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Finitely presented lattice-ordered abelian groups with order-unit

Leonardo Cabrer [a], Daniele Mundici [b,*]

[a] Departamento de Matemáticas, Facultad de Ciencias Exactas, Universidad Nacional del Centro, Pinto 399, 7000 Tandil, Argentina
[b] Dipartimento di Matematica "Ulisse Dini", Università degli Studi di Firenze, viale Morgagni 67/A, I-50134 Firenze, Italy

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ABSTRACT

Let G be an ℓ-group (which is short for "lattice-ordered abelian group"). Baker and Beynon proved that G is finitely presented iff it is finitely generated and projective. In the category $\mathcal{U}$ of *unital* ℓ-groups, those ℓ-groups having a distinguished order-unit u, only the $(\Leftarrow)$-direction holds in general. We show that a unital ℓ-group (G,u) is finitely presented iff it has a basis. A large class of projectives is constructed from bases having special properties.

1. Introduction

We refer to [3,7,8] for background on ℓ-groups. A unital ℓ-group (G,u) is an abelian group G equipped with a translation-invariant lattice-order and a distinguished *order-unit* u, i.e., an element

* Corresponding author.
E-mail addresses: lmcabrer@yahoo.com.ar (L. Cabrer), mundici@math.unifi.it (D. Mundici).

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whose positive integer multiples eventually dominate each element of G. Unital ℓ-groups are a recent mathematization of the euclidean magnitudes with an archimedean unit. By [13, Theorem 3.9], the category $\mathcal{U}$ of unital ℓ-groups is equivalent to the equational class of MV-algebras. Thus, while the archimedean property of order-units is not definable in first-order logic, $\mathcal{U}$ is endowed with all the typical properties of equational classes: in particular, $\mathcal{U}$ has free algebras, finitely presented algebras, subalgebras, quotients and products—which in general do not coincide with cartesian products. Morphisms in $\mathcal{U}$ are lattice-preserving unit-preserving homomorphisms.

For a geometric investigation of finitely presented unital ℓ-groups, in [12] the notion of basis (see Definition 2.1) was introduced as a purely algebraic counterpart of Schauder bases. In [12, Theorem 4.5] it is proved that an *archimedean* unital ℓ-group (G,u) is finitely presented iff it has a basis. The archimedean condition means that G is isomorphic to an ℓ-group of real-valued functions defined on some set X. In Theorem 3.1 we will prove that the archimedean assumption can be dropped, thus obtaining a characterization of finitely presented unital ℓ-groups that does not mention free objects and their universal property.

A unital ℓ-group (G,u) is *projective* if whenever $\psi:(A,a)\to(B,b)$ is a surjective morphism and $\phi:(G,u)\to(B,b)$ is a morphism, there is a morphism $\theta:(G,u)\to(A,a)$ such that $\phi=\psi\circ\theta$. For ℓ-groups, Baker [1] and Beynon [2, Theorem 3.1] (also see [7, Corollary 5.2.2]) gave the following characterization: *An ℓ-group G is finitely generated projective iff it is finitely presented.* For unital ℓ-groups the $(\Rightarrow)$-direction holds. The converse direction fails in general. From Theorem 3.1 it follows that every finitely generated projective unital ℓ-group has a basis. In Section 4 bases will be used to construct large classes of projective unital ℓ-groups.

In Theorem 5.3 it is proved that if (G,u) has a basis then its bases provide a direct system of simplicial groups with 1–1 positive unital homomorphisms, whose limit is (G,u). Thus the Effros–Handelman–Shen representation theorem [4], Grillet's theorem [10, 2.1], and Marra's theorem [11] have a very simple proof for (G,u).

2. Preliminaries

A *lattice-ordered abelian group* (*ℓ-group*) is a structure $(G,+,-,0,\vee,\wedge)$ such that $(G,+,-,0)$ is an abelian group, $(G,\vee,\wedge)$ is a lattice, and $x+(y\vee z)=(x+y)\vee(x+z)$ for all $x,y,z\in G$. An *order-unit* in G is an element $u\in G$ with the property that for every $g\in G$ there is $n\in\{1,2,3,\ldots\}$ such that $g\leqslant nu$. A *unital ℓ-group* (G,u) is an ℓ-group G with a distinguished order-unit u. A map $h:(G,u)\to(G',u')$ is said to be a *unital ℓ-homomorphism* if it preserves the lattice as well as the group structure, and $h(u)=u'$. By an *ideal* $\mathfrak{i}$ of a unital ℓ-group (G,u) we mean the kernel of a unital ℓ-homomorphism of (G,u). We denote by $\operatorname{MaxSpec}(G,u)$ (or even, $\operatorname{MaxSpec} G$ if there is no danger of confusion) the set of maximal ideals of (G,u) equipped with the *spectral* topology [3, §10]: a basis of closed sets for $\operatorname{MaxSpec} G$ is given by sets of the form $\{\mathfrak{p}\in\operatorname{MaxSpec} G\mid g\in\mathfrak{p}\}$, where g ranges over all elements of G. Since G has an order-unit, $\operatorname{MaxSpec} G$ is a nonempty compact Hausdorff space [3, 10.2.2].

