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The Brauer–Clifford group of G -rings

Alexandre Turull¹

Department of Mathematics, University of Florida, Gainesville, FL 32611, USA

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ABSTRACT

In previous work, the author introduced the Brauer–Clifford group of certain G -algebras. This group is useful because to every irreducible character of a normal subgroup of a finite group, one can associate a unique element of a specific Brauer–Clifford group, and this element controls the Clifford theory of this character in its ambient group. In the present paper, we define the Brauer–Clifford group of G -rings. This new definition only requires us to discuss tensor products over the underlying G -ring, and it is simpler than the earlier one. We prove that the new definition yields a group which is canonically isomorphic to the Brauer–Clifford group of a corresponding suitable G -algebra.

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1. Introduction

Clifford theory is central in the representation theory of finite groups, and many attempts to codify it have been made. Clifford theory can be said to begin with Clifford's paper [3]. In this paper, Clifford acknowledges similar ideas having been discussed earlier by Frobenius, Speiser, van der Waerden, Burnside, Brauer, Weyl, Nakayama, and Shoda. Since then, many papers on representation theory of finite groups use Clifford theory in some way or another. A way to codify Clifford theory so as to make it more useful was proposed by Dade in the early 1970's [4–6]. This has had a number of important applications, and its consequences, refinements, and extensions continue to be developed [7,8]. Dade's approach emphasizes graded algebras. A different codification (named “character triple isomorphism”) was proposed by Isaacs in his very influential book [9], and it is now the language of choice for many applications of Clifford theory to the theory of finite solvable groups. Isaacs's approach emphasizes bijections between sets of characters.

E-mail address: turull@math.ufl.edu.

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The author introduced Clifford classes as a convenient way to describe Clifford theory over small fields. Clifford classes are equivalence classes of G -algebras. A generalization of these equivalence classes to group graded algebras has recently been investigated by Marcus [1,2]. In a development in a different direction, the present author introduced the concept of the Brauer–Clifford group of G -algebras in [10,11]. The elements of the Brauer–Clifford group are refinements of Clifford classes. The Brauer–Clifford group provides a finer way to describe Clifford theory over small fields. The elements of the Brauer–Clifford group are equivalence classes of G -algebras, which is sufficient for the applications to Clifford theory of finite groups that we have in mind. The Brauer–Clifford group has already proved useful to study the rationality and the Schur indices of characters that lie above characters that are connected by the Glauberman correspondence [12], and appears to be a suitable framework to establish more rationality properties for characters related under some Clifford correspondences. The Brauer–Clifford group was defined in [10,11] using preexisting concepts such as the concept of *central* G -algebra, and tensor products over some related fields which emphasized the relationship between the new concept of the Brauer–Clifford group, and Clifford classes.

In the present paper, we concentrate instead in defining as simply as possible the Brauer–Clifford group of a group G over a G -ring Z , and its related concepts. We modify as appropriate the existing definitions of the concepts needed for this in a way which is more suitable to discuss the Brauer–Clifford group. Whereas in [10,11] most objects are at least algebras over fields, and the definition of equivalence of G -algebras is defined in terms of tensor products over certain fields, we avoid the use of these fields almost entirely in these new definitions. This simplifies the concepts considerably, and, in particular, avoids any need to have two different concepts of *central* G -algebra. While the concepts presented in this paper are different and simpler than those in [10,11], the final concept of the Brauer–Clifford group that we present in the present paper is equivalent, with suitable identifications, to the Brauer–Clifford group given in [10,11].

The ingredients to define the Brauer–Clifford group are a finite group G and an object Z . In [10,11], Z is a *commutative central simple G -algebra over some field*. Here we only require that Z be a commutative simple G -ring. The definition is given in Definition 2.10. In Proposition 2.12 we discuss the basic structure of such a G -ring. The elements of the Brauer–Clifford group are equivalence classes of central simple G -algebras of finite rank over Z . These objects are defined and studied in Section 4, including the fact (Theorem 4.17) that the tensor product over Z of two central simple G -algebras of finite rank over Z is also a central simple G -algebra of finite rank over Z . Perhaps the most notable difference between the present definitions and those of [10,11] is that in the present paper trivial G -algebras (Definition 5.2) are central simple G -algebras over Z , whereas under the old definitions, trivial G -algebra always where required (among other things) to have a field as their center. In order to define this more general concept, we need to study representations of G with coefficients over a G -ring Z , where Z need not be fixed under the action of G . This is done in Section 3, and these concepts are more generally applicable to other situations in the paper. Using this new definition of trivial G -algebra over Z , the complete definition of the Brauer–Clifford group can be given in way that only involves tensor products over Z , instead of involving tensor products over Z and tensor products over a field, as the original definition did [10,11]. Our new definition is given in Definition 6.1. An alternative description of the equivalence relation is described in Theorem 7.4. In Theorem 8.8 we prove that the new definition is essentially equivalent to the old one. We conclude with Section 9, where we see how the full matrix Brauer–Clifford subgroup is defined in the new context, and we recover its description in terms of a second cohomology group (Theorem 9.6).

Note that we systematically write all functions on the left, and compose them from right to left. Furthermore, we view the elements of Galois groups as functions on their underlying field.

2. Rings and G -rings

We begin by describing our conventions about common terms.

Definition 2.1. Unless otherwise explicitly stated, we have the following conventions about rings and their homomorphisms. We say that R is a *ring* if R is an associative ring with identity. The identity of a ring is not necessarily assumed to be different from zero. A *subring* will have an identity as well,

but it need not be the same as the identity of the parent ring. A *ring homomorphism* need not send the identity to the identity. A *unital* subring of R is one that contains the identity of R , and a *unital* ring homomorphism is one that sends the identity of the first ring to the identity of the second ring.

Definition 2.2. Let R be a ring. By an R -module we mean a unital left R -module unless otherwise explicitly stated. All (left or right) modules are assumed to be unital, unless otherwise explicitly stated.

Definition 2.3. Let R be any ring. Then, we denote by R^\times the multiplicative group of all units of R .

Definition 2.4. Let G be a finite group. A G -ring is a ring R together with a group homomorphism $\phi : G \rightarrow \text{Aut}(R)$ from G to the group of ring automorphisms of R . If R is a G -ring and $r \in R$, and $g \in G$, then we set ${}^g r = \phi(g)(r)$ to be the result of applying the automorphism corresponding to g to the element r . This convention allows us to dispense from having to always have an explicit name for the structural homomorphism ϕ . Often, we will denote the G -ring simply by R .

Definition 2.5. Let G be a finite group, and let R be a G -ring. Let A be an element or a set of elements of R , and let H be a subset of G . We denote by $C_H(A)$ the set of all elements of H which fix every element of A . We denote by A^H or by $C_A(H)$ the set of elements of A which are fixed by every element of H .

Definition 2.6. Let G be a finite group, and let R be a G -ring. A *subring* (resp. *ideal*) of R means a subring (resp. ideal) of the underlying ring R . A G -subring of R is a subring of R that is invariant as a set under the action of G . A G -ideal of R is an ideal of R that is invariant as a set under the action of G .

