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Endoproperties of modules and local duality

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Abstract

Let R be any ring and $N = \bigoplus_{i \in I} N_i$ be a direct sum of finitely presented left R-modules N_i. Suppose that $D(N)$ and $D(N_i)$ are the local duals of N and N_i for each $i \in I$. We prove that the lattice of endosubmodules of N is anti-isomorphic to the lattices of matrix subgroups of $D(N)$ and of $M = \bigoplus_{i \in I} D(N_i)$. As consequences, N is endoartinian if and only if M (or $D(N)$) is endonoetherian, and N is endonoetherian if and only if M (or $D(N)$) is Σ-pure-injective. We obtain, in particular, that if R is a Krull–Schmidt ring, and M is an indecomposable pure-injective endonoetherian right R-module which is the source of a left almost split morphism in $\mathrm{Mod}(R)$, then M is endofinite. As an application, a ring R is of finite representation type if and only if every pure-injective right R-module is endonoetherian.

Keywords: Local duality; Endonoetherian module; Endofinite module; Pure semisimple ring; Ring of finite representation type

1. Introduction

Modules that are of finite length over their endomorphism rings, also called *endofinite* modules, are of significant importance in general module theory and representation theory. Every endofinite module M, over any ring R, is known to be Σ-pure-injective, hence M is a direct sum of modules with local endomorphism rings, and the number of isomorphism classes of indecomposable summands of M is finite [11]. A ring R is of finite representation type if and only if every

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right R-module is endofinite (see [10,30,35]). Endofinite modules and modules with related endoproperties have attracted much attention in recent years, particularly because of their close relationships with classical notions of module theory, and their special role in the interaction between finitely generated and infinitely generated modules (see, e.g., [3,4,10,11,14,18,31–33]).

For a left R-module N, with $S = \mathrm{End}_R N$, the local dual of N is defined as the right R-module $D(N) = \mathrm{Hom}_S(N_S, C_S)$, where C_S is a minimal injective cogenerator of $\mathrm{Mod}(S)$. The local duality was used by Auslander [9] in the construction of almost split sequences over a general ring. Moreover, the local duality provides an important tool for passing information from left modules to right modules, or vice versa (for example, in Herzog's solution of the pure semisimplicity conjecture for PI-rings and quasi-Frobenius rings [25]). If R is an Artin algebra, then it is well known that the local duality restricted to finitely generated modules coincides with the usual Morita duality between finitely generated left and right R-modules. For an arbitrary ring R, Huisgen-Zimmermann and Zimmermann [30] have shown that the lattice of finite matrix subgroups of a left R-module M is anti-isomorphic to the lattice of finite matrix subgroups of the local dual $D(M)$ of M.

In this paper, we consider local duals of modules that are direct sums of finitely presented modules (or more generally, local duals of pure-projective modules), and show that, in this situation, the above mentioned result of Huisgen-Zimmermann and Zimmermann [30] can be strengthened. More precisely, we prove that if R is any ring and $N = \bigoplus_{i\in I} N_i$ is a direct sum of finitely presented left R-modules N_i, then the lattice of endosubmodules of N is anti-isomorphic to the lattices of matrix subgroups of $D(N)$ and of $M = \bigoplus_{i\in I} D(N_i)$, where $D(N)$ and $D(N_i)$ are the local duals of N and N_i for each $i \in I$ (Theorem 4.1). As a consequence, we deduce that, for any ring R, the left R-module $N = \bigoplus_{i\in I} N_i$ is endoartinian if and only if the right R-module $M = \bigoplus_{i\in I} D(N_i)$ is endonoetherian (Theorem 4.2). This generalizes a similar statement obtained by Huisgen-Zimmermann and Saorín [31, Proposition L] for Artin algebras, with a different method. Our arguments also yield, as a by-product, that every finitely presented endoartinian left R-module is endofinite, for any ring R. Another consequence is that $N = \bigoplus_{i\in I} N_i$ is endonoetherian if and only if the right R-module $M = \bigoplus_{i\in I} D(N_i)$ is Σ-pure-injective (Theorem 4.3). This may be regarded as a "local version" of the well-known result that a ring R is right pure semisimple if and only if every pure-projective left R-module is endonoetherian (see [30,43]). We also discuss the transferring of the finendo and cofinendo properties of modules in connection with the local duality.

There is another line of research that motivates our work. A classical result asserts that a right self-injective left noetherian ring R is quasi-Frobenius. A module-theoretic version of this result is the well-known Teply–Miller theorem [42] stating that, over any ring R, an endonoetherian injective right R-module is endofinite (see also [1,19]). Endofinite modules need not be injective, but they are always pure-injective, and it appears to be a natural and interesting question to study the structure of endonoetherian pure-injective modules. In general, an endonoetherian pure-injective module need not be endofinite (Example 5.1), and may not have even an indecomposable decomposition (Example 5.2; cf. [36]).

Note that the endonoetherian condition occurs naturally also in the context of pure semisimple rings. As applications of our main results, we first show that if R is a Krull–Schmidt ring and M is an indecomposable pure-injective endonoetherian right R-module that is the source of a left almost split morphism in $\mathrm{Mod}(R)$, then M is endofinite (Theorem 5.3). Consequently, if R is a left pure semisimple ring, then an endonoetherian pure-injective right R-module M is endofinite if and only if M has an indecomposable decomposition. Moreover, if R is not of finite representation type, then there is an endonoetherian pure-injective right R-module that is not endofinite (Corollary 5.7). We also obtain that a ring R is of finite representation type if and only if every pure-injective right R-module is endonoetherian (Theorem 5.8). This generalizes the well-known characterization of rings of finite representation type in terms of the endofiniteness of all right modules, proved in [10,30,35]. Another somewhat surprising consequence is that, if R is a left pure semisimple hereditary ring, then any direct sum of preinjective left R-modules is endoartinian (Corollary 5.11).

The paper is organized as follows. In Section 2 we study torsion theories cogenerated or determined by families of injective modules and finitely generated projective modules, respectively, over rings with enough idempotents. Section 3 is devoted to matrix subgroups and endosubmodules, and their relationships with certain torsion theories over the functor ring. In Section 4, we present our main results on transferring endoproperties of modules through the local duality. Finally, in Section 5, we give applications for endonoetherian pure-injective modules and pure semisimple rings.

We refer to [1,2,41,44] for ring and module-theoretic background, and for all undefined notions used in the text.

2. Modules over rings with enough idempotents

Throughout this section, A will be a *ring with enough idempotents*, i.e. $A = \bigoplus_{\delta\in\Delta} A\theta_\delta = \bigoplus_{\delta\in\Delta} \theta_\delta A$ for a family of pairwise orthogonal idempotents $\{\theta_\delta\}_{\delta\in\Delta}$ in A (see, e.g., [44]). A right A-module M will always mean a unitary right A-module (i.e., $MA = M$), and $\mathrm{Mod}(A)$ will denote the category of unitary right A-modules. The ring A is *right locally coherent* if every finitely generated submodule of a finitely presented right A-module is finitely presented.

We shall use torsion theories over rings with enough idempotents (see [41, Chapter VI.2, 3], to which we also refer for the notation on this topic). If A is a ring with enough idempotents, we represent a hereditary torsion theory of $\mathrm{Mod}(A)$ as $(\mathbf{T}, \mathbf{F})$, where $\mathbf{T}$ and $\mathbf{F}$ denote the classes of torsion and torsion-free modules, respectively. The associated quotient category of $\mathrm{Mod}(A)$ will be denoted as $\mathrm{Mod}(A, \mathbf{T})$. The localization functor $\mathbf{L} : \mathrm{Mod}(A) \to \mathrm{Mod}(A, \mathbf{T})$ is an exact functor [41, Theorem X.1.6]. The torsion theory $(\mathbf{T}, \mathbf{F})$ is said to be *cogenerated* by a right A-module M in case the torsion class $\mathbf{T}$ consists of all the modules X such that $\mathrm{Hom}(X, M) = 0$. A torsion theory is hereditary precisely when it is cogenerated by an injective module. On the other hand, a projective right A-module P induces a hereditary torsion theory $(\mathbf{T}, \mathbf{F})$ where the class $\mathbf{T}$ consists of all the modules X such that $\mathrm{Hom}(P, X) = 0$. We will say that $(\mathbf{T}, \mathbf{F})$ is the torsion theory *determined* by P.

If X is a unitary right A-module, and L is a submodule of X, we say that L is a *saturated submodule* when the quotient module $\frac{X}{L}$ is torsion-free. Given an arbitrary submodule N of X, there always exists a smallest saturated submodule L of X such that $N \subseteq L$. In such case $L = N^c$ is called the *saturation* of N in X, and $\frac{N^c}{N}$ is torsion. We know from [41, Chapter IX.4] that the lattice of saturated submodules of X is isomorphic to the lattice of subobjects of the localization $\mathbf{L}(X)$ in the quotient category $\mathrm{Mod}(A, \mathbf{T})$.

Lemma 2.1. *Let A be a ring with enough idempotents and $\{E_i\}_{i\in I}$ a family of injective right A-modules. Let $E = \bigoplus_{i\in I} E_i$, U be the injective hull of E, and $Q = \prod_{i\in I} E_i$ be the direct product of the E_i. Then the hereditary torsion theory $(\mathbf{T}, \mathbf{F})$ cogenerated by U is also cogenerated by each of the modules Q and E.*

Proof. It is enough to see that the three torsion classes coincide. Since $E \subseteq U \subseteq Q$, two of the inclusions are obvious. Then, suppose that X is a right A-module and $\mathrm{Hom}(X, E) = 0$. If we had $\mathrm{Hom}(X, Q) \neq 0$, then $\mathrm{Hom}(X, E_i) \neq 0$ for some index $i \in I$. But then $\mathrm{Hom}(X, E) \neq 0$, which gives a contradiction. $\square$

Therefore we have that a direct sum $E = \bigoplus_{i \in I} E_i$ of injective right A-modules cogenerates a hereditary torsion theory. We use this observation in the next lemma. We also introduce the following notation. For any homomorphism $f : X \to Y$, where X, Y are right A-modules, we set $\mathcal{S}_X(E, f) = \mathrm{Hom}(Y, E) \circ f \subseteq \mathrm{Hom}(X, E)$. Moreover, S will be the endomorphism ring of E and we write $\mathcal{S}_X(E)$ to denote the set of all left S-submodules of $\mathrm{Hom}(X, E)$ of the form $\mathcal{S}_X(E, f)$.

Lemma 2.2. *Let A be a ring with enough idempotents, E a direct sum of injective right A-modules which cogenerates the torsion theory* $(\mathbf{T}, \mathbf{F})$, *and let X be a finitely presented right A-module. Then there exists an order-inverting bijection between $\mathcal{S}_X(E)$ and the lattice* $\mathrm{Sat}(X)$ *of saturated A-submodules of X.*

Proof. Note that each $\mathcal{S}_X(E, f)$ is an S-submodule of $\mathrm{Hom}(X, E)$. We define $\phi : \mathrm{Sat}(X) \to \mathcal{S}_X(E)$ by setting $\phi(K) = \mathrm{Hom}(\frac{X}{K}, E) \subseteq \mathrm{Hom}(X, E)$.

We start by showing that $\phi(K) \in \mathcal{S}_X(E)$. Take $Y = \frac{X}{K}$ and $f : X \to Y$ the canonical projection. Then it is clear that $\mathrm{Hom}(Y, E) \circ f = \phi(K)$.

