questions.

PROPOSITION 6.1. *Assume that A is in general position. Then*

$$J_A = \sqrt{J_A} \cap \big(J_A + I_p(X) + I_p(Y) + I_m(Z)\big).$$

Thus every embedded component of J_A contains $\big(J_A + I_p(X) + I_p(Y) + I_m(Z)\big)$.

Proof. By Lemma 2.1 and multihomogeneity,

$$\begin{aligned}
&\sqrt{J_A} \cap \big(J_A + I_p(X) + I_p(Y) + I_m(Z)\big) \\
&\quad = J_A + \underline{Y} \cdot \underline{Z} \cap \big(J_A + I_p(X) + I_p(Y) + I_m(Z)\big) \\
&\quad \subseteq J_A + \underline{Y} A_Y + \underline{Z} A_Z + \underline{Y} \cdot \underline{Z} I_p(X) + \underline{Z} I_p(Y) + \underline{Y} I_m(Z).
\end{aligned}$$

Thus, $J_A = \sqrt{J_A} \cap \big(J_A + I_p(X) + I_p(Y) + I_m(Z)\big)$, as wanted. ∎

As A is in general position, the radical $I_p(X) + \underline{Y} + \underline{Z}$ of $J_A + I_p(X) + I_p(Y) + I_m(Z)$ is a prime ideal by Lemma 4.8. However, $J_A + I_p(X) + I_p(Y) + I_m(Z)$ is in general not primary to this prime. In fact, as shown in [BBG] and in [BG], there are cases of trilinear forms in general position where the maximal irrelevant ideal is an associated prime, so that for those the ideal $J_A + I_p(X) + I_p(Y) + I_m(Z)$ could not be primary to the nonmaximal ideal $I_p(X) + \underline{Y} + \underline{Z}$.

To simplify notation, we next introduce several admissible changes of variables, admissible in the sense that the primary decompositions stay the same. We admit linear changes of variables among the X_i, the Y_j, and the Z_k separately. Such a change is an automorphism of $K[\underline{X}, \underline{Y}, \underline{Z}]$ and it maps isomorphically the Jacobian ideal J_A to the corresponding new Jacobian ideal J_A. Thus the primary decomposition of J_A is unaffected by these changes.

Some specific changes we can use are as follows:

1. renaming of the X_i, in other words, a linear change of variables among the X_i;

2. elementary row operation on X: this corresponds to a linear change of variables among the Z_k and an elementary row operation on Y;

3. elementary column operation on X: this corresponds to a linear change of variables among the Y_j and an elementary row operation on Z.

Note that none of these changes affects the 1-genericity of X, and so by Proposition 3.6 it also does not affect the 1-genericity of Y and Z.

When $p = 2$, by Eisenbud [E2, Theorem 5.1 (iii)] these admissible changes transform X into the scrollar space form $\mathbf{M}(a_1, \ldots, a_d)$ with $a_1 \geq a_2 \geq \cdots \geq a_d \geq 1$, $\sum_i a_i = n$, $d = n - m$. Explicitly, X has the form

$$\left[\begin{array}{cccc|ccc|c|ccc} X_1 & X_2 & \cdots & X_{a_1-1} & X_{a_1+1} & \cdots & X_{a_1+a_2-1} & \cdots & X_{a_1+\cdots+a_{d-1}+1} & \cdots & X_{a_1+\cdots+a_d-1} \\ X_2 & X_3 & \cdots & X_{a_1} & X_{a_1+2} & \cdots & X_{a_1+a_2} & & X_{a_1+\cdots+a_{d-1}+2} & \cdots & X_{a_1+\cdots+a_d} \end{array}\right].$$

For example, when $n = m + 1$ (smallest possible),

$$X = \mathbf{M}(m+1) = \begin{bmatrix} X_1 & X_2 & \cdots & X_{m-1} & X_m \\ X_2 & X_3 & \cdots & X_m & X_{m+1} \end{bmatrix},$$

and when $n > m + 1$, X is a juxtaposition of $d = n - m$ such matrices, with no overlaps among the variables in these submatrices. We will use the name *scroll* to indicate a single block of $\mathbf{M}(a_1, \ldots, a_d)$.

We will calculate the primary decomposition of J_A when $p = 2$. We first explicitly do the case $m = p = 2$ separately for the sake of clarity. Here, $n \geq m + p - 1 = 3$, and as every variable X_i is used, necessarily $4 = mp \geq n$.