Remark 2.7. Since under our conventions rings have an identity, it is not necessarily true that ideals or G -ideals are subrings of G -rings.

Definition 2.8. Let G be a finite group, and let R and S be G -rings. A *homomorphism* from R to S is a ring homomorphism $\psi : R \rightarrow S$ such that, for all $g \in G$, $r \in R$, we have $\psi({}^g r) = {}^g \psi(r)$. An *isomorphism* is a bijective homomorphism.

Remark 2.9. The kernels of the homomorphisms from a G -ring R are exactly the G -ideals of R .

Definition 2.10. Let G be a finite group, and let R be a G -ring. We say that R is a *simple G -ring* if R is not zero and it has no non-trivial proper two-sided G -ideals.

Remark 2.11. Of course, in the case when $G = 1$, the above definition coincides with the usual definition of a simple ring R .

Proposition 2.12. Let G be a finite group and let Z be a commutative simple G -ring. Let e_1, \dots, e_α be the primitive idempotents of Z (viewed here simply as a ring). For $i = 1, \dots, \alpha$, we set $K_i = e_i Z$, we set $I_i = C_G(K_i)$, we set $G_i = C_G(e_i)$, and we set $F_i = K_i^{G_i}$. Furthermore, we set $F_0 = Z^G$. Then, α is necessarily finite, and we have the following.

- (1) For $i = 1, \dots, \alpha$, we have that K_i is a field, F_i is a subfield of K_i , G_i is a subgroup of G , I_i is a normal subgroup of G_i , and K_i/F_i is a Galois extension with Galois group G_i/I_i .
- (2) G acts transitively on $\{e_1, \dots, e_\alpha\}$.
- (3) Viewed as rings, we have $Z = K_1 \oplus \dots \oplus K_\alpha$.
- (4) F_0 is a field, and the map $F_0 \rightarrow F_i$ given by $f \mapsto e_i f$ is an isomorphism of fields for $i = 1, \dots, \alpha$.
- (5) Z is a vector space over F_0 , and its dimension is the index $[G : I_i]$ of I_i in G , for any $i = 1, \dots, \alpha$.

Proof. Let M_1 be a maximal ideal of Z . Then M_1 has a finite set of G -conjugates, and we let M_1, \dots, M_n be the complete set of G -conjugates of M_1 . The ideal $\bigcap_{i=1}^n M_i$ is G -invariant, and therefore, $\bigcap_{i=1}^n M_i = 0$, since Z is a simple G -ring. Since each M_i is a maximal ideal of Z , the rings Z/M_i are all fields, and if $i \neq j$ we have $M_i + M_j = Z$. Hence, by the Chinese Remainder Theorem, we have that

$$Z \simeq (Z/M_1) \oplus \cdots \oplus (Z/M_n)$$

as rings. Since the Z/M_i are fields, it follows that there are exactly n primitive idempotents of Z , and that these can be numbered e_1, \dots, e_n in such a way that $e_i Z$ is isomorphic under projection to Z/M_i for $i = 1, \dots, n$. Hence, we have that K_i is a field for $i = 1, \dots, n$, that $\alpha = n$ is finite, and that (3) holds. If O is an orbit of G in its action on $\{e_1, \dots, e_\alpha\}$, then the sum S of all the $e_i Z$ with $e_i \in O$ is a G -invariant ideal of Z , and since Z is a simple G -ring, we have $S = Z$. This implies that $O = \{e_1, \dots, e_\alpha\}$, so that G is transitive, and (2) holds. In particular, $\alpha = [G : G_1]$.

Let $i \in \{1, \dots, \alpha\}$. It is clear that G_i is a subgroup of G , that I_i is a normal subgroup of G_i , and that G_i acts on the field K_i with kernel I_i , so that G_i/I_i can be identified with a finite subgroup of the automorphism group of the field K_i . Since F_i is the fixed subfield of K_i under G_i/I_i , it follows that F_i is a field, and that K_i/F_i is a Galois extension with Galois group G_i/I_i . Hence, (1) holds. Of course, F_0 is a unital subring of Z . If $f \in F_0$, then $e_i f \in F_i = K_i^{G_i}$. Hence, we may define a map $\phi_i : F_0 \rightarrow F_i$ by $\phi_i(f) = e_i f$, for all $f \in F_0$. Then, ϕ_i is a ring homomorphism. By (2), we can find $g_1, \dots, g_\alpha \in G$ such that $e_j = g_j e_i$ for $j = 1, \dots, \alpha$. Let $k \in \ker(\phi_i)$. Then, $e_j k = g_j(e_i k) = 0$ for $j = 1, \dots, \alpha$, and since $\sum_{j=1}^\alpha e_j = 1$, this implies that $k = 0$. Hence, ϕ_i is injective. Let $f_i \in F_i$. Then, set $f = \sum_{j=1}^\alpha g_j f_i$. Then, $f \in F_0$, and $\phi_i(f) = f_i$. Hence, ϕ_i is a bijection, and so a ring isomorphism. In particular, F_0 is a field, and (4) holds.

Since F_0 is a unital subfield of Z , Z is naturally a vector space over F_0 . Under this vector space structure, the K_1, \dots, K_α are subspaces, and the direct sum of (3) is a direct sum as vector spaces. Furthermore, the action of each $g \in G$ on Z is a vector space automorphism. It then follows from (2) that there exist vector space isomorphisms from K_1 to any K_i for $i = 1, \dots, \alpha$. Hence, $\dim_{F_0}(Z) = \alpha \dim_{F_0}(K_1)$. Now K_1 is a field, and it has a unital subfield F_1 so that K_1 has a vector space structure both over F_0 and over F_1 . It follows from (4) that the dimension of K_1 is the same under both vector space structures. By (1) and Galois theory we have that

$$\dim_{F_0}(K_1) = \dim_{F_1}(K_1) = |G_1/I_1|.$$

It then follows that $\dim_{F_0}(Z) = \alpha |G_1/I_1|$, and, since $\alpha = [G : G_1]$, it follows that $\dim_{F_0}(Z) = [G : I_1]$. By (2), the groups I_1, \dots, I_α are all G -conjugate, and their indices in G are all equal. Hence, (5) follows. This completes the proof of the proposition. \square

Remark 2.13. It follows from Proposition 2.12 that, under the definitions of [10,11], Z is a central simple G -algebra over F_0 , and a central simple G -algebra over Z itself. Viewing Z simply as a G -ring eliminates this ambiguity.

Corollary 2.14. Let G be a finite group and let Z be a commutative simple G -ring. Suppose M is a free Z -module of finite rank r , and N is a free Z -submodule of M of rank r . Then $M = N$.

Proof. By Proposition 2.12, Z is a vector space of finite dimension, say d , over a unital subfield F_0 of Z . Hence, both M and N are vector spaces of dimension dr over F_0 . It follows that $M = N$, as desired. \square

3. Group rings and group representations

Given a finite group and a ring, it is standard to discuss the corresponding group ring, and the representations and modules of the group over the ring. Analogous concepts also exist when we are

given a finite group G and G -ring. The explicit use of modules over such group rings allows us to simplify some of the definitions for the Brauer–Clifford group.