We show next that if $K \subseteq X$ is a saturated submodule, then K is the intersection L of the kernels of all the homomorphisms $g : X \to E$ such that $g \in \phi(K)$. Suppose it is not, and let $L_0 \subseteq L$ be such that $\frac{L_0}{K}$ is nonzero and finitely generated. Since K is saturated, $\frac{L_0}{K}$ is torsion-free, and hence there exists some nonzero homomorphism $h : L_0 \to E$ such that $h(K) = 0$. But then h factors through $\frac{L_0}{K}$ which is finitely generated and therefore the image of h is contained in an injective direct summand of E. Consequently, h can be extended to a homomorphism $X \to E$, which belongs to $\phi(K)$. But this implies that h should be zero on L, which gives a contradiction.

Thus, let K, L be saturated submodules of X. If $K \subseteq L$, then clearly $\phi(L) \subseteq \phi(K)$. Suppose now that $\phi(K) = \phi(L)$. We have seen that K is the intersection of all the kernels of the homomorphisms in $\phi(K)$. So is also L, hence $K = L$.

Finally, let $f : X \to Y$ be any homomorphism and consider $\mathcal{S}_X(E, f)$. Let K be the kernel of f, and take K^c the saturation of K in X. We claim that $\mathcal{S}_X(E, f) = \phi(K^c)$. One inclusion is clear, because any homomorphism in $\mathcal{S}_X(E, f)$ is zero on K and hence it is zero on K^c. For the other direction, let $g : X \to E$ be zero on K^c. Now, g can be factored through $\frac{X}{K}$ and the image of this factorization is inside an injective direct summand of E. This implies that g can be extended to a homomorphism $Y \to E$, and thus it is factored through f. $\square$

Note that, by the definition of ϕ above, if $K \subseteq X$ is saturated in X and $L \subseteq K$ is such that $\frac{K}{L}$ is torsion, then $\phi(K) = \mathrm{Hom}(\frac{X}{L}, E)$. If L is finitely generated, then $\frac{X}{L}$ is finitely presented, and $\phi(K) = \mathcal{S}_X(E, f)$ for a homomorphism $f : X \to Y$ with $Y = \frac{X}{L}$ finitely presented. Conversely, assume that Y is finitely presented and $f : X \to Y$ is a homomorphism. If L is the kernel of f, we know that $\mathcal{S}_X(E, f) = \phi(L^c)$. If A is right locally coherent, we have that L is finitely generated, whence L^c is the saturation of a finitely generated submodule. This gives the following result.

Lemma 2.3. *Let A be a right locally coherent ring with enough idempotents. Then the bijection of Lemma* 2.2 *restricts to a bijection between the saturated submodules K of X such that $K = L^c$*

for some finitely generated submodule L, and the sets of the form $\mathcal{S}_X(E, f)$ with $f : X \to Y$ and Y finitely presented.

We consider now a kind of dual situation to the one above. Again A is a ring with enough idempotents, and let $P = \bigoplus_{j \in J} P_j$ be a direct sum of finitely generated projective right A-modules P_j. Then P determines a hereditary torsion theory of $\mathrm{Mod}(A)$, say $(\mathbf{T}_0, \mathbf{F}_0)$. Let B be the endomorphism ring of P, and consider the subring B_f consisting of all the endomorphisms α such that $\alpha(P_j) = 0$ for almost all $j \in J$. B_f is a ring with enough idempotents.

Lemma 2.4. *Let A, P, B and B_f be as above. There is an equivalence Φ of categories between the quotient category $\mathrm{Mod}(A, \boldsymbol{T}_0)$ and $\mathrm{Mod}(B_f)$. The equivalence is afforded by the functor $\bigoplus_{j \in J} \mathrm{Hom}_A(P_j, -)$, in the sense that for any right A-module M, we have that $\bigoplus_{j \in J} \mathrm{Hom}_A(P_j, M) \cong \Phi(\mathbf{L}(M))$ in $\mathrm{Mod}(B_f)$, where $\mathbf{L}(M)$ is the localization of M. In particular, if P is finitely generated then the functor $\mathrm{Hom}_A(P, -)$ induces an equivalence between $\mathrm{Mod}(A, \mathbf{T}_0)$ and $\mathrm{Mod}(B)$.*

Proof. Let B' be the subset of B consisting of all endomorphisms α of P such that $\alpha(P)$ is contained in a finite direct sum $\bigoplus_{j \in F} P_j$. B' is both a two-sided ideal of B and a ring with enough idempotents. By [22, Theorem 1.3] (see also [23, Theorem 1.3]), which is also valid for rings with enough idempotents, the quotient category $\mathrm{Mod}(A, \mathbf{T}_0)$ is equivalent to the quotient category of $\mathrm{Mod}(B)$ with respect to the Gabriel filter generated by B', and the equivalence is induced by the functor $\mathrm{Hom}_A(P, -)$. Note that the torsion class of $\mathrm{Mod}(B)$ corresponding to this Gabriel filter is formed with those right B-modules L such that $L \cdot B' = 0$. Since $B_f \cdot B' = B'$, that torsion class contains precisely the modules L with $L \cdot B_f = 0$. The functor $\mathrm{Hom}_{B_f}(B_f, -) : \mathrm{Mod}(B_f) \to \mathrm{Mod}(B)$ is a full and faithful embedding, and hence the functor $(-) \otimes_B B_f : \mathrm{Mod}(B) \to \mathrm{Mod}(B_f)$ is an exact left adjoint to the above inclusion. By the preceding observation on the torsion class of $\mathrm{Mod}(B)$, an application of [41, Theorem X.2.1] shows that the quotient category of $\mathrm{Mod}(B)$ is equivalent to $\mathrm{Mod}(B_f)$, and thus $\mathrm{Mod}(A, \mathbf{T}_0)$ is also equivalent to $\mathrm{Mod}(B_f)$, via the composed functor $\mathrm{Hom}_A(P, -) \otimes_B B_f$. Finally, note that this functor is equivalent to $\mathrm{Hom}_A(P, -) \cdot B_f \cong \bigoplus_{j \in J} \mathrm{Hom}_A(P_j, -)$. □

We may deduce from this a result that corresponds to Lemma 2.2.

Corollary 2.5. *Let A, P, B and B_f be as above, with P determining the torsion theory $(\mathbf{T}_0, \mathbf{F}_0)$. For any finitely presented right A-module X, there is an order-preserving bijection between the lattice $\mathrm{Sat}(X)$ of saturated submodules of X and the lattice of B_f-submodules of $\bigoplus_{j \in J} \mathrm{Hom}(P_j, X)$.*

Proof. The first lattice is isomorphic to the lattice of subobjects of the localization $\mathbf{L}(X)$ of X in the quotient category $\mathrm{Mod}(A, \mathbf{T}_0)$, by [41, Corollary IX.4.4]. By Lemma 2.4, this is isomorphic to the lattice of subobjects of $\Phi(\mathbf{L}(X)) \cong \bigoplus_{j \in J} \mathrm{Hom}(P_j, X)$ in the category $\mathrm{Mod}(B_f)$. □

We find applications of these lemmas when the torsion theories constructed above are the same. In that case, the bijections of the preceding results can be combined. Following [12], we say that a right A-module M is *endonoetherian* (respectively, *endoartinian*, *endofinite*) when $\mathrm{Hom}(X, M)$ is a noetherian (respectively, artinian, of finite length) left module over the endomorphism ring of M, for any finitely presented right A-module X.

We are now ready to prove our first main result on endonoetherian injective modules over a ring with enough idempotents.

Theorem 2.6. *Let A be a ring with enough idempotents, and let E be an injective right A-module that cogenerates the torsion theory* $(\mathbf{T}, \mathbf{F})$. *Suppose that there exists a finitely generated projective right A-module P, such that P determines the same torsion theory* $(\mathbf{T}, \mathbf{F})$. *If E is endonoetherian, then E is endofinite.*

Proof. Consider the torsion theory $(\mathbf{T}, \mathbf{F})$ of $\mathrm{Mod}(A)$ cogenerated by the injective E, and assume that it is also determined by the finitely generated projective module P. Let Y be any finitely presented right A-module. Then $\mathrm{Hom}(Y, E)$ is noetherian over the endomorphism ring of E. We want to show that $\mathrm{Hom}(Y, E)$ is of finite length over the endomorphism ring of E.

Since $\mathrm{Hom}(Y, E)$ has the ascending chain condition on endosubmodules, we have that it has also the ascending chain condition on the lattice $\mathcal{S}_Y(E)$, as defined in Lemma 2.2. By using this lemma, we see that the lattice $\mathrm{Sat}(Y)$ of saturated submodules of Y has the descending chain condition. Then we have that $\mathrm{Hom}(P, Y)$ satisfies the descending chain condition on submodules over the endomorphism ring of P, by Corollary 2.5. Since this holds for every finitely presented right A-module Y, we have that $\mathrm{Hom}(P, P)$ is also right artinian. This implies that it is right noetherian, so that $\mathrm{Hom}(P, Y)$ is of finite length. By using again Corollary 2.5, we infer that the lattice $\mathrm{Sat}(Y)$ is of finite length.

Then we have a chain of saturated submodules $0 = L_0 \subset L_1 \subset L_2 \subset \cdots \subset L_k = Y$ where the quotients are simple, in the sense that there are no saturated submodules between two consecutive terms. We obtain the submodules $\phi(L_j) \subseteq \mathrm{Hom}(Y, E)$, which give again a finite chain. Note that $\phi(L_j) = \mathrm{Hom}(\frac{Y}{L_j}, E)$.

Now the quotients in this chain may be calculated from the exact sequences $0 \to \frac{L_{j+1}}{L_j} \to \frac{Y}{L_j} \to \frac{Y}{L_{j+1}} \to 0$. By applying the $\mathrm{Hom}(-, E)$ functor, we see that the quotients of this chain are isomorphic to $\mathrm{Hom}(\frac{L_{j+1}}{L_j}, E)$.

We know that $\frac{L_{j+1}}{L_j}$ is torsion-free, but any of its proper quotients is torsion. Therefore, each nonzero homomorphism $\frac{L_{j+1}}{L_j} \to E$ is a monomorphism. By the injectivity of E, we deduce that $\mathrm{Hom}(\frac{L_{j+1}}{L_j}, E)$ is simple as a left module over the endomorphism ring of E. This shows that $\mathrm{Hom}(Y, E)$ is of finite length over the endomorphism ring of E and thus E is endofinite. □

We will need the following simple, but quite useful, lemma.

Lemma 2.7. *Let A be a ring with enough idempotents, E an injective right A-module with a simple and essential socle Z, and suppose that Z has a (finitely generated) projective cover P. For any right A-module M, we have that* $\mathrm{Hom}(P, M) = 0$ *if and only if* $\mathrm{Hom}(M, E) = 0$.

Proof. Suppose that $\mathrm{Hom}(M, E) \neq 0$. Then there exists a nonzero homomorphism $f : M \to E$, and the image of f contains Z. Therefore, there exists a submodule $M_0 \subseteq M$ so that there is a nonzero homomorphism $M_0 \to Z$. Since this is an epimorphism and P is projective, we get a homomorphism $P \to M_0$ which is nonzero. Therefore $\mathrm{Hom}(P, M) \neq 0$.

For the converse, note that the existence of a nonzero homomorphism $g : P \to M$ implies that there exists an epimorphism $\mathrm{Im}(g) \to Z$. By the injectivity of E, we get a nonzero homomorphism $M \to E$ and $\mathrm{Hom}(M, E) \neq 0$. $\square$

The following result gives some sufficient conditions for an endonoetherian injective module over a ring with enough idempotents to be endofinite.

