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ScienceDirect
Journal of Algebra 320 (2008) 1963–1982
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Finitely generated antisymmetric graph monoids

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Received 20 June 2007
Available online 10 July 2008
Communicated by Kent R. Fuller

Abstract

A *graph monoid* is a commutative monoid for which there is a particularly simple presentation, given in terms of a quiver. Such monoids are known to satisfy various nonstable K-theoretical representability properties for either von Neumann regular rings or C^*-algebras. We give a characterization of graph monoids within finitely generated antisymmetric refinement monoids. This characterization is formulated in terms of the *prime elements* of the monoid, and it says that each free prime has at most one free lower cover. We also characterize antisymmetric graph monoids of finite quivers. In particular, the monoid $\mathbb{Z}^\infty = \{0, 1, 2, \dots\} \cup \{\infty\}$ is a graph monoid, but it is not the graph monoid of any finite quiver.

Keywords: Monoid; Commutative; Quiver; Graph monoid; Separative; Refinement; Primely generated; Primitive; Antisymmetric; Prime element; Free element; Regular element; Von Neumann regular; C^*-algebra

1. Introduction

Many module-theoretical properties of a ring R can be expressed in terms of the so-called *nonstable K-theory of* R, which can be encoded in the commutative monoid $\mathcal{V}(R)$ defined, in

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1 The first and second authors were partially supported by the DGI and European Regional Development Fund, jointly, through Project MTM2005-00934, and by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2008.06.013

the unital case, as the monoid of all isomorphism types of finitely generated projective right R-modules. Of particular interest is the case where the ring R is *von Neumann regular*, in which case the fundamental, still unsolved, open problem is the characterization problem of all monoids of the form $\mathcal{V}(R)$, published for the first time in [10]. While the original guess—namely, "all conical refinement monoids" (cf. Section 2 for the basic definitions)—got disproved in [15], with a counterexample of size $\aleph_2$, the following fundamental question is still open:

Is every countable, conical refinement monoid representable*, that is, isomorphic to* $\mathcal{V}(R)$*, for some von Neumann regular ring* R?

An important positive partial solution was recently obtained by Ara and Brustenga [2], where the authors prove that the representation problem above has a positive solution for the so-called *graph monoids*—in fact, the regular ring solving the problem can be taken an algebra over any given field, see [2, Theorem 4.4]. Graph monoids are a special class of refinement monoids for which there is a particularly simple presentation, given in terms of a row-finite quiver (see Section 3). The graph monoid of a row-finite quiver E is denoted by $\mathrm{M}(E)$. We refer the reader to [2] for more information on the problem above and its relationship with the Separativity Problem of [3].

For any row-finite quiver E, there is a C^*-algebra $C^*(E)$ associated to it, called the *Cuntz–Krieger graph C^*-algebra* of E. These graph C^*-algebras provide a wide generalization of the ubiquitous *Cuntz algebras* $\mathcal{O}_n$, introduced by Cuntz in [8]. We refer the reader to [14] for the basic theory of graph C^*-algebras. For any field K, the *Leavitt path K-algebra* of the row-finite quiver E, denoted by $L_K(E)$, has been defined in [1] and [4], as a purely algebraic analogue of the C^*-algebra $C^*(E)$. Indeed, it turns out that $L_{\mathbb{C}}(E)$ can be identified with a dense $*$-subalgebra of $C^*(E)$. It was proven in [4, Theorem 3.5] that $\mathcal{V}(L_K(E)) \cong \mathrm{M}(E)$ for every field K and every row-finite quiver E, and likewise the monoid $\mathrm{M}(E)$ is isomorphic to $\mathcal{V}(C^*(E))$ by [4, Theorem 7.1]. The algebras $L_K(E)$ are not in general von Neumann regular, and the main goal of the paper [2] is to build an appropriate von Neumann regular algebra of fractions $Q_K(E)$ of $L_K(E)$ in such a way that the corresponding monoid is not altered: $\mathcal{V}(Q_K(E)) \cong \mathrm{M}(E)$; see [2, Theorem 4.4].

Although graph monoids have a simple combinatorial definition, it is *a priori* difficult to determine whether a given finitely generated monoid (given, say, by generators and relations) is a graph monoid. In this paper we solve that particular problem in the *antisymmetric* case, see Theorem 5.1. Our characterization is formulated in terms of the so-called *prime elements* of our monoid, and it says, for a given primely generated, antisymmetric refinement monoid (we say *primitive* monoid) whose set of primes is *lower finite* (cf. Section 2), that *each free prime has at most one free lower cover* (*among the primes*). Our main preliminary result is the discovery of a finitely generated primitive monoid that is not even a *retract* of any graph monoid, see Lemma 4.1. As another surprise, there are finitely generated graph monoids that are not the graph monoid of any *finite* quiver, the simplest of them being $\mathbb{Z}^{\infty} = \mathbb{Z}^{+} \cup \{\infty\}$. A characterization of all antisymmetric graph monoids of finite quivers is given in Theorem 6.1.

2. Basic concepts

All commutative monoids will be written additively. For elements x and y in a commutative monoid M, we put

$$\begin{aligned}
x \leqslant y \quad &\Leftrightarrow \quad (\exists z)(x + z = y),\\
x < y \quad &\Leftrightarrow \quad (x \leqslant y \text{ and } y \nleqslant x),\\
x \equiv y \quad &\Leftrightarrow \quad (x \leqslant y \text{ and } y \leqslant x),\\
x \lhd y \quad &\Leftrightarrow \quad x + y = y,\\
x \ll y \quad &\Leftrightarrow \quad x + y \leqslant y.
\end{aligned}$$

An element x of M is

- *free*, if $(n+1)x \nleqslant nx$ for any $n \in \mathbb{Z}^+$;
- *regular*, if $2x \leqslant x$;
- *idempotent*, if $2x = x$;
- *an atom*, if $x \nleqslant 0$ and $x = y + z$ implies that either $y \leqslant 0$ or $z \leqslant 0$, for all $y, z \in M$;
- *prime*, if $p \nleqslant 0$ and, further, $p \leqslant x + y$ implies that either $p \leqslant x$ or $p \leqslant y$, for all $x, y \in M$.

We denote by $\mathbb{P}(M)$ the set of all prime elements in M. We denote by $\mathbb{P}_{\mathrm{free}}(M)$ (resp., $\mathbb{P}_{\mathrm{reg}}(M)$) the set of all free primes (resp., regular primes) in M. We say that M is

- *conical*, if $x \leqslant 0$ implies that $x = 0$, for any $x \in M$;
- *antisymmetric*, if its algebraic preordering $\leqslant$ is antisymmetric;
- *separative*, if $2x = x + y = 2y$ implies that $x = y$, for all $x, y \in M$;
- *strongly separative*, if $2x = x + y$ implies that $x = y$, for all $x, y \in M$;
- *primely generated*, if M is generated, as a monoid, by $\mathbb{P}(M)$. (*This is not equivalent to the definition given in* [5], *as primes may there be below zero, however, for conical monoids the two definitions are equivalent*.)

For a monoid N and a homomorphism $f : M \to N$, the *kernel* of f, defined as

$$\ker f = \{(x, y) \in M \times M \mid f(x) = f(y)\},$$

is a monoid congruence of M. A particular sort of congruence is obtained when we start with an *o-ideal* of M, that is, a nonempty subset I of M such that $x + y \in I$ if and only if $x \in I$ and $y \in I$, for all $x, y \in M$. Namely, the equivalence relation $\equiv_I$ defined on M by the rule

$$x \equiv_I y \quad \Leftrightarrow \quad (\exists u, v \in I)(x + u = y + v), \quad \text{for all } x, y \in M$$

is a monoid congruence of M. We put $M/I = M/{\equiv_I}$ and we denote by $x_{/I}$ the $\equiv_I$-equivalence class of any element x of M. Observe that $x_{/I} \leqslant y_{/I}$ in M/I if and only if the relation $x \leqslant_I y$ defined as

$$x \leqslant_I y \quad \Leftrightarrow \quad (\exists h \in I)(x \leqslant y + h)$$

holds, for any $x, y \in M$. We shall say that M/I is an *ideal quotient* of M. We denote by

$$M \mid a = \{x \in M \mid (\exists n \in \mathbb{Z}^+)(x \leqslant na)\}$$

the o-ideal generated by an element $a \in M$.

The *antisymmetrisation* of M is the quotient $M/{\equiv}$. We put

$$\begin{aligned}
\mathrm{L}(M,p) &= \{q \in \mathbb{P}(M) \mid q < p \text{ and there is no } r \in \mathbb{P}(M) \text{ with } q < r < p\},\\
\mathrm{L}_{\mathrm{free}}(M,p) &= \{q \in \mathrm{L}(M,p) \mid q \text{ is free}\},\\
\mathrm{L}_{\mathrm{reg}}(M,p) &= \{q \in \mathrm{L}(M,p) \mid q \text{ is regular}\},
\end{aligned}$$

for any $p \in \mathbb{P}(M)$. We say that M is a *refinement monoid* [9,16], if for all elements $a_0, a_1, b_0, b_1 \in M$ such that $a_0 + a_1 = b_0 + b_1$, there are elements $c_{i,j} \in M$, for $i, j < 2$, such that $a_i = c_{i,0} + c_{i,1}$ and $b_i = c_{0,i} + c_{1,i}$ for all $i < 2$. It is well known that every o-ideal and every ideal quotient of a refinement monoid is a refinement monoid. It is established in [5, Corollary 6.8] that every finitely generated refinement monoid is primely generated. A monoid is *primitive* [13, Section 3.4], if it is an antisymmetric, primely generated, refinement monoid. For example, $\mathbb{Z}^{\infty} = \mathbb{Z}^{+} \cup \{\infty\}$, endowed with its natural addition, is a primitive monoid. For any prime element p in a refinement monoid M, the map

$$\phi_p : M \to \mathbb{Z}^{\infty}, \quad x \mapsto \sup(n \in \mathbb{Z}^{+} \mid np \leqslant x)$$

is a monoid homomorphism from M to $\mathbb{Z}^{\infty}$, see [5, Theorem 5.4]. Furthermore, if M is primitive, then the map

$$\phi : M \to (\mathbb{Z}^{\infty})^{\mathbb{P}(M)}, \quad x \mapsto (\phi_p(x) \mid p \in \mathbb{P}(M)) \tag{2.1}$$

is a monoid embedding as well as an order-embedding, see [5, Theorem 5.11] or [16, Corollary 6.14].