Definition 3.1. Let G be a finite group and R be a G -ring. Then the *group ring* RG is the set of all formal linear combinations of G with coefficients in R ,

$$RG = \left\{ \sum_{g \in G} \lambda_g g : \lambda_g \in R \text{ for all } g \in G \right\},$$

with addition defined by, for all $\lambda_g, \mu_g \in R$,

$$\sum_{g \in G} \lambda_g g + \sum_{h \in G} \mu_h h = \sum_{g \in G} (\lambda_g + \mu_g) g$$

and multiplication defined by

$$\left(\sum_{g \in G} \lambda_g g \right) \left(\sum_{h \in G} \mu_h h \right) = \sum_{z \in G} \left(\sum_{\substack{g, h \in G \\ gh=z}} (\lambda_g {}^g \mu_h) \right) z$$

for all $\lambda_g, \mu_h \in R$, where the operations inside the big parenthesis on the right are in R . The group ring RG is provided with two structural maps $i : G \rightarrow RG^\times$ and $\rho : R \rightarrow RG$ as follows. For $g, h \in G$, we set $\delta_{g,h} \in R$ to be 0 if $g \neq h$, and $\delta_{g,h} = 1$ if $g = h$. Then, i is given by, for all $g \in G$,

$$i(g) = \sum_{h \in G} \delta_{g,h} h$$

and ρ is given by, for all $r \in R$,

$$\rho(r) = \sum_{g \in G} (r \delta_{g,1}) g.$$

Remark 3.2. It is well known that the group ring RG is a ring, a fact that one obtains from a set of straightforward calculations. Furthermore, it is easy to see that the map i is an injective group homomorphism, and ρ is an injective ring homomorphism. In practice, one often dispenses with notation for the maps i and ρ , and instead claims that each $g \in G$ is identified with $i(g)$, and each $r \in R$ is identified with $\rho(r)$.

Proposition 3.3. Let G be a finite group and R be a G -ring. Let \mathcal{C} be the following category. The objects of \mathcal{C} are the triples (S, i, ρ) where S is a ring, $i : G \rightarrow S^\times$ is a group homomorphism, and $\rho : R \rightarrow S$ is a unital ring homomorphism such that for all $g \in G, r \in R$, we have $\rho({}^g r) = i(g)\rho(r)i(g)^{-1}$. A morphism in \mathcal{C} from (S, i, ρ) to (S', i', ρ') is a (unital) ring homomorphism $\phi : S \rightarrow S'$ such that, for all $g \in G$, we have $i'(g) = \phi(i(g))$, and $\rho' = \phi\rho$.

Then, with the notation of Definition 3.1, (RG, i, ρ) is an initial object in \mathcal{C} . Furthermore, i is injective, and setting left multiplication on RG by $r \in R$ to be left multiplication by $\rho(r)$ makes RG into a free R -module with basis $i(G)$.

Proof. The fact that \mathcal{C} is a category is straightforward, and left to the reader. The fact that (RG, i, ρ) is an object in \mathcal{C} follows from Remark 3.2 and a short calculation. Let (S, i', ρ') be an arbitrary object in \mathcal{C} and suppose ϕ is a morphism from (RG, i, ρ) to (S, i', ρ') . Then, for all $\lambda_g \in R$, we have

$$\phi\left(\sum_{g \in G} \lambda_g g\right) = \sum_{g \in G} \phi(\rho(\lambda_g))\phi(i(g)) = \sum_{g \in G} \rho'(\lambda_g)i'(g).$$

Now let (S, i', ρ') be any object in \mathcal{C} . It follows, from what we just proved, that there is at most one morphism from (RG, i, ρ) to (S, i', ρ') . Define a map $\phi : RG \rightarrow S$ by setting, for all $\lambda_g \in R$,

$$\phi\left(\sum_{g \in G} \lambda_g g\right) = \sum_{g \in G} \rho'(\lambda_g)i'(g).$$

A short computation shows that ϕ is a morphism in \mathcal{C} and completes the proof of the proposition. \square

The modules over the group ring ZG can be studied by combining G -module structure with the Z -module structure.

Proposition 3.4. *Let G be a finite group and R be a G -ring. Suppose first that M is an RG -module. Then M is simultaneously an R -module and a G -module. Furthermore, these two structures satisfy the condition:*

$$\text{For all } g \in G, \lambda \in R, m \in M, \quad \text{we have } g(\lambda m) = ({}^g\lambda)gm. \quad (1)$$

Conversely, suppose that M is an abelian group and simultaneously an R -module and a G -module, and the two structures satisfy (1). Then M has a unique structure as an RG -module which extends the two given module structures on M .

Proof. Suppose first that M is an RG -module. Then M is simultaneously an R -module and a G -module. Since, for all $g \in G$, and $\lambda \in R$, we have $g\lambda = {}^g\lambda g$ in the group ring RG , we have that (1) holds.

Conversely, suppose that M is an abelian group and simultaneously an R -module and a G -module, and the two structures satisfy (1). We denote by $\text{End}(M)$ the ring of endomorphisms of the abelian group M . Since M is an R -module, we have a unital ring homomorphism $\rho : R \rightarrow \text{End}(M)$. Since M is a G -module, we have a group homomorphism $\gamma : G \rightarrow \text{End}(M)^\times$. Then, our conditions imply that $(\text{End}(M), \gamma, \rho)$ is an object in \mathcal{C} of Proposition 3.3. Since RG is an initial object in this category, it follows that there is a unique unital ring homomorphism $RG \rightarrow \text{End}(M)$ extending ρ and γ . This makes M into an RG -module, and this is the only module structure that extends the two given ones. Hence, the proposition holds. \square

In the case when Z is a commutative simple G -ring, the ZG -modules share some of the properties of G -modules over fields.

Proposition 3.5. *Let G be a finite group, let Z be a commutative simple G -ring, and let M be a ZG -module. Then M is free as a Z -module.*

Proof. The hypotheses of Proposition 2.12 are satisfied, so we also assume the notation of Proposition 2.12. By (2) in that proposition, we may also let $g_i \in G$ be such that ${}^{g_i}e_1 = e_i$ for $i = 1, \dots, \alpha$, with $g_1 = 1$. Now $e_1 M$ is a vector space over K_1 and we let B_1 be a basis for it. We define a map $f : e_1 M \rightarrow M$ by

$$f(v) = \sum_{i=1}^{\alpha} g_i v \quad \text{for all } v \in e_1 M.$$

Since, for every $v \in e_1 M$, we have $v = e_1 f(v)$, the map f is injective. More generally, if $v \in e_1 M$ and $i \in \{1, \dots, \alpha\}$, then $v = e_1 g_i^{-1} f(v)$. We let $B = f(B_1)$.