Theorem 2.8. *Let $A = \bigoplus_{\delta\in\Delta} A\theta_\delta = \bigoplus_{\delta\in\Delta} \theta_\delta A$ be a ring with enough idempotents, and let E be an endonoetherian injective right A-module with an essential socle $X = \bigoplus_{i\in I} X_i$, where each X_i is simple. Suppose that each X_i has a (finitely generated) projective cover. Assume, moreover, that for each $\delta \in \Delta$, there are only finitely many non-isomorphic X_i such that* $\mathrm{Hom}(\theta_\delta A, E(X_i))$ *is nonzero. Then E is endofinite.*

Proof. First, let N be an indecomposable summand of E. Since the property of being endonoetherian is preserved under taking direct summands, N is endonoetherian injective with an essential simple socle X_i. Then $N \cong E(X_i)$ for some $i \in I$, and since each X_i has a finitely generated projective cover P_i, Lemma 2.7 yields that the torsion theory cogenerated by $E(X_i)$ is also determined by P_i. Hence $E(X_i)$ is endofinite by Theorem 2.6. Thus, each indecomposable summand of E is endofinite.

Now, we consider the module E in the theorem, and note that each direct summand K of E contains a simple submodule X, hence K contains an indecomposable summand N (e.g. N is any injective envelope of X in K). By Zorn's lemma, E contains a local direct summand $U = \bigoplus_{\alpha\in\Omega} U_\alpha$ maximal with respect to the property that each U_α is indecomposable. By the special case dealt with above, each U_α is endofinite.

Note that each U_α is the injective envelope $E(X)$ of a simple module X, X being isomorphic to some of the X_i. By hypothesis, for each $\delta \in \Delta$, there are only finitely many non-isomorphic modules U_α such that $\mathrm{Hom}(\theta_\delta A, U_\alpha)$ is nonzero. Since $\{\theta_\delta A, \delta \in \Delta\}$ is a generating set of finitely generated projective right A-modules, it follows from [16, Proposition 1.4] that $U = \bigoplus_{\alpha\in\Omega} U_\alpha$ is endofinite, hence pure-injective. Note that, being a local direct summand of E, $U = \bigoplus_{\alpha\in\Omega} U_\alpha$ is a pure submodule of E. So U splits in E, i.e. $E = U \oplus L$. If L is nonzero, L contains an indecomposable summand, contradicting the maximality of the local direct summand $U = \bigoplus_{\alpha\in\Omega} U_\alpha$. Hence $E = U$ is endofinite in $\mathrm{Mod}(A)$. $\square$

Suppose now that A is a semiperfect ring, meaning that every finitely generated right (or left) R-module has a projective cover (see e.g. [44, 49.10]).

Corollary 2.9. *Let $A = \bigoplus_{\delta\in\Delta} A\theta_\delta = \bigoplus_{\delta\in\Delta} \theta_\delta A$ be a semiperfect ring with enough idempotents, and let E be an endonoetherian injective right A-module with an essential socle $X = \bigoplus_{i\in I} X_i$, where each X_i is simple. Suppose that for each $\delta \in \Delta$, there are only finitely many non-isomorphic X_i such that* $\mathrm{Hom}(\theta_\delta A, E(X_i))$ *is nonzero. Then E is endofinite.*

In particular, if E is the injective hull of a simple right A-module, then E is endofinite.

Proof. The result follows immediately by Theorem 2.8, keeping in mind that each simple right A-module X_i has a projective cover. $\square$

The following corollary is a version of the Teply–Miller theorem [42] for indecomposable right modules over left perfect rings with enough idempotents.

Corollary 2.10. *Let R be a left perfect ring with enough idempotents. If E is an indecomposable injective endonoetherian right R-module, then E is endofinite.*

Proof. Since R is left perfect, it implies that R is right semiartinian (see [44, 49.9]), hence in particular E contains an essential simple submodule. Then apply Corollary 2.9 $\square$

3. Matrix subgroups and torsion theories

We start now with a unital ring R and construct the *left functor ring* A of R. We take a module in each of the isomorphism classes of finitely presented left R-modules, giving the family $\{V_\delta \mid \delta \in \Delta\}$. Then the functor ring A consists of the endomorphisms α of $V = \bigoplus_{\delta\in\Delta} V_\delta$ such that $\alpha(V_\delta) = 0$ for almost all $\delta \in \Delta$. In particular, θ_μ will denote the endomorphism of V which is the identity on V_μ and 0 over V_δ if $\delta \neq \mu$. When $V_\mu = {}_RR$, we write $\theta = \theta_\mu$. A is a ring with enough idempotents which is right locally coherent.

It is well known that $-\otimes_R V : \mathrm{Mod}(R) \to \mathrm{Mod}(A)$ defines a full and faithful functor, which we denote as T. The functor T restricts to an equivalence between $\mathrm{Mod}(R)$ and the FP-injective right A-modules, and $T(M)$ is injective in $\mathrm{Mod}(A)$ if and only if M is pure-injective in $\mathrm{Mod}(R)$. Moreover, T preserves direct sums and direct products, preserves and reflects finitely presented modules, and every finitely presented right A-module embeds in $T(X)$ for some finitely presented right R-module X. By [12, Lemma 1], $T(M)$ is endofinite in $\mathrm{Mod}(A)$ if and only if M is endofinite in $\mathrm{Mod}(R)$, and similarly, $T(M)$ is endonoetherian in $\mathrm{Mod}(A)$ if and only if M is endonoetherian in $\mathrm{Mod}(R)$.

We will also use the full and faithful functor

$$H = \bigoplus_{\delta\in\Delta} \mathrm{Hom}(V_\delta, -) : \mathrm{Mod}\big(R^{\mathrm{op}}\big) \to \mathrm{Mod}\big(A^{\mathrm{op}}\big).$$

The functor H restricts to an equivalence between $\mathrm{Mod}(R^{\mathrm{op}})$ and the flat left A-modules and $H(N)$ is projective in $\mathrm{Mod}(A^{\mathrm{op}})$ if and only if N is pure-projective in $\mathrm{Mod}(R^{\mathrm{op}})$. We refer to [12] and [44, Chapter 10] for more information on the functor category $\mathrm{Mod}(A)$ and the functors T and H.

We recall here the notion of a (finite) matrix subgroup for a given module M over a unital ring R.

Definition 3.1. (See [30,45].) Let M be a right R-module. A subgroup L of the Abelian group M is called a *matrix subgroup* of M if it is of the form $L = \mathrm{Hom}(Y, M)(x) = \{f(x) \mid f \in \mathrm{Hom}(Y, M)\}$, where Y is a right R-module, and $x \in Y$. If Y is finitely presented, then L is called a *finite matrix subgroup* of M.

It is clear that matrix subgroups of a module M are endosubmodules of M, that is, submodules of M viewed as a left $\mathrm{End}_R M$-module. The canonical isomorphism $\mathrm{Hom}(R_R, M) \cong M$ is obviously an isomorphism of left $\mathrm{End}_R M$-modules. Therefore the endosubmodules of M may be identified with the $\mathrm{End}_R M$-submodules of $\mathrm{Hom}(R_R, M)$. In particular, the (finite) matrix subgroups of M correspond, under this identification, with the submodules of $\mathrm{Hom}(R_R, M)$ which are of the form $\mathrm{Hom}(Y, M) \circ f$ for any (finitely presented) right R-module Y and any homomorphism $f : R \to Y$. This gives an equivalent definition for the notion of (finite) matrix subgroups of M, which we shall use freely in the sequel. We say that the (finite) matrix subgroup $\mathrm{Hom}(Y, M) \circ f$ is the matrix subgroup *determined by f*.

There is a relationship between the matrix subgroups of any pure-injective right R-module M and the torsion theory $(\mathbf{T},\mathbf{F})$ of $\mathrm{Mod}(A)$ cogenerated by the injective module $T(M)$. In a more general form, this relationship is given in the next result.

Proposition 3.2. *Let R be a unital ring, A its left functor ring and $M=\bigoplus_{i\in I} M_i$ a direct sum of pure-injective right R-modules M_i, so that $(\mathbf{T},\mathbf{F})$ is the torsion theory of $\mathrm{Mod}(A)$ cogenerated by $T(M)=\bigoplus_{i\in I} T(M_i)$. Then there is a bijective mapping which reverses the inclusions between the lattice of matrix subgroups of M and the lattice $\mathrm{Sat}(\theta A)$ of saturated submodules of θA.*

Furthermore, this bijection restricts to an order-inverting bijection between the finite matrix subgroups of M and the saturated submodules K of θA such that $K=L^c$ for some finitely generated submodule L.

Proof. Note first that $T(R_R)=R\otimes_R V\cong V\cong\theta A$, so we identify from now on $T(R_R)$ as θA. If $f:R\to Y$ is a homomorphism of right R-modules, then we have $T(f):\theta A\to T(Y)$. As in the previous section, we have the submodule $\mathcal{S}_{\theta A}(T(M),T(f))\subseteq \mathrm{Hom}_A(\theta A,T(M))$, which is obtained by applying the functor T to all the elements of the matrix subgroup of M determined by f. Note that, if $p:\theta A\to Z$ is an epimorphism of $\mathrm{Mod}(A)$, and $u:Z\to Y$ is a monomorphism, then $\mathcal{S}_{\theta A}(T(M),p)=\mathcal{S}_{\theta A}(T(M),u\circ p)$, because Z is finitely generated and $T(M)$ is a direct sum of injective modules. This fact and the property that the functor T is full and faithful imply that $\mathcal{S}_{\theta A}(T(M))$ is a lattice isomorphic to the lattice of matrix subgroups of M. Now, $\mathcal{S}_{\theta A}(T(M))$ is anti-isomorphic to $\mathrm{Sat}(\theta A)$ by Lemma 2.2. This shows the first assertion of our proposition.

To prove the second part, we note that, since the right R-module Y is finitely presented if and only if $T(Y)$ is finitely presented, the finite matrix subgroups correspond, by applying T, to the sets of the form $\mathcal{S}_{\theta A}(T(M),T(f))$ for $T(f):\theta A\to T(Y)$ with $T(Y)$ finitely presented. Then, the result follows from Lemma 2.3. $\square$

We deal also in this setting with the torsion theory determined by a projective right A-module. We start now with a unital ring R and $N=\bigoplus_{i\in I} N_i$ will be a direct sum of finitely presented left R-modules N_i. As before, we denote by A the left functor ring of R, and $\theta\in A$ is the element which is the identity on the summand ${}_RR$, and zero elsewhere. Also, $\theta_i\in A$ will denote the idempotent which is the identity on the module isomorphic to N_i, and zero elsewhere. Then we set $P_i=\theta_i A$ and $P=\bigoplus_{i\in I} P_i$ is a projective right A-module. The class $\mathbf{T}_0$ of those right A-modules X such that $\mathrm{Hom}(P,X)=0$ gives a hereditary torsion theory of $\mathrm{Mod}(A)$, say $(\mathbf{T}_0,\mathbf{F}_0)$. Our next result gives a relationship between this torsion theory and the endosubmodules of N.

Proposition 3.3. *Let R be a unital ring and $N=\bigoplus_{i\in I} N_i$ a direct sum of finitely presented left R-modules N_i, with endomorphism ring $S=\mathrm{End}_R(N)$. A is the left functor ring of R, and $\theta,\theta_i\in A$ are as indicated above. $P=\bigoplus_{i\in I}\theta_i A$ determines a hereditary torsion theory $(\mathbf{T}_0,\mathbf{F}_0)$ of $\mathrm{Mod}(A)$. Then there is an order-preserving bijection between the submodules of N as a right S-module and the saturated submodules of θA.*

This bijection restricts to a bijection between the finite matrix subgroups of N and the saturated submodules K of θA such that $K=L^c$ for some finitely generated submodule L.

