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Journal of Algebra 273 (2004) 686–699

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

# Rank ideals and Capelli identities

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Received 14 January 2003

Communicated by J.T. Stafford

**Abstract**

We define and study a family of completely prime rank ideals in the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$. A rank ideal is a noncommutative analogue of a determinantal ideal, the defining ideal for the closure of the set of $n \times n$ matrices of fixed rank. We introduce a notion of rank for $\mathfrak{gl}_n$-modules and determine the rank of simple highest weight modules and of simple finite-dimensional modules. The main tools are Capelli-type identities and filtered algebra.


*Keywords:* Universal enveloping algebra; Rank ideal; Determinantal ideal; Capelli identity; Reductive dual pair

## 1. Introduction

In this paper we define and study a family of completely prime *rank ideals* in the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$ of the complex Lie algebra $\mathfrak{gl}_n$ of $n \times n$ matrices. Rank ideals are noncommutative analogues of *determinantal ideals* in the algebra of polynomial functions on $n \times n$ matrices. The definition of rank ideals is motivated by the theory of reductive dual pairs. The situation considered in this paper corresponds to the irreducible dual pair of Lie algebras $(\mathfrak{gl}_n, \mathfrak{gl}_k)$ (cf. [9,23]).

Our starting point is a classical representation of $\mathfrak{gl}_n$ and $\mathrm{U}(\mathfrak{gl}_n)$ by polynomial coefficient differential operators on the complex vector space $\mathsf{M}_{k,n}$ of $k \times n$ matrices (see [24]). Let $\{\mathsf{E}_{ij}\colon 1 \leqslant i, j \leqslant n\}$ be the standard basis of $\mathfrak{gl}_n$ consisting of the matrix units. The differential operator corresponding to $\mathsf{E}_{ij}$ is the *polarization operator* $\mathsf{D}_{ij}$, explicitly given as

$$\mathsf{D}_{ij} = \sum_{r=1}^{k} x_{ri}\,\partial_{rj}, \tag{1.1}$$

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doi:10.1016/S0021-8693(03)00431-9

where we denoted by $\partial_{rj}$ the partial derivative $\partial/\partial x_{rj}$. The polarization operators $\mathsf{D}_{ij}$ commute with the action of the group $GL_k$ on $\mathsf{M}_{k,n}$ by left multiplication. By construction, the assignment $\mathsf{E}_{ij} \mapsto \mathsf{D}_{ij}$ defines an embedding $R$ of the Lie algebra $\mathfrak{gl}_n$ into the algebra $\mathcal{PD}(\mathsf{M}_{k,n})$ of the polynomial coefficient differential operators, and $R$ is a Lie algebra homomorphism. Therefore, the map $R$ extends canonically to an associative algebra homomorphism $R : \mathrm{U}(\mathfrak{gl}_n) \to \mathcal{PD}(\mathsf{M}_{k,n})$.

**Definition 1.2.** The $k$th rank ideal $\mathcal{J}_k \subseteq \mathrm{U}(\mathfrak{gl}_n)$ is the kernel of the homomorphism $R$.

It is well known that $R$ is injective if (and only if) $k \geqslant n$. We describe a natural generating set for the ideal $\mathcal{J}_k$ when $0 \leqslant k \leqslant n-1$. Let $I_k$ be the ideal of the algebra of polynomial functions on $n \times n$ matrices generated by the minors of order $k+1$, the $k$th *determinantal ideal*. The ideal $I_k$ is prime and defines the affine algebraic variety of matrices whose rank is at most $k$ (see [3]). We prove (Proposition 3.9) that the associated graded ideal of the $k$th rank ideal is the $k$th determinantal ideal, $\operatorname{gr} \mathcal{J}_k = I_k$. We can get a system of generators of $\mathcal{J}_k$ by lifting the minors of order $k+1$ from $\operatorname{gr} \mathrm{U}(\mathfrak{gl}_n)$ to the noncommutative algebra $\mathrm{U}(\mathfrak{gl}_n)$. Thus, we need to find the elements $\mathsf{E}_{IJ} \in \mathcal{J}_k$ whose symbols are order $k+1$ minors $m_{IJ} \in I_k$. It turns out that the appropriate $\mathsf{E}_{IJ}$, which we call *quantum minors* of order $k+1$, are given by the following formula:

$$\mathsf{E}_{IJ} = \operatorname{col.det} \begin{vmatrix} \mathsf{E}_{i_1 j_1}(k) & \mathsf{E}_{i_1 j_2}(k-1) & \dots & \mathsf{E}_{i_1 j_{k+1}}(0) \\ \mathsf{E}_{i_2 j_1}(k) & \mathsf{E}_{i_2 j_2}(k-1) & \dots & \mathsf{E}_{i_2 j_{k+1}}(0) \\ \vdots & \vdots & \ddots & \vdots \\ \mathsf{E}_{i_{k+1} j_1}(k) & \mathsf{E}_{i_{k+1} j_2}(k-1) & \dots & \mathsf{E}_{i_{k+1} j_{k+1}}(0) \end{vmatrix}. \tag{1.3}$$

Here $I$ and $J$ are $k+1$-term sequences of row and column indices and $\mathsf{E}_{ij}(a) = \mathsf{E}_{ij} + a\delta_{ij}$, and col.det is the so-called column determinant (see [11,24] and Definition 2.1). In the special case when $I = J$ is an increasing sequence of $k+1$ indices between 1 and $n$, the elements $\mathsf{E}_{II}$ were introduced by A. Capelli. The sum $C_{k+1} = \sum_I \mathsf{E}_{II}$ of the principal quantum minors belongs to the center of $\mathrm{U}(\mathfrak{gl}_n)$ and appears in the Capelli identity (see [1, 2,11]). M. Nazarov and A. Okounkov introduced a family of noncentral higher Capelli-type elements of $\mathrm{U}(\mathfrak{gl}_n)$ depending on a partition $\lambda$ and proved the corresponding higher Capelli identities (see [20,21]). For $\lambda = (1^{k+1})$ the Capelli-type elements are the quantum minors $\mathsf{E}_{IJ}$ with arbitrary $I$, $J$ and an appropriate Capelli-type identity shows that $R(\mathsf{E}_{IJ}) = 0$. Hence, $\mathsf{E}_{IJ} \in \mathcal{J}_k$, and we obtain the following theorem.