Suppose $\lambda_b \in Z$ for all $b \in B$, almost all $\lambda_b = 0$, but not all $\lambda_b = 0$, and

$$\sum_{b \in B} \lambda_b b = 0.$$

Suppose that $b_0 \in B$ is such that $\lambda_{b_0} \neq 0$. Then, for some i , we have $e_i \lambda_{b_0} \neq 0$. Applying g_i^{-1} , we obtain that $e_1 g_i^{-1} \lambda_{b_0} \neq 0$. For each $b_1 \in B_1$ set $\mu_{b_1} = e_1 g_i^{-1} \lambda_{f(b_1)} \in K_1$. Conjugating the above equation by g_i^{-1} , and multiplying the result by e_1 , we get that $\sum_{b_1 \in B_1} \mu_{b_1} b_1 = 0$. This contradiction proves that B is linearly independent over Z .

Let S be the span of B as a Z -module. Multiplying B by e_1 we get B_1 , and it follows that $e_1 M \subseteq e_1 S$, as B_1 is a K_1 -basis for $e_1 M$. Similarly, every element of $e_i M = g_i(e_1 M)$ is a Z -linear combination of elements of $g_i B_1$. It then follows from the definition of f that $e_i M \subseteq e_i S$. Hence, $S = M$. This completes the proof of the proposition. \square

Remark 3.6. Let G be a finite group, let Z be a commutative simple G -ring, and let M be a free Z -module. Then the rank of M is uniquely determined. We denote this rank by $\text{rank}_Z(M)$. When N is any G -module over Z , we denote by $\text{rank}_Z(N)$ its rank as a free Z -module.

A standard property of RG -modules where R is a commutative ring (with trivial G -action on R) is that one can take the tensor product of two of them and obtain a new RG -module. The tensor product of ZG -modules is similarly a ZG -module when Z is a commutative simple G -ring.

Definition 3.7. Let G be a finite group, let Z be a commutative simple G -ring, and suppose that M and N are ZG -modules. Let $L = M \otimes_Z N$. Then L has a Z -module structure induced by, for $z \in Z$, $m \in M$, and $n \in N$, we have $z(m \otimes_Z n) = (zm) \otimes_Z n$, and a G -action induced from, for all $g \in G$, $m \in M$, and $n \in N$, we have $g(m \otimes_Z n) = (gm) \otimes_Z (gn)$.

Proposition 3.8. Assume the hypothesis of Definition 3.7. Then, L is a ZG -module. Furthermore, $\text{rank}_Z(L) = \text{rank}_Z(M) \text{rank}_Z(N)$.

Proof. It is a standard result that the defined multiplication by Z makes L uniquely into a Z -module. By Proposition 3.5, M and N are free Z -modules. Standard results then tell us that L is a free Z -module and that $\text{rank}_Z(L) = \text{rank}_Z(M) \text{rank}_Z(N)$. It follows from Proposition 3.4, that the defined action of G on L makes it uniquely into a G -module. Furthermore, (1) of Proposition 3.4 also holds in this case. Hence, we have defined uniquely on L the structure of a ZG -module, as desired. \square

Most authors assume that modules for finite groups are finitely generated, but modules for arbitrary rings are not. Hence, we do not assume that ZG -modules are finitely generated. We reserve the following expression for the finitely generated ones.

Definition 3.9. Let G be a finite group and R be a G -ring. A G -module over R is a finitely generated RG -module.

4. G -algebras

While the ingredients to define the Brauer–Clifford group are a finite group G and a commutative simple G -ring R , its objects are equivalence classes of G -algebras over Z . We now define what we mean by G -algebras over Z .

Definition 4.1. Let G be a finite group, and Z be a commutative G -ring. A G -algebra over Z is a G -ring A (not necessarily with identity) together with an additional structure on A of Z -module, which uses the ring addition on A , and satisfies the conditions that, for all $a, b \in A$, $w, z \in Z$, and $g \in G$, we have $(wa)(zb) = (wz)(ab)$, and ${}^g(wa) = {}^g w {}^g a$. By saying that A is a G -algebra we will also mean that A has an identity unless we say explicitly that it may not have it.

Remark 4.2. Even though a G -algebra A over Z is also ZG -module in a natural way (see Proposition 3.4), the conditions on Z and on G are different, and so A may not be a G -algebra over ZG , nor a 1-algebra over ZG .

For G -rings with identity, the algebra structure can be determined simply by the multiplication by Z on the center of the G -ring.

Definition 4.3. Let G be a finite group, and let R be a G -ring. Then the *center* of R is

$$Z(R) = \{r \in R : rs = sr \text{ for all } s \in R\}.$$

$Z(R)$ is a unital G -subring of R .

Proposition 4.4. Let G be a finite group, let Z be a commutative G -ring, and let A be a G -ring. Then, if A is a G -algebra over Z , the map $u : Z \rightarrow Z(A)$ defined by, for all $z \in Z$, $u(z) = z1_A$, is a unital G -ring homomorphism. Conversely, given any unital G -ring homomorphism $u : Z \rightarrow Z(A)$ there is a unique G -algebra over Z structure on A that yields u .

Proof. Suppose first that A is a G -algebra over Z . If $z \in Z$, then, for all $a \in A$, we have $a(z1_A) = (1za)(z1_A) = (z1_A)a$. Hence, the map $u : Z \rightarrow Z(A)$ defined by, for all $z \in Z$, $u(z) = z1_A$ is well defined. It is straightforward to show that u is a unital G -ring homomorphism.

Conversely, suppose that $u : Z \rightarrow Z(A)$ is a unital G -ring homomorphism. Then, we can define on A a structure of module over Z by setting $za = u(z)a$ for all $z \in Z$, and $a \in A$. It is straightforward to show that this provides to A the structure of a G -algebra over Z , that this structure has u as its structural map, and that this algebra structure on A is the only such. \square

Definition 4.5. Let G be a finite group, let Z be a commutative G -ring, and let A be a G -algebra over Z . Let $u : Z \rightarrow Z(A)$ be the unital G -ring homomorphism given in Proposition 4.4. We will call the map u the *structural G -homomorphism* of A .

Definition 4.6. Let G be a finite group, let Z be a commutative G -ring, let A be a G -algebra over Z , and let $u : Z \rightarrow Z(A)$ be the structural G -ring homomorphism. We will say that A is a *central G -algebra* over Z if u is an isomorphism from Z onto $Z(A)$. In the case of central G -algebras over Z , and only if the context does not lead to confusion, we will often dispense with a specific name for the structural isomorphism u , and instead we will use it to identify Z and $Z(A)$.

Remark 4.7. The concepts of G -algebra and central G -algebra coincide with the usual ones in the case when G acts trivially on a commutative ring Z . However, not all results from the classical case carry over ipso facto to the more general case. For example, when G acts non-trivially on a commutative G -ring Z then the group ring ZG (see Definition 3.1) is *not* a G -algebra over Z since Z is not in its center.

Definition 4.8. Let G be a finite group, Z be a commutative G -ring, and let A be a G -algebra over Z . A G -subalgebra of A over Z is a subset B of A which is a G -subring of A and is also a Z -submodule of A . We say that a G -subalgebra B of A is *unital* if the identity of A is in B .

Definition 4.9. Let G be a finite group, and Z be a commutative G -ring. Let A and B be G -algebras over Z . Then a *homomorphism* from A to B of G -algebras over Z is a map $\phi : A \rightarrow B$ such that ϕ is a homomorphism of G -rings (see Definition 2.8) and ϕ is also a homomorphism of modules over Z .