We shall need the following lemma.

Lemma 2.1. *Let a, b, c be elements in a refinement monoid M, with c primely generated. If $a + c = b + c$, then there are $x, y \lhd c$ such that $a + x = b + y$.*

Proof. By [5, Theorem 4.1], there are $d, a', b', c' \in M$ such that $a = d + a'$, $b = d + b'$, $c = a' + c' = b' + c'$, and $c \leqslant c'$. Let $h \in M$ such that $c' = c + h$. The elements $x = b' + h$ and $y = a' + h$ are as required. □

A partially ordered set P is *lower finite*, if the subset $P \downarrow p = \{q \in P \mid q \leqslant p\}$ is finite, for any $p \in P$. We say that P is a *forest*, if $P \downarrow p$ is a chain for any $p \in P$.

3. Graph monoids

We first recall some definitions from [4]. A *quiver* (in some other references, a *graph*) consists of a 'vertex set' E^0, an 'edge set' E^1, together with maps r and s from E^1 to E^0 describing, respectively, the range and source of edges; so we write $e : s(e) \to r(e)$, for any $e \in E^1$. We say that $u \in E^0$ *emits edges*, if $s^{-1}\{u\}$ is nonempty; otherwise we say that u is a *sink*. We say that E is *row-finite*, if any $u \in E^0$ emits only finitely many edges, that is, $s^{-1}\{u\}$ is finite. We say that E is *finite*, if both E^0 and E^1 are finite.

The *graph monoid* of a row-finite quiver E, denoted by $\mathrm{M}(E)$, is the commutative monoid defined by generators $\bar{u}$, for $u \in E^0$, and relations

$$\bar{u} = \sum \bigl(\overline{r(e)} \bigm| e \in s^{-1}\{u\}\bigr), \quad \text{for any } u \in E^0 \text{ not a sink.} \tag{3.1}$$

The restriction that u is not a sink in (3.1) may seem artificial at first sight. However, this is inessential, as adding one more edge $u \to u$ for each sink u adds the corresponding relation $\bar{u} = \bar{u}$ to the presentation (3.1), hence it does not affect the monoid defined by that presentation. Hence we shall mainly work with quivers with no sink.

Conversely, with any set Σ, any doubly indexed family $(k_{u,v} \mid (u, v) \in \Sigma \times \Sigma)$ of natural numbers such that $\{v \in \Sigma \mid k_{u,v} \neq 0\}$ is finite for any $u \in \Sigma$, and any set of (formal) relations of the form

$$u = \sum (k_{u,v} \cdot v \mid v \in \Sigma), \quad \text{for any } u \in \Sigma \text{ for which some } k_{u,v} \text{ is nonzero,} \tag{3.2}$$

one can associate a row-finite quiver E such that (3.2) is a system of defining relations for $\mathrm{M}(E)$: just take $E^0 = \Sigma$ and put $k_{u,v}$ edges with source u and range v in E^1, for any $u, v \in \Sigma$. We will say that E is the *quiver associated with the equation system* (3.2). This quiver has no sink if and only if $(\forall u)(\exists v)(k_{u,v} \neq 0)$.

We shall denote by $\mathrm{Fr}(X)$ the free commutative monoid on X, for any set X; we identify X with its canonical image in $\mathrm{Fr}(X)$. For a row-finite quiver E and $\alpha, \beta \in \mathrm{Fr}(E^0)$, let $\alpha \to_1 \beta$ hold, if there are $\gamma \in \mathrm{Fr}(E^0)$ and $x \in E^0$ emitting edges such that

$$\alpha = \gamma + x \quad \text{and} \quad \beta = \gamma + \sum \bigl(r(e) \bigm| e \in s^{-1}\{x\}\bigr).$$

Furthermore, we put $\to_n = (\to_1) \circ \cdots \circ (\to_1)$ (n times), for all $n \in \mathbb{Z}^+$, and we denote by $\to$ the union of all the $\to_n$, for $n \in \mathbb{Z}^+$. We denote by $\pi_E : \mathrm{Fr}(E^0) \to \mathrm{M}(E)$ the unique monoid homomorphism such that $\pi_E(x) = \bar{x}$ for all $x \in E^0$. Of course, π_E is surjective, and solving the word problem for $\mathrm{M}(E)$ amounts to finding a convenient description of the kernel $\sim_E$ of π_E, defined by

$$\alpha \sim_E \beta \quad \Leftrightarrow \quad \pi_E(\alpha) = \pi_E(\beta), \quad \text{for all } \alpha, \beta \in \mathrm{Fr}\bigl(E^0\bigr).$$

Such a description is item (3) of the following lemma, established in [4, Section 4].

Lemma 3.1. *Let E be a row-finite quiver.*

(1) *The relation $\to$ is* right refining*, that is, for all $\alpha_0, \alpha_1, \beta \in \mathrm{Fr}(E^0)$, if $\alpha_0 + \alpha_1 \to \beta$, then there are $\beta_0, \beta_1 \in \mathrm{Fr}(E^0)$ such that $\alpha_0 \to \beta_0$, $\alpha_1 \to \beta_1$, and $\beta = \beta_0 + \beta_1$.*
(2) *The relation $\to$ is* confluent*, that is, for all $\alpha, \beta_0, \beta_1 \in \mathrm{Fr}(E^0)$, if $\alpha \to \beta_0$ and $\alpha \to \beta_1$, then there exists $\gamma \in \mathrm{Fr}(E^0)$ such that $\beta_0 \to \gamma$ and $\beta_1 \to \gamma$.*
(3) *For all $\alpha, \beta \in \mathrm{Fr}(E^0)$, $\pi_E(\alpha) = \pi_E(\beta)$ if and only if there exists $\gamma \in \mathrm{Fr}(E^0)$ such that $\alpha \to \gamma$ and $\beta \to \gamma$.*

The following result is established in [4, Proposition 4.4].

Proposition 3.2. *The monoid $\mathrm{M}(E)$ is a conical refinement monoid, for any row-finite quiver E.*

In case E is *finite*, $\mathrm{M}(E)$ is finitely generated. As finitely generated refinement monoids are primely generated [5, Corollary 6.8], it follows that $\mathrm{M}(E)$ is primely generated.

The following few definitions about quivers can be found in [4]. For $u, v \in E^0$, let $u \geqslant_1 v$ hold, if $v \in r(s^{-1}\{u\})$; denote by $\geqslant$ the reflexive, transitive closure of $\geqslant_1$. A subset H of E^0 is *hereditary*, if $u \in H$ and $u \geqslant v$ implies that $v \in H$, for all $u, v \in E^0$. Then we put $H^1 = s^{-1}(H)$ and $E \restriction H = (H, H^1)$, the *restriction of E to H*. As H is hereditary, H^1 is contained in $r^{-1}(H)$. Then we define a quiver, denoted by $E \setminus H$, by $(E \setminus H)^0 = E^0 \setminus H$ and $(E \setminus H)^1 = \{e \in E^1 \mid r(e) \notin H\}$. A subset H of E^0 is *saturated*, if $s^{-1}\{v\} \neq \emptyset$ and $r(s^{-1}\{v\}) \subseteq H$ implies that $v \in H$, for each $v \in E^0$.

A *subquiver* of a quiver F is a pair $E = (E^0, E^1)$ with $E^0 \subseteq F^0$, $E^1 \subseteq F^1$, and $s_F(E^1) \cup r_F(E^1) \subseteq E^0$. Of course, then we denote by s_E and r_E the restrictions of s_F and r_F from E^1 to E^0, respectively. We say that E is a *complete subquiver* of F, if $s_F^{-1}\{v\} \cap E^1 \neq \emptyset$ implies that $s_F^{-1}\{v\} \subseteq E^1$, for all $v \in E^0$.

A *quiver homomorphism* from a quiver E to a quiver F consists of a pair $f = (f^0, f^1)$ of maps $f^0 : E^0 \to F^0$ and $f^1 : E^1 \to F^1$ such that $r_F \circ f^1 = f^0 \circ r_E$ and $s_F \circ f^1 = f^0 \circ s_E$. We say that f is *complete*, if both f^0 and f^1 are injective and $(f^0(E^0), f^1(E^1))$ is a complete subquiver of F. If F is row-finite and $f : E \to F$ is a complete quiver embedding, then there exists a unique monoid homomorphism $\mathrm{M}(f) : \mathrm{M}(E) \to \mathrm{M}(F)$ such that $\mathrm{M}(f)(\bar{v}) = \overline{f^0(v)}$ for all $v \in E^0$. The assignment $E \mapsto \mathrm{M}(E)$, $f \mapsto \mathrm{M}(f)$ is a functor.

An easy application of Lemma 3.1 yields the following.