**Theorem 1.4.** *The quantum minors $\mathsf{E}_{IJ}$ of order $k+1$ generate the $k$th rank ideal $\mathcal{J}_k$.*

The algebra $\mathrm{U}(\mathfrak{gl}_n)$ is a noncommutative deformation of the commutative algebra of polynomial functions of $n \times n$ matrices. Thus, the theorem may be interpreted as saying that the rank ideal $\mathcal{J}_k$ and the quantum minors of order $k+1$ "quantize" the determinantal ideal $I_k$ together with its natural generating set. A noncommutative analogue of the First Fundamental Theorem of Classical Invariant Theory for $GL_k$ due to R. Howe (see [9,10]) asserts that the algebra of $GL_k$-invariants in $\mathcal{PD}(\mathsf{M}_{k,n})$ is generated by the polarization operators $\mathsf{D}_{ij}$, and hence isomorphic to $\mathrm{U}(\mathfrak{gl}_n)/\mathcal{J}_k$. Theorem 1.4 provides explicit defining relations for this algebra analogous to the Second Fundamental Theorem of Classical Invariant Theory for $GL_k$ (see [4,10]).

We use rank ideals to introduce the notion of rank for modules over the Lie algebra $\mathfrak{gl}_n$ as follows.

**Definition 1.5.** A $\mathfrak{gl}_n$-module $M$ has rank at most $k$ if its annihilator $\mathrm{Ann}_{\mathrm{U}(\mathfrak{gl}_n)} M$ contains the $k$th rank ideal $\mathcal{J}_k$.

The notion of rank for $\mathfrak{gl}_n$-modules is analogous to the $N$-rank (or *singular rank*) for unitary representations of classical Lie groups defined by R. Howe and J.-S. Li, see [7,16, 17]. In these papers the $N$-rank was tied with the *dual pair correspondence.* Our notion of rank reflects the dual pair correspondence algebraically. If $(G, G')$ is a reductive dual pair of real Lie groups with complexified Lie algebras $\mathfrak{gl}_n$ and $\mathfrak{gl}_k$ and $M$ is a $\mathfrak{gl}_n$-module that occurs in the correspondence, then $M$ has rank at most $k$.

We use explicit determinantal formulas for the generators of rank ideals to determine the rank of simple highest weight modules $L(\lambda)$. Our main result, Theorem 4.6, gives simple necessary and sufficient conditions for $L(\lambda)$ to be of rank at most $k$, expressed in terms of the highest weight $\lambda$. In particular, we determine all finite-dimensional simple $\mathfrak{gl}_n$-modules that have rank at most $k$. It turns out that these modules may be parametrized by pairs of Young tableaux with the combined depth at most $k$. The exact statement is given in Theorem 4.11. Theorem 4.6 may also be applied to the description of the primitive ideals of $\mathrm{U}(\mathfrak{gl}_n)$ containing the $k$th rank ideal (see [22]).

## 2. Quantum minors

In this section we define the quantum minors in $\mathrm{U}(\mathfrak{gl}_n)$ and discuss their properties.

As in the introduction, for any $a \in \mathbb{C}$ we let $\mathsf{E}_{ij}(a) = \mathsf{E}_{ij} + a\delta_{ij} \in \mathrm{U}(\mathfrak{gl}_n)$ ($\delta_{ij}$ is the Kronecker delta). For a fixed $a$ the assignment $\mathsf{E}_{ij} \mapsto \mathsf{E}_{ij}(a)$ extends to an algebra automorphism of $\mathrm{U}(\mathfrak{gl}_n)$ called the $a$-twist.

**Definition 2.1.** Suppose $0 \leqslant k \leqslant n-1$. Let $I$ and $J$ be two $k+1$-element sequences of indices between 1 and $n$. A *quantum minor* $\mathsf{E}_{IJ}$ of order $k+1$ with the row indices $I$ and the column indices $J$ is an element of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$ which is the column determinant of the $(k+1)\times(k+1)$ matrix with the entries $\mathsf{E}_{i_r j_s}(k+1-s)$:

$$\begin{aligned}\mathsf{E}_{IJ} &= \mathrm{col.det}\begin{vmatrix} \mathsf{E}_{i_1 j_1}(k) & \mathsf{E}_{i_1 j_2}(k-1) & \dots & \mathsf{E}_{i_1 j_{k+1}}(0) \\ \mathsf{E}_{i_2 j_1}(k) & \mathsf{E}_{i_2 j_2}(k-1) & \dots & \mathsf{E}_{i_2 j_{k+1}}(0) \\ \vdots & \vdots & \ddots & \vdots \\ \mathsf{E}_{i_{k+1} j_1}(k) & \mathsf{E}_{i_{k+1} j_2}(k-1) & \dots & \mathsf{E}_{i_{k+1} j_{k+1}}(0) \end{vmatrix} \\ &= \sum_{\sigma\in S_{k+1}} \mathrm{sgn}(\sigma) \prod_{s=1}^{k+1} \mathsf{E}_{i_{\sigma(r)} j_s}(k+1-s) \qquad (2.2)\end{aligned}$$

(the order of the factors in each product is determined by the order of their columns: first $j_1$, then $j_2$, and so on). Similarly, the row quantum minor is the row determinant of the $(k+1)\times(k+1)$ matrix with the entries $\mathsf{E}_{i_r j_s}(r-1)$:

$$\begin{aligned}\mathsf{E}_{IJ}^{\mathrm{row}} &= \mathrm{row.det}\begin{vmatrix} \mathsf{E}_{i_1 j_1}(0) & \mathsf{E}_{i_1 j_2}(0) & \dots & \mathsf{E}_{i_1 j_{k+1}}(0) \\ \mathsf{E}_{i_2 j_1}(1) & \mathsf{E}_{i_2 j_2}(1) & \dots & \mathsf{E}_{i_2 j_{k+1}}(1) \\ \vdots & \vdots & \ddots & \vdots \\ \mathsf{E}_{i_{k+1} j_1}(k) & \mathsf{E}_{i_{k+1} j_2}(k) & \dots & \mathsf{E}_{i_{k+1} j_{k+1}}(k) \end{vmatrix} \\ &= \sum_{\sigma\in S_{k+1}} \mathrm{sgn}(\sigma)\prod_{r=1}^{k+1} \mathsf{E}_{i_r j_{\sigma(s)}}(r-1) \end{aligned} \tag{2.3}$$

(the order of the factors in each product is determined by the order of the rows).

**Remarks.**

1. Quantum minors are defined via noncommutative determinants and are clearly analogous to ordinary matrix minors. However, due to the presence of twists in (2.2) and (2.3), the quantum minors of order $k+1$ cannot be viewed as subdeterminants of order $k+1$ of a *single* matrix. Unlike in the case of the Capelli determinant, the properties of quantum minors do not follow directly from the general theory of quantum determinants and quasi-determinants developed by D. Krob, G. Leclerc, I. Gelfand, V. Retakh and their coauthors (see [5,14]).
2. The row and column quantum minors arise from *different* matrices. Nevertheless, $\mathsf{E}_{IJ} = \mathsf{E}_{IJ}^{\mathrm{row}}$ (Proposition 3.10).
3. Quantum minors in the sense of Definition 2.1 are elements of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$. Note that in the literature the term "quantum minor" is also used for certain elements in two other algebras, the *quantized coordinate ring of $n\times n$ matrices* and the *quantized enveloping algebra of upper triangular matrices* (see [6,15] and references therein). The definition of the latter elements does not involve twists. It would be interesting to study analogues of quantum minors in the quantized universal enveloping algebra of $\mathfrak{gl}_n$.