Remark 4.10. Let A and B be G -algebras over Z , and let $u : Z \rightarrow Z(A)$ and $v : Z \rightarrow Z(B)$ be their respective structural homomorphisms. If a homomorphism $\phi : A \rightarrow B$ of G -rings sends the identity to the identity (as it will do, for example, whenever ϕ is surjective) then ϕ is a G -algebra over Z homomorphism if and only if we have $v(z) = \phi(u(z))$ for all $z \in Z$.

Definition 4.11. Let G be a finite group, and Z be a commutative G -ring, and let A be a G -algebra over Z . An *ideal* of A is an ideal of A viewed as a ring. A G -*ideal* of A is an ideal of A viewed as a G -ring.

Remark 4.12. The kernels of the homomorphisms from a G -algebra over Z are exactly its G -ideals.

Definition 4.13. Let G be a finite group, and Z be a commutative G -ring, and let A be a G -algebra over Z . We say that A is a *simple G -algebra* over Z if the underlying G -ring A is simple (see Definition 2.10).

Definition 4.14. Let G be a finite group, and Z be a commutative simple G -ring, and let A be a G -algebra over Z . By the *rank* of A over Z we will mean the rank $\text{rank}_Z(A)$ of A as a free Z -module (see Proposition 3.5 and Remark 3.6).

Remark 4.15. We will usually say that A is a *central simple G -algebra* over Z to mean that A is central G -algebra over Z which is simple. Of particular interest to us are the central simple G -algebras of finite rank over Z , by which we mean the central simple G -algebras over Z whose rank over Z is finite.

Theorem 4.16. Let G be a finite group, and Z be a commutative simple G -ring. Let A be a central simple G -algebra of finite rank over Z . Then, for each primitive idempotent e of Z , we have that eA is a central simple algebra of finite dimension over the field eZ . Furthermore, if e_1, \dots, e_α are the primitive idempotents of Z , then

$$A = e_1 A \oplus \dots \oplus e_\alpha A,$$

as rings.

Proof. By Proposition 2.12, we know that eZ is a field. As a ring, A is Artinian since Z is Artinian and the rank of A is finite. Since A has an identity, it follows that the radical $J(A)$ of A is a proper G -invariant two-sided ideal. As A is simple as a G -ring, it follows that $J(A) = 0$ and A is semisimple as a ring. Hence, as a ring, A has a finite number of minimal ideals, its minimal ideals are central simple algebras over their centers, and A is the direct sum of its minimal ideals. Since the center of A is identified with Z , this implies that the theorem holds. \square

Theorem 4.17. Let G be a finite group, and Z be a commutative simple G -ring. Let A and B be central simple G -algebras of finite rank over Z . Then $A \otimes_Z B$ is a central simple G -algebra of finite rank over Z .

Proof. The tensor product $A \otimes_Z B$ has naturally the structure of a G -algebra over Z . Let e_1, \dots, e_α be the primitive idempotents of Z . By Theorem 4.16 applied to A and to B , we see that $e_i A$ and $e_i B$ are central simple algebras over the field $e_i Z$ for $i = 1, \dots, \alpha$. It follows that $e_i(A \otimes_Z B)$ is a central simple algebra over $e_i Z$. Hence,

$$A \otimes_Z B = e_1(A \otimes_Z B) \oplus \dots \oplus e_\alpha(A \otimes_Z B)$$

as a ring. It then follows that $A \otimes_Z B$ is a central G -algebra over Z . Since G acts transitively on primitive idempotents of Z , it also acts transitively on the minimal ideals of $A \otimes_Z B$. Hence, $A \otimes_Z B$ is simple as a G -ring. The theorem follows easily from this. \square

5. Trivial G -algebras

A natural way to construct G -algebras over Z is to use representations. In this section we describe this construction. We call the G -algebras obtained in this way *trivial*. Since we have expanded in Section 3 the concept of representation of a finite group over Z to the case when Z is a G -ring, the concept of trivial G -algebra that we obtain in this section is more general than the one in [10,11]. Recall that, under our convention, a G -module over Z is a finitely generated ZG -module.

Proposition 5.1. *Let G be a finite group, and let Z be a commutative simple G -ring. Let M be a non-zero G -module over Z . Set $T = \text{End}_Z(M)$ to be the ring of all endomorphisms of M as a Z -module. For each $g \in G$ and each $f \in T$, we set ${}^g f : M \rightarrow M$ to be defined by, for all $m \in M$, we have $({}^g f)(m) = gf(g^{-1}m)$. Let $u : Z \rightarrow Z(T)$ be the map that assigns to each $z \in Z$ the map $M \rightarrow M$ that is multiplication by z . This makes T into a central simple G -algebra of finite rank over Z .*

Proof. Since M is a G -module over Z , for all $g \in G$, g normalizes the action of Z on M . Hence, T is actually a G -ring. By Proposition 2.12, we know that Z has a finite number (say α) of primitive idempotents, e_1, \dots, e_α . For $i = 1, \dots, \alpha$, we set $K_i = e_i Z$. Then, for $i = 1, \dots, \alpha$, we have that K_i is a field, G acts transitively on $\{e_1, \dots, e_\alpha\}$, and, viewed as rings, we have $Z = K_1 \oplus \dots \oplus K_\alpha$. Furthermore, by Proposition 3.5, M is a free Z -module of finite rank. Since M is not zero, multiplication on M provides a ring isomorphism from Z to a subring of T , and we use it to identify Z with a subring of T . We note that because M is a G -module over Z , this identification is compatible with the action of G . For $i = 1, \dots, \alpha$, we set $T_i = e_i T$, and $M_i = e_i M$. Now, $T_i \simeq \text{End}_{K_i}(M_i)$, so that T_i is isomorphic, as rings, to a full matrix algebra over the field K_i , so that, in particular, T_i is simple, and its center is K_i . Furthermore, $M = M_1 \oplus \dots \oplus M_\alpha$, and $T = T_1 \oplus \dots \oplus T_\alpha$. Since each T_i is simple, and G acts transitively on $\{T_1, \dots, T_\alpha\}$, we have that T is simple as a G -ring. Furthermore, the center of T is $K_1 \oplus \dots \oplus K_\alpha = Z$. Hence, the result holds. \square

Definition 5.2. Let G be a finite group and Z a commutative simple G -ring. We say that a central simple G -algebra A over Z is *trivial* if there exists a non-zero G -module M over Z such that $\text{End}_Z(M)$ is isomorphic to A as central simple G -algebras over Z .