Lemma 3.3. *Let E be a row-finite quiver. Then the following statements hold*:

1. *For every hereditary subset H of E^0, the restriction $E \restriction H$ is a complete subquiver of E, and the canonical homomorphism* $\mathrm{M}(E \restriction H) \to \mathrm{M}(E)$ *is an embedding, whose image is an o-ideal of* $\mathrm{M}(E)$.
2. *Conversely, for every o-ideal J of* $\mathrm{M}(E)$, *the set $H = \{u \in E^0 \mid \bar{u} \in J\}$ is a hereditary subset of E^0, and $J \cong \mathrm{M}(E \restriction H)$.*

In the context of Lemma 3.3, we shall identify $\mathrm{M}(E \restriction H)$ with its canonical image in $\mathrm{M}(E)$. Although the hereditary set H obtained in Lemma 3.3(ii) is saturated, saturation is not required in the proof of Lemma 3.3(i). Further, we observe the following result, established in [4, Lemma 3.1].

Lemma 3.4. *Let E be a row-finite quiver. Then every finite subquiver of E is a subquiver of some finite complete subquiver of E. Consequently, E is a direct limit of finite quivers with complete embeddings.*

As the functor $E \mapsto \mathrm{M}(E)$, $f \mapsto \mathrm{M}(f)$ preserves direct limits [4, Lemma 3.4], it follows that the graph monoid of any row-finite quiver is a direct limit of graph monoids of finite quivers. Denote by $\mathcal{G}$ the category of all monoids isomorphic to graph monoids of finite quivers with monoid homomorphisms, and by $\overline{\mathcal{G}}$ the category of all commutative monoids that are direct limits of members of $\mathcal{G}$ with monoid homomorphisms. In particular, the graph monoid of any row-finite quiver is an object of $\overline{\mathcal{G}}$.

Observe now that $\mathcal{G}$ is closed under finite direct products (take the disjoint union of the corresponding quivers). Hence, it follows from Corollary 4.2, Remark 4.3, and Lemma 4.4 in [11] that $\overline{\mathcal{G}}$ is closed under direct limits and retracts. The finitely generated monoids from $\overline{\mathcal{G}}$ can be characterized as follows.

Lemma 3.5. *A finitely generated commutative monoid M belongs to $\overline{\mathcal{G}}$ if and only if it is a retract of some member of $\mathcal{G}$.*

Proof. As $\overline{\mathcal{G}}$ is closed under retracts, it suffices to prove that if M belongs to $\overline{\mathcal{G}}$, then M is a retract of some member of $\mathcal{G}$. So let $M = \varinjlim_{i \in I} M_i$, with a directed partially ordered set I, monoids M_i in $\mathcal{G}$, transition morphisms $f_i^j : M_i \to M_j$ for $i \leqslant j$ in I, and limiting morphisms $f_i : M_i \to M$ for $i \in I$. As M is finitely generated, there exists $i \in I$ such that f_i is surjective. As $M \cong M_i / \ker f_i$ is finitely generated, it is, by Redei's Theorem, finitely presented, thus $\ker f_i$ is a finitely generated monoid congruence of M_i. As $\ker f_i = \bigcup_{j \geqslant i} \ker f_i^j$ (directed union), there exists $j \geqslant i$ such that $\ker f_i = \ker f_i^j$. For all $y \in M$, there exists $x \in M_i$ such that $y = f_i(x)$, and then $f_i^j(x)$ does not depend of the choice of $x \in f_i^{-1}\{y\}$; denote it by $e(y)$. Then e is a homomorphism from M to M_j, and $f_j \circ e = \mathrm{id}_M$. Therefore, M is a retract of M_j. □

We shall also need the following simple observation.

Lemma 3.6. *Both classes $\mathcal{G}$ and $\overline{\mathcal{G}}$ are closed under o-ideals and ideal quotients.*

Proof. Closure of $\mathcal{G}$ under o-ideals follows from Lemma 3.3, while closure of $\mathcal{G}$ under ideal quotients follows from Theorem 5.3 and Lemma 6.6 in [4].

Now we deal with $\overline{\mathcal{G}}$. Any member M of $\overline{\mathcal{G}}$ can be written as a direct limit

$$(M, f_i \mid i \in I) = \varinjlim\big(M_i, f_i^j \;\big|\; i \leqslant j \text{ in } I\big),$$

for some directed partially ordered set I, monoids $M_i \in \mathcal{G}$, and monoid homomorphisms $f_i^j : M_i \to M_j$, $f_i : M_i \to M$. Let N be an o-ideal of M. The subset $N_i = f_i^{-1}(N)$ is an o-ideal of M_i, for all $i \in I$. We can define g_i^j (resp., g_i) as the restriction of f_i^j from N_i to N_j (resp., from N_i to N), and then it is straightforward to verify that

$$(N, g_i \mid i \in I) = \varinjlim\big(N_i, g_i^j \;\big|\; i \leqslant j \text{ in } I\big),$$

and so N belongs to $\overline{\mathcal{G}}$. Furthermore, for all $i \leqslant j$ in I, there exists a unique monoid homomorphism $h_i^j : M_i/N_i \to M_j/N_j$ (resp., $h_i : M_i/N_i \to M/N$) such that $h_i(x_{/N_i}) = f_i(x)_{/N}$ for any $x \in M_i$, and it is straightforward to verify that

$$(M/N, h_i \mid i \in I) = \varinjlim\big(M_i/N_i, h_i^j \;\big|\; i \leqslant j \text{ in } I\big),$$

and so M/N belongs to $\overline{\mathcal{G}}$. □

4. A strongly separative primitive monoid not in $\overline{\mathcal{G}}$

In this section, we denote by M_0 the commutative monoid defined by generators p, a, b and relations $p = p + a = p + b$. It can be described as

$$M_0 = \left(\mathbb{Z}^+ a + \mathbb{Z}^+ b\right) \cup \{p, 2p, 3p, \ldots\}, \tag{4.1}$$

where $a = (1, 0)$, $b = (0, 1)$, and $p + x = p$ for any element x in $\mathbb{Z}^+ a + \mathbb{Z}^+ b = \mathbb{Z}^+ \times \mathbb{Z}^+$. So M_0 is a strongly separative, finitely generated, primitive monoid: this can either be verified by hand from the description (4.1), or by applying [13, Proposition 3.5.2] (for "primitive") and the comments following the proof of [5, Corollary 5.9] (for "strongly separative").

Lemma 4.1. *The monoid M_0 does not belong to $\overline{\mathcal{G}}$. That is, M_0 is not a direct limit of graph monoids.*

Proof. Suppose that M_0 belongs to $\overline{\mathcal{G}}$. By Lemma 3.5, M_0 is a submonoid of some monoid N in $\mathcal{G}$ with a retraction $\rho : N \twoheadrightarrow M_0$. Let E be a finite quiver such that $N \cong \mathrm{M}(E)$, with E^0 of minimal cardinality.

As M_0 is conical, $\rho^{-1}\{0\}$ is an o-ideal of N. By Theorem 5.3 and Lemma 6.6 in [4], the monoid $N' = N/\rho^{-1}\{0\}$ belongs to $\mathcal{G}$, *via* a subquiver E' of E, with $(E')^0 = E^0 \setminus H$, where $H = \{u \in E^0 \mid \rho(\bar{u}) = 0\}$. As M_0 is also a retract of N' and by the minimality assumption on E, we obtain that $H = \emptyset$, and so $\rho^{-1}\{0\} = \{0\}$. In particular, as both a and b are atoms of M_0, they are also atoms of N.

Denote by I the o-ideal of M_0 generated by $\{a, b\}$. Hence $J = \rho^{-1}(I)$ is an o-ideal of N. We denote by $\bar{\rho}$ the unique monoid homomorphism from $\mathrm{Fr}(E^0)$ to M_0 that sends u to $\rho(\bar{u})$, for all $u \in E^0$.

Claim 1. *Every element of J is cancelable in N.*

Proof. Let $x + z = y + z$ hold, where $x, y \in N$ and $z \in J$. By Lemma 2.1, there are $u, v \lhd z$ such that $x + u = y + v$. As $\rho(u), \rho(v) \lhd \rho(z)$ and $\rho(z) \in I$, it follows that $\rho(u) = \rho(v) = 0$, thus $u = v = 0$, and thus $x = y$. □

As N is a finitely generated graph monoid, it is primely generated. So there are $n \in \mathbb{N}$ and primes $q_0, \ldots, q_n$ in N such that $p = \sum_{i=0}^{n} q_i$. Applying ρ gives $p = \sum_{i=0}^{n} \rho(q_i)$, thus, up to permutation of the indices and putting $h = \sum_{i=1}^{n} q_i$, we get $\rho(h) \in I$ and $\rho(q_0) = p$. As $p + a = p$, we get $q_0 + a + h = q_0 + h$, hence, by Claim 1, $q_0 + a = q_0$. Similarly, $q_0 + b = q_0$. Therefore, by keeping the same ρ and by replacing the inclusion map from M_0 into N by the unique homomorphism fixing both a and b and sending p to q_0, we reduce the problem to the case where $p = q_0$, that is, p is prime in N. Hence there exists $q \in E^0$ such that $\bar{q} \equiv p$. As both a and b are atoms of N, there are $x, y \in E^0$ such that $\bar{x} = a$ and $\bar{y} = b$.