We record several properties of quantum minors. Several related statements appear in the literature (see, for example, [9,12,21]). Our point of view is close to that of Itoh's paper.

**Proposition 2.4.** *Quantum minors are skew-symmetric in the row and column indices.*

**Proof.** The skew-symmetry in the row indices is an immediate corollary of the definition (2.2). To establish the skew-symmetry in the column indices, it is enough to consider the case of two *consecutive* columns with the column indices $j$ and $m$, namely to show that

$$\mathrm{col.det}\begin{vmatrix} \mathsf{E}_{ij}(a+1) & \mathsf{E}_{im}(a) \\ \mathsf{E}_{lj}(a+1) & \mathsf{E}_{lm}(a) \end{vmatrix} = -\,\mathrm{col.det}\begin{vmatrix} \mathsf{E}_{im}(a+1) & \mathsf{E}_{ij}(a) \\ \mathsf{E}_{lm}(a+1) & \mathsf{E}_{lj}(a) \end{vmatrix}. \tag{2.5}$$

From the commutation relations between the generators of $\mathfrak{gl}_n$ we obtain the identity

$$\operatorname{col.det}\begin{vmatrix} \mathsf{E}_{ij}(a+1) & \mathsf{E}_{im}(a) \\ \mathsf{E}_{lj}(a+1) & \mathsf{E}_{lm}(a) \end{vmatrix} = \operatorname{row.det}\begin{vmatrix} \mathsf{E}_{ij}(a) & \mathsf{E}_{im}(a) \\ \mathsf{E}_{lj}(a+1) & \mathsf{E}_{lm}(a+1) \end{vmatrix}, \tag{2.6}$$

which shows that each column subdeterminant of (2.2) of order 2 involving these columns is skew-symmetric in $j$ and $m$. The skew-symmetry of $\mathsf{E}_{IJ}$ follows by the Laplace expansion of the column determinant (2.2) in the chosen two consecutive columns. □

The statement and the proof of the proposition remain valid for the row quantum minors.

The universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$ is a filtered algebra and its associated graded algebra $\operatorname{gr}\mathrm{U}(\mathfrak{gl}_n)$ is naturally identified with the algebra of polynomials in $n^2$ variables $m_{ij}$, $1 \leqslant i, j \leqslant n$, by letting $\operatorname{gr}\mathsf{E}_{ij} = m_{ij}$. We view $m_{ij}$ as the coordinate functions on the space $M_{n,n}$ of the square matrices of order $n$ and arrange them into a generic $n \times n$ matrix $M$. For any sequences $I$ and $J$ of indices between 1 and $n$ of the same length $d$, denote by $m_{IJ}$ the determinant of the matrix with the entries $m_{i_r j_s}$, where the row indices $i_r \in I$ and the column indices $j_s \in J$, $1 \leqslant r, s \leqslant d$. If $I$ and $J$ are increasing sequences, $m_{IJ}$ is the minor of $M$ with the rows $I$ and the columns $J$.

**Proposition 2.7.** *Under the identification above, the symbol of a quantum minor is the corresponding minor*: $\operatorname{gr}\mathsf{E}_{IJ} = m_{IJ}$. *Similarly,* $\operatorname{gr}\mathsf{E}^{\mathrm{row}}_{IJ} = m_{IJ}$.

**Proof.** The proposition follows immediately from the skew-symmetry of ordinary and quantum minors and the standard properties of the symbol map. □

Let $e_i$ and $e_j^*$ be the standard bases of $\mathbb{C}^n$ and $\mathbb{C}^{n*}$. Set $e_I = e_{i_1} \wedge \cdots \wedge e_{i_{k+1}} \in \bigwedge^{k+1}\mathbb{C}^n$ and $e_J^* = e_{j_1}^* \wedge \cdots \wedge e_{j_{k+1}}^* \in \bigwedge^{k+1}\mathbb{C}^{n*}$.

**Proposition 2.8.** *The linear span $S$ in $\mathrm{U}(\mathfrak{gl}_n)$ of the quantum minors of order $k+1$ is stable under the adjoint action of $\mathfrak{gl}_n$ on its universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$. Moreover, the assignment $e_I \wedge e_J^* \mapsto \mathsf{E}_{IJ}$ extends to a $\mathfrak{gl}_n$-module isomorphism between $\bigwedge^{k+1}\mathbb{C}^n \otimes \bigwedge^{k+1}\mathbb{C}^{n*}$ and $S$.*

**Proof.** Propositions 2.4 and 2.7 yield a vector space isomorphism between the linear span of the minors of order $k+1$ and $S$, hence between $\bigwedge^{k+1}\mathbb{C}^n \otimes \bigwedge^{k+1}\mathbb{C}^{n*}$ and $S$. The equivariance of this isomorphism under the action of $\mathfrak{gl}_n$ is established by a direct computation as in the proof of Theorem 5.1 in [21]. □

**Proposition 2.9.** *Each quantum minor $\mathsf{E}_{IJ}$ and row quantum minor $\mathsf{E}^{\mathrm{row}}_{IJ}$ of order $k+1$ belongs to the $k$th rank ideal.*

**Proof.** Let us denote $\mathsf{D}_{IJ} = R(\mathsf{E}_{IJ})$, $\mathsf{D}^{\mathrm{row}}_{IJ} = R(\mathsf{E}^{\mathrm{row}}_{IJ})$. We need to prove that these expressions are identically zero for any $k+1$-element sequences $I$ and $J$. The generalized Capelli identity [21, Theorem 1.3 and Eq. 1.4] essentially shows that

$$\mathsf{D}^{\mathrm{row}}_{IJ} = \sum_{\sigma\in S_{k+1}} \sum_{a_1,\dots,a_{k+1}} \operatorname{sgn}(\sigma) x_{a_1 i_{\sigma(1)}} \dots x_{a_{k+1} i_{\sigma(k+1)}} \partial_{a_1 j_1} \dots \partial_{a_{k+1} j_{k+1}}. \tag{2.10}$$

Here the second summation is over a $k+1$-element sequence $A$ of indices between 1 and $k$. Identity (2.10) was proved in [21] only when $R$ is an injection ($k \geqslant n$), whereas we are interested in the opposite case when $R$ has a non-trivial kernel. Nevertheless, (2.10) holds in general. As pointed out by the referee, a short proof of the generalized Capelli identity and hence of (2.10) for arbitrary $k$ was found by A. Molev, see [18]. An easy modification of the argument in [18] shows that (2.10) also holds with $\mathsf{D}_{IJ}$ in place of $\mathsf{D}^{\mathrm{row}}_{IJ}$. The summands in the right-hand side of (2.10) are skew-symmetric with respect to permutations of the sequence $a_1, \dots, a_{k+1}$. Since $1 \leqslant a_1, \dots, a_{k+1} \leqslant k$, the right-hand side of (2.10) is identically zero. □