Lemma 5.3. *Let G be a finite group and Z a commutative simple G -ring, and let T and S be trivial central simple G -algebras over Z . Then, $T \otimes_Z S$ is a trivial central simple G -algebra over Z .*

Proof. Let M and N be non-zero G -modules over Z such that $\text{End}_Z(M)$ is isomorphic to T and $\text{End}_Z(N)$ is isomorphic to S as central simple G -algebras over Z . Then, $M \otimes_Z N$ is a non-zero finitely generated ZG -module (see Proposition 3.8), and $\text{End}_Z(M \otimes_Z N)$ is isomorphic to $T \otimes_Z S$. Hence, $T \otimes_Z S$ is a trivial G -algebra over Z , as desired. \square

Lemma 5.4. *Let G be a finite group and Z a commutative simple G -ring. Let A be a central simple G -algebra of finite rank over Z . Then, A^{op} is a central simple G -algebra of finite rank over Z , and $A \otimes_Z A^{op} \simeq \text{End}_Z(A)$, as G -algebras over Z , where we view A as a finitely generated G -module over Z . In particular, $A \otimes_Z A^{op}$ is a trivial central simple G -algebra over Z .*

Proof. By A^{op} , we mean the *opposite* G -algebra over Z to A . That is A^{op} has the same underlying set, the same addition and the same action from G as A , the same map identifying Z with the center of A^{op} , the only difference being that the product on A^{op} is \cdot_{op} given by $a \cdot_{op} b = ba$ for all $a, b \in A$. Since Z is commutative, then A^{op} is a central simple G -algebra over Z of the same rank as A . Let r be the rank of A , so that A is a free Z -module of rank r . Now $A \otimes_Z A^{op}$ is a central simple G -algebra

over Z of rank r^2 . Furthermore, $\text{End}_Z(A)$ is free of rank r^2 as a Z -module. Hence, $\text{End}_Z(A)$ is a trivial G -algebra over Z of rank r^2 . We define a map

$$\phi : A \otimes_Z A^{op} \rightarrow \text{End}_Z(A),$$

$$\phi(a \otimes_Z b) : A \rightarrow A,$$

$$\phi(a \otimes_Z b)(c) = acb$$

for all $a \in A$, $b \in A^{op}$, and $c \in A$. This defines a unique unital homomorphism ϕ of G -algebras over Z . Since $A \otimes_Z A^{op}$ is a central simple G -algebra, it follows that ϕ is injective. Since the rank of $A \otimes_Z A^{op}$ and the rank of $\text{End}_Z(A)$ are both r^2 , it follows from Corollary 2.14 that ϕ is an isomorphism. \square

6. The Brauer–Clifford group

We are now ready to define the Brauer–Clifford group. We have simplified the definition of the objects needed to define the Brauer–Clifford group, mainly by no longer requiring that they be algebras over fields. The definition that we get in this section is more general than that in [10,11], but it is essentially *equivalent*.

Definition 6.1. Let G be a finite group, and Z be a commutative simple G -ring. We define the *Brauer–Clifford group* of G over Z to be the set

$$\text{BrClif}(G, Z)$$

together with a binary operation. The elements of $\text{BrClif}(G, Z)$ are the equivalence classes of central simple G -algebras of finite rank over Z , under the equivalence given as follows. Suppose A , and B are central simple G -algebras of finite rank over Z . Then, we say that A is equivalent to B if and only if there exist trivial central simple G -algebras T_1 and T_2 over Z such that

$$A \otimes_Z T_1 \simeq B \otimes_Z T_2$$

as central G -algebras over Z . The binary operation on $\text{BrClif}(G, Z)$ is that induced by the tensor product over Z of central simple G -algebras over Z .

The next theorem can be compared to [10, Theorem 3.10].

Theorem 6.2. Let G be a finite group, and Z be a commutative simple G -algebra. Then, the Brauer–Clifford group $\text{BrClif}(G, Z)$ of G over Z is an abelian group.

Proof. The cardinality of the underlying set of any central simple G -algebra of finite rank over Z is bounded above by $\aleph_0|Z|$. Hence, we may take a set S of representatives of the central simple G -algebras of finite rank over Z up to isomorphism. Since Z is a trivial G -algebra over Z , the tensor product is associative and commutative up to isomorphism, and the tensor product of trivial G -algebras over Z is a trivial G -algebra over G (see Lemma 5.3), the relation described in the statement defines an equivalence relation on S . Hence the underlying set of $\text{BrClif}(G, Z)$ can be thought of as a set. It is straightforward to check that the product on $\text{BrClif}(G, Z)$ is well defined, associative and commutative. Furthermore, one can check that the class of Z is the identity of $\text{BrClif}(G, Z)$, and that, given any central simple G -algebra A of finite rank over Z , the class of the central simple G -algebra A^{op} of finite rank over Z is the inverse of the class of A (see Lemma 5.4). Hence, $\text{BrClif}(G, Z)$ is an abelian group, as desired. \square

7. Free trivial G -algebras over Z

As we have seen, discussing free representations is not necessary to give a definition of the Brauer–Clifford group. Theorem 7.4 below shows that the use of free ZG -modules helps to provide an alternative equivalent definition of the Brauer–Clifford group.

Definition 7.1. Let G be a finite group and let Z be a commutative simple G -ring. Then by a *free G -module over Z* we simply mean a finitely generated free ZG -module.

Proposition 7.2. Let G be a finite group and let Z be a commutative simple G -ring. Let F be a free G -module over Z , and let M be any G -module over Z . Then, $F \otimes_Z M$ is a free G -module over Z . Furthermore, if the rank of F as a free G -module over Z is r , then the rank of $F \otimes_Z M$ as a free G -module over Z is $r \operatorname{rank}_Z(M)$.

Proof. Let B be a free basis for F as a G -module over Z , and let C be a free Z -basis for M , see Proposition 3.5. This implies that there are exactly $|G||B|$ elements of the form gb for $g \in G$, and $b \in B$ and that they form a Z -basis for F . It then follows that there are exactly $|G||B||C|$ elements of the form $gb \otimes_Z gc \in F \otimes_Z M$, for $g \in G$, $b \in B$, and $c \in C$, and that they are linearly independent over Z . Hence, the space they generate over Z is a free Z -submodule of rank $|G||B||C|$. It then follows from Corollary 2.14 that they are a Z -basis for $F \otimes_Z M$. Since G permutes semiregularly the elements of this basis, it follows that $F \otimes_Z M$ is a finitely generated free ZG -module, as desired. \square

Definition 7.3. Let G be a finite group and Z a commutative simple G -ring. We say that a central simple G -algebra A over Z is *free trivial* if there exists a non-zero free G -module M over Z such that $\operatorname{End}_Z(M)$ is isomorphic to A as central simple G -algebras over Z .

Theorem 7.4. Let G be a finite group, and Z be a commutative simple G -ring. Suppose A , and B are central simple G -algebras of finite rank over Z . Then, A is equivalent to B under Definition 6.1 if and only if there exist free trivial G -algebras T_1 and T_2 over Z such that

$$A \otimes_Z T_1 \simeq B \otimes_Z T_2$$

as central G -algebras over Z .