Proof. Let B_f be the ring of the endomorphisms α of P such that $\alpha(P_i)=0$ for almost all $i\in I$. Note that $\mathrm{Hom}_R(N_i,N_j)\cong\theta_i A\theta_j\cong\mathrm{Hom}_A(P_j,P_i)$. Thus, B_f is isomorphic to the ring of all the endomorphisms α of $N=\bigoplus_{i\in I} N_i$ such that $\alpha(N_i)=0$ for almost all $i\in I$. In particular,

$B_f \cong \mathrm{End}_R(N)$ when the set I is finite. With the obvious identification, we will assume that N is a unitary right B_f-module.

By applying Corollary 2.5 to the finitely presented module θA, we get that the lattice of saturated submodules of θA is isomorphic to the lattice of B_f-submodules of $\bigoplus_{i\in I}\mathrm{Hom}(P_i,\theta A)$ and the isomorphism takes K to $\bigoplus_{i\in I}\mathrm{Hom}(P_i,K)$. Now, we have

$$\mathrm{Hom}(P_i,\theta A)\cong \theta A\theta_i \cong \mathrm{Hom}_R(R,N_i)\cong N_i$$

so that the isomorphism occurs between $\mathrm{Sat}(\theta A)$ and the lattice of B_f-submodules of N.

We show now that the (finitely generated) S-submodules of N are exactly the unitary (and finitely generated) B_f-submodules of N and the first part of the result follows. First, note that B_f can be seen as a right ideal of the ring S, that is, $B_f S = B_f$. Thus, if $L = LB_f$ is any unitary B_f-submodule, then $L = LB_f S = LS$ is an S-submodule. It is also clear that if L is finitely generated as a B_f-submodule, then it is also finitely generated (with the same system of generators) as a right S-module.

Then, let L be a (finitely generated) S-submodule of N. Now, for each element $x\in L$, we know that x is inside a finite direct sum of the N_i, from which it follows that it is left invariant by some element of B_f. Therefore it belongs to LB_f and this shows that $L = LB_f$ is also a unitary B_f-submodule. Finally, the above equation $B_f S = B_f$ implies that a finite system of generators of L as a right S-module is also a generating system as a right B_f-module.

For the second part, suppose that K is a saturated submodule of θA containing a finitely generated submodule L, such that $\frac{K}{L}$ is torsion. This implies that $\frac{K}{L}\cdot\theta_i = 0$ for each $i\in I$, and hence the endosubmodule of N corresponding to K will be $\bigoplus_{i\in I} L\cdot\theta_i$. We check that this gives a finite matrix subgroup.

Since L is finitely generated and the $\theta_\delta A$ $(\delta\in\Delta)$ form a system of finitely generated projective generators of $\mathrm{Mod}(A)$, there is an epimorphism $h:\bigoplus_{\delta\in F}\theta_\delta A\to L$, with F finite. Then $h(\theta_\delta) = h_\delta\theta_\delta = \theta h_\delta\theta_\delta\in L\subseteq\theta A$, and $h_0 = (\theta h_\delta\theta_\delta)_{\delta\in F}$ defines a homomorphism $h_0: R\to\bigoplus_F V_\delta$. We claim that the endosubmodule of N corresponding to K (i.e., $\bigoplus_{i\in I} L\cdot\theta_i$) is the finite matrix subgroup determined by this homomorphism h_0.

Take any element of the form $f\cdot\theta_i$ with $f = \theta\cdot f\in L$. Since h is an epimorphism, there exist elements $g_\delta\in\theta_\delta A$, so that $h(\sum_{\delta\in F}\theta_\delta g_\delta) = f\theta_i = \sum_F h_\delta\cdot\theta_\delta\cdot g_\delta$. Therefore $f\cdot\theta_i = \sum_F(\theta h_\delta\cdot\theta_\delta)\cdot(\theta_\delta\cdot g_\delta\cdot\theta_i)$. Each $\theta_\delta\cdot g_\delta\cdot\theta_i$ is a homomorphism $V_\delta\to N_i$, and thus $f\cdot\theta_i: R\to N_i$ may be factored through $h_0: R\to\bigoplus_F V_\delta$. So $f\cdot\theta_i$ belongs to the matrix subgroup determined by h_0.

Conversely, let us take an element in this finite matrix subgroup, say $h_0\cdot f$, where $f:\bigoplus_F V_\delta\to N$. Then there is a finite subset $J\subseteq I$ such that $f = \sum_F\sum_J \theta_\delta f_{\delta,j}\theta_j$, with $f_{\delta,j}: V_\delta\to N_j$. Then $h_0\cdot f = \sum h_\delta\theta_\delta f_{\delta,j}\cdot\theta_j$, and belongs to $\bigoplus_{i\in I} L\cdot\theta_i$.

It remains to see that every finite matrix subgroup of N can be put as $\bigoplus_{i\in I} L\cdot\theta_i$ for some finitely generated submodule L of θA. So, take the matrix subgroup determined by $f_0: R\to Y$, with Y finitely presented. If we assume that $Y\cong V_\delta$, then $f_0 = \theta f_0\theta_\delta$. This gives a homomorphism $\theta_\delta A\to\theta A$ taking θ_δ to $\theta f_0\theta_\delta$. Let L be the finitely generated image of this homomorphism. The same proof as above shows that the endosubmodule of N corresponding to L^c is the finite matrix subgroup determined by f_0. This completes the proof of the proposition. □

4. Applications to local duality

Throughout this section, we will assume that R is a unital ring. For a left R-module N with its endomorphism ring S, recall that the *local dual* of N is defined as the right R-module $D(N) =$

$\mathrm{Hom}_S(N, C)$, where C is a minimal injective cogenerator of the category $\mathrm{Mod}(S)$ of all right S-modules. It is well known that the local dual $D(N)$ is a pure-injective right R-module (see, e.g., [27]). According to [30, Proposition 3], the lattice of finite matrix subgroups of N is anti-isomorphic to the lattice of finite matrix subgroups of $D(N)$. We refer the reader also to [35] for a model-theoretic approach to the local duality.

Under the hypotheses that will follow, the bijections of Propositions 3.2 and 3.3 of the preceding section may be combined and applied to local dual modules. Let $N = \bigoplus_{i\in I} N_i$ be a direct sum of finitely presented left R-modules. We will examine relationships between endoproperties of the left R-module N and the following right R-modules: the local dual $D(N)$ of N, the direct product $\prod_{i\in I} D(N_i)$ of the local duals $D(N_i)$, and the direct sum $\bigoplus_{i\in I} D(N_i)$ of the local duals $D(N_i)$.

The following is our main result in this section.

Theorem 4.1. *Let R be a ring and $N = \bigoplus_{i\in I} N_i$ a direct sum of finitely presented left R-modules N_i. Let $D(N)$ and $D(N_i)$ be the local duals of N and of N_i, for each $i \in I$. Suppose that M is any of the following modules*:

(a) $M = D(N)$;
(b) $M = \prod_{i\in I} D(N_i)$;
(c) $M = \bigoplus_{i\in I} D(N_i)$.

Then, in each of these cases, the lattice of matrix subgroups of M is anti-isomorphic to the lattice of endosubmodules of N, and the lattice of finite matrix subgroups of M is anti-isomorphic to the lattice of finite matrix subgroups of N.

More generally, if Q is any pure-projective left R-module, then the lattice of matrix subgroups of $D(Q)$ is anti-isomorphic to the lattice of endosubmodules of Q.

Proof. We develop the proof in several steps.

(1) We start by showing that, for any left R-module X, we have $T(D(X)) \cong D(H(X))$, where T and H are the canonical functors described at the beginning of Section 3.

A will be the left functor ring of R, and V_δ, θ_δ have the same meaning therein. Let us call $Y = D(X)$. We have

$$T(Y) = Y \otimes_R V \cong \bigoplus_{\delta\in\Delta} \mathrm{Hom}_E(X, C) \otimes_R V_\delta,$$

where E is the endomorphism ring of X and C is a minimal injective cogenerator of $\mathrm{Mod}(E)$. The canonical isomorphism shows that

$$T(Y) \cong \bigoplus_{\delta\in\Delta} \mathrm{Hom}_E\bigl(\mathrm{Hom}_R(V_\delta, X), C\bigr).$$

By the definition of the functor H, we have $\mathrm{Hom}_R(V_\delta, X) \cong \theta_\delta H(X)$. Therefore $T(Y) \cong \bigoplus_{\delta\in\Delta} \mathrm{Hom}_E(\theta_\delta H(X), C)$.

As is implicit above, we may assume that E is also the endomorphism ring of $H(X)$, since H is full and faithful. According to [17, p. 77], the local dual of $H(X)$ is the right A-module $\mathrm{Hom}_E(H(X), C)A$. Since $H(X) = \bigoplus_{\delta\in\Delta} \theta_\delta H(X)$, we have that $\mathrm{Hom}_E(H(X), C) \cong \prod_{\delta\in\Delta} \mathrm{Hom}_E(\theta_\delta H(X), C)$.

In order to calculate $D(H(X))$, i.e., the unitary part of $\mathrm{Hom}_E(H(X), C)$, we observe that each element of A annihilates on the left almost all the elements θ_δ. Therefore $D(H(X))$ is contained in the direct sum $\bigoplus_{\delta\in\Delta} \mathrm{Hom}_E(\theta_\delta H(X), C) \cong T(Y)$, which is unitary. So we have $T(Y) = T(D(X)) \cong D(H(X))$.

(2) Again X is any left R-module and $D(X)$ is its local dual. Let $(\mathbf{T}_X, \mathbf{F}_X)$ be the torsion theory of $\mathrm{Mod}(A)$ cogenerated by $T(D(X))$. Then any right A-module L belongs to the torsion class $\mathbf{T}_X$ if and only if $L \otimes_A H(X) = 0$.

To see this, we use the isomorphism $T(D(X)) \cong D(H(X))$ established in (1). Thus $L \in \mathbf{T}_X$ if and only if $\mathrm{Hom}_A(L, D(H(X))) = 0$. But this means precisely $\mathrm{Hom}_A(L, \mathrm{Hom}_E(H(X), C)) = 0$, where E is the endomorphism ring of X and C is a minimal injective cogenerator of $\mathrm{Mod}(E)$. This, in turn, is equivalent to the condition $\mathrm{Hom}_E(L \otimes_A H(X), C) = 0$. Since C is an injective cogenerator, the above condition holds true if and only if $L \otimes_A H(X) = 0$.

(3) We may now prove case (a) of the first part of the theorem. Let $N = \bigoplus_{i\in I} N_i$ be a direct sum of finitely presented left R-modules N_i, and $M = D(N)$. Assume that $\theta, \theta_i, P, B_f, S$ have the same meaning as in Proposition 3.3, i.e. $\theta \in A$ is the element which is the identity on the summand ${}_RR$ and zero elsewhere. Also, $\theta_i \in A$ will denote the idempotent which is the identity on N_i and zero elsewhere, and $P = \bigoplus_{i\in I} \theta_i A$, with B_f being the subring of endomorphisms of P indicated in the proof of Proposition 3.3, and S is the endomorphism ring of N. Then $(\mathbf{T}_0, \mathbf{F}_0)$ is the torsion theory of $\mathrm{Mod}(A)$ determined by the projective right R-module P, while $(\mathbf{T}_N, \mathbf{F}_N)$ is the torsion theory cogenerated by $T(D(N))$, as in step (2). We show that both torsion classes do coincide.

We know from (2) that $L \in \mathbf{T}_N$ if and only if $L \otimes_A H(N) = 0$. Since $H(N_i) \cong A\theta_i$, we have that L belongs to that torsion class if and only if $L \otimes_A A\theta_i \cong L\theta_i = 0$, for each $i \in I$. On the other hand, $L \in \mathbf{T}_0$ if and only if $\mathrm{Hom}_A(P, L) = 0$. But $\mathrm{Hom}_A(P, L) = \mathrm{Hom}_A(\bigoplus_I \theta_i A, L) \cong \prod_I L\theta_i$. Thus L belongs to this torsion class if and only if $L\theta_i = 0$ for each $i \in I$. This shows that both torsion theories coincide.

