Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Algebra 278 (2004) 127–133
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

A partial order on partitions and the generalized Vandermonde determinant

Loring W. Tu

Department of Mathematics, Tufts University, Medford, MA 02155-7049, USA

Received 17 July 2003
Available online 2 April 2004
Communicated by Michel Broué

Abstract

We introduce a partial order on partitions which permits an inductive proof on partitions. As an example of this technique, we reprove the discriminant formula for the generalized Vandermonde determinant.

Keywords: Partial order; Partitions of integers; Vandermonde determinant

1. A partial order on partitions

A *partition* of a positive integer n is a nonincreasing sequence of positive integers $m_1 \geqslant \cdots \geqslant m_r$ that sum to n. For a partition of n other than $(1, 1, \ldots, 1)$ we define a unique *predecessor* as follows. Suppose $(m_1, \ldots, m_r) \neq (1, \ldots, 1)$ is a partition. Let m_s be the last element > 1; thus,

$$(m_1, \ldots, m_r) = (m_1, \ldots, m_s, 1, \ldots, 1).$$

The predecessor of m is the partition of length $r + 1$,

$$\widetilde{m} = (m_1, \ldots, m_{s-1}, m_s - 1, 1, 1, \ldots, 1),$$

E-mail address: loring.tu@tufts.edu.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2003.09.048

$$\begin{tikzcd}
 & & 1+1+1+1+1 \arrow[d] & \\
 & & 2+1+1+1 \arrow[dl] \arrow[dr] & \\
 & 3+1+1 \arrow[dl] \arrow[dr] & & 2+2+1 \\
4+1 \arrow[d] & & 3+2 & \\
5 & & &
\end{tikzcd}$$

Fig. 1. The partial order on P_5.

obtained from m by decomposing m_s into two terms $(m_s - 1) + 1$. In other words,

$$\widetilde{m}_i = \begin{cases} m_i, & \text{for } 1 \leqslant i \leqslant s-1; \\ m_s - 1, & \text{for } i = s; \\ 1, & \text{for } s+1 \leqslant i \leqslant r+1. \end{cases}$$

This relation generates a partial order on the set P_n of all partitions of n.

If a partition α is a predecessor of a partition β, we say that β is a *successor* of α. The successors of $(m_1, \ldots, m_s, 1, 1, 1, \ldots, 1)$, $m_s > 1$, are

$$(m_1, \ldots, m_s + 1, 1, 1, \ldots, 1) \quad \text{or} \quad (m_1, \ldots, m_s, 2, 1, \ldots, 1),$$

if these are partitions.

This partial order is best illustrated with an example.

Example. For $n = 5$, the partial order on the set of partitions of 5 is as in Fig. 1.

In Fig. 1 we write

$$(m_1, \ldots, m_r) = m_1 + \cdots + m_r.$$

The partition $(2, 2, 1)$ has no successors because $(2, 3)$ is not a partition.

2. The generalized Vandermonde determinant

Given n distinct numbers $a_1, \ldots, a_n$ the Vandermonde determinant

$$\Delta(a_1, \ldots, a_n) = \det \begin{bmatrix} a_1^{n-1} & a_1^{n-2} & \ldots & a_1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ a_n^{n-1} & a_n^{n-2} & \ldots & a_n & 1 \end{bmatrix}$$

is ubiquitous in mathematics. It is computable from the well-known discriminant formula (see, for example, [1, Chapter III, §8.6, p. 99], or [3, §24, Exercice 14, p. 563])

$$\Delta(a_1, \ldots, a_n) = \prod_{i<j} (a_i - a_j). \tag{1}$$

For a variable x, define $R(x)$ to be the row vector of length n,

$$R(x) = \begin{bmatrix} x^{n-1} & x^{n-2} & \dots & x & 1 \end{bmatrix}.$$

Denote the kth derivative of $R(x)$ by $R^{(k)}(x)$. For a positive integer ℓ, define $M_\ell(x)$ to be the ℓ by n matrix whose first row is $R(x)$ and each row thereafter is the derivative with respect to x of the preceding row,

$$M_\ell(x) = \begin{bmatrix} R(x) \\ R'(x) \\ \vdots \\ R^{(\ell-1)}(x) \end{bmatrix}.$$

If $a = (a_1, \dots, a_r)$ is an r-tuple of distinct real numbers and $m = (m_1, \dots, m_r)$ a partition of n, the *generalized Vandermonde matrix* $M_m(a)$ and the *generalized Vandermonde determinant* $D_m(a)$ are defined to be

$$M_m(a) = \begin{bmatrix} M_{m_1}(a_1) \\ \vdots \\ M_{m_r}(a_r) \end{bmatrix}, \qquad D_m(a) = \det M_m(a).$$

We say that m_i is the *multiplicity* of a_i. When the multiplicities m_i are all 1, the generalized Vandermonde determinant $D_m(a)$ reduces to the usual Vandermonde determinant $\Delta(a_1, \dots, a_n)$.