Claim 2. *The inequality $\bar{u} \leqslant p$ holds for each $u \in E^0$.*

Proof. The set $H = \{u \in E^0 \mid \bar{u} \leqslant p\}$ is a hereditary subset of E^0, thus, by Lemma 3.3, the canonical map $j : \mathrm{M}(E \restriction H) \to \mathrm{M}(E)$ is a monoid embedding. As p, a, and b are finite sums of images of elements of H, M_0 is a submonoid of $\mathrm{M}(E \restriction H)$, and so the restriction of ρ

to $\mathrm{M}(E \upharpoonright H)$ defines a retraction from $\mathrm{M}(E \upharpoonright H)$ onto M_0. By the minimality assumption on E, we obtain that $H = E^0$. □

Now we put $P = \{u \in E^0 \mid \bar{u} = \bar{u} + a = \bar{u} + b\}$. As $\bar{q} \equiv p$ and $p = p + a = p + b$, we obtain that q belongs to P, thus P is nonempty. For $u \in P$, if $\rho(\bar{u}) < p$, then, by Claim 1, $\bar{u}$ is cancelable, a contradiction as $\bar{u} = \bar{u} + a$; hence $\rho(\bar{u}) = p$.

Claim 3. *Every element $u \in P$ emits exactly one edge $e_{(u)}$ such that $r(e_{(u)}) \in P$. Every edge $e \in s^{-1}\{u\} \setminus \{e_{(u)}\}$ satisfies $\overline{r(e)} \in J$.*

Proof. As $\bar{u} = \bar{u} + a$ and by Lemma 3.1, there exists $\alpha \in \mathrm{Fr}(E^0)$ such that $u \to \alpha$ and $u + x \to \alpha$. If u emits no edges, then $\alpha = u$, thus $u + x \to u$, a contradiction by Lemma 3.1(1); hence u emits edges. From $\bar{u} = \sum(\overline{r(e)} \mid e \in s^{-1}\{u\})$ it follows that $p = \sum(\rho(\overline{r(e)}) \mid e \in s^{-1}\{u\})$. Hence there exists exactly one $e_{(u)} \in s^{-1}\{u\}$ such that $\rho(\overline{r(e_{(u)})}) = p$, and $\rho(\overline{r(e)}) \in I$ for all other $e \in s^{-1}\{u\}$. If X denotes the set of all those other edges, then $t = \sum(\overline{r(e)} \mid e \in X)$ is cancelable in N and $\bar{u} = \overline{r(e_{(u)})} + t$. As $\bar{u} = \bar{u} + a = \bar{u} + b$, we obtain that $\overline{r(e_{(u)})} = \overline{r(e_{(u)})} + a = \overline{r(e_{(u)})} + b$, so $r(e_{(u)}) \in P$. □

Now we fix $q_0 \in P$, and we put $e_n = e_{(q_n)}$ and $q_{n+1} = r(e_{(q_n)})$, for every natural number n. So all elements q_n belong to P. As P is finite, there are natural numbers $k < m$ such that $q_k = q_m$. By taking the pair $(m, m-k)$ minimal with respect to the lexicographical ordering and truncating the sequences $(q_n)_n$ and $(e_n)_n$ at k, we may assume without loss of generality that $k = 0$, so $q_0, \dots, q_{m-1}$ are pairwise distinct, $q_{m+n} = q_n$, and $e_{m+n} = e_n$, for all $n \in \mathbb{Z}^+$. We put $E_n = s^{-1}\{q_n\} \setminus \{e_n\}$ and $c_n = \sum(\rho(\overline{r(e)}) \mid e \in E_n)$, an element of I, for all $n \in \mathbb{Z}^+$. Furthermore, we put $c = \sum_{i<m} c_i$.

Claim 4. *Let $i \in \mathbb{Z}^+$ and $\alpha \in \mathrm{Fr}(E^0)$ such that $q_i \to \alpha$. Then there are an integer $j \geqslant i$ and $\beta \in \mathrm{Fr}(E^0)$ such that $\alpha = q_j + \beta$ with $\bar{\rho}(\beta) = \sum_{i \leqslant k < j} c_k$.*

Proof. By induction on l such that $q_i \to_l \alpha$. For $l = 0$ it is trivial, so suppose the claim established at stage l, and let $q_i \to_{l+1} \alpha$. So there exists $\alpha' \in \mathrm{Fr}(E^0)$ such that $q_i \to_l \alpha' \to_1 \alpha$. By the induction hypothesis, there are $j \geqslant i$ and $\beta' \in \mathrm{Fr}(E^0)$ such that $\bar{\rho}(\beta') = \sum_{i \leqslant k < j} c_k$ and $\alpha' = q_j + \beta'$. By the definition of $\to_1$, either there exists $\gamma \in \mathrm{Fr}(E^0)$ such that $\beta' \to_1 \gamma$ and

$$\alpha = q_j + \gamma,$$

or

$$\alpha = q_{j+1} + \sum \bigl(r(e) \bigm| e \in E_j\bigr) + \beta'.$$

In the first case, $\bar{\rho}(\gamma) = \bar{\rho}(\beta') = \sum_{i \leqslant k < j} c_k$, so the result holds. In the second case, put $\beta = \sum(r(e) \mid e \in E_j) + \beta'$. Then

$$\bar{\rho}(\beta) = c_j + \bar{\rho}(\beta') = \sum_{i \leqslant k < j+1} c_k,$$

so $j + 1$ and β are as required. □

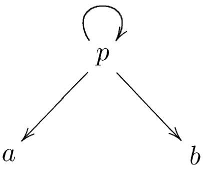

Fig. 1. The quiver E corresponding to $p = p + a + b$.

Now we can conclude the proof. As $\overline{q_0} = \overline{q_0} + a$, there exists $\alpha \in \mathrm{Fr}(E^0)$ such that $q_0 \to \alpha$ and $q_0 + x \to \alpha$. The second relation implies the existence of $\alpha', \gamma \in \mathrm{Fr}(E^0)$ such that $q_0 \to \alpha'$, $x \to \gamma$, and $\alpha = \alpha' + \gamma$. By Claim 4, there are natural numbers i, j and elements $\beta, \beta' \in \mathrm{Fr}(E^0)$ such that

$$\alpha = q_i + \beta, \quad \alpha' = q_j + \beta', \quad \bar{\rho}(\beta) = \sum_{k<i} c_k, \quad \text{and} \quad \bar{\rho}(\beta') = \sum_{k<j} c_k. \tag{4.2}$$

As all the c_ks belong to I, so do $\bar{\rho}(\beta)$ and $\bar{\rho}(\beta')$. From $x \to \gamma$ it follows that $\bar{\rho}(\gamma) = a$ belongs to I. From $\alpha = \alpha' + \gamma$ it follows that $q_i + \beta = q_j + \beta' + \gamma$. As $\bar{\rho}(\beta)$, $\bar{\rho}(\beta')$, and $\bar{\rho}(\gamma)$ belong to I, the elements β, β', and γ have no component in P, thus $q_i = q_j$, so

$$i \equiv j \pmod{m} \quad \text{and} \quad \beta = \beta' + \gamma. \tag{4.3}$$

As we have seen, $\bar{\rho}(\gamma) = a$, thus, applying $\bar{\rho}$ to the equation in (4.3) and using (4.2), we obtain

$$\sum_{k<i} c_k = a + \sum_{k<j} c_k.$$

Hence, as $I \cong \mathbb{Z}^+ \times \mathbb{Z}^+$ is cancellative, $i > j$ and $\sum_{j \leqslant k < i} c_k = a$. Furthermore, as $i - j = \ell m$ for some $\ell > 0$ and the sequence $(c_l \mid l \in \mathbb{Z}^+)$ is periodical with period m, we get $a = \ell c$. A similar argument gives $b = \ell' c$, for some positive integer ℓ', which forces $c = 0$, a contradiction as $a = \ell c$. $\square$

Observe that the monoid M_0 is the antisymmetrisation of the commutative monoid M_0' defined by generators p, a, b and relation $p = p + a + b$. As $M_0' = \mathrm{M}(E)$ for the quiver E represented in Fig. 1, this implies that *the antisymmetrisation of a finitely generated graph monoid is not necessarily a graph monoid*.

By Lemmas 3.6 and 4.1, no commutative monoid M such that M_0 is an ideal quotient of an o-ideal of M can belong to $\overline{\mathcal{G}}$. In particular, we obtain the following result.

Theorem 4.2. *Let M be a primely generated refinement monoid with free prime elements p, a, b such that $p = p + a = p + b$ and a and b are incomparable in $\mathrm{L}(M, p)$. Then M does not belong to $\overline{\mathcal{G}}$. That is, M is not a direct limit of graph monoids.*

Proof. It follows from [5, Theorem 5.8] and the comments following it (about uniqueness of the decomposition) that our assumptions imply that M_0 is isomorphic to the submonoid of M

generated by $\{p, a, b\}$, so we may identify those two monoids. The o-ideal $N = M \mid p$ obviously contains M_0. As both a and b are prime elements in M, the subset

$$I = \{x \in N \mid a \nleqslant x \text{ and } b \nleqslant x\}$$

is an o-ideal of N. We shall prove that $M_0 \cong N/I$.