## 3. Generators of rank ideals

The purpose of this section is to prove that the quantum minors of order $k+1$ generate the $k$th rank ideal $\mathcal{J}_k$ (Theorem 1.4). Recall that the ideal $\mathcal{J}_k$ was defined as the kernel of a certain homomorphism $R : \mathrm{U}(\mathfrak{gl}_n) \to \mathcal{PD}(\mathsf{M}_{k,n})$. Both $\mathrm{U}(\mathfrak{gl}_n)$ and $\mathcal{PD}(\mathsf{M}_{k,n})$ are filtered algebras, and we define and explicitly compute the corresponding graded map $\overline{R}$. We establish that $\operatorname{gr} \mathcal{J}_k = \ker \overline{R}$ and identify this ideal with the $k$th determinantal ideal $I_k$. We then deduce that $\mathcal{J}_k$ is generated by the set of quantum minors of order $k+1$ from the corresponding generation property of determinantal ideals.

For a filtered algebra $A = \bigcup_{p\geqslant 0} \mathcal{F}^p A$ we let $\operatorname{gr} A = \bigoplus \operatorname{gr}^p A$ be the associated graded algebra of $A$ and $\operatorname{gr} : A \to \operatorname{gr} A$ be the symbol map (see [13,19]). Let $A = \mathrm{U}(\mathfrak{gl}_n)$, $B = \mathcal{PD}(\mathsf{M}_{k,n})$ endowed with their standard filtrations.

**Proposition 3.1.** *For any $p \geqslant 0$ we have*

$$R\bigl(\mathcal{F}^p A\bigr) \subseteq \mathcal{F}^{2p} B.$$

**Proof.** For $p = 0$ both sides are equal to $\mathbb{C}$. The subspace $\mathcal{F}^1 A$ is spanned by 1 and $\mathsf{E}_{ij}$, and $\mathcal{F}^p A$ is spanned by $p$-fold products of the elements of $\mathcal{F}^1 A$. Since $R(\mathsf{E}_{ij})$ is a quadratic expression in the elements $x_{ai}$ and $\partial_{aj}$ which have degree 1 in $B$, $R(\mathsf{E}_{ij})$ belongs to $\mathcal{F}^2 B$. This establishes the proposition for $p = 1$. For general $p$ the subspace $R(\mathcal{F}^p A)$ coincides with the linear span of $p$-fold products of the elements of $R(\mathcal{F}^1 A) \subseteq \mathcal{F}^2 B$. Each such $p$-fold product is contained in $\mathcal{F}^{2p} B$. □

Generalizing the well-known concept of a degree-preserving filtered map, we say that $R$ is a 2-*filtered map*. Intuitively, we think of $R$ as a map that "doubles the degree." This is not quite correct, since the degree of $R(a)$ may be strictly smaller than twice the degree of $a$. Nevertheless, the proof of Proposition 3.5 below shows that for any $a \in A$ there exists $a' \in A$ such that $R(a') = R(a)$ and $\deg R(a') = 2\deg(a')$.

Proposition 3.1 allows us to introduce the associated graded map of $R$ as follows. The map $\overline{R} : \operatorname{gr} A \to \operatorname{gr} B$ is the homogeneous degree 2 homomorphism of graded algebras obtained from restrictions of $R$ to the subspaces in the filtration:

$$\overline{R}\big(\operatorname{gr}^p A\big) \subseteq \operatorname{gr}^{2p} B, \qquad \overline{R}\big(\operatorname{gr}^p a\big) = \operatorname{gr}^{2p} R(a) \quad \text{for any } a \in \mathcal{F}^p A.$$

Let us compute $\overline{R}$ and its kernel explicitly. Identify $\operatorname{gr}\mathcal{PD}(\mathsf{M}_{k,n})$ with the commutative algebra $\mathcal{P}(\mathsf{M}_{k,n} \oplus \mathsf{M}_{n,k})$ of polynomial functions of a pair $(X, Y)$ of matrices by $\operatorname{gr} x_{ai} = x_{ai}$, $\operatorname{gr} \partial_{ai} = y_{ia}$. Here $x_{ai}$ are the coordinate functions on $\mathsf{M}_{k,n}$, $y_{ia}$ are the coordinate functions on $\mathsf{M}_{n,k}$, $1 \leqslant a \leqslant k$, $1 \leqslant i \leqslant n$.

**Proposition 3.2.** *The associated graded map $\overline{R}$ is the homomorphism of commutative algebras dual to the map of algebraic varieties*

$$f : \mathsf{M}_{k,n} \oplus \mathsf{M}_{n,k} \to \mathsf{M}_{n,n}, \quad (X, Y) \mapsto X^{\mathrm{t}} Y^{\mathrm{t}}.$$

**Proof.** The map $r$ written in coordinates is

$$r(X, Y)_{ij} = \big(X^{\mathrm{t}} Y^{\mathrm{t}}\big)_{ij} = \sum_{a=1}^{k} x_{ai} y_{ja} = \operatorname{gr} \mathsf{D}_{ij}, \tag{3.3}$$

and $\operatorname{gr} \mathsf{D}_{ij} = \overline{R}(m_{ij})$. Thus, the dual map $r^* : \mathcal{P}(\mathsf{M}_{n,n}) \to \mathcal{P}(\mathsf{M}_{k,n} \oplus \mathsf{M}_{n,k})$ pulls back the coordinate function $m_{ij}$ on $\mathsf{M}_{n,n}$ to $\overline{R}(m_{ij})$, and $r^* = \overline{R}$. □

The map $r$ is the factorization map with respect to a natural $GL_k$-action on $\mathsf{M}_{k,n} \oplus \mathsf{M}_{n,k}$ (see [4]). The image of $r$ is the set of $n \times n$ matrices of rank at most $\min(k, n)$. Moreover, this set is closed in Zariski topology. Its defining ideal is called the $k$th *determinantal ideal* $I_k$. For $k \geqslant n$ we have $I_k = 0$. The ideal $I_k$ is generated by the minors of order $k + 1$ (see [3]). We have thus obtained the following description of the kernel of $\overline{R}$:

**Proposition 3.4.** $\ker \overline{R}$ *is the $k$th determinantal ideal $I_k$.*

**Proposition 3.5.** *The map $R$ has the property that* $\ker \overline{R} = \operatorname{gr}(\ker R)$.

**Proof.** The right inclusion is obvious. To prove that $\ker \overline{R} \subseteq \operatorname{gr} \ker R$ we use descriptions of $\operatorname{im} R$ and $\operatorname{im} \overline{R}$ due to R. Howe. The group $GL_k$ acts on $\mathcal{PD}(\mathsf{M}_{k,n})$ by conjugation. Let us denote by $\mathcal{PD}(\mathsf{M}_{k,n})^{GL_k}$ the subalgebra of invariants. It consists of the polynomial coefficient differential operators on $\mathsf{M}_{k,n}$ commuting with the action of $GL_k$ on $\mathsf{M}_{k,n}$ by left multiplication. By [9, Theorem 7] we have that

$$\operatorname{gr} \operatorname{im} R = \operatorname{gr}\big(R\big(\mathsf{U}(\mathfrak{gl}_n)\big)\big) = \big(\operatorname{gr}\mathcal{PD}(\mathsf{M}_{k,n})\big)^{GL_k} = \mathcal{P}(\mathsf{M}_{k,n} \oplus \mathsf{M}_{n,k})^{GL_k}. \tag{3.6}$$

Moreover, this algebra is generated by the elements $\overline{R}(m_{ij})$, which form a basis in the homogeneous component of degree 2. The inductive argument of [9] then shows that any $b \in R(a)$ of degree less than $2p$ belongs to the linear span of $q$-fold products, $q < p$, of $\mathsf{D}_{ij} = R(\mathsf{E}_{ij})$. In particular,

$$\operatorname{im} R = \mathcal{PD}(\mathsf{M}_{k,n})^{GL_k}, \tag{3.7}$$

and this algebra is generated by $\mathsf{D}_{ij}$, $1 \leqslant i, j \leqslant n$. If $\bar{a} \in \ker \overline{R}$ is a homogeneous element of degree $p$, then there exists $a \in A$ such that $\bar{a} = \operatorname{gr} a$, $\deg a = p$, and since $\overline{R}(a) = 0$, we have $\deg R(a) < 2p$. Expressing $R(a)$ as a linear combination of $q$-fold products of $\mathsf{D}_{ij}$ with $q < p$, we find that $R(a) = R(a')$ for some $a' \in A$ with $\deg a' < p$. We conclude that $a - a' \in \ker R$ and $\deg(a - a') = p$, whence $\operatorname{gr}(a - a') = \operatorname{gr} a = \bar{a}$ and $\bar{a}$ belongs to $\operatorname{gr}\ker R$. $\square$

**Remark 3.8.** The proof shows that if $R : A \to B$ is a 2-filtered map of filtered algebras, then the equality $\ker \overline{R} = \operatorname{gr}\ker R$ is equivalent to $\operatorname{im} \overline{R} = \operatorname{gr}\operatorname{im} R$. This fact is analogous to a well-known property of degree-preserving filtered maps [19, Theorem D.III.3].

Combining Propositions 3.4 and 3.5, we arrive at the relationship between the $k$th rank ideal $\mathcal{J}_k = \ker R$ and the $k$th determinantal ideal.

**Proposition 3.9.** *The associated graded ideal* $\operatorname{gr}\mathcal{J}_k$ *is the* $k$*th determinantal ideal.*

We are now in a position to prove that the set of quantum minors of order $k+1$ generates the $k$th rank ideal.

**Proof of Theorem 1.4.** To prove that a system of elements of a filtered algebra generates a given ideal, it is sufficient to show that their symbols generate the associated graded ideal. By Proposition 2.9, the quantum minors $\mathsf{E}_{IJ}$ of order $k+1$ belong to the $k$th rank ideal $\mathcal{J}_k$. Proposition 2.7 asserts that their symbols are ordinary minors of order $k+1$, which generate the $k$th determinantal ideal $I_k$. According to Proposition 3.9, $\operatorname{gr}\mathcal{J}_k = I_k$. Therefore, the quantum minors of order $k+1$ generate $\mathcal{J}_k$. $\square$

**Proposition 3.10.** *For any index sequences* $I$ *and* $J$ *of length* $k+1$, *the corresponding row and column quantum minors in* $\mathrm{U}(\mathfrak{gl}_n)$ *coincide*:

$$\mathsf{E}_{IJ} = \mathsf{E}_{IJ}^{\mathrm{row}}. \tag{3.11}$$

**Proof.** If $I$ or $J$ contains a repeating index then both sides are equal to zero by Proposition 2.4. The symbol map is injective on the elements of the lowest degree in $\mathcal{J}_k$. Observe that non-zero minors $m_{IJ}$ are elements of the lowest degree in $\operatorname{gr}\mathcal{J}_k = I_k$. Since $\mathsf{E}_{IJ}$ and $\mathsf{E}_{IJ}^{\mathrm{row}}$ both belong to $\mathcal{J}_k$ and have the same symbol $m_{IJ}$, we must have $\mathsf{E}_{IJ} = \mathsf{E}_{IJ}^{\mathrm{row}}$. $\square$

**Corollary 3.12.** *The rank ideals form a descending filtration on* $\mathrm{U}(\mathfrak{gl}_n)$:

$$\mathrm{U}(\mathfrak{gl}_n) \supseteq \mathcal{J}_0 \supseteq \mathcal{J}_1 \supseteq \cdots \supseteq \mathcal{J}_n = 0.$$

**Proof.** Let us expand a quantum minor $\mathsf{E}_{IJ}$ of order $k+1$ in the first column. We see that $\mathsf{E}_{IJ}$ is a sum of the products of $\mathsf{E}_{i_r j_1}(k)$ and a quantum minor of order $k$. By Theorem 1.4 we conclude that the generators $\mathsf{E}_{IJ}$ of the ideal $\mathcal{J}_k$ belong to $\mathcal{J}_{k-1}$. Therefore, $\mathcal{J}_{k-1} \supseteq \mathcal{J}_k$. For $k \geqslant n$ the associated graded ideal of $\mathcal{J}_k$ is 0, hence $\mathcal{J}_k = 0$. □

## 4. Rank of highest weight modules

One of the main applications of the rank ideals is a notion of rank for $\mathfrak{gl}_n$-modules introduced in Definition 1.5. Recall that a $\mathfrak{gl}_n$-module $M$ has rank at most $k$ if its annihilator $\mathrm{Ann}_{\mathrm{U}(\mathfrak{gl}_n)} M$ contains the $k$th rank ideal $\mathcal{J}_k$. In this section we determine which simple highest weight $\mathfrak{gl}_n$-modules $L(\lambda)$ have rank at most $k$. This determination relies on the explicit formulas for the generators of $\mathcal{J}_k$ and proceeds as follows. By Theorem 1.4, the quantum minors of order $k+1$ generate $\mathcal{J}_k$. We analyze their action on the highest weight vector and obtain in Theorem 4.6 the necessary and sufficient conditions on the highest weight $\lambda$ assuring that a simple highest weight module $L(\lambda)$ has rank at most $k$.