Definition 2.1. Let (G,u) be a unital ℓ-group. A *basis* of (G,u) is a set $\mathcal{B}=\{b_1,\ldots,b_n\}$ of elements $\neq 0$ of the positive cone $G^+=\{g\in G\mid g\geqslant 0\}$ such that

(i) $\mathcal{B}$ generates G using the group and lattice operations;
(ii) for each $k=1,2,\ldots$ and k-element subset C of $\mathcal{B}$ with $0\neq\bigwedge\{b\mid b\in C\}$, the set $\{\mathfrak{m}\in\operatorname{MaxSpec}(G)\mid\mathfrak{m}\supseteq\mathcal{B}\setminus C\}$ is homeomorphic to a $(k-1)$-simplex;
(iii) there are integers $1\leqslant m_1,\ldots,m_n$ such that $\sum_{i=1}^n m_ib_i=u$.

This is an equivalent simplified reformulation of [12, Definition 4.3]. From (ii)–(iii) it follows that the *multiplicity* m_i of each $b_i\in\mathcal{B}$ is uniquely determined.

For each $i=1,\ldots,n$ we let $\pi_i:[0,1]^n\to\mathbb{R}$ denote the ith coordinate map. The standard basis of $\mathbb{R}^n$ is denoted $E=\{e_1,\ldots,e_n\}$. For any subset S of $\mathcal{B}$ we define the simplex $\mathcal{T}_S\subseteq[0,1]^n$ by

$$\mathcal{T}_S=\operatorname{conv}\{e_i/m_i\mid b_i\in S\}. \tag{1}$$

Let $k = 1, 2, \dots, n$. Then by a *k-cluster* of $\mathcal{B}$ we understand a k-element subset C of $\mathcal{B}$ such that $\bigwedge C \neq 0$. We denote by $\mathcal{B}^{\bowtie}$ the set of all clusters of $\mathcal{B}$. For each $C \in \mathcal{B}^{\bowtie}$, displaying the complementary set $\mathcal{B} \setminus C$ as $\{b_{j_1}, \dots, b_{j_s}\}$, we define the function $a_C : [0,1]^n \to \mathbb{R}$ by

$$a_C = \pi_{j_1} \vee \cdots \vee \pi_{j_s} \quad (a_C = 0 \text{ in case } C = \mathfrak{B}). \tag{2}$$

We have the identity

$$\mathcal{T}_C = \mathcal{T}_{\mathcal{B}} \cap a_C^{-1}(0). \tag{3}$$

For $n = 1, 2, \dots$ we let $\mathcal{M}_n$ denote the unital ℓ-group of all continuous functions $f : [0,1]^n \to \mathbb{R}$ satisfying: there are (affine) linear polynomials $p_1, \dots, p_m$ with integer coefficients, such that for all $x \in [0,1]^n$ there is $i \in \{1, \dots, m\}$ with $f(x) = p_i(x)$. $\mathcal{M}_n$ is equipped with the pointwise operations $+, -, \wedge, \vee$ of $\mathbb{R}$, and with the constant function 1 as the distinguished order-unit. The characteristic universal property of $\mathcal{M}_n$ is as follows:

Proposition 2.2. *(See [13, 4.16].) $\mathcal{M}_n$ is generated by the maps $\pi_i : [0,1]^n \to \mathbb{R}$ together with the order-unit 1. For every unital ℓ-group (G, u) and elements $g_1, \dots, g_n$ in the unit interval $[0, u]$ of G, if the set $\{g_1, \dots, g_n, u\}$ generates G, then there is a unique unital ℓ-homomorphism ψ of $\mathcal{M}_n$ onto G such that $\psi(\pi_i) = g_i$ for each $i = 1, \dots, n$.*

A unital ℓ-group (G, u) is *finitely presented* if for some $n = 1, 2, \dots,$ it is isomorphic to the quotient of $\mathcal{M}_n$ by a finitely generated ($=$ singly generated $=$ principal) ideal.

Given $f \in \mathcal{M}_n$ let $\mathcal{Z} f$ denote the *zeroset* of f. More generally, for every ideal $\mathfrak{j}$ of $\mathcal{M}_n$ we will write $\mathcal{Z}\mathfrak{j