Claim. *For any $x \in N$, there exists $y \in M_0$ such that $x \equiv_I y$.*

Proof. The subset $N_1 = \{x \in N \mid (\exists y \in M_0)(x \equiv_I y)\}$ is a submonoid of N. We must prove that $x \in N_1$, for any $x \in N$. As M is primely generated, it suffices to consider the case where x is prime, and thus, by the definition of N, $x \leqslant p$. Obviously we can assume that either $a \leqslant x$ or $b \leqslant x$. Now assume, say, that $a \leqslant x$. If $x < p$, then, as $a \in \mathrm{L}(M, p)$, we get $a \equiv x$, so, as a is free, $x = a + y$ for some $y < a$; by the previous case, $y \in N_1$, so $x \in N_1$. The remaining case is where $x \equiv p$. As p is free, there exists $y < p$ such that $x = p + y$. By the previous case, $y \in N_1$, and so $x \in N_1$. $\square$

It follows from the claim above that the monoid homomorphism $\varepsilon : M_0 \to N/I$, $x \mapsto x_{/I}$ is surjective. To prove that it is one-to-one, it suffices to prove the following statements:

The element $p_{/I}$ is free in N/I. Suppose, to the contrary, that $(n + 1)p_{/I} \leqslant np_{/I}$, for some $n \in \mathbb{N}$. This means that there exists $x \in I$ such that $(n + 1)p \leqslant np + x$. By applying the homomorphism $\phi_p : N \to \mathbb{Z}^\infty$ and using the freeness of p, we obtain that $n+1 \leqslant n+\phi_p(x)$, thus $p \leqslant x$, a contradiction as $x \in I$.

By using ϕ_a and ϕ_b instead of ϕ_p, we obtain in a similar manner that *both elements $a_{/I}$ and $b_{/I}$ are free in N/I.*

The elements $a_{/I}$ and $b_{/I}$ are incomparable in N/I. Suppose, say, that $a_{/I} \leqslant b_{/I}$, that is, there exists $x \in I$ such that $a \leqslant b + x$. As a is prime and $a \nleqslant b$, we get that $a \leqslant x$, a contradiction as $x \in I$.

Both elements $a_{/I}$ and $b_{/I}$ are prime in N/I. Let $x, y \in N$ such that $a_{/I} \leqslant x_{/I} + y_{/I}$, that is, there exists $u \in I$ such that $a \leqslant x + y + u$. From $u \in I$ it follows that $a \nleqslant u$, thus $\phi_a(u) = 0$, hence, by applying the homomorphism ϕ_a to the inequality $a \leqslant x + y + u$, we obtain that either $\phi_a(x) \geqslant 1$ or $\phi_a(y) \geqslant 1$, so either $a \leqslant x$ or $a \leqslant y$. Hence $a_{/I}$ is prime in N/I. Similarly, $b_{/I}$ is prime.

By using (4.1), it follows that ε is an isomorphism, and so $M_0 \cong N/I$. By Lemma 4.1, M_0 does not belong to $\overline{\mathcal{G}}$. Therefore, by Lemma 3.6, neither does M. $\square$

5. A characterization of graph monoids among primitive monoids with lower finite set of primes

The main goal of the present section is to characterize graph monoids within finitely generated primitive monoids. As every finitely generated primitive monoid has a finite set of primes (which is the smallest generating subset), the following result is slightly more general.

Theorem 5.1. *Let M be a primitive monoid such that $\mathbb{P}(M)$ is lower finite. Then the following statements are equivalent*:

(i) *M is a graph monoid.*

(ii) *M is a direct limit of graph monoids.*
(iii) $|\mathrm{L}_{\mathrm{free}}(M,p)| \leqslant 1$ *for each* $p \in \mathbb{P}_{\mathrm{free}}(M)$.

Proof. (i) ⇒ (ii) is trivial, while (ii) ⇒ (iii) follows immediately from Theorem 4.2. It remains to prove the direction (iii) ⇒ (i).

So assume that $|\mathrm{L}_{\mathrm{free}}(M,p)| \leqslant 1$ for each $p \in \mathbb{P}_{\mathrm{free}}(M)$. We shall construct a row-finite quiver E with vertex set

$$E^0 = \mathbb{P}(M) \sqcup \left\{ b^p_{i,j} \;\middle|\; 0 \leqslant i < m_p,\ 0 \leqslant j,\ p \in \mathbb{P}_{\mathrm{reg}}(M) \right\},$$

where $m_p = |\mathrm{L}_{\mathrm{free}}(M,p)|$.

For $p \in \mathbb{P}(M)$ we define elements z_p and w_p in $\mathrm{Fr}(E^0)$ by

$$z_p = \sum \big(q \mid q \in \mathbb{P}_{\mathrm{reg}}(M),\ q < p \big),$$
$$w_p = \sum \big(q \mid q \in \mathbb{P}(M),\ q < p \big),$$

and we consider the following relations:

Case 1: If p is free, write down the relation

$$p = p + w_p. \tag{5.1}$$

Case 2: If p is regular and all the elements of $\mathrm{L}(M,p)$ are regular, write down the relation

$$p = 2p + \sum \big(q \mid q \in \mathrm{L}(M,p) \big). \tag{5.2}$$

Case 3: If p is regular and $p_0, \dots, p_{m-1}$ are the elements of $\mathrm{L}_{\mathrm{free}}(M,p)$, with $m > 0$, write down the relations

$$p = b^p_{0,0}, \tag{5.3}$$
$$b^p_{i,0} = 2b^p_{i,0} + b^p_{i,1} + b^p_{i,2} + p_i, \tag{5.4}$$
$$b^p_{i,1} = b^p_{i,0} + 2b^p_{i,1} + b^p_{i,2}, \tag{5.5}$$

for $i < m$. Furthermore, define $\alpha : \mathbb{Z}^+ \to \mathbb{Z}^+$ as $\alpha(n) = 2nm + 2$. For $k \in \mathbb{Z}$ set $\epsilon(k) = 1$ if k is even and $\epsilon(k) = 2$ if k is odd, and put $\beta(k) = \lceil \frac{k-1}{2} \rceil$, where $\lceil x \rceil$ denotes the least integer greater than or equal to x, for any real number x.

For $n \geqslant 0$ and $0 \leqslant k \leqslant 2m-1$, write down the relation

$$b^p_{i,\alpha(n)+k} = b^p_{i,\alpha(n)+k} + \epsilon(k) b^p_{i,nm+\beta(k)+1} + b^p_{[\beta(k),m-1]\times[\alpha(n)+1,\alpha(n+1)]} + z_p \tag{5.6}$$

for $i < m$, where we set

$$b^p_X = \sum_{(i,j)\in X} b^p_{i,j}, \quad \text{for any subset } X \subseteq [0, m-1] \times \mathbb{Z}^+.$$

For example, for $n = 0$, this yields the relations

$$b^p_{i,2} = b^p_{i,2} + b^p_{i,1} + b^p_{[0,m-1]\times[3,2m+2]} + z_p, \tag{5.7}$$
$$b^p_{i,3} = b^p_{i,3} + 2b^p_{i,1} + b^p_{[0,m-1]\times[3,2m+2]} + z_p, \tag{5.8}$$
$$\cdots \quad \cdots$$
$$b^p_{i,2m} = b^p_{i,2m} + b^p_{i,m} + b^p_{[m-1,m-1]\times[3,2m+2]} + z_p,$$
$$b^p_{i,2m+1} = b^p_{i,2m+1} + 2b^p_{i,m} + b^p_{[m-1,m-1]\times[3,2m+2]} + z_p.$$

Let E be the quiver associated with the relations (5.1)–(5.6) above (cf. Section 3), so that $\mathrm{M}(E)$ is the commutative monoid defined by generators E^0 and defining relations (5.1)–(5.6). We shall prove that $\mathrm{M}(E)$ and M are isomorphic. There is a surjective monoid homomorphism

$$\varphi : \mathrm{M}(E) \to M$$

such that $\varphi(p) = p$ for all $p \in \mathbb{P}(M)$ and $\varphi(b^p_{i,j}) = p$ for all $p \in \mathbb{P}_{\mathrm{reg}}(M)$ and all i, j. To prove that φ is well defined we need to check that the relations (5.1)–(5.6) are satisfied by the images of E^0 under φ, which is obvious.

It remains to prove that φ is one-to-one. As M is a primitive monoid, the relations $p = p + q$, for $p, q \in \mathbb{P}(M)$ such that $p = p + q$ holds in M, are defining relations of M (see [13, Section 3.5]). As the subset $\mathbb{P}(M) \cup \{b^p_{i,j} \mid p \in \mathbb{P}_{\mathrm{reg}}(M)\}$ generates $\mathrm{M}(E)$, it suffices to prove that the following relations hold in $\mathrm{M}(E)$:

$$p = p + q, \qquad \text{for all } p, q \in \mathbb{P}(M) \text{ with } q < p, \tag{5.9}$$
$$p = b^p_{i,j} = 2p, \qquad \text{for all } i < m,\ \text{all } j \in \mathbb{Z}^+, \text{ and all } p \in \mathbb{P}_{\mathrm{reg}}(M),\ \text{with } m = \left|\mathrm{L}_{\mathrm{free}}(M, p)\right|. \tag{5.10}$$

Strictly speaking, we should write $\bar{p} = \bar{p} + \bar{q}$, and so on, but we shall drop the bars for clarity of notation, choosing instead to specify the monoid where the relations should be verified—in particular, $\mathrm{M}(E)$ in the case of (5.9), (5.10). We first prove that $b^p_{i,j} \equiv b^p_{i',j'}$ in $\mathrm{M}(E)$, for any pair of indices (i, j) and (i', j'). We argue by induction. Note that (5.4) gives $b^p_{i,1} \leqslant b^p_{i,0}$ and that (5.5) gives $b^p_{i,0} \leqslant b^p_{i,1}$, so that we get $b^p_{i,0} \equiv b^p_{i,1}$. Also $b^p_{i,2} \leqslant b^p_{i,0} \equiv b^p_{i,1}$, and (5.7) implies $b^p_{i,1} \leqslant b^p_{i,2}$, hence

$$b^p_{i,2} \equiv b^p_{i,1} \equiv b^p_{i,0}, \quad \text{for all } i < m.$$

Now (5.8) gives $b^p_{0,3} \leqslant b^p_{i,3} \leqslant b^p_{0,3}$, whence $b^p_{i,3} \equiv b^p_{0,3}$, for all $i < m$. Using again (5.7), (5.8), we get $b^p_{i,3} \leqslant b^p_{i,2}$ and $b^p_{i,1} \leqslant b^p_{i,3}$, so we obtain $b^p_{i,3} \equiv b^p_{i,1} \equiv b^p_{i,2}$, for all $i < m$. By further using (5.7), we obtain that $b^p_{0,3} \leqslant b^p_{i,2}$ and $b^p_{i,3} \leqslant b^p_{0,2}$ for each $i < m$. It follows that $b^p_{0,1} \equiv b^p_{i,j}$, for all $i < m$ and for all $j < 4$.