Let us fix the choice of Borel and Cartan subalgebras of $\mathfrak{gl}_n$ and the notation concerning the weights. The diagonal matrices form a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{gl}_n$, and the upper triangular matrices form a Borel subalgebra $\mathfrak{b} \subseteq \mathfrak{gl}_n$. The nilradical $\mathfrak{n}$ of this Borel subalgebra consists of the upper triangular matrices with zeros on the diagonal and is spanned by the generators $\mathsf{E}_{ij} \in \mathfrak{gl}_n$ with $1 \leqslant i < j \leqslant n$. Let $\mathbb{C}^n$ be the standard $n$-dimensional vector space with the basis $\{\epsilon_i \mid 1 \leqslant i \leqslant n\}$. The space $\mathfrak{h}^*$ of linear functionals on $\mathfrak{h}$ is identified with $\mathbb{C}^n$ via $\epsilon_i(\mathsf{E}_{jj}) = \delta_{ij}$. A weight $\mu \in \mathfrak{h}^*$ is represented by an $n$-tuple of complex numbers $\mu_1, \ldots, \mu_n$ which are its components in the basis $\epsilon$. A vector $v$ in a $\mathfrak{gl}_n$-module $M$ is a *highest weight vector* of weight $\lambda$ if $v$ is non-zero, $\mathsf{E}_{ij} v = 0$ for $i < j$ and $\mathsf{E}_{ii} v = \lambda_i v$.

Suppose that $M$ is a highest weight module over $\mathfrak{gl}_n$ with the highest weight vector $v$ of weight $\lambda$. Let $I = (i_1, \ldots, i_k, i_{k+1})$ be an *increasing* sequence of indices between 1 and $n$, and let us compute the effect of the principal quantum minor $\mathsf{E}_{II}$ on $v$. Using the full expansion of $\mathsf{E}_{II}$ as a column determinant, we see that

$$\mathsf{E}_{II} v = \sum_{\sigma \in S_{k+1}} \mathrm{sgn}(\sigma) \prod_{1 \leqslant s \leqslant k+1} \mathsf{E}_{i_{\sigma(s)} i_s}(k+1-s) v. \tag{4.1}$$

We claim that in this sum all the terms with $\sigma \neq \mathrm{id}$ are equal to 0. Indeed, suppose that $\sigma \neq \mathrm{id}$ and let $r$ be the largest integer between 1 and $k+1$ such that $\sigma(r) \neq r$. Since $\sigma$ is a bijection and $\sigma(s) = s$ for $r+1 \leqslant s \leqslant k+1$, we have $\sigma(r) < r$, hence $i_{\sigma(r)} < i_r$. Thus, $\mathsf{E}_{i_{\sigma(s)} i_s} = \mathsf{E}_{i_s i_s} \in \mathfrak{h}$ for $r+1 \leqslant s \leqslant k+1$ and $\mathsf{E}_{i_{\sigma(r)} i_r}(k+1-r) = \mathsf{E}_{i_{\sigma(r)} i_r} \in \mathfrak{n}$. Since $v$ is a highest weight vector, elements of $\mathfrak{h}$ multiply $v$ by a constant, and any element of $\mathfrak{n}$ annihilates $v$. We see that the last $k+1-r$ terms in the product in (4.1) multiply $v$ by a constant, and the term with $s = r$ maps the result to 0, so that the entire product is 0. Therefore, the sum in (4.1) reduces to $\prod_{1 \leqslant s \leqslant k+1} \mathsf{E}_{i_s i_s}(k+1-s) v$ corresponding to $\sigma = \mathrm{id}$. We have proved the following proposition.

**Proposition 4.2.** *Suppose that $M$ is a highest weight module with the highest weight vector $v$ of weight $\lambda$. Let $1 \leqslant i_1 < \cdots < i_k < i_{k+1} \leqslant n$. Then the principal quantum minor $\mathsf{E}_{II}$ of order $k+1$ multiplies $v$ by the scalar $\prod_{1\leqslant s\leqslant k+1}(\lambda_{i_s} + k + 1 - s) = (\lambda_{i_1} + k)\cdots(\lambda_{i_k} + 1)\lambda_{i_{k+1}}$.*

Let us see how the proposition works for $k = 1$. We set $I = (i, j)$ with $i < j$ and find that

$$\mathsf{E}_{II} v = \text{col.det}\begin{vmatrix} \mathsf{E}_{ii}(1) & \mathsf{E}_{ij} \\ \mathsf{E}_{ji}(1) & \mathsf{E}_{jj} \end{vmatrix} v = \mathsf{E}_{ii}(1)\mathsf{E}_{jj} v - \mathsf{E}_{ji}(1)\mathsf{E}_{ij} v = (\lambda_i + 1)\lambda_j v. \tag{4.3}$$

The last equality holds because $v$ is a highest weight vector of weight $\lambda$, hence $\mathsf{E}_{ii}(1)v = (\mathsf{E}_{ii} + 1)v = (\lambda_i + 1)v$, $\mathsf{E}_{jj} v = \lambda_j v$, and $\mathsf{E}_{ij} v = 0$.

Let $L(\lambda)$ be the simple highest weight $\mathfrak{gl}_n$-module with highest weight $\lambda$. We want to describe all values of $\lambda$ for which $L(\lambda)$ has rank at most $k$. Since any quantum minor $\mathsf{E}_{IJ}$ of order $k+1$ belongs to the $k$th rank ideal, the action of $\mathsf{E}_{IJ}$ on a module of rank at most $k$ is identically zero. In particular, $\mathsf{E}_{II} v = 0$ for the highest weight vector $v$. Proposition 4.2 thus implies a number of necessary conditions on a highest weight $\lambda$ such that $L(\lambda)$ has rank at most $k$.

**Example 4.4.** Suppose that $L(\lambda)$ has rank at most 1. Equation (4.3) shows that for any $i < j$ we must have $(\lambda_i + 1)\lambda_j = 0$. This translates easily into the condition that for some $1 \leqslant p \leqslant n$ the first $p - 1$ entries of $\lambda$ are equal to $-1$ and the last $n - p$ entries are equal to 0. Therefore, $\lambda$ must have the form

$$\lambda = (-1, \ldots, -1, \lambda_p, 0, \ldots, 0). \tag{4.5}$$

It turns out that for any $\lambda$ of the form (4.5) the module $L(\lambda)$ has rank at most 1. Thus, the conditions provided by Proposition 4.2 are also sufficient. In fact, for an arbitrary $k$, we are going to completely describe the highest weights $\lambda$ such that $L(\lambda)$ has rank at most $k$.