Proof. Of course, if two central simple G -algebras of finite rank over Z are equivalent under the condition of the theorem, then they are equivalent under the condition of Definition 6.1. Hence, we suppose that A and B are central simple G -algebras of finite rank over Z and they are equivalent under the condition of Definition 6.1. Then, there exist trivial G -algebras T_1 and T_2 over Z such that

$$A \otimes_Z T_1 \simeq B \otimes_Z T_2$$

as central G -algebras over Z . We let F be a free trivial G -algebra over Z . Then, by the proof Lemma 5.3 and Proposition 7.2, we have that $T_1 \otimes_Z F$ and $T_2 \otimes_Z F$ are free trivial G -algebras over Z . Since

$$A \otimes_Z T_1 \otimes_Z F \simeq B \otimes_Z T_2 \otimes_Z F$$

as central G -algebras over Z , it follows that A and B are equivalent under the relation of our theorem, as desired. \square

8. Extending the G -ring

One can use tensor products to extend the G -ring that is acting on a module.

Definition 8.1. Let G be a finite group, let Z be a commutative simple G -ring, and suppose that Z_0 is a unital G -subring of Z , and Z_0 is a simple G -ring. Let M be a G -module over Z_0 . Then $Z \otimes_{Z_0} M$ has a Z -module structure induced by, for $z, z' \in Z$, and $m \in M$, we have $z(z' \otimes_{Z_0} m) = (zz') \otimes_{Z_0} m$, and a G -action induced from, for all $g \in G$, $z \in Z$ and $m \in M$, we have $g(z \otimes_{Z_0} m) = ({}^g z) \otimes_{Z_0} (gm)$.

Proposition 8.2. Assume the hypothesis of Definition 8.1. Then, $Z \otimes_{Z_0} M$ is a G -module over Z . Furthermore, $\text{rank}_Z(Z \otimes_{Z_0} M) = \text{rank}_{Z_0}(M)$.

Proof. The proof is analogous to the proof of Proposition 3.8. \square

One advantage of free modules is that they can be obtained by extension of scalars from modules over fields which are fixed by the action of the group.

Lemma 8.3. Let G be a finite group, let Z be a commutative simple G -ring, and let M be a free G -module over Z . Let $F_0 = Z^G$ be the fixed field of Z (see Proposition 2.12). Then, there exists a free G -module M_0 over F_0 such that $Z \otimes_{F_0} M_0$ is isomorphic to M .

Proof. It is enough to show that a free G -module over Z of free rank one can be obtained, so we assume that $M = ZG$ is such a module. We let $M_0 = F_0G$ be viewed as a free G -module over F_0 of rank one. One can then check that $M \simeq Z \otimes_{F_0} M_0$ as G -modules over Z . Hence, the lemma holds. \square

One can also use tensor products to extend the G -ring at the center of a trivial G -algebra.

Definition 8.4. Let G be a finite group, let Z be a commutative simple G -ring, and suppose that Z_0 is a unital G -subring of Z , and Z_0 is a simple G -ring. Let T be a trivial central simple G -algebra over Z_0 . Then $Z \otimes_{Z_0} T$ has a standard algebra structure, a Z -module structure induced by, for $z \in Z$, $z_0 \in Z_0$ and $t \in T$, we have $z(z_0 \otimes_{Z_0} t) = (zz_0) \otimes_{Z_0} t$, and a G -action induced from, for all $g \in G$, $z_0 \in Z_0$ and $t \in T$, we have ${}^g(z_0 \otimes_{Z_0} t) = ({}^g z_0) \otimes_{Z_0} ({}^g t)$.

Proposition 8.5. Assume the hypothesis of Definition 8.4. Then, $Z \otimes_{Z_0} T$ is a trivial central simple G -algebra over Z . Furthermore, $\text{rank}_Z(Z \otimes_{Z_0} T) = \text{rank}_{Z_0}(T)$.

Proof. It is routine to check that the indicated properties do define uniquely on $Z \otimes_{Z_0} T$ the structure of a G -algebra over Z , and that $\text{rank}_Z(Z \otimes_{Z_0} T) = \text{rank}_{Z_0}(T)$. By Definition 5.2, there exists a finitely generated non-zero G -module M_0 over Z_0 such that $\text{End}_{Z_0}(M_0)$ is isomorphic to T as G -algebras over Z_0 . We set $M = Z \otimes_{Z_0} M_0$. By Proposition 8.2, M is a G -module over Z and $\text{rank}_Z(M) = \text{rank}_{Z_0}(M_0)$. Now $Z \otimes_{Z_0} T$ is isomorphic to $\text{End}_Z(M)$ as a G -algebra over Z , so that it is a trivial G -algebra over Z , as desired. \square

This process of extension can be used to obtain all free trivial algebras from free trivial algebras over fixed fields.

Proposition 8.6. Let G be a finite group, let Z be a commutative simple G -ring, and let T be a free trivial G -algebra over Z . Let $F_0 = Z^G$ be the fixed field of Z (see Proposition 2.12). Then, there exists a free trivial G -algebra T_0 over F_0 such that $Z \otimes_{F_0} T_0$ is isomorphic to T .

Proof. By Definition 7.3, there exists a non-zero free G -module M over Z such that $\text{End}_Z(M)$ is isomorphic to T as central simple G -algebras over Z . By Lemma 8.3, there is a non-zero free G -module

M_0 over F_0 such that $M \simeq Z \otimes_{F_0} M_0$. We set $T_0 = \text{End}_{F_0}(M_0)$. Hence, T_0 is a free trivial G -algebra over F_0 . By the proof of Proposition 8.5, T is isomorphic as a G -algebra over Z to $Z \otimes_{F_0} T_0$, and the proposition follows. \square

With the results of this section, we can now see that the new definition of the Brauer–Clifford group is *equivalent* to the one given in [10,11].

Lemma 8.7. *Let G be a finite group. Suppose first that Z is a commutative simple G -ring. Set $F_0 = Z^G$ to be the fixed field of Z (see Proposition 2.12). Then F_0 is a field, and Z is naturally a commutative central simple G -algebra over F_0 in the sense of [10,11]. Conversely, suppose Z is a commutative central simple G -algebra over a field F in the sense of [10,11], then Z can also be viewed as a commutative simple G -ring, and setting $F_0 = Z^G$, we have that $F \simeq F_0$ as fields, and structure of Z as a commutative central simple G -algebra over F_0 in the sense of [10,11] is recovered from the structure of Z as a commutative simple G -ring.*

Proof. Suppose first that Z is a commutative simple G -ring, and we set $F_0 = Z^G$. By Proposition 2.12, F_0 is a unital subfield of Z , and it follows that Z is naturally a finite dimensional algebra over F_0 . Hence, Z is naturally a commutative central simple G -algebra over F_0 in the sense of [10,11]. Conversely, suppose now that Z is a commutative central simple G -algebra over a field F in the sense of [10,11]. Since Z is *central* this implies that setting $F_0 = Z^G$, we have that $F \simeq F_0$ as fields. Since Z is a simple G -algebra, we have that if we forget the algebra over F structure on it, Z is a commutative simple G -ring. Furthermore, the isomorphism $F_0 = Z^G$ and the G -ring structure of Z give back to Z its original structure as a G -algebra over F , as desired. \square

Theorem 8.8. *Let Z be a commutative simple G -ring (or equivalently, by Lemma 8.7, a commutative central simple G -algebra in the sense of [10,11]). Then, there is a natural isomorphism from the group $\text{BrClif}(G, Z)$ in the sense of the present paper, and the group $\text{BrClif}(G, Z)$ in the sense of [10,11].*

Proof. Set $F_0 = Z^G$. If A is a central simple G -algebra of finite rank over Z , then A is naturally also a central simple G -algebra over F_0 in the sense of [10,11], and conversely. Hence, our theorem will follow as soon as we prove that the equivalence relation defined in the present paper and the equivalence relation defined in [10,11] coincide. Let A and B be central simple G -algebras over Z .