Now assume that $b^p_{i,j} \equiv b^p_{0,1}$ for all $i < m$ and all $j < \ell$, with $\ell \geqslant 4$. We shall check that $b^p_{i,\ell} \equiv b^p_{0,1}$ for all $i < m$. Write $\ell = \alpha(n) + k$ for some $n \geqslant 0$ and $0 \leqslant k \leqslant 2m - 1$. Since $nm + \beta(k) + 1 < \alpha(n) + k = \ell$, we get by induction that $b^p_{0,1} \equiv b^p_{i,nm+\beta(k)+1}$. Observe also that $b^p_{i,nm+\beta(k)+1} \leqslant b^p_{i,\ell}$ by (5.6). Now assume that $k > 0$. Then we get from the relation

$$b^p_{i,\alpha(n)} = b^p_{i,\alpha(n)} + b^p_{i,nm+1} + b^p_{[0,m-1]\times[\alpha(n)+1,\alpha(n+1)]} + z_p$$

that $b^p_{i,\ell} \leqslant b^p_{i,\alpha(n)}$ so that

$$b^p_{0,1} \equiv b^p_{i,nm+\beta(k)+1} \leqslant b^p_{i,\ell} \leqslant b^p_{i,\alpha(n)} \equiv b^p_{0,1}.$$

We conclude that $b^p_{i,\ell} \equiv b^p_{0,1}$.

Assume finally that $k = 0$. Then $\ell = \alpha(n)$ with $n \geqslant 1$, and we get from the relation

$$b^p_{i,\alpha(n-1)} = b^p_{i,\alpha(n-1)} + b^p_{i,nm-m+1} + b^p_{[0,m-1]\times[\alpha(n-1)+1,\alpha(n)]} + z_p$$

that $b^p_{i,\ell} \leqslant b^p_{i,\alpha(n-1)} \equiv b^p_{0,1}$ and so

$$b^p_{0,1} \equiv b^p_{i,nm+1} \leqslant b^p_{i,\ell} \leqslant b^p_{i,\alpha(n-1)} \equiv b^p_{0,1},$$

which proves that $b^p_{i,\ell} \equiv b^p_{0,1}$.

As $\mathrm{M}(E)$ is separative (cf. [4, Theorem 6.3]), it embeds into a product of monoids of the form $G \cup \{\infty\}$, for abelian groups G (this follows immediately from Hewitt and Zuckermann's result [7, Theorem 5.59]). Hence, to prove that φ is one-to-one, it is sufficient to establish the following claim.

Claim. *For any abelian group G and any set of elements*

$$\bigl\{\tilde{p} \bigm| p \in \mathbb{P}(M)\bigr\} \cup \bigl\{\tilde{b}^p_{i,j} \bigm| p \in \mathbb{P}_{\mathrm{reg}}(M),\ i < \bigl|\mathrm{L}_{\mathrm{free}}(M,p)\bigr|,\ j \in \mathbb{Z}^+\bigr\}$$

in $G \cup \{\infty\}$ satisfying the relations (5.1)–(5.6)*, the relations* (5.9) *and* (5.10) *are also satisfied.*

Proof. We prove the claim by induction on the height of p in $\mathbb{P}(M)$. If p is a minimal prime then (5.9) holds vacuously and (5.10) follows from (5.2). Assume now that p is a prime of height $h + 1$ and the result holds for all primes of height at most h. Assume first that p is a free prime. Note that the induction hypothesis together with (5.1) gives us $\tilde{p} = \tilde{p} + \tilde{u}_p$, where $\tilde{u}_p = \sum_{q\in \mathrm{L}(M,p)} \tilde{q}$. If $\mathrm{L}_{\mathrm{free}}(M,p) = \emptyset$, then all the elements of $\mathrm{L}(M,p)$ are regular and so $\tilde{u}_p + \tilde{q} = \tilde{u}_p$ for all $q \in \mathrm{L}(M,p)$ by the induction hypothesis. We get that

$$\tilde{p} + \tilde{q} = \tilde{p} + \tilde{u}_p + \tilde{q} = \tilde{p} + \tilde{u}_p = \tilde{p}.$$

This proves (5.9) for all $q \in \mathrm{L}(M,p)$, and thus for all primes $q < p$ by the induction hypothesis. Assume now that $\mathrm{L}_{\mathrm{free}}(M,p) \neq \emptyset$. By assumption, $\mathrm{L}_{\mathrm{free}}(M,p) = \{p_*\}$ for some p_*. Since $\tilde{p} = \tilde{p} + \tilde{p}_* + \sum_{q\in \mathrm{L}_{\mathrm{reg}}(M,p)} \tilde{q}$, the induction hypothesis gives again that $\tilde{p} = \tilde{p} + \tilde{q}$ for all $q \in \mathrm{L}_{\mathrm{reg}}(M,p)$, so that $\tilde{p} = \tilde{p} + \sum_{q\in \mathrm{L}_{\mathrm{reg}}(M,p)} \tilde{q}$. From this we get $\tilde{p} + \tilde{p}_* = \tilde{p} + \sum_{q\in \mathrm{L}_{\mathrm{reg}}(M,p)} \tilde{q} + \tilde{p}_* = \tilde{p}$. As before, this gives (5.9) at p.

Assume now that p is a regular prime. Suppose first that $\mathrm{L}_{\mathrm{free}}(M,p) = \emptyset$. It follows from (5.9) that $\tilde{p} = \tilde{p} + \tilde{q}$ for each $q \in \mathrm{L}(M,p)$, thus, by (5.2), $\tilde{p} = 2\tilde{p}$. So assume that $|\mathrm{L}_{\mathrm{free}}(M,p)| = m > 0$. As $\tilde{b}^p_{i,j} \equiv \tilde{b}^p_{0,0} = \tilde{p}$, some $\tilde{b}^p_{i,j} = \infty$ if and only if all $\tilde{b}^p_{i,j} = \infty$. In this case, (5.10) holds trivially. So we can assume throughout that $\tilde{p}$ and all the $\tilde{b}^p_{i,j}$ belong to G. Now (5.4) and (5.5) give

$$\begin{tikzcd} p_0 \arrow[loop above] \arrow[r] & p_1 \arrow[loop above] \arrow[r] & p_2 \arrow[loop above] \arrow[r] & \cdots \end{tikzcd}$$

Fig. 2. A quiver in whose graph monoid the set of primes is not lower finite.

$$\begin{aligned} 0 &= \tilde{b}^p_{i,0} + \tilde{b}^p_{i,1} + \tilde{b}^p_{i,2} + \tilde{p}_i, \\ 0 &= \tilde{b}^p_{i,0} + \tilde{b}^p_{i,1} + \tilde{b}^p_{i,2} \end{aligned} \tag{5.11}$$

and so $\tilde{p}_i = 0$ for all $i < m$. Furthermore, (5.6) gives that $\tilde{z}_p \in G$, and thus $\tilde{q} \in G$ for all $q \in \mathrm{L}_{\mathrm{reg}}(M, p)$; from $\tilde{q} = 2\tilde{q}$ for all such q, together with all $\tilde{p}_i = 0$, we finally get that $\tilde{q} = 0$ for any $q \in \mathrm{L}(M, p)$, and thus, by the induction hypothesis, $\tilde{q} = 0$ for any $q < p$ in $\mathbb{P}(M)$.

Now let j be a positive integer. There exists a unique natural number n such that $nm < j \leqslant (n+1)m$. The integer $k = 2j - 2nm - 2$ lies in the interval $[0, 2m-2]$, and $j = nm + \beta(k) + 1$. Therefore, by applying (5.6) with the consecutive values k and $k+1$, we obtain that $\tilde{b}^p_{i,j} = 0$. In particular, $\tilde{b}^p_{i,1} = \tilde{b}^p_{i,2} = 0$, whence, by (5.11), $\tilde{b}^p_{i,0} = 0$, and therefore $\tilde{b}^p_{i,j} = 0$ for all $i < m$ and all $j \in \mathbb{Z}^+$. Also, $\tilde{p} = \tilde{b}^p_{0,0} = 0$, so (5.10) holds at p. This concludes the proof of the claim. □

This concludes the proof of Theorem 5.1. □

From Theorem 5.1, together with the observation that $\overline{\mathcal{G}}$ is closed under retracts (noted before Lemma 3.5), we can deduce immediately the following closure result for the class of graph monoids.

Corollary 5.2. *Let M be a primitive monoid with $\mathbb{P}(M)$ lower finite. Then M is a retract of some graph monoid if and only if M is a graph monoid.*

Observe that even in case M is finitely generated, the quiver constructed in the proof of Theorem 5.1 may not be finite. That in some cases that quiver cannot be made finite will be established in Theorem 6.1.

The analogue of Corollary 5.2 for graph monoids of *finite* quivers does not hold, see Example 6.5.