**Theorem 4.6.** *Let $\lambda \in \mathbb{C}^n$ and $1 \leqslant k \leqslant n - 1$. Then the following are equivalent*:

(1) *The simple highest weight module $L(\lambda)$ is of rank at most $k$.*

(2) *There exists a sequence of indices $i_0 = 0 < i_1 < i_2 < \cdots < i_k < i_{k+1}$ with $i_{k+1} > n$ such that for any $1 \leqslant s \leqslant k+1$, $i_{s-1} < i < i_s$ implies $\lambda_i = -k - 1 + s$. In other words, $\lambda$ has the form*

$$\lambda = (-k, \ldots, -k, *, -k+1, \ldots, -k+1, *, \ldots, -1, \ldots, -1, *, 0, \ldots, 0), \tag{4.7}$$

*where the stars represent the entries of $\lambda$ with indices $i_1, i_2, \ldots, i_k$.*

(3) *The sequence $(\lambda_1 + n - 1, \lambda_2 + n - 2, \ldots, \lambda_n)$ contains a subsequence $(n - k - 1, n - k - 2, \ldots, 0)$, the arithmetic progression with the first term $n - k - 1$ and the difference $-1$.*

**Proof.** To prove that (2) implies (3), let us assume that $\lambda$ satisfies the condition of (2). Denote by $\{a_i\}$ the sequence $a_i = \lambda_i + n - i$ and by $\{b_j\}$ its subsequence consisting of the first $n-k$ terms with indices not equal to $i_1, i_2, \ldots, i_k$. We claim that $\{b_j\}$ is the arithmetic progression described in (3). Indeed, suppose that $i_{s-1} < i < i_s$, then $\lambda_i = -k - 1 + s$, $a_i = n - k - 1 + s - i$, and $a_i = b_j$ with $j = i - s + 1$. Therefore, $b_j = n - k - 1 + s - i = n - k - (s - i + 1) = n - k - j$. Clearly, this argument is reversible, proving the equivalence of (2) and (3).

Assume that the module $L(\lambda)$ is of rank at most $k$. Denote by $v$ its highest weight vector. We want to show that $\lambda$ has the special form (4.7), establishing the implication (1) $\Rightarrow$ (2). Let us define inductively a sequence of indices $i_0 < i_1 < \cdots < i_k < i_{k+1}$: $i_0 = 0$, $i_s = \min\{i > i_{s-1} \mid \lambda_i \neq -k-1+s\}$. In order for this definition to always work, we declare that the condition on $\lambda_i$ holds for all $i > n$. If in the resulting sequence $i_{k+1} > n$ then $\lambda$ has the form (4.7). Let us show that the other possibility, namely that $i_{k+1} \leqslant n$, leads to a contradiction. Let $I = (i_1, \ldots, i_k, i_{k+1})$. By construction, $\lambda_{i_s} + k + 1 - s \neq 0$ for any $1 \leqslant s \leqslant k+1$, hence the product of these numbers is non-zero. On the other hand, by Proposition 4.2 this product is exactly the scalar by which the principal quantum minor $\mathsf{E}_{II}$ of order $k+1$ multiplies the highest weight vector $v$. Since $\mathsf{E}_{II}$ belongs to the $k$th rank ideal and does not act by zero on $v \in L(\lambda)$, the module $L(\lambda)$ cannot be of rank at most $k$, a contradiction.

In order to prove the converse implication (2) $\Rightarrow$ (1), for any $\lambda$ satisfying condition (2) of the theorem we construct a module $M$ over the algebra of polynomial coefficient differential operators $\mathcal{PD}(\mathsf{M}_{k,n})$ and a vector $v$ in $M$ which is a highest weight of weight $\lambda$ with respect to the ensuing action of $\mathfrak{gl}_n$ on $M$. Here it is necessary to recall that the polarization operators $\mathsf{D}_{ij}$ span a Lie subalgebra $\mathfrak{gl}_n$ inside $\mathcal{PD}(\mathsf{M}_{k,n})$, and a module over $\mathcal{PD}(\mathsf{M}_{k,n})$ becomes a module over $\mathfrak{gl}_n$ by restriction. A $\mathfrak{gl}_n$-module arising in this fashion necessarily has rank at most $k$. Let us see how the existence of $M$ and $v$ with these properties implies that $L(\lambda)$ has rank at most $k$ (condition (1) of the theorem). Denote by $N \subseteq M$ the $\mathfrak{gl}_n$-submodule of $M$ generated by $v$. The module $N$ is a cyclic highest weight module of highest weight $\lambda$ and admits $L(\lambda)$ as a quotient. Therefore, $\operatorname{Ann} L(\lambda) \supseteq \operatorname{Ann} N \supseteq \operatorname{Ann} M \supseteq \mathcal{J}_k$, with all annihilators taken in $\mathrm{U}(\mathfrak{gl}_n)$, so that $L(\lambda)$ is of rank at most $k$.

**Proposition 4.8.** *For any* $1 \leqslant i_1 < i_2 < \cdots < i_k$ *there exists a module* $M$ *over* $\mathcal{PD}(\mathsf{M}_{k,n})$ *and a non-zero vector* $v$ *in* $M$ *with the following properties* $(1 \leqslant t \leqslant k, 1 \leqslant i \leqslant n)$:

(i) $x_{ti} v = 0$ *for* $1 \leqslant i < i_t$;
(ii) $\partial_{ti} v = Q$ *for* $i > i_t$;
(iii) $x_{ti} \partial_{ti} v = (\lambda_i + k - t) v$ *for* $i = i_t$.

We defer the proof of the proposition, and check that the vector $v$ is indeed a highest weight vector of weight $\lambda$ with respect to the $\mathfrak{gl}_n$-action on $M$. We need to show that $\mathsf{E}_{ij} v = 0$ for any $i < j$ and that $\mathsf{E}_{ii} v = \lambda_i v$. Suppose that $i < j$. Then for any $t$ we have either $i < i_t$ or $j > i_t$. In the first case $x_{ti}v = 0$, and so $x_{ti}\partial_{tj}v = \partial_{tj}x_{ti}v = 0$. In the second case $\partial_{tj}v = 0$, hence $x_{ti}\partial_{tj}v = 0$. Therefore,

$$\mathsf{E}_{ij}v = \sum_{t=1}^{k} x_{ti}\partial_{tj}v = 0.$$

To find the weight of $v$, we consider the following two cases. Suppose that $i_{s-1} < i < i_s$. If $t < s$ then $i > i_t$ and $\partial_{ti}v = 0$, hence $x_{ti}\partial_{ti}v = 0$. If $t \geqslant s$ then $i < i_t$ and $x_{ti}v = 0$, hence $x_{ti}\partial_{ti}v = (\partial_{ti}x_{ti} - 1)v = -v$. Therefore,

$$\mathsf{E}_{ii}v = \sum_{t=1}^{k} x_{ti}\partial_{ti}v = \sum_{t=1}^{s-1} 0 + \sum_{t=s}^{k} -v = -(k-s+1)v = \lambda_i v.$$