Theorem 1 [6]. *Let $a = (a_1, \dots, a_r)$ be an r-tuple of distinct real numbers and $m = (m_1, \dots, m_r)$ a partition of n. Then*

$$D_m(a) = \left(\prod_{i=1}^{r} (-1)^{m_i(m_i-1)/2}\right)\left(\prod_{i=1}^{r}\prod_{k=1}^{m_i-1}(k!)\right)\prod_{1\leqslant i<j\leqslant r}(a_i - a_j)^{m_i m_j}.$$

Remarks.

(1) In keeping with the convention that a product over the empty set is 1, in case a multiplicity $m_i = 1$, define

$$\prod_{k=1}^{m_i-1}(k!) = 1.$$

Similarly, in case $r = 1$, define

$$\prod_{1\leqslant i<j\leqslant r}(a_i - a_j)^{m_i m_j} = 1.$$

(2) When all the multiplicities m_i are 1, Theorem 1 reduces to formula (1).

Theorem 1 has a long history. Muir [5, pp. 178–180] attributes it to Schendel ([6], article dated 1891, published in 1893), but Muir says of this paper that "in no case is there any hint of a proof" and that special cases had appeared earlier in the work of Weihrauch (1889) and Besso (1882). More recent proofs may be found in van der Poorten [7] and Krattenthaler [4]. Krattenthaler [4] discusses many variants and generalizations of the Vandermonde determinant and gives extensive references.

The classic Vandermonde determinant occurs naturally in the Lagrange interpolation problem of finding a polynomial $p(z)$ of degree $n-1$ with specified values at n distinct numbers $a_1, \ldots, a_n$. The Hermite interpolation problem is the generalization where one specifies not only the values of the polynomial but also the values of its derivatives up to order m_i at the points a_i for $i = 1, \ldots, r$ (see, for example, [2]). The discriminant formula (Theorem 1) gives a direct proof that the Hermite interpolation problem has a unique solution.

3. A relation among Vandermonde determinants

Lemma 2. *Let $\widetilde{m}$ be the predecessor of the r-tuple*

$$m = (m_1, \ldots, m_{s-1}, \ell + 1, 1, \ldots, 1).$$

Thus, $\widetilde{m}$ is the $(r+1)$-tuple

$$\widetilde{m} = (m_1, \ldots, m_{s-1}, \ell, 1, 1, \ldots, 1).$$

For $t \neq 0$ in $\mathbb{R}$, suppose

$$a = (a_1, \ldots, a_{s-1}, \lambda, a_{s+1}, \ldots, a_r) \quad \textit{and}$$
$$\tilde{a}(t) = (a_1, \ldots, a_{s-1}, \lambda, \lambda + t, a_{s+1}, \ldots, a_r)$$

have multiplicity vectors m and $\widetilde{m}$, respectively. Then

$$D_m(a) = \lim_{t \to 0} \left(\frac{\partial}{\partial t} \right)^{\ell} D_{\widetilde{m}}\big(\tilde{a}(t)\big).$$

Proof. The Vandermonde matrix $M_m(a)$ is obtained from $M_{\widetilde{m}}(\tilde{a})$ by replacing the submatrix

$$\begin{bmatrix} M_\ell(\lambda) \\ R(\lambda + t) \end{bmatrix}$$

by the submatrix $M_{\ell+1}(\lambda)$. Note that

$$M_{\ell+1}(\lambda) = \begin{bmatrix} M_\ell(\lambda) \\ R^{(\ell)}(\lambda) \end{bmatrix} = \begin{bmatrix} M_\ell(\lambda) \\ \lim_{t \to 0} (\partial/\partial t)^{\ell} R(\lambda + t) \end{bmatrix}. \tag{2}$$

Since the determinant can be expanded about any row,

$$\frac{\partial}{\partial t} D_{\widetilde{m}}\big(\tilde{a}(t)\big) = \frac{\partial}{\partial t} \det \begin{bmatrix} \vdots \\ M_\ell(\lambda) \\ R(\lambda+t) \\ \vdots \end{bmatrix} = \det \begin{bmatrix} \vdots \\ M_\ell(\lambda) \\ (\partial/\partial t) R(\lambda+t) \\ \vdots \end{bmatrix}$$

and therefore,

$$\left(\frac{\partial}{\partial t}\right)^{\ell} D_{\widetilde{m}}\big(\tilde{a}(t)\big) = \det \begin{bmatrix} \vdots \\ M_\ell(\lambda) \\ (\partial/\partial t)^{\ell} R(\lambda+t) \\ \vdots \end{bmatrix}. \tag{3}$$

By (2) and (3),

$$D_m(a) = \det \begin{bmatrix} \vdots \\ M_\ell(\lambda) \\ R^{(\ell)}(\lambda) \\ \vdots \end{bmatrix} = \lim_{t\to 0} \det \begin{bmatrix} \vdots \\ M_\ell(\lambda) \\ (\partial/\partial t)^{\ell} R(\lambda+t) \\ \vdots \end{bmatrix} = \lim_{t\to 0} \left(\frac{\partial}{\partial t}\right)^{\ell} D_{\widetilde{m}}\big(\tilde{a}(t)\big).$$

$\square$

4. Proof of Theorem 1

The proof is by induction on the partial order on the set of partitions of n. The initial case $(1, 1, \ldots, 1)$ corresponds to the usual Vandermonde determinant, for which we know the theorem holds.