Suppose first that A and B are equivalent in the sense of [10,11]. Then, there exist two non-zero G -modules M_1 and M_2 over F_0 such that

$$A \otimes_{F_0} \text{End}_{F_0}(M_1) \simeq B \otimes_{F_0} \text{End}_{F_0}(M_2)$$

as G -algebras over Z . It follows that

$$A \otimes_Z Z \otimes_{F_0} \text{End}_{F_0}(M_1) \simeq B \otimes_Z Z \otimes_{F_0} \text{End}_{F_0}(M_2)$$

as G -algebras over Z . By Proposition 8.5, $Z \otimes_{F_0} \text{End}_{F_0}(M_1)$ and $Z \otimes_{F_0} \text{End}_{F_0}(M_2)$ are trivial G -algebras over Z . Hence, A and B are equivalent in the sense of the present paper.

Suppose now that A and B are equivalent in the sense of the present paper. By Theorem 7.4, there exists free trivial G -algebras T_1 and T_2 over Z such that

$$A \otimes_Z T_1 \simeq B \otimes_Z T_2$$

as G -algebras over Z . It follows from Proposition 8.6 that there exist free trivial G -algebras T_3 and T_4 over F_0 such that $T_1 \simeq Z \otimes_Z T_3$ and $T_2 \simeq Z \otimes_Z T_4$. This implies that

$$A \otimes_{F_0} T_3 \simeq B \otimes_{F_0} T_4$$

as G -algebras over Z , so that A and B are equivalent in the sense of [10,11], as desired. \square

9. The full matrix Brauer–Clifford group

In [10,11], a group homomorphism from the Brauer–Clifford group to a subgroup of a certain Brauer group is defined, and its kernel is called the full matrix Brauer–Clifford group. This is easy to do in our new simplified context.

Lemma 9.1. *Let G be a finite group, and Z be a commutative simple G -ring. Let e_1 be a primitive idempotent of Z , and set $K_1 = e_1 Z$, $I_1 = C_G(e_1)$, and $F_1 = K_1^{I_1}$. Let A be a central simple G -algebra of finite rank over Z . Then, K_1 is an extension field of the field F_1 , K_1/F_1 is a finite Galois extension, $e_1 A$ is a central simple algebra over the field K_1 , and its class $[e_1 A]$ in the Brauer group $\text{Br}(K_1)$ is invariant under the action of I_1 , we write $[e_1 A] \in \text{Br}(K_1)^{I_1}$. Furthermore, the map*

$$\phi : \text{BrClif}(G, Z) \rightarrow \text{Br}(K_1)^{I_1}$$

defined by $\phi([A]) = [e_1 A]$, for all central simple G -algebra A of finite rank over Z (where $[A]$ is the class in $\text{BrClif}(G, Z)$ of A), is a group homomorphism. Finally, the kernel of ϕ does not depend on the choice of the idempotent e_1 .

Proof. By Proposition 2.12, we know that K_1 is an extension field of the field F_1 , and that K_1/F_1 is a finite Galois extension. If A is any central simple G -algebra of finite rank over Z , then by Theorem 4.16, we have that $e_1 A$ is a central simple algebra over K_1 . Furthermore, if T is any trivial G -algebra over Z , then $e_1 T$ is a full matrix algebra over K_1 . Hence, the class $[e_1 A] \in \text{Br}(K_1)$ depends only on the class of A in $\text{BrClif}(G, Z)$. If $g \in I_1$, since g fixes e_1 , the action of g provides an automorphism of the ring $e_1 A$ onto itself which when restricted to the center gives the action of g on K_1 . This proves that $[e_1 A] \in \text{Br}(K_1)^{I_1}$. Hence, ϕ is well defined and it is straightforward to see that ϕ is a group homomorphism. Suppose e_2 is another primitive idempotent of Z . Then there exists some $g \in G$ such that ${}^g e_1 = e_2$. Furthermore, for every central simple G -algebra A of finite rank over Z , conjugation by g provides an isomorphism of $e_1 A$ onto $e_2 A$. Hence, $[A]$ is in the kernel of ϕ if and only if $e_2 A$ is a full matrix algebra over the field $e_2 Z$. Hence, the kernel of ϕ does not depend on our choice of idempotent e_1 , as desired. \square

Remark 9.2. By Proposition 2.12, the field F_1 is canonically isomorphic to the field $F_0 = Z^G$, and the extension K_1/F_1 is uniquely defined up to isomorphism. Taking K_0/F_0 to be a representative of this isomorphism class of finite Galois field extensions, we get that the previous proposition defines uniquely a map

$$\phi : \text{BrClif}(G, Z) \rightarrow \text{Br}(K_0)^{\text{Gal}(K_0/F_0)}.$$

Definition 9.3. Let G be a finite group, and Z be a commutative simple G -ring. We call the kernel of any one of the homomorphisms ϕ of Lemma 9.1 the *full matrix subgroup of the Brauer–Clifford group* of G over Z , and we denote it by $\text{FMBrClif}(G, Z)$.

Proposition 9.4. *Let G be a finite group, and Z be a commutative simple G -ring. Let A be a central simple G -algebra over Z . Then the class of A is in $\text{FMBrClif}(G, Z)$ if and only if, viewed simply as an algebra over Z , A is isomorphic to a full matrix algebra over Z .*

Proof. Let e_1, \dots, e_α be the primitive idempotents of Z . Suppose that A is isomorphic as an algebra over Z to a full matrix algebra over Z . Then, $e_1 A$ is a full matrix algebra over $e_1 Z$, so that

$$[A] \in \text{FMBrClif}(G, Z).$$

Conversely, suppose now that $[A] \in \text{FMBrClif}(G, Z)$. Then, $e_1 A$ is a full matrix algebra over $e_1 Z$. Since G acts transitively on the set of primitive idempotents of Z , if e is any idempotent of Z , then

eA is a full matrix algebra over eZ , and its dimension as a vector space over eZ is the same as the dimension as a vector space of e_1A over e_1Z . Hence, as a ring, we have

$$A = e_1A \oplus \cdots \oplus e_\alpha A,$$

where each summand is a full matrix algebra of the same dimension. It follows that A is isomorphic to a full matrix algebra over Z , as desired. \square

Remark 9.5. From the proof of Theorem 8.8, it is easy to see that the full matrix Brauer–Clifford group defined here corresponds exactly to the one defined in [10,11] under the canonical isomorphism.

Theorem 9.6. *Let G be a finite group, and Z be a commutative simple G -ring. Then $\text{FMBrcClf}(G, Z)$ is isomorphic to $H^2(G, Z^\times)$.*

Proof. By Remark 9.5, this just follows from [11, Theorem 3.12]. \square

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