Not all antisymmetric graph monoids have lower finite set of primes. For example, letting E be the row-finite quiver represented in Fig. 2, the monoid $\mathrm{M}(E)$ is defined by the generators p_n and the relations $p_n = p_n + p_{n+1}$, for $n \in \mathbb{Z}^+$. Observe that $\mathrm{M}(E)$ is antisymmetric. As all the p_ns are prime in $\mathrm{M}(E)$ and $p_0 > p_1 > p_2 > \cdots$, the subset $\mathbb{P}(\mathrm{M}(E))$ is not lower finite.

6. A characterization of antisymmetric graph monoids of finite quivers

We characterize in this section those antisymmetric finitely generated refinement monoids M which are isomorphic to a graph monoid $\mathrm{M}(E)$ for a *finite* quiver E.

Theorem 6.1. *Let M be a finitely generated primitive monoid. Then there exists a finite quiver E such that $M \cong \mathrm{M}(E)$ if and only if the set $\mathbb{P}_{\mathrm{reg}}(M)$ of regular primes is a lower subset of $\mathbb{P}(M)$ and $|\mathrm{L}_{\mathrm{free}}(M, p)| \leqslant 1$ for each $p \in \mathbb{P}_{\mathrm{free}}(M)$. Equivalently, the set R of regular elements of M is an o-ideal of M and $\mathbb{P}(M/R)$ is a forest.*

One implication follows easily from Theorem 5.1. Indeed, assume that the set $\mathbb{P}_{\mathrm{reg}}(M)$ of regular primes is a lower subset of $\mathbb{P}(M)$ and that $|\mathrm{L}_{\mathrm{free}}(M, p)| \leqslant 1$ for each $p \in \mathbb{P}_{\mathrm{free}}(M)$. Then the quiver E built in the proof of Theorem 5.1 is finite because Case 3 of the proof of Theorem 5.1 does not occur. Thus $M \cong \mathrm{M}(E)$ for the finite quiver E.

The other implication will be proved at the end of this section. We start with a crucial observation.

Proposition 6.2. *Let E be a finite quiver. Assume that* $\mathrm{M}(E)$ *is an antisymmetric monoid. Then* $\mathbb{P}_{\mathrm{reg}}(\mathrm{M}(E))$ *is a lower subset of* $\mathbb{P}(\mathrm{M}(E))$.

Proof. As observed in Section 3, we may assume that E has no sink. Observe that if q is a regular prime then $\phi_{q'}(q) = 0$ or $\phi_{q'}(q) = \infty$ for all $q' \in \mathbb{P}(M)$, according to whether $q' \nleqslant q$ or $q' \leqslant q$ respectively. We proceed by way of contradiction. Let p be a minimal element in $\mathbb{P}_{\mathrm{reg}}(M)$ (with respect to $\leqslant$) with the property that $q \leqslant p$ for some free prime q. Since p is prime, $p = \bar{v}$ for some $v \in E^0$. Let H be the hereditary subset of E^0 generated by v. Then $\mathrm{M}(E \restriction H)$ is isomorphic to the o-ideal generated by p, so that

$$\mathrm{M}(E \restriction H) \cong \mathrm{M}(E) \mid p = \bigl\{x \in \mathrm{M}(E) \bigm| x \leqslant p\bigr\},$$

where the latter equality follows from the regularity of p. Replacing E by $E \restriction H$, we can assume that $\phi_q(p') = 0$ for every free prime q and for every regular prime p' such that $p' \neq p$. Put

$$U = \bigl\{z \in E^0 \bigm| \bar{z} \neq p\bigr\} = \{x_i \mid 1 \leqslant i \leqslant m\},$$
$$V = \bigl\{z \in E^0 \bigm| \bar{z} = p\bigr\} = \{y_j \mid 1 \leqslant j \leqslant n\}.$$

For $1 \leqslant i, i' \leqslant m$ and $1 \leqslant j, j' \leqslant n$, set

$$\alpha_{i,i'} = \bigl|\bigl\{e \in E^1 \bigm| e : x_i \to x_{i'}\bigr\}\bigr|,$$
$$\beta_{j,j'} = \bigl|\bigl\{e \in E^1 \bigm| e : y_j \to y_{j'}\bigr\}\bigr|,$$
$$\gamma_{j,i} = \bigl|\bigl\{e \in E^1 \bigm| e : y_j \to x_i\bigr\}\bigr|.$$

As E has no sink, a presentation of $\mathrm{M}(E)$ is obtained in matricial form as follows:

$$X = AX, \qquad Y = BY + CX, \tag{6.1}$$

where $X = (x_1, \dots, x_m)^{\mathrm{t}}$, $Y = (y_1, \dots, y_n)^{\mathrm{t}}$, $A = (\alpha_{i,i'})_{1 \leqslant i,i' \leqslant m}$, $B = (\beta_{j,j'})_{1 \leqslant j,j' \leqslant n}$, and $C = (\gamma_{j,i})_{1 \leqslant i \leqslant m,\, 1 \leqslant j \leqslant n}$.

Claim. *For any abelian group G, the only $X \in M_{m\times 1}(G)$ and $Y \in M_{n\times 1}(G)$ that satisfy* (6.1) *are $X = 0$ and $Y = 0$.*

Proof. Let M be the submonoid of G generated by $\{x_1, \dots, x_m, y_1, \dots, y_n\}$. There is a unique monoid homomorphism $\psi : \mathrm{M}(E) \to M$ such that $\psi(\bar{x}_i) = x_i$ and $\psi(\bar{y}_j) = y_j$ for all i, j. But since $\bar{x}_i + \bar{y}_j = \bar{y}_j = 2\bar{y}_j$ in $\mathrm{M}(E)$, we get that $x_i + y_j = y_j = 2y_j$ in G so that $x_i = y_j = 0$ for all i, j. $\square$

We can assume that $\mathbb{P}_{\mathrm{free}}(M) = \{\bar{x}_1, \dots, \bar{x}_k\}$, so that $\bar{x}_1, \dots, \bar{x}_k$ are the different free primes in the collection $\bar{x}_1, \dots, \bar{x}_m$. We can suppose that $\bar{x}_1$ is a maximal element in $\mathbb{P}_{\mathrm{free}}(M)$ (with respect to $\leqslant$). Then $\phi_{\bar{x}_1}(\bar{x}_i) < \infty$ for every free prime $\bar{x}_i$, and $\phi_{\bar{x}_1}(\bar{x}_j) = 0$ for every regular prime $\bar{x}_j$, because $\bar{x}_j \neq p$ and $\phi_q(p') = 0$ for every $q \in \mathbb{P}_{\mathrm{free}}(M)$ and every regular prime p' with $p' \neq p$. It follows that $\phi_{\bar{x}_1}(\bar{x}_i) < \infty$ for $i = 1, \dots, m$ and so we obtain that the column matrix

$$X_0 = \big(\phi_{\bar{x}_1}(\bar{x}_1), \dots, \phi_{\bar{x}_1}(\bar{x}_m)\big)^{\mathrm{t}} \in M_{m\times 1}\big(\mathbb{Z}^+\big)$$

satisfies $X_0 = AX_0$. Observe that $\phi_{\bar{x}_1}(\bar{x}_1) = 1$, so that $X_0 \neq 0$. Take $G = \mathbb{Q}$ and any $Y \in M_{n\times 1}(\mathbb{Q})$ such that $Y = BY$. Then $X = 0$ and Y give a solution to (6.1), and so $Y = 0$ by Claim. Thus $(I - B)Y = 0$ implies $Y = 0$, and so $I - B$ is an invertible matrix in $M_n(\mathbb{Q})$. Consider now the above column matrix $0 \neq X_0 \in M_{m\times 1}(\mathbb{Z}^+)$, and set $Y_0 = (I - B)^{-1}CX_0 \in M_{n\times 1}(\mathbb{Q})$. Then we get a solution (X_0, Y_0) over $\mathbb{Q}$ of Eq. (6.1) with $X_0 \neq 0$, a contradiction. □

Lemma 6.3. *Let N be an o-ideal in a primitive monoid M. Then the following properties hold*:

(1) $\mathbb{P}(N) = \mathbb{P}(M) \cap N$.
(2) *M/N is a primitive monoid and the canonical map $\pi : M \twoheadrightarrow M/N$ induces an $\lhd$-isomorphism from $\mathbb{P}(M) \setminus \mathbb{P}(N)$ onto $\mathbb{P}(M/N)$. Moreover,*

$$\phi^{M/N}_{\pi(p)}\big(\pi(a)\big) = \phi^M_p(a),$$

for every $p \in \mathbb{P}(M) \setminus \mathbb{P}(N)$ and every $a \in M$.

Proof. (1) Straightforward.

(2) As N is an o-ideal of the refinement monoid M, the quotient M/N is also a refinement monoid. It is straightforward to verify that for each $p \in \mathbb{P}(M)$, either $p_{/N} = 0$ (i.e., $p \in \mathbb{P}(N)$) or $p_{/N}$ is prime in M/N. Conversely, as M is primely generated and antisymmetric, every prime element of M/N belongs to the image under π of $\mathbb{P}(M) \setminus \mathbb{P}(N)$. Therefore, the image of $\mathbb{P}(M) \setminus \mathbb{P}(N)$ under π is equal to $\mathbb{P}(M/N)$ and the monoid M/N is generated by its prime elements.