We now apply this observation along with Proposition 3.3. If we denote as $\mathrm{Sat}_N(\theta A)$ the lattice of saturated submodules of θA with respect to the torsion theory $(\mathbf{T}_N, \mathbf{F}_N)$, then we see that the map which takes $K \in \mathrm{Sat}_N(\theta A)$ to $\bigoplus_I K\theta_i$ is an order-preserving bijection from $\mathrm{Sat}_N(\theta A)$ to the lattice of the endosubmodules of N. By applying Proposition 3.2 and the coincidence of the torsion theories, we obtain that the lattice of the endosubmodules of N is anti-isomorphic to the lattice of the matrix subgroups of M.

(4) We may prove cases (b) and (c) in an analogous way, by showing that in each case, the torsion theory of $\mathrm{Mod}(A)$ cogenerated by $T(M)$ coincides with the torsion theory determined by P, so that the assertions follow immediately from Propositions 3.2 and 3.3.

(b) We have $N = \bigoplus_{i\in I} N_i$ and $M = \prod_{i\in I} M_i$ with $M_i = D(N_i)$. By part (a) applied to N_i, we know that for any right A-module L, one has that $\mathrm{Hom}(\theta_i A, L) = 0$ if and only if $\mathrm{Hom}(L, T(M_i)) = 0$. It follows from this that the torsion theory determined by $P = \bigoplus_{i\in I} \theta_i A$ is the same as the torsion theory cogenerated by $T(M) = \prod_{i\in I} T(M_i)$, as it was to be seen.

(c) This follows from (b) and Lemma 2.1.

(5) Suppose now that Q is a pure-projective left R-module. So, there is a left R-module $N = \bigoplus_I N_i$, a direct sum of finitely presented modules N_i, so that Q is isomorphic to a direct summand of N, i.e., there exists an idempotent element $e \in S = \mathrm{End}_R(N)$, such that $Q \cong Ne$. Let $\mathrm{Sat}_Q(\theta A)$ be the lattice of the saturated submodules of θA with respect to the torsion theory $(\mathbf{T}_Q, \mathbf{F}_Q)$. By step (2), a right A-module L belongs to $\mathbf{T}_Q$ if and only if $L \otimes_A H(Q) = L \otimes_A H(N)e = 0$, and this implies that $\mathbf{T}_N \subseteq \mathbf{T}_Q$. Therefore $\mathrm{Sat}_Q(\theta A) \subseteq \mathrm{Sat}_N(\theta A)$.

We use the bijection between $\mathrm{Sat}_N(\theta A)$ and the lattice of the S-submodules of N obtained in step (3). We will show that this bijection induces, by restriction, a bijection between $\mathrm{Sat}_Q(\theta A)$ and the lattice of the endosubmodules of Q. This bijection φ is defined by taking $K \in \mathrm{Sat}_Q(\theta A)$ to $(\bigoplus_I K\theta_i)e$.

To see that we obtain indeed a bijection in this way, take $L \subseteq K$, with $L, K \in \mathrm{Sat}_N(\theta A)$. We have that $\bigoplus_I L\theta_i$ and $\bigoplus_I K\theta_i$ are endosubmodules of N. Thus $(\bigoplus_I L\theta_i)e$ and $(\bigoplus_I K\theta_i)e$ are endosubmodules of Q. We show that these two endosubmodules of Q are equal if and only if $K/L \in \mathbf{T}_Q$.

We have that $K/L \in \mathbf{T}_Q$ if and only if $(K/L) \otimes_A H(N)e \cong (\bigoplus_I (K/L)\theta_i)e = 0$. But this means precisely that $(\bigoplus_I K\theta_i)e = (\bigoplus_I L\theta_i)e$. It follows that the images of the above map φ range over all Ze, for all endosubmodules Z of N. This entails that φ is a surjection. But it also implies that if $L \subseteq K$ belong to $\mathrm{Sat}_Q(\theta A)$ and $\varphi(L) = \varphi(K)$, then $L = K$. From this, it follows easily that φ is also injective.

Thus we have an order-preserving bijection between $\mathrm{Sat}_Q(\theta A)$ and the lattice of the endosubmodules of Q. If we now apply Proposition 3.2 to $D(Q)$, we deduce the last statement of the theorem. □

We may now relate some endo-chain conditions of modules in connection with the local duality. See also [30, Corollary 12] for related ideas in the context of Artin algebras.

Theorem 4.2. *Let R be a ring, and $N = \bigoplus_{i\in I} N_i$ be a direct sum of finitely presented left R-modules N_i. The following conditions are equivalent*:

(a) *N is endoartinian.*
(b) *The local dual $D(N)$ is endonoetherian.*
(c) *$Q = \prod_{i\in I} D(N_i)$ is endonoetherian.*
(d) *$M = \bigoplus_{i\in I} D(N_i)$ is endonoetherian.*

Moreover, under any of these conditions, the modules N_i are all endofinite.

More generally, if N is any pure-projective left R-module, then N is endoartinian if and only if $D(N)$ is endonoetherian.

Proof. If N is endoartinian, then by the anti-isomorphism of Theorem 4.1, each of the modules $D(N)$, Q, M has the ascending chain condition (ACC) on matrix subgroups. Since finitely generated endosubmodules of any module are matrix subgroups (see [28]), it follows that the modules $D(N)$, Q, M have the ACC on finitely generated endosubmodules, hence they are endonoetherian. So the endoartinian property for N implies any of the other conditions.

Conversely, if any of the modules $D(N)$, Q, M is endonoetherian, or equivalently, the ACC on matrix subgroups holds, then Theorem 4.1 implies that the other two modules also have the same property, and N has the DCC on endosubmodules, i.e. N is endoartinian.

To get the assertion on the summands N_i, let $T : \mathrm{Mod}(R) \to \mathrm{Mod}(A)$ be the canonical full and faithful functor. For each $i \in I$, $\theta_i \in A$ denotes the idempotent which is the identity on N_i and zero elsewhere. Note that each N_i has a local endomorphism ring, so by [17, Lemma 2.3] we have that $E_i = T(D(N_i))$ is the injective hull of a simple right A-module X and $\theta_i A$ is a projective cover of X in $\mathrm{Mod}(A)$. Then E_i is an endonoetherian injective right A-module that satisfies the hypotheses of Theorem 2.8, thus E_i is endofinite, implying that $D(N_i)$ is endofinite, hence N_i is endofinite.

The last part of the statement is proved in exactly the same form as (a) $\Leftrightarrow$ (b) above. $\square$

Recall that a module M is *Σ-pure-injective* if every direct sum of copies of M is pure-injective. By Zimmermann [45], a module M is Σ-pure-injective if and only if M satisfies the descending chain condition (DCC) on (finite) matrix subgroups. We obtain the following dual result to the previous one.

Theorem 4.3. *Let R be a ring, and $N = \bigoplus_{i\in I} N_i$ be a direct sum of finitely presented left R-modules N_i. Then the following conditions are equivalent*:

(a) *N is endonoetherian.*
(b) *$D(N)$ is Σ-pure-injective.*
(c) *$Q = \prod_{i\in I} D(N_i)$ is Σ-pure-injective.*
(d) *$M = \bigoplus_{i\in I} D(N_i)$ is Σ-pure-injective.*

More generally, if N is any pure-projective left R-module, then N is endonoetherian if and only if $D(N)$ is Σ-pure-injective.

Proof. Suppose first that (a) holds, i.e. N is endonoetherian. Then Theorem 4.1 implies that the modules $D(N)$, Q and M satisfy the DCC on matrix subgroups, hence they are Σ-pure-injective, so (a) implies (b)–(d). Now suppose that (b) holds, i.e. $D(N)$ is Σ-pure-injective. It follows then that $D(N)$ has the DCC on matrix subgroups, thus an application of Theorem 4.1 gives that N is endonoetherian, and Q and M have the DCC on matrix subgroups, thus they are Σ-pure-injective, proving (a), (c) and (d), respectively. The cases when (c) or (d) holds are proved similarly.

The last assertion is proved in exactly the same form as (a) $\Leftrightarrow$ (b) above. $\square$

We proceed with some consequences of our results. The first corollary may be regarded as a module-theoretic version of the classical Hopkins–Levitzki theorem.

Corollary 4.4. *Let R be a ring, and N be any finitely presented endoartinian left R-module. Then N is endofinite.*

Proof. This follows directly from Theorem 4.2. $\square$

The following corollary strengthens Theorem 4.2 in case each finitely presented left R-module N_i has a finitely presented local dual $D(N_i)$.

Corollary 4.5. *Let R be a ring, and $N = \bigoplus_{i\in I} N_i$ a direct sum of finitely presented left R-modules. Suppose further that the local dual $D(N_i)$ is finitely presented for each $i \in I$. Then the following conditions are equivalent*:

(a) *N is Σ-pure-injective.*
(b) *N is endoartinian.*
(c) *$M = \bigoplus_{i\in I} D(N_i)$ is endonoetherian.*

Moreover, in this case, the modules N_i are all endofinite.

Proof. (a) $\Rightarrow$ (c). Suppose that (a) holds. For each $i \in I$, we have that N_i is (Σ-)pure-injective, and since $D(N_i)$ is finitely presented, it follows by Zimmermann [47, Lemma 5] that $D(D(N_i)) \cong N_i$. Because N is Σ-pure-injective and $N \cong \bigoplus_{i\in I} D(D(N_i))$, Theorem 4.3 ((d) $\Rightarrow$ (a)) yields that $M = \bigoplus_{i\in I} D(N_i)$ is endonoetherian.

(c) $\Rightarrow$ (b). This was proven in Theorem 4.2.

(b) $\Rightarrow$ (a). This is immediate, because if N is endoartinian, then N has the DCC on matrix subgroups, implying that N is Σ-pure-injective.

That each N_i is endofinite was also proven in Theorem 4.2. $\square$

We note the following special case of Corollary 4.5 that might be of independent interest. This is also a generalization of [17, Proposition 3.18] in the unital ring case, that was proved using a different method.

Corollary 4.6. *Let R be a ring, and N a finitely presented Σ-pure-injective left R-module. If the local dual $D(N)$ is finitely presented as a right R-module, then N is endofinite.*

Let R be an Artin algebra, with center C, and $D : R\text{-mod} \to \text{mod-}R$ is the usual Morita duality between finitely generated left and right R-modules, i.e. $D = \mathrm{Hom}_C(-, E)$ where E is the C-injective envelope of $C/J(C)$. It is well known that if ${}_RX$ is a finitely generated left R-module, then $D(X)$ coincides with the local dual of X. As a consequence of Corollary 4.5, we rediscover the following result, due to Huisgen-Zimmermann and Saorín [31].

Corollary 4.7. (See [31, Proposition L].) *Let R be an Artin algebra, and $D : R\text{-mod} \to \text{mod-}R$ the usual Morita duality. If $N = \bigoplus_{i\in I} N_i$ is a direct sum of finitely generated left R-modules, then the following conditions are equivalent*:

(a) *N is Σ-pure-injective.*
(b) *N is endoartinian.*
(c) *$M = \bigoplus_{i\in I} D(N_i)$ is endonoetherian.*

In the final part of this section, we study finendo and cofinendo modules in connection also to the local duality. Following [19], a right (or left) R-module M is called *finendo* if M is finitely generated as a module over its endomorphism ring. Dually, we say that an R-module N is *cofinendo* if N is finitely cogenerated as a module over its endomorphism ring. Rings over which right modules are finendo or cofinendo were recently discussed in [15].