Let the r-tuple

$$a = (a_1, \ldots, a_{s-1}, \lambda, a_{s+1}, \ldots, a_r)$$

have multiplicity vector

$$m = (m_1, \ldots, m_{s-1}, \ell+1, 1, \ldots, 1), \quad \text{with } \ell \geqslant 1.$$

By the induction hypothesis, we assume that the theorem holds for the predecessor $\widetilde{m}$ of m:

$$\widetilde{m} = (m_1, \ldots, m_{s-1}, \ell, 1, 1, \ldots, 1).$$

Take $\tilde{a}(t)$ to be

$$\tilde{a}(t) = (a_1, \ldots, a_{s-1}, \lambda, \lambda+t, a_{s+1}, \ldots, a_r)$$

and assign to $\tilde{a}(t)$ the multiplicity vector $\widetilde{m}$. By the induction hypothesis,

$$\begin{aligned} D_{\widetilde{m}}\bigl(\tilde{a}(t)\bigr) &= C \cdot \Biggl(\prod_{\substack{1 \leqslant i<j \leqslant r \\ i,j \neq s}} (a_i - a_j)^{m_i m_j} \Biggr) \cdot \bigl(\lambda - (\lambda + t)\bigr)^{\ell} \\ &\quad \times \Biggl(\prod_{i<s} (a_i - \lambda)^{m_i \ell} \bigl(a_i - (\lambda + t)\bigr)^{m_i} \Biggr) \Biggl(\prod_{s<j} (\lambda - a_j)^{\ell m_j} (\lambda + t - a_j)^{m_j} \Biggr), \end{aligned} \tag{4}$$

where

$$C = \Biggl(\prod_{i=1}^{s-1} (-1)^{m_i(m_i-1)/2} \Biggr) (-1)^{\ell(\ell-1)/2} \Biggl(\prod_{i=1}^{s-1} \prod_{k=1}^{m_i-1} (k!) \Biggr) \prod_{k=1}^{\ell-1} (k!).$$

We write this more simply as

$$D_{\widetilde{m}}\bigl(\tilde{a}(t)\bigr) = (-1)^{\ell} t^{\ell} f(t),$$

where $f(t)$ is the obvious function defined by Eq. (4).

By Lemma 2,

$$\begin{aligned} D_m(a) &= \lim_{t \to 0} \left(\frac{\partial}{\partial t} \right)^{\ell} (-1)^{\ell} t^{\ell} f(t) \\ &= \lim_{t \to 0} (-1)^{\ell} \ell! f(t) + (-1)^{\ell} \lim_{t \to 0} \sum_{k=0}^{\ell-1} \binom{\ell}{k} \left(\left(\frac{\partial}{\partial t} \right)^{k} t^{\ell} \right) \cdot f^{(\ell-k)}(t) \\ &\qquad \text{(product rule for the derivative)} \\ &= (-1)^{\ell} \ell! f(0) \\ &= (-1)^{\ell} \ell! C \prod_{\substack{1 \leqslant i<j \leqslant r \\ i,j \neq s}} (a_i - a_j)^{m_i m_j} \prod_{i<s} (a_i - \lambda)^{m_i(\ell+1)} \prod_{s<j} (\lambda - a_j)^{(\ell+1)m_j} \\ &= \Biggl(\prod_{i=1}^{s} (-1)^{m_i(m_i-1)/2} \Biggr) \Biggl(\prod_{i=1}^{s} \prod_{k=1}^{m_i-1} (k!) \Biggr) \prod_{1 \leqslant i<j \leqslant r} (a_i - a_j)^{m_i m_j} \\ &\qquad \text{(since } m_s = \ell + 1 \text{ and } a_s = \lambda). \end{aligned}$$

In this last expression, the product $\prod_{i=1}^{s}$ may be replaced by $\prod_{i=1}^{r}$, since for $s + 1 \leqslant i \leqslant r$, the multiplicity $m_i = 1$.

Acknowledgments

The author acknowledges the hospitality and support of the Institut Henri Poincaré and the Institut de Mathématiques de Jussieu during the preparation of this article. He also thanks Albert Tarantola for stimulating his interest in this subject and Bartholomé Coll for several preliminary calculations.

References

[1] N. Bourbaki, Algèbre, Chapitre III, Hermann, Paris, 1971.
[2] M. Crouzeix, A.L. Mignot, Analyse numérique des équations différentielles, Masson, Paris, 1984.
[3] R. Godement, Cours d'algèbre, Hermann, Paris, 1963.
[4] C. Krattenthaler, Advanced determinant calculus, Sém. Lothar. Combin. 42 (1999) 67, Article B42q.
[5] T. Muir, The Theory of Determinants in the Historical Order of Development, vol. 3, Macmillan & Co., London, 1920.
[6] L. Schendel, Das alternirende Exponentialdifferenzenproduct, Zeitschrift f. Math. u. Phys. xxxviii (1893) 84–87.
[7] A.J. van der Poorten, Some determinants that should be better known, J. Austral. Math. Soc. Ser. A 21 (1976) 278–288.