Suppose now that $i = i_s$. Then by the preceding argument and property (iii) of $v$, $x_{ti}\partial_{ti}v$ is equal to 0, $(\lambda_i + k - s)v$, or $-v$ according to whether $t < s$, $t = s$, or $t > s$. Therefore, in this case we have

$$\mathsf{E}_{ii}v = \sum_{t=1}^{k} x_{ti}\partial_{ti}v = \sum_{t=1}^{s-1} 0 + (\lambda_i + k - s)v + \sum_{t=s+1}^{k} -v = \lambda_i v.$$

Thus, $\mathsf{E}_{ii}v = \lambda_i v$ for all $i$, and $v$ is a highest weight vector of weight $\lambda$. To complete the proof of the theorem, it only remains to prove Proposition 4.8. $\square$

**Proof of Proposition 4.8.** The algebra $\mathcal{PD}(\mathsf{M}_{k,n})$ is a tensor product over $1 \leqslant t \leqslant k$, $1 \leqslant i \leqslant n$ of its subalgebras $\mathbb{C}[x_{ti}, \partial_{ti}]$ of differential operators acting on only *one* variable $x_{ti}$. The conditions of the proposition involve only *one* variable $x_{ti}$ and the corresponding partial derivative $\partial_{ti}$. We may therefore let $M = \bigotimes M_{ti}$ and $v = \bigotimes v_{ti}$, where $M_{ti}$ is a module over the subalgebra $\mathbb{C}[x_{ti}, \partial_{ti}]$, and $v_{ti} \in M_{ti}$ has the relevant property (i), (ii), or (iii). We have reduced the proof of the proposition to its special case $k = n = 1$. Let us omit $t$ and $i$ from the notation, so that $x = x_{ti}$, $\partial = \partial_{ti}$. In case (ii) we let $M = \mathbb{C}[x]$ with the standard action of $\mathbb{C}[x, \partial]$ and $v = 1$. In case (i) we let $M = \mathbb{C}[x]$ where $x$ acts by $\partial/\partial x$ and $\partial$ acts by multiplication by $-x$, and $v = 1$ (this module is obtained from the previous one by twisting with the automorphism of the algebra of polynomial coefficient differential operators that maps $x$ to $\partial$ and $\partial$ to $-x$). In case (iii) we let $M = x^p\mathbb{C}[x, x^{-1}]$ with the standard action of $\mathbb{C}[x, \partial]$ and $v = x^p$. The assertion about $v$ is immediate in all three cases. $\square$

**Remark 4.9.** The argument used in the proof of Proposition 4.8 allows one to establish the following property of rank: *if $M$ is a $\mathfrak{gl}_n$-module of rank at most $k$ and $N$ is a $\mathfrak{gl}_n$-module of rank at most $l$, then $M \otimes N$ has rank at most $k + l$.*

Let us apply Theorem 4.6 to determination of the finite-dimensional simple $\mathfrak{gl}_n$-modules that have rank at most $k$. By the highest weight theory, a simple highest weight module $L(\lambda)$ is finite-dimensional if and only if $\lambda$ is a *dominant integral* weight. Conversely, each finite-dimensional simple $\mathfrak{gl}_n$-module is isomorphic to a unique $L(\lambda)$ whose highest weight $\lambda$ is dominant and integral. Therefore, we need to see which dominant integral $\lambda$ satisfy the condition (2) or (3) of Theorem 4.6.

A weight $\lambda$ is *dominant* if it is a decreasing sequence: $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. It is *integral* if $\lambda_i - \lambda_j$ is an integer for all $i$ and $j$. Equivalently, the sequence $(\lambda_1 + n - 1, \lambda_2 + n - 2, \ldots, \lambda_n)$ is strictly decreasing and integral. Then there can be at most one subsequence of $\lambda$ described in condition (3) of Theorem 4.6. Since the subsequence is an arithmetic progression with difference $-1$, it must occur in $n - k$ consecutive terms of $\lambda$. Suppose that the indices of these terms run from $r + 1$ to $n - s$, where $r$ and $s$ are nonnegative integers such that $r + s = k$. Equating the terms, we arrive at the condition $\lambda_i + n - i = n - k - (i - r)$, or $\lambda_i = r - k = -s$, for all $r + 1 \leqslant i \leqslant n - s$. In addition we must have $\lambda_1 \geqslant \cdots \geqslant \lambda_r \geqslant -s$ and $-s \geqslant \lambda_{n-s+1} \geqslant \cdots \geqslant \lambda_n$.

Thus $\lambda$ is a sequence of the form

$$\begin{aligned}\lambda &= (d_1 - s, \ldots, d_r - s, -s, \ldots, -s, -s - e_s, \ldots, -s - e_1)\\ &= (d_1, \ldots, d_r, 0, \ldots, 0, -e_s, \ldots, -e_1) - s(1, 1, \ldots, 1),\end{aligned} \tag{4.10}$$

where $D = (d_1 \geqslant \cdots \geqslant d_r \geqslant 0)$ and $E = (e_1 \geqslant \cdots \geqslant e_s \geqslant 0)$. Note that $D$ is a partition with at most $r$ parts and $E$ is a partition with at most $s = k - r$ parts. Thus, the simple finite-dimensional $\mathfrak{gl}_n$-module $L(\lambda)$ is the $\mathfrak{gl}_n$-module $\varrho^{D,E} \otimes \det^{r-k}$ in the "mixed tensor" notation (see [10, Section 1.2.8]). If we pass from the partitions $D$ and $E$ to their Young diagrams then the diagram corresponding to $D$ has at most $r$ rows and the diagram corresponding to $E$ has at most $k - r$ rows. We summarize the preceding discussion in a theorem.

**Theorem 4.11.** *The simple finite-dimensional $\mathfrak{gl}_n$-modules of rank at most $k$ are parametrized by triples $(r, D, E)$ consisting of an integer $r$ between $0$ and $k$, a Young diagram $D$ with at most $r$ rows and a Young diagram $E$ with at most $k - r$ rows. The $\mathfrak{gl}_n$-module corresponding to $(r, D, E)$ is the module $\varrho^{D,E} \otimes \det^{r-k}$.*

The modules appearing in Theorem 4.11 are well-known in the theory of reductive dual pairs and theta correspondence. Up to a minor normalization, they are the modules appearing on the $U_n$ side of the theta correspondence for the reductive dual pair $(U_n, U_{r,s})$ (see [8]). Their explicit realizations in the space of polynomial functions on $\mathsf{M}_{r,n} \oplus \mathsf{M}^*_{s,n}$, where the Lie algebra $\mathfrak{gl}_n$ acts by natural analogues of the polarization operators $\mathsf{D}_{ij}$, are described in [10].

## Acknowledgments

This paper is based on a part of my PhD thesis [22]. I thank Roger Howe for his guidance during my studies at Yale, Mara Neusel for many constructive comments, and the referee for providing several relevant references.

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