Proposition 4.8. *Let R be a ring, and $N = \bigoplus_{i\in I} N_i$ be a direct sum of finitely presented left R-modules. Let $M_i = D(N_i)$ be the local dual of N_i and $M = \bigoplus_{i\in I} M_i$. Then M is cofinendo if and only if N is finendo and $N/\mathrm{Rad}(N_S)$ is a semisimple right S-module, where S is the endomorphism ring of N.*

Proof. By definition, M is cofinendo if and only if the lattice of endosubmodules of M is finitely cogenerated in the following sense: it contains a finite set of minimal elements $Z_1, \dots, Z_k$ with their sum Z, so that the following condition holds:

- Any nonzero element of the lattice has a nonzero intersection with Z.

We know that all finitely generated endosubmodules of M are matrix subgroups. Z and the Z_i are finitely generated, and hence they are also matrix subgroups of M, hence the lattice of the matrix subgroups of M is finitely cogenerated in the above sense if the lattice of endosubmodules of M is finitely cogenerated. Conversely, if the lattice of matrix subgroups of M has a finite set of minimal elements with the above property, then these minimal elements are necessarily simple endosubmodules of M, and this property still holds for the lattice of endosubmodules of M, which is thus finitely cogenerated.

By Theorem 4.1, M is cofinendo if and only if the lattice of endosubmodules of N satisfies the dual property. Hence there exist maximal endosubmodules of N, say $L_1, \dots, L_k$ with intersection L and with the property:

- The sum of L and any proper endosubmodule of N is again proper.

This means that the endoradical $\mathrm{Rad}(N_S)$ of N is superfluous in N and is a finite intersection of maximal endosubmodules. This, in turn, is equivalent to the fact that $\mathrm{Rad}(N_S)$ is superfluous in N and $N/\mathrm{Rad}(N_S)$ is finitely generated and semisimple over S. By [2, Theorem 10.4], we have that this property holds if and only if N_S is finitely generated and semisimple modulo its radical. □

Proposition 4.9. *Let R be a ring, and $N = \bigoplus_{i\in I} N_i$ be a direct sum of finitely presented left R-modules each with a local endomorphism ring. Let $M_i = D(N_i)$ be the local dual of N_i and $M = \bigoplus_{i\in I} M_i$. If M is finendo, then N is cofinendo.*

Proof. Let $A, T(M), \theta A$ be as in Proposition 3.2. Our hypothesis entails that $\mathrm{Hom}_A(\theta A, T(M))$ is finitely generated over the endomorphism ring of $T(M)$. This means that there exists a homomorphism $\alpha : \theta A \to T(M)^k$ such that every homomorphism $\theta A \to T(M)$ can be factored through α.

Let $\mathbf{t}$ be the torsion radical corresponding to the torsion theory $(\mathbf{T}, \mathbf{F})$ cogenerated by $T(M)$. Thus $\mathbf{t}(\theta A)$ is the intersection of the kernels of all the homomorphisms $\theta A \to T(M)$, as seen in the proof of Lemma 2.2. The hypothesis shows that $\mathbf{t}(\theta A)$ is the intersection of $K_1, \dots, K_s$, where each K_i is the kernel of one of the homomorphisms $\theta A \to T(M)$ induced by α.

Therefore $\frac{\theta A}{\mathbf{t}(\theta A)}$ is isomorphic to a finitely generated submodule of $T(M)^k$. Now, $T(M) = \bigoplus_{i\in I} T(M_i)$, and thus $\frac{\theta A}{\mathbf{t}(\theta A)}$ is isomorphic to a finitely generated submodule of a finite sum of the $T(M_i)$. By [17, Lemma 2.3], each $T(M_i)$ has a simple and essential socle, so that $\frac{\theta A}{\mathbf{t}(\theta A)}$ has also a finitely generated and essential socle.

Thus $\frac{\theta A}{\mathbf{t}(\theta A)}$ contains a finite family of simples, $L_1, \dots, L_r$, which are necessarily torsion-free, and whose sum is essential. Suppose that for each $i = 1, \dots, r$, we have $L_i = \frac{Y_i}{t(\theta A)}$. Then let $Z_i = Y_i^c$, the saturation of Y_i in θA. Z_i is a minimal (nonzero) element of the lattice of saturated submodules of θA, and the join in this lattice of those minimal elements is essential.

We remark that the proof of Theorem 4.1 shows that the torsion theory $(\mathbf{T}, \mathbf{F})$ is precisely the torsion theory $(\mathbf{T}_0, \mathbf{F}_0)$ of Proposition 3.3. Now, by this same proposition we have that the lattice of endosubmodules of N is isomorphic to the lattice of saturated submodules of θA. This implies that N contains a finite set of minimal (hence, simple) endosubmodules whose sum is essential in N. Therefore, N has an essential and finitely generated endosocle, that is, N is finitely cogenerated, as a module over its endomorphism ring. □

The converse of this last result also holds when we assume the additional condition that the left functor ring A of R is right semiartinian. This happens, for example, when the ring R is left pure semisimple (see Section 5 for further discussions on these rings).

Proposition 4.10. *Let R be a ring, and $N = \bigoplus_{i\in I} N_i$ be a direct sum of indecomposable finitely presented left R-modules. Let $M_i = D(N_i)$ be the local dual of N_i and $M = \bigoplus_{i\in I} M_i$. Suppose moreover that the left functor ring A of R is right semiartinian. If N is cofinendo, then M is finendo.*

Proof. By hypothesis, the lattice of endosubmodules of N is finitely cogenerated. We use again the torsion theory $(\mathbf{T}_0, \mathbf{F}_0)$ of Proposition 3.3, and so we have that the lattice of saturated submodules of θA is also finitely cogenerated. This lattice is isomorphic to the lattice of saturated submodules of $\frac{\theta A}{\mathbf{t}(\theta A)}$. Therefore there is a finite number of independent minimal saturated submodules, say $K_1, \dots, K_s$ of $\frac{\theta A}{\mathbf{t}(\theta A)}$, such that their join is essential in that lattice. Note that this join is the saturation K^c of the direct sum $K = \bigoplus_{i=1}^{s} K_i$.

Since any submodule of a torsion-free module is essential in its saturation, we deduce that K is essential in $\frac{\theta A}{\mathbf{t}(\theta A)}$. Now, A is right semiartinian. Hence each K_i contains a simple submodule L_i, and the minimality of K_i implies that L_i is essential in K_i, so that $L = \bigoplus_{i=1}^{s} L_i$ is the essential socle of K and of $\frac{\theta A}{\mathbf{t}(\theta A)}$.

Again by the proof of Theorem 4.1, the torsion theory $(\mathbf{T}_0, \mathbf{F}_0)$ is cogenerated by $T(M)$. Then each L_i, being torsion-free, is isomorphic to a submodule of $T(M)$ and L can be embedded as a submodule of $T(M)^s$. Since L is finitely generated and $T(M)$ is a direct sum of injective modules, the embedding $L \to T(M)^s$ can be extended to a homomorphism $\frac{\theta A}{\mathbf{t}(\theta A)} \to T(M)^s$, which is necessarily a monomorphism. This defines a homomorphism $f : \theta A \to T(M)^s$ whose kernel is precisely $\mathbf{t}(\theta A)$. Thus any homomorphism $\theta A \to T(M)$ can be factored through f and it follows that the homomorphisms f_i generate $\mathrm{Hom}(\theta A, T(M))$ over the endomorphism ring of $T(M)$. This shows that M is finendo, taking into account the isomorphism $\mathrm{Hom}(\theta A, T(M)) \cong \mathrm{Hom}(R, M)$. $\square$

5. Endonoetherian pure-injective modules and pure semisimple rings

Throughout this section, R will be a unital ring. In view of the Teply–Miller theorem [42] on endonoetherian injective modules, it is natural to ask if every endonoetherian pure-injective right module over a ring R is always endofinite. It is known that every Σ-pure-injective endonoetherian module over any ring is endofinite (see [11, Proposition 4.1], [33, Lemma 4.3]). Since any countable pure-injective module is Σ-pure-injective (see [46, Proposition 3]), it follows that any countable pure-injective endonoetherian module over any ring is endofinite. However, the following example shows that the Teply–Miller theorem does not extend from injective modules to pure-injective modules, in general.

Example 5.1. Let $R = K[[X_1, \dots, X_n]]$ be the power series ring over a field K, then R is a commutative noetherian linearly compact ring, thus R is endonoetherian pure-injective as a module over itself, but R is not even semiprimary, so R is not artinian (see [28, Proposition 1]). Hence R as a right module over itself is an endonoetherian pure-injective module which is not endofinite.

It is well known that any endofinite module is a direct sum of indecomposable submodules with local endomorphism rings [11]. One may ask if the same is true for endonoetherian pure-injective modules. Puninskaya and Toffalori [36] have recently constructed an example of an endonoetherian pure-injective module M over the ring $\mathbf{Z}$ of integers such that M is not a direct sum of indecomposable modules.

The following is another example of an endonoetherian pure-injective module, over an Artin algebra, that admits no indecomposable decompositions.

Example 5.2. Let R be an Artin algebra, and $N = \bigoplus_{i \in I} N_i$ be an infinite direct sum of non-isomorphic finitely generated indecomposable left R-modules such that N is Σ-pure-injective. (For example, R is a hereditary tame Artin algebra of infinite representation type, and $N = \bigoplus_{i \in I} N_i$ is the direct sum of all non-isomorphic preinjective left R-modules; see Lenzing [34, 4.6]). Let $D : R\text{-mod} \to \text{mod-}R$ be the standard Morita duality between finitely generated left and right R-modules. Then $M = \bigoplus_{i \in I} D(N_i)$ and $Q = \prod_{i \in I} D(N_i)$ are both endonoetherian as right R-modules (see [30, Corollary 12]; cf. Theorem 4.2). Moreover Q is pure-injective because each $D(N_i)$ is pure-injective. Let $E(M)$ be a pure-injective envelope of M in Q, then $E(M)$ is a direct summand of Q, hence $E(M)$ is endonoetherian pure-injective. Suppose that $E(M)$ has an indecomposable decomposition $E(M) = \bigoplus_{j \in J} L_j$. Since $E(M)$ is pure-injective, it has the exchange property [29, Theorem 11, Example 2, p. 431], so the family $\{L_j\}_{j \in J}$ is locally semi-T-nilpotent (see [29, Corollary 6]), and because $E(M)$ is endonoetherian, it follows from [31, Theorem F] that there are only finitely many non-isomorphic modules among the modules L_j. Each $D(N_i)$, being a pure-injective pure submodule of $E(M)$, is a direct summand of $E(M)$, and by the Krull–Schmidt–Azumaya theorem, each $D(N_i)$ is isomorphic to some L_j. It follows that there are only finitely many non-isomorphic modules among the modules $D(N_i)$, giving a contradiction. Therefore $E(M)$ has no indecomposable decompositions.

Our next result gives a sufficient condition for an indecomposable pure-injective endonoetherian module over a Krull–Schmidt ring to be endofinite. Recall that a ring R is *Krull–Schmidt* if every finitely presented right (or left) R-module is a direct sum of modules with local endomorphism rings. Following [8], a *left almost split morphism* in $\text{Mod}(R)$ is a morphism $f : M \to N$ of the category $\text{Mod}(R)$, with M indecomposable, such that f is not a split monomorphism and for every morphism $g : M \to X$ in $\text{Mod}(R)$ that is not a split monomorphism, there exists $h : N \to X$ such that $h \circ f = g$. In this case, we say that M is the *source* of a left almost split morphism in $\text{Mod}(R)$. We define similarly the concepts of (the source of) a left almost split morphism in the category $\text{mod}(R)$ (or $\text{mod}(R^{\text{op}})$) of finitely presented right (or left) R-modules.