THEOREM 6.2. *Let $m = p = 2$. If $n = 4$, J_A has no embedded components. If $n = 3$, an irredundant primary decomposition is*

$$J_A = \big((\underline{Y}, A_Y) + I_2(X)\big) \cap \big((\underline{Z}, A_Z) + I_2(X)\big) \\ \cap (\underline{Y}, \underline{Z}) \cap (\underline{X}, A_X, I_2(Y), I_2(Z)).$$

Proof. In case $n = 4$, after renaming the X_i, the matrix X is

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}.$$

It is easy to check that in this case, $A_X = \underline{Y} \cdot \underline{Z}$, so that J_A has no embedded components, and so Theorem 5.2 calculates the primary decomposition of J_A.

Now suppose instead that $n = 3$. The first three ideals in the intersection in the statement of the theorem are the minimal primes and $(\underline{X}, A_X, I_2(Y), I_2(Z))$ is the claimed unique embedded component. Clearly it is primary to the maximal homogeneous ideal.

We know that $d = n - m = 1$, so that

$$X = \mathbf{M}(3) = \begin{bmatrix} X_1 & X_2 \\ X_2 & X_3 \end{bmatrix}.$$

Since $A = X_1Y_1Z_1 + X_2Y_1Z_2 + X_2Y_2Z_1 + X_3Y_2Z_2$, we explicitly obtain

$$\begin{aligned} A_{X_1} &= Y_1Z_1 \\ A_{X_2} &= Y_1Z_2 + Y_2Z_1 \\ A_{X_3} &= Y_2Z_2 \end{aligned} \qquad \begin{aligned} A_{Y_1} &= X_1Z_1 + X_2Z_2 \\ A_{Y_2} &= X_2Z_1 + X_3Z_2 \end{aligned}$$

$$\begin{aligned} A_{Z_1} &= X_1Y_1 + X_2Y_2 \\ A_{Z_2} &= X_2Y_1 + X_3Y_2 \end{aligned}$$

We have $Y_1Z_2 \notin J_A$, but

$$\begin{aligned} X_1Y_1Z_2 &= Z_2\frac{\partial A}{\partial Z_1} - X_2\frac{\partial A}{\partial X_3}, \\ X_2Y_1Z_2 &= Z_2\frac{\partial A}{\partial Z_2} - X_3\frac{\partial A}{\partial X_3}, \\ X_3Y_1Z_2 &= X_3\frac{\partial A}{\partial X_2} - Z_1\frac{\partial A}{\partial Z_2} - X_2\frac{\partial A}{\partial X_1}. \end{aligned}$$

Thus $\underline{X} \cdot (Y_1Z_2) \subseteq J_A$. As $\underline{Y} \cdot \underline{Z} = A_X + (Y_1Z_2)$, this means that $\underline{X} \cdot \underline{Y} \cdot \underline{Z} \subseteq J_A$; hence

$$\begin{aligned} J_A &\subseteq \sqrt{J_A} \cap (\underline{X} + I_2(Y) + I_2(Z) + A_X) \\ &= J_A + \underline{Y} \cdot \underline{Z} \cap (\underline{X} + I_2(Y) + I_2(Z) + A_X) \\ &\subseteq J_A + \underline{Y} \cdot \underline{Z} \cdot \underline{X} + \underline{Z}I_2(Y) + \underline{Y}I_2(Z) + A_X \\ &\subseteq J_A, \end{aligned}$$

which was to be proved. ∎

Thus the primary decompositions depend on n.

Before we start the general $p = 2$ case, we renumber the variables Y_j to be

$$Y_1, Y_2, \ldots, Y_{a_1-1}, Y_{a_1+1}, \ldots, Y_{a_1+a_2-1}, \ldots,$$
$$Y_{a_1+a_2+\cdots+a_{d-1}+1}, \ldots, Y_{a_1+a_2+\cdots+a_d-1}.$$

Thus the subscripts of the Y_j correspond to the subscripts of the variables X_i in the first row of the scrollar matrix $X = \mathbf{M}(a_1, \ldots, a_d)$.

LEMMA 6.3. *With notation as above, let X_i appear in the hth column of a scroll, and let X_j appear in the kth column, first row, of a possibly different scroll. If $h \geq k-1$, then $X_i Y_j \underline{Z} \subseteq J_A$.*

Proof. We first reduce to showing that $X_i Y_j Z_2 \in J_A$. If X_j is the first variable in its scroll (appearing in the top left corner of that scroll), then $Y_j Z_1 = A_{X_j} \in J_A$. Since X_j is in the first row of its scroll, it is not the last variable there, so $A_{X_j} = Y_j Z_1 + Y_{j-1} Z_2$. Thus $X_i Y_j Z_1 = X_i A_{X_j} - X_i Y_{j-1} Z_2$. Thus in order to finish the proof, it suffices to prove that $X_i Y_j Z_2 \in J_A$.