Now let us prove that M/N is antisymmetric. It suffices to prove that $a + x \leqslant_N a$ implies that $a + x \equiv_N a$, for all $a, x \in M$. Let $h \in N$ such that $a + x \leqslant a + h$. It follows from [5, Corollary 4.2] that there exists $u \ll a$ such that $x \leqslant h + u$, and hence there are $h' \leqslant h$ and $u' \leqslant u$ such that $x = h' + u'$. From $h' \leqslant h$ it follows that $h' \in N$, while from $u' \leqslant u$ it follows that $u' \ll a$. As M is antisymmetric, $u' + a = a$, and thus

$$a + x = (a + u') + h' = a + h' \equiv_N a.$$

Therefore, M/N is primitive.

It remains to prove that $\pi(p) \lhd \pi(q)$ implies that $p \lhd q$, for all $p, q \in \mathbb{P}(M) \setminus \mathbb{P}(N)$. By assumption, there are $u, v \in N$ such that $p + q + u = q + v$. From $p \notin N$ and $u, v \in N$ it follows that $\phi_p(u) = \phi_p(v) = 0$, hence, applying the homomorphism ϕ_p to the equality $p + q + u = q + v$, we obtain that $\phi_p(p) + \phi_p(q) = \phi_p(q)$, thus, as $\phi_p(p) \geqslant 1$, we get $\phi_p(q) = \infty$, and so, as M is primitive and by applying [5, Theorem 5.5], $p \lhd q$. □

Corollary 6.4. *Let E be a finite quiver. Assume that $M = \mathrm{M}(E)$ is an antisymmetric monoid. Let N be the submonoid of M generated by $\mathbb{P}_{\mathrm{reg}}(\mathrm{M}(E))$. Then*:

(1) *N is an o-ideal of M.*
(2) *N is the set of regular elements of M.*
(3) *Let $H = \{v \in E^0 \mid \bar{v} \in N\}$. Then H is a hereditary saturated subset of E^0 and the quiver $E \setminus H$ satisfies that $\mathrm{M}(E \setminus H) \cong M/N$ is an antisymmetric graph monoid in which all nonzero elements are free, and $\mathbb{P}(M/N) = \mathbb{P}_{\mathrm{free}}(M)$* via *the identification of $\mathbb{P}(M/N)$ with $\mathbb{P}(M) \setminus \mathbb{P}(N)$ provided by Lemma* 6.3(2).

Proof. (1) Let $p_1, \ldots, p_r \in \mathbb{P}_{\mathrm{reg}}(M)$ and $a \in M$ such that $a \leqslant p_1 + \cdots + p_r$. Then $a = q_1 + \cdots + q_\ell$, where all $q_i \in \mathbb{P}(M)$ so that $q_i \leqslant p_1 + \cdots + p_r$ and, by primeness of q_i, we get $q_i \leqslant p_j$ for some j. By Proposition 6.2 we get that all q_i are regular, hence $a \in N$.

(2) It is obvious that every element of N is regular. Conversely, for each $a \in M$ we can write, by the decomposition result given in [13, Proposition 3.4.4] or [5, Theorem 5.8],

$$a = p_1 + \cdots + p_r + n_1 q_1 + \cdots + n_\ell q_\ell,$$

where $p_1, \ldots, p_r$ are regular primes, $q_1, \ldots, q_\ell$ are free primes, $p_1, \ldots, p_r, q_1, \ldots, q_\ell$ are pairwise incomparable, $n_1, \ldots, n_\ell$ are nonzero, and this expression is unique (up to the obvious permutations). If a is a regular element, then $2a = a$ and it follows from the equality

$$2a = p_1 + \cdots + p_r + 2n_1 q_1 + \cdots + 2n_\ell q_\ell,$$

that $\ell = 0$, so $a \in N$.

(3) Most of this is clear from (1), (2), and Lemma 6.3. By [4, Lemma 6.6] we obtain that $M/N \cong \mathrm{M}(E \setminus H)$ is a graph monoid. By Lemma 6.3(2), all primes in M/N are free, thus, by [13, Proposition 3.4.4] or [5, Theorem 5.8], all nonzero elements of M/N are free. □

We are now ready to complete the proof of Theorem 6.1. Assume that E is a finite quiver such that $\mathrm{M}(E)$ is an antisymmetric monoid. Then we get from Corollary 6.4 that the set R of all regular elements of $\mathrm{M}(E)$ is an o-ideal and the monoid $\mathrm{M}(E)/R$ is an antisymmetric graph monoid with no regular primes. Theorem 5.1 gives that $\mathbb{P}(\mathrm{M}(E)/R)$ is a forest. We finally deal with the equivalence in the last statement of Theorem 6.1. By applying the argument of the proof of Corollary 6.4(1), $\mathbb{P}_{\mathrm{reg}}(M)$ is a lower subset of $\mathbb{P}(M)$ if and only if R is an o-ideal of M. If this is satisfied, then, by Lemma 6.3, the condition that $\mathbb{P}(\mathrm{M}(E)/R)$ is a forest is equivalent to the condition that $|\mathrm{L}_{\mathrm{free}}(M, p)| \leqslant 1$ for each $p \in \mathbb{P}_{\mathrm{free}}(M)$.

Example 6.5. It follows from Theorem 5.1 that the monoid $\mathbb{Z}^{\infty}$ is a graph monoid. On the other hand, by Theorem 6.1, $\mathbb{Z}^{\infty}$ is not the graph monoid of any finite quiver.

Nevertheless, $\mathbb{Z}^{\infty}$ is a *retract* of the graph monoid of a finite quiver. Indeed, consider the quiver E represented by Fig. 3.

A presentation of $\mathrm{M}(E)$ is given by the two equations

$$a = a + 1, \qquad b = 2b + a.$$

As $a + b$ is idempotent and absorbs 1 in $\mathrm{M}(E)$, there are unique monoid homomorphisms $\varepsilon : \mathbb{Z}^{\infty} \to \mathrm{M}(E)$ and $\rho : \mathrm{M}(E) \to \mathbb{Z}^{\infty}$ such that

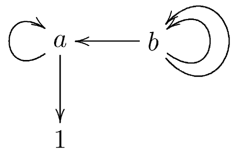

Fig. 3. A quiver whose graph monoid retracts to $\mathbb{Z}^{\infty}$.

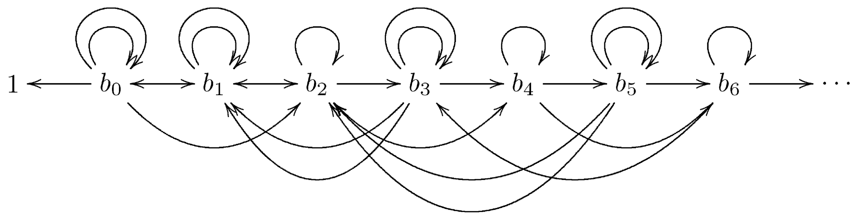

Fig. 4. A quiver that represents $\mathbb{Z}^{\infty}$.

$$\varepsilon(1) = 1, \qquad \varepsilon(\infty) = a + b,$$
$$\rho(1) = 1, \qquad \rho(a) = \rho(b) = \infty.$$

In particular, $\rho \circ \varepsilon = \mathrm{id}_{\mathbb{Z}^{\infty}}$, so $\mathbb{Z}^{\infty}$ is a retract of $\mathrm{M}(E)$.

The proof of Theorem 5.1 gives the following infinite presentation for the monoid $\mathbb{Z}^{\infty}$: the generators are $1, b_0, b_1, b_2, \dots$, and the relations are

$$\begin{aligned}
b_0 &= 2b_0 + b_1 + b_2 + 1;\\
b_1 &= b_0 + 2b_1 + b_2;\\
b_2 &= b_2 + b_1 + b_3 + b_4;\\
b_3 &= 2b_3 + 2b_1 + b_4;\\
b_4 &= b_4 + b_2 + b_5 + b_6;\\
b_5 &= 2b_5 + 2b_2 + b_6;\\
b_6 &= b_6 + b_3 + b_7 + b_8;\\
\dots &\quad \dots
\end{aligned}$$

The corresponding quiver is represented in Fig. 4.

7. Open problems

Problem 1. Is it decidable whether a given finitely generated monoid is isomorphic to the graph monoid of some row-finite (resp., finite) quiver?

As every finitely generated commutative monoid is finitely presented, Problem 1 is well posed. The two main results of the present paper, Theorems 5.1 and 6.1, solve the analogue of Problem 1 for *antisymmetric* monoids.

Problem 2. Is any retract of a graph monoid also a graph monoid?

By Corollary 5.2, the answer to Problem 2 for primitive monoids with lower finite set of primes is positive. On the other hand, by Example 6.5, its analogue for graph monoids of *finite* quivers fails. Also, observe that the class of graph monoids is not closed under direct limits (with respect to monoid homomorphisms). Indeed, the results of [15,17] show that there exists a distributive bounded semilattice S, of cardinality $\aleph_2$, that is not representable (i.e., it is not isomorphic to $\mathcal{V}(R)$ for any von Neumann regular ring R). As, by [2, Theorem 4.4], all graph monoids are representable, we see that S cannot be a graph monoid. On the other hand, S, as every distributive semilattice, is a direct limit of finite Boolean semilattices [6,12], and by Theorem 6.1, every finite distributive semilattice is the graph monoid of a finite quiver, thus representable. This shows that S is a direct limit of graph monoids, without being itself a graph monoid.

Acknowledgment

Part of this work was completed during the third author's visit at Universitat Autònoma de Barcelona in March 2007. Excellent conditions provided by the CRM are greatly appreciated.

The authors are very grateful to the anonymous referee for his or her very careful reading of the paper, which led to the correction of many embarrassing oversights and in numerous improvements both in form and substance.

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