Theorem 5.3. *Let R be a Krull–Schmidt ring, and suppose that M is an indecomposable pure-injective endonoetherian right R-module which is the source of a left almost split morphism in* $\text{Mod}(R)$. *Then M is endofinite.*

Proof. Let A be the left functor ring of R, and let $T : \text{Mod}(R) \to \text{Mod}(A)$ be the canonical full and faithful functor (see Section 3). Then A is a semiperfect ring with enough idempotents (see, e.g., [25]). Moreover, $E = T(M)$ is an indecomposable injective endonoetherian right A-module. Since M is the source of a left almost split morphism in $\text{Mod}(R)$, it follows by [11, Theorem 2.3] that E is the injective envelope of a simple right A-module. By Corollary 2.9, E is an endofinite right A-module. It follows that M is an endofinite right R-module. $\square$

Recall that a ring R is called *left pure semisimple* if every left R-module is a direct sum of finitely generated modules (see, e.g., [20,38,39]). It is well known that left and right pure semisimple rings are precisely the rings of finite representation type, i.e. artinian rings with finitely many isomorphism classes of finitely generated indecomposable left and right modules (see [7, 21,37]). However it has been a long-standing open problem, known as the *pure semisimplicity conjecture*, whether left pure semisimple rings are also right pure semisimple (see, e.g., [27,40] for historical surveys). It is thus of interest to have a better understanding of the category of right modules over a left pure semisimple ring. It is known that a ring R is left pure semisimple if and only if every pure-projective right R-module is endonoetherian, if and only if every right R-module has the ACC on finite matrix subgroups (see [30, Theorems 6 and 9], [43, Theorem 3.1]). Moreover, a left pure semisimple ring R is representation-infinite if and only if there is a *generic* right R-module M, i.e. M is non-finitely presented indecomposable endofinite (see [17, Corollary 3.20]). We will consider endonoetherian pure-injective right modules M over a left pure semisimple ring R, and examine necessary and sufficient conditions for such modules M to be endofinite.

First, we note the following useful lemma.

Lemma 5.4. *Let R be a left pure semisimple ring, and M be any nonzero pure-injective right R-module. Then M contains an indecomposable summand which is the source of a left almost split morphism in* $\mathrm{Mod}(R)$.

Proof. Again, let $T:\mathrm{Mod}(R)\to\mathrm{Mod}(A)$ be the canonical full and faithful functor. Since R is left pure semisimple, the left functor ring A of R is left perfect, hence A is right semiartinian (see, e.g., [44, 49.9]), so $E=T(M)$ is an injective module in $\mathrm{Mod}(A)$ with a simple submodule X. Let $E(X)$ be an injective envelope of X in E, then there is a nonzero indecomposable summand K of M such that $T(K)\cong E(X)$. By [11, Theorem 2.3], K is the source of a left almost split morphism in $\mathrm{Mod}(R)$. □

We obtain the following immediate consequence.

Corollary 5.5. *Let R be a left pure semisimple ring, and M be any indecomposable pure-injective endonoetherian right R-module. Then M is endofinite.*

Proof. Since M is indecomposable pure-injective, Lemma 5.4 shows that M is the source of a left almost split morphism in $\mathrm{Mod}(R)$. Because M is furthermore endonoetherian, Theorem 5.3 yields that M is endofinite. □

The next result describes an arbitrary pure-injective right module M over a left pure semisimple ring R.

Proposition 5.6. *Let R be a left pure semisimple ring, and M be any pure-injective right R-module. Then the following statements hold*:

(a) *M is the pure-injective envelope of a direct sum of indecomposable pure-injective modules $\bigoplus_{i\in I} M_i$, where $M_i\cong D(N_i)$, N_i is an indecomposable finitely generated left R-module, for each $i\in I$.*

(b) *The lattice of matrix subgroups of M is anti-isomorphic to the lattice of endosubmodules of* $N = \bigoplus_{i\in I} N_i$.
(c) *M is endonoetherian if and only if $\bigoplus_{i\in I} M_i$ is endonoetherian. In this case each module M_i is endofinite.*

Proof. (a) Note that, by Lemma 5.4, any nonzero direct summand of M contains an indecomposable summand. Consider the non-empty set $\mathcal{A}$ of all local direct summands $U = \bigoplus_{\alpha\in\Omega} U_\alpha$ of M with each U_α indecomposable (i.e., each finite subsum of $\bigoplus_{\alpha\in\Omega} U_\alpha$ is a summand of M). By Zorn's lemma, M contains a maximal local direct summand $K = \bigoplus_{i\in I} M_i$ in $\mathcal{A}$. Then K is a pure submodule of M. Let $E = E(K)$ be a pure-injective envelope of K in M, then $M = E \oplus L$ for some submodule L, and if L is nonzero, L must contain an indecomposable summand, which is a contradiction to the maximality of the local direct summand $K = \bigoplus_{i\in I} M_i$. Thus $L = 0$, yielding that $M = E$ is the pure-injective envelope of K. By Lemma 5.4, each indecomposable pure-injective module M_i is the source of a left almost split morphism in $\mathrm{Mod}(R)$. It follows, by Krause [32, Theorem 3.1, Proposition 4.2], that $M_i \cong D(N_i)$, for some indecomposable finitely generated left R-module N_i, proving (a).

(b) Let $N = \bigoplus_{i\in I} N_i$ and $K = \bigoplus_{i\in I} M_i$, then by applying Theorem 4.1, there is an anti-isomorphism between the lattice of matrix subgroups of K and the lattice of endosubmodules of N. Now, if A is the left functor ring of R, and $T : \mathrm{Mod}(R) \to \mathrm{Mod}(A)$ is the canonical full and faithful functor, then it follows from Lemma 2.1 that $T(M)$ and $T(K)$ cogenerate the same hereditary torsion theory in $\mathrm{Mod}(A)$. Then Proposition 3.2 shows that the lattices of matrix subgroups of M and of $K = \bigoplus_{i\in I} M_i$ are isomorphic. Hence we can conclude, by Theorem 4.1, that the lattice of matrix subgroups of M is anti-isomorphic to the lattice of endosubmodules of $N = \bigoplus_{i\in I} N_i$.

(c) Since the endonoetherian property of a module is equivalent to the ACC on matrix subgroups, the first assertion of (c) follows immediately from the fact, shown in the proof of (b), that M and $K = \bigoplus_{i\in I} M_i$ have isomorphic lattices of matrix subgroups. Moreover, if M is endonoetherian, then each M_i is indecomposable pure-injective endonoetherian, hence M_i is endofinite by Corollary 5.5. $\square$

We obtain now necessary and sufficient conditions for an endonoetherian pure-injective right module over a left pure semisimple ring to be endofinite.

Corollary 5.7. *Let R be a left pure semisimple ring, and M be any endonoetherian pure-injective right R-module. Then the following conditions are equivalent*:

(a) *M has an indecomposable decomposition.*
(b) *M contains only finitely many non-isomorphic indecomposable summands.*
(c) *M is endofinite.*

Moreover, if R is not of finite representation type, then there exists an endonoetherian pure-injective right R-module M that is not endofinite.

Proof. (a) $\Rightarrow$ (b). Suppose that (a) holds, i.e. $M = \bigoplus_{i\in I} M_i$ is an indecomposable decomposition of M. Since M is pure-injective, it has the exchange property, so the family $\{M_i\}$ is locally semi-T-nilpotent (see [29]). As M is endonoetherian, [31, Theorem F] yields that there are only finitely many non-isomorphic modules among the modules M_i. By the Krull–Schmidt–Azumaya theorem, each indecomposable summand of M is isomorphic to some M_i, hence we get that M contains only finitely many non-isomorphic indecomposable summands.

(b) $\Rightarrow$ (c). Suppose that (b) holds. By Proposition 5.6, M is the pure-injective envelope of a direct sum $U = \bigoplus_{\alpha\in\Omega} U_\alpha$ of indecomposable endofinite modules U_α. Since there are only finitely many non-isomorphic modules among the modules U_α, it follows that U is endofinite [11, Proposition 4.5], so in particular U is pure-injective, thus $U = M$, proving that M is endofinite.

(c) $\Rightarrow$ (a). This was proved in [11, Proposition 4.5].

Finally, assume that R is not of finite representation type, and let $\{M_i \mid i \in I\}$ be any infinite family of non-isomorphic finitely presented indecomposable right R-modules. As R is left pure semisimple, the pure-projective right R-module $\bigoplus_{i\in I} M_i$ is endonoetherian [30, Theorem 9], hence it follows from Proposition 5.6(c) that the pure-injective envelope M of $\bigoplus_{i\in I} M_i$ is endonoetherian. If M is endofinite, M contains only finitely many non-isomorphic indecomposable summands [11], which is a contradiction because each module M_i is an indecomposable summand of M. Thus M is not endofinite, completing our proof. $\square$

Our next application is a characterization of rings of finite representation type in terms of endonoetherian modules. This might be regarded as a generalization of a result, due to Prest [35], Huisgen-Zimmermann and Zimmermann [30] and Crawley-Boevey [10], stating that a ring R is of finite representation type if and only if every right R-module is endofinite.

Theorem 5.8. *A ring R is of finite representation type if and only if every pure-injective right R-module is endonoetherian.*

Proof. The "only if" part is clear. For the "if" part, suppose that every pure-injective right R-module is endonoetherian. Let N be any left R-module, and let $M = D(N)$ be the local dual of N. Then M is a pure-injective right R-module, so M is endonoetherian by hypothesis. In particular, M has the ACC on finite matrix subgroups. On the other hand, by [30, Proposition 3], there is an anti-isomorphism between the lattices of finite matrix subgroups of N and of M. It follows that N has the DCC on finite matrix subgroups, so N is Σ-pure-injective [45]. Since this holds for every left R-module N, this implies that R is left pure semisimple. Now, let X be any finitely generated indecomposable left R-module, then its local dual $Y = D(X)$ is an indecomposable pure-injective right R-module. By hypothesis, Y is endonoetherian, hence by Corollary 5.5, Y is endofinite, implying that X is endofinite. It follows that every finitely generated indecomposable left R-module is endofinite, hence R is of finite representation type by [18, Theorem 4.1]. $\square$

Our next result shows that the finite representation type of a Krull–Schmidt ring R is determined by the endonoetherian property of a single right R-module.

Proposition 5.9. *Let R be a Krull–Schmidt ring, and let $\{N_i \mid i \in I\}$ be a complete family of all non-isomorphic finitely presented indecomposable left R-modules. Let $M_i = D(N_i)$ be the local dual of N_i $(i \in I)$, and let $M = \bigoplus_{i\in I} M_i$. Then R is of finite representation type if and only if the right R-module M is endonoetherian.*

Proof. If R is of finite representation type then it is well known that every right R-module is endofinite. Conversely, suppose that the right R-module $M = \bigoplus_{i\in I} D(N_i)$ is endonoetherian.

First we show that R is a left pure semisimple ring. Set $N = \bigoplus_{i \in I} N_i$. By Theorem 4.2, it follows that N is endoartinian, hence in particular N is Σ-pure-injective, so that any direct sum of copies of N is pure-injective. Let $\{L_j \mid j \in J\}$ be any family of finitely presented indecomposable left R-modules. Then the direct sum $L = \bigoplus_{j \in J} L_j$ is a direct summand of a direct sum of copies of N, implying that L is pure-injective. Then L has the exchange property [29], and there does not exist an infinite sequence of non-isomorphisms $\{f_n : L_{j_n} \to L_{j_{n+1}}\}_{n=1}^{\infty}$, with distinct j_n in J, such that the composition $f_n \circ f_{n-1} \circ \cdots \circ f_1$ is nonzero for any positive integer n (see [29, Corollary 6]). As is well known, this shows that R is left pure semisimple (see, e.g., [25, Lemma 3.2]).