If X_j is in the last column of its scroll, then $A_{X_{j+1}} = Y_j Z_2$. So we may assume that X_j is not in the last column of its scroll. If X_i is not the last variable in its block, then

$$X_i Y_j Z_2 = X_i A_{X_{j+1}} - Y_{j+1} A_{Y_i} + X_{i+1} Y_{j+1} Z_2,$$

so that it suffices to prove that $X_{i+1} Y_{j+1} Z_2$ lies in J_A. Notice that in this step we increased by one the indices of both X_i and Y_j. This means that we increased the column numbers of X_i and X_j by one, or if X_i was already in the last column, then we made the new X_i the last variable in its scroll.

As $h \geq k-1$, we have thus reduced the proof to showing that $X_i Y_j Z_2$ lies in J_A, where X_i is the last variable in its scroll and X_j does not lie in the last column of its scroll in X. If X_j is the first variable in its scroll, then whenever X_i is not the first variable in its scroll, $X_i Y_j Z_2 = Y_j A_{Y_{i-1}} - X_{i-1} A_{X_j}$, and we are done. So we may assume that X_j is not the first variable in its scroll. But then

$$X_i Y_j Z_2 = Y_j A_{Y_{i-1}} - X_{i-1} A_{X_j} + X_{i-1} Y_{j-1} Z_2,$$

which is the reverse operation of what we just did: here we shift back the indices of the columns. Now, as $h \geq k$, this procedure ensures that, in at most $k-1$ steps, the X_j gets pushed into the first entry of its scroll, whence $X_i Y_j Z_2$ lies in J_A. ∎

COROLLARY 6.4. *With notation as above, assume that $a_h \geq a_k - 2$. Then for all X_i taken from the scroll corresponding to a_h and all Y_j such that X_j is from a scroll corresponding to a_k,*

$$X_i Y_j \underline{Z} \subseteq J_A.$$

Proof. As in the proof of the lemma, it suffices to prove that $X_i Y_j Z_2$ lies in J_A, where X_j does not lie in the last column of its scroll and X_i is the last variable in its block. But then by the reduction of indices procedure as at the end of the previous proof, X_j reduces to the first variable in its scroll in at most $a_k - 2 \leq a_h$ steps, whence $X_i Y_j Z_2 \in J_A$. ∎

THEOREM 6.5. *Suppose that* $p = 2$, $n < 2p$, *and* $X = \mathbf{M}(a_1, \ldots, a_d)$ *with* $a_1, \ldots, a_d \in \{a, a+1, a+2\}$ *for some integer* a. *Then* J_A *has exactly one embedded prime, namely, the maximal homogeneous ideal. As the embedded component one can take* $J_A + (X_1, \ldots, X_n) + I_2(Y) + I_m(Z)$.

Proof. By the previous corollary, $\underline{X}\,\underline{Y}\,\underline{Z} \subseteq J_A$. Then

$$J_A = (J_A + \underline{Y}\,\underline{Z}) \cap (J_A + I_2(Y) + I_m(Z) + \underline{X}),$$

so that the only embedded prime is the homogeneous maximal ideal, with the displayed embedded component. ∎

Remark 6.6. This gives precisely the primary decomposition in the case $p = 2$ and $n = m + 1$, since in that case there is only one scroll in X.

In the next result we tackle the general $p = 2$ case. The ideas of the proof are similar to the ideas of the proof of Lemma 6.3, however, the two proofs accomplish slightly different things.

THEOREM 6.7. *Let* $p = 2$, m *and* n *arbitrary. If* $n = 2m$, J_A *has no embedded components. When instead* $n < 2m$, *then* J_A *has only one embedded component, and that one is primary to the maximal homogeneous ideal. The embedded component may be taken to be*

$$J_A + (X_1, \ldots, X_n)^{m-1} + I_2(Y) + I_m(Z).$$

Proof. The first statement holds by the remark after Theorem 5.2.

We use the notation of the previous few results. It is easy to see that it suffices to prove that every $X_{i_1} \cdots X_{i_{m-1}} Y_j Z_k$ lies in J_A. As in the proof of Lemma 6.3, it suffices to prove this for $k = 2$.

If X_j is in the last column of its scroll, then $A_{X_{j+1}} = Y_j Z_2$. So we may assume that X_j is not in the last column of its scroll. If for some s, say $s = 1$, X_{i_s} is not the last variable in its block, then

$$\begin{aligned} &X_{i_1} \cdots X_{i_{m-1}} Y_j Z_2 \\ &\quad = X_{i_1} \cdots X_{i_{m-1}} A_{X_{j+1}} - X_{i_2} \cdots X_{i_{m-1}} Y_{j+1} A_{Y_{i_1}} \\ &\quad\quad + X_{i_1+1} X_{i_2} \cdots X_{i_{m-1}} Y_{j+1} Z_2, \end{aligned}$$

so that it suffices to prove that $X_{i_1+1} X_{i_2} \cdots X_{i_{m-1}} Y_{j+1} Z_2$ lies in J_A. By raising the indices more if necessary we have thus reduced the proof to showing that $X_{i_1} X_{i_2} \cdots X_{i_{m-1}} Y_j Z_2$ lies in J_A, where all X_{i_s} are the last variables in their scroll and where X_j does not lie in the last column of its scroll in X. If X_j is the first variable in its scroll, then

$$X_{i_1} \cdots X_{i_{m-1}} Y_j Z_2 = X_{i_2} \cdots X_{i_{m-1}} Y_j A_{Y_{i_1 - 1}} - X_{i_2} \cdots X_{i_{m-1}} X_{i_1 - 1} A_{X_j},$$

and we are done. So we may assume that X_j is not the first variable in its scroll. But then

$$\begin{aligned} &X_{i_1} \cdots X_{i_{m-1}} Y_j Z_2 \\ &\quad = X_{i_2} \cdots X_{i_{m-1}} Y_j A_{Y_{i_1-1}} - X_{i_2} \cdots X_{i_{m-1}} X_{i_1-1} A_{X_j} \\ &\qquad + X_{i_2} \cdots X_{i_{m-1}} X_{i_1-1} Y_{j-1} Z_2, \end{aligned}$$

so it suffices to prove that $X_{i_2} \cdots X_{i_{m-1}} X_{i_1-1} Y_{j-1} Z_2$ lies in J_A. We can play "reduce the indices game" on the i_s as long as possible. Now, as Y_j is not the last variable in its scroll and there are m variables Y_k, the index reduction procedure ensures that, in some step, we arrive at an element of the form $X_{i_1} \cdots X_{i_{m-1}} Y_j Z_2$, where X_j is the first variable in its scroll but some X_{i_s} is not. But then $X_{i_s} Y_j Z_2 = Y_j A_{Y_{i_s-1}} - X_{i_s-1} A_{X_j}$, and so we are done. ∎

The class of Jacobian ideals of trilinear forms considered in [BBG] always had the maximal irrelevant ideal as an associated prime. Theorems 6.5 and 6.7 are further evidence of this behavior. We do not know if the same holds more generally for arbitrary $n \geq m \geq p$:

QUESTION 6.8. Is the maximal irrelevant ideal an associated prime whenever K is algebraically closed and A is a trilinear form in general position?

We now briefly discuss the general case $n \geq m \geq p$. The main reason for lack of positive results for $p \geq 3$ is that there is unfortunately no natural description of p by m 1-generic matrices. Unlike in the $p = 2$ case, for larger p a trilinear form in general position need not be of the form studied in [BBG] with all coefficients equal to 1. In fact, when $p = m = 3$, $n = 5$, then for

$$A = [\, Z_1 \quad Z_2 \quad Z_3 \,] \begin{bmatrix} X_1 & X_2 & X_3 \\ X_2 & X_3 & X_4 \\ 2X_3 & X_4 & X_5 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix},$$

$I_2(X) + (\underline{Y}, \underline{Z})$ is not a prime ideal, whereas for

$$A = [\, Z_1 \quad Z_2 \quad Z_3 \,] \begin{bmatrix} X_1 & X_2 & X_3 \\ X_2 & X_3 & X_4 \\ X_3 & X_4 & X_5 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix},$$

$I_2(X) + (\underline{Y}, \underline{Z})$ is a prime ideal associated to J_A. Thus X for the first A is not equivalent via admissible changes to the second X. This shows that when $p = 3$, the primary decompositions are much more difficult to get at than when $p = 2$.

Note that both of the trilinear forms above are of the form studied in [BBG], but the second one is symmetric and the first one is not.

With the help of the computer algebra system Singular [GPS] we have calculated primary decompositions for several cases when $m = p = 3$. If

$$X = \begin{bmatrix} X_1 & X_2 & X_3 \\ X_2 & X_3 & X_4 \\ X_3 & X_4 & X_5 \end{bmatrix} \qquad \text{or} \qquad X = \begin{bmatrix} X_1 & X_2 & X_3 \\ X_2 & X_4 & X_5 \\ X_3 & X_5 & X_2 \end{bmatrix},$$

and a few other symmetric matrices, Singular returns $(\underline{X}, \underline{Y}, \underline{Z})$ and $J_A + I_2(X) + (\underline{Y}, \underline{Z})$ as the embedded primes. In general, Singular finishes the primary decomposition calculation for symmetric matrices within a day via its Gianni et al. [GTZ] algorithm and within half an hour via its Shimoyama and Yokoyama [SY] algorithm. For non-symmetric 1-generic 3 by 3 matrices, however, we have not gotten a single primary decomposition via Singular.

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