Note also that each local dual $D(N_i)$ of N_i is an indecomposable pure-injective endo-noetherian right R-module. Then $D(N_i)$ is endofinite by Corollary 5.5, yielding that N_i is also endofinite. Thus every finitely generated indecomposable left R-module is endofinite, so we get that R is of finite representation type by [18, Theorem 4.1]. □

Using our results on the local duality of Section 4, we can now shed some new light on the endo-structure of left modules over a left pure semisimple ring.

Proposition 5.10. *Let R be a left pure semisimple ring.*

(a) *If N is any left R-module, and $D(N)$ is the local dual of N, then there is an anti-isomorphism between the lattice of endosubmodules of N and the lattice of matrix subgroups of $D(N)$.*
(b) *Suppose that $\{N_i \mid i \in I\}$ is a family of finitely generated indecomposable left R-modules such that each N_i is the source of a left almost split morphism in* $\mathrm{mod}(R^{\mathrm{op}})$. *Then the left R-module $N = \bigoplus_{i \in I} N_i$ is endoartinian.*

Proof. (a) This follows immediately from Theorem 4.1 and the well-known facts that the left pure semisimple ring R is left artinian, and every left R-module is a direct sum of finitely presented left R-modules.

(b) For a finitely generated indecomposable left R-module N_i, since N_i is the source of a left almost split morphism in $\mathrm{mod}(R^{\mathrm{op}})$, it follows from [17, Proposition 2.5] that $M_i = D(N_i)$ contains a finitely presented indecomposable pure submodule X_i, and since R is left pure semisimple, X_i is endofinite (see [24]; cf. [13, Lemma 3.7]), hence pure-injective. Thus X_i is a direct summand of M_i, and since $M_i = D(N_i)$ is indecomposable, we get that $X_i = M_i$, implying that M_i is finitely presented (cf. also [32, Theorem 3.1]). Let $M = \bigoplus_{i \in I} M_i$, then M is a pure-projective right R-module, and so M is endonoetherian (see [30, Theorem 9]). By Theorem 4.2, we get that the left R-module $N = \bigoplus_{i \in I} N_i$ is endoartinian. □

Recall that, for a left artinian hereditary ring R, a finitely generated indecomposable left R-module M is called *preinjective* if there are only finitely many pairwise non-isomorphic finitely generated indecomposable left R-modules X such that $\mathrm{Hom}_R(M, X) = 0$ (see, e.g., [39]). Note that, over hereditary rings, this notion coincides with that of a preinjective left R-module, defined for any left pure semisimple ring R in [26]. Preinjective left R-modules have played an important role in the study of left pure semisimple rings (see, e.g., [5,14,26,40]). Our next result adds an interesting feature on the endo-structure of direct sums of these modules.

Corollary 5.11. *Let R be a left pure semisimple hereditary ring. Let $N = \bigoplus_{i \in I} N_i$ be a direct sum of preinjective left R-modules. Then N is endoartinian.*

Proof. Let N_i be a preinjective left R-module. Because R is left pure semisimple hereditary, it follows from [6, Proposition 8.2] that N_i is the source of a left almost split morphism in $\mathrm{mod}(R^{\mathrm{op}})$. Now Theorem 5.10(b) yields that $N = \bigoplus_{i \in I} N_i$ is endoartinian. $\square$

We conclude the paper with an observation, and some related questions that might be of interest. Let R be a left pure semisimple ring and M a Σ-pure-injective right R-module, then M is endofinite. Indeed, we know by [30, Theorem 6] that any right R-module has the ACC on finite matrix subgroups, so if M is Σ-pure-injective, it has the DCC on finite matrix subgroups [45], hence M is endofinite by [11, Proposition 4.1].

It is natural to ask if the converse of this statement also holds. We formalize this as the following question.

Question 1. Let R be a left artinian ring, and suppose that every Σ-pure-injective right R-module is endofinite. Is R left pure semisimple?

If R is an Artin algebra, then by Auslander's theorem [8], R is left pure semisimple if and only if R is of finite representation type. Thus the above question has the following equivalent form.

Question 2. Let R be an Artin algebra, and suppose that every Σ-pure-injective right R-module is endofinite. Is R of finite representation type?

Question 2 has a positive answer if R is a hereditary tame Artin algebra. Indeed, let M be the direct sum of all non-isomorphic preinjective right R-modules. Then it is well known (see [34, 4.6]) that M is Σ-pure-injective, hence M is endofinite by hypothesis, so there are finitely many non-isomorphic preinjective right R-modules, implying that R is finite representation type.

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References

[1] T. Albu, C. Nastasescu, Relative Finiteness in Module Theory, Dekker, New York, 1984.
[2] F.W. Anderson, K.R. Fuller, Rings and Categories of Modules, Springer, New York, 1992.
[3] L. Angeleri Hügel, Direct summands of products, Arch. Math. 78 (2002) 12–23.
[4] L. Angeleri Hügel, Covers and envelopes via endoproperties of modules, Proc. London Math. Soc. 86 (2003) 649–665.
[5] L. Angeleri Hügel, A key module over pure-semisimple hereditary rings, J. Algebra 307 (2007) 361–376.
[6] L. Angeleri Hügel, H. Valenta, A duality result for almost split sequences, Colloq. Math. 80 (1999) 267–292.
[7] M. Auslander, Representation theory of Artin algebras II, Comm. Algebra 1 (1974) 269–310.
[8] M. Auslander, Large modules over Artin algebras, in: Algebra, Topology and Categories, Academic Press, New York, 1976, pp. 1–17.
[9] M. Auslander, Functors and morphisms determined by objects, in: Representation Theory of Algebras: Proceedings, Philadelphia Conference, Marcel Dekker, New York, 1978, pp. 1–244.
[10] W.W. Crawley-Boevey, Tame algebras and generic modules, Proc. London Math. Soc. 63 (1991) 241–265.
[11] W.W. Crawley-Boevey, Modules of finite length over their endomorphism rings, in: S. Brenner, H. Tachikawa (Eds.), Representations of Algebras and Related Topics, in: London Math. Soc. Lecture Note Ser., vol. 168, Cambridge Univ. Press, Cambridge, 1992, pp. 127–184.

[12] W.W. Crawley-Boevey, Locally finitely presented additive categories, Comm. Algebra 22 (1994) 1644–1674.
[13] N.V. Dung, Preinjective modules and finite representation type of artinian rings, Comm. Algebra 27 (1999) 3921–3947.
[14] N.V. Dung, Strong preinjective partitions and almost split morphisms, J. Pure Appl. Algebra 158 (2001) 131–150.
[15] N.V. Dung, Contravariant finiteness and pure semisimple rings, in: D.V. Huynh, S.K. Jain, S.R. López-Permouth (Eds.), Algebra and Its Applications, in: Contemp. Math., vol. 419, Amer. Math. Soc., Providence, RI, 2006, pp. 111–124.
[16] N.V. Dung, J.L. García, Additive categories of locally finite representation type, J. Algebra 238 (2001) 200–238.
[17] N.V. Dung, J.L. García, Copure semisimple categories and almost split maps, J. Pure Appl. Algebra 188 (2004) 73–94.
[18] N.V. Dung, J.L. García, Endofinite modules and pure semisimple rings, J. Algebra 289 (2005) 574–593.
[19] C. Faith, Injective Modules and Injective Quotient Rings, Dekker, New York, 1982.
[20] K.R. Fuller, On rings whose left modules are direct sums of finitely generated modules, Proc. Amer. Math. Soc. 54 (1976) 39–44.
[21] K.R. Fuller, I. Reiten, Note on rings of finite representation type and decompositions of modules, Proc. Amer. Math. Soc. 50 (1975) 92–94.
[22] J.L. García, J.L. Gómez Pardo, On endomorphism rings of quasiprojective modules, Math. Z. 196 (1987) 87–108.
[23] J.L. García, J.L. Gómez Pardo, Self-injective and PF endomorphism rings, Israel J. Math. 58 (1987) 324–350.
[24] I. Herzog, Finitely presented right modules over a left pure semisimple ring, Bull. London Math. Soc. 26 (1994) 333–338.
[25] I. Herzog, A test for finite representation type, J. Pure Appl. Algebra 95 (1994) 151–182.
[26] B. Huisgen-Zimmermann, Strong preinjective partitions and representation type of artinian rings, Proc. Amer. Math. Soc. 109 (1990) 309–322.
[27] B. Huisgen-Zimmermann, Purity, algebraic compactness, direct sum decompositions, and representation type, in: Infinite Length Modules, Bielefeld, 1998, in: Trends Math., Birkhäuser, Basel, 2000, pp. 331–367.
[28] B. Huisgen-Zimmermann, W. Zimmermann, Algebraically compact rings and modules, Math. Z. 161 (1978) 81–93.
[29] B. Huisgen-Zimmermann, W. Zimmermann, Classes of modules with the exchange property, J. Algebra 88 (1984) 416–434.
[30] B. Huisgen-Zimmermann, W. Zimmermann, On the sparsity of representations of rings of pure global dimension zero, Trans. Amer. Math. Soc. 320 (1990) 695–711.
[31] B. Huisgen-Zimmermann, M. Saorín, Direct sums of representations as modules over their endomorphism rings, J. Algebra 250 (2002) 67–89.
[32] H. Krause, Generic modules over Artin algebras, Proc. London Math. Soc. 76 (1998) 276–306.
[33] H. Krause, M. Saorín, On minimal approximations of modules, in: E.L. Green, B. Huisgen-Zimmermann (Eds.), Trends in the Representation Theory of Finite Dimensional Algebras, in: Contemp. Math., vol. 229, Amer. Math. Soc., Providence, RI, 1998, pp. 227–236.
[34] H. Lenzing, Homological transfer from finitely presented to infinite modules, in: R. Göbel, L. Lady, A. Mader (Eds.), Abelian Group Theory, in: Lecture Notes in Math., vol. 1006, Springer, Berlin, 1983, pp. 734–761.
[35] M.Y. Prest, Duality and pure-semisimple rings, J. London Math. Soc. 38 (1988) 403–409.
[36] V. Puninskaya, C. Toffalori, preprint, 2006.
[37] C.M. Ringel, H. Tachikawa, QF-3 rings, J. Reine Angew. Math. 272 (1975) 49–72.
[38] D. Simson, Pure semisimple categories and rings of finite representation type, J. Algebra 48 (1977) 290–296; Corrigendum: J. Algebra 67 (1980) 254–256.
[39] D. Simson, Partial Coxeter functors and right pure semisimple hereditary rings, J. Algebra 71 (1981) 195–218.
[40] D. Simson, A class of potential counter-examples to the pure semisimplicity conjecture, in: M. Droste, R. Göbel (Eds.), Advances in Algebra and Model Theory, Algebra, Logic and Applications, Gordon & Breach, Amsterdam, 1997, pp. 345–373.
[41] B. Stenström, Rings of Quotients, Springer, Heidelberg, 1975.
[42] M.L. Teply, R.W. Miller, The descending chain condition relative to a torsion theory, Pacific J. Math. 83 (1979) 207–219.
[43] R. Wisbauer, On modules with the Kulikov property and pure semisimple modules and rings, J. Pure Appl. Algebra 70 (1991) 315–320.
[44] R. Wisbauer, Foundations of Module and Ring Theory, Algebra, Logic and Applications, vol. 3, Gordon & Breach, Reading, MA, 1991.
[45] W. Zimmermann, Rein-injektive direkte Summen von Moduln, Comm. Algebra 5 (1977) 1083–1117.
[46] W. Zimmermann, (Σ-)algebraic compactness of rings, J. Pure Appl. Algebra 23 (1982) 319–328.
[47] W. Zimmermann, Existenz von Auslander–Reiten-Folgen, Arch. Math. (Basel) 40 (1983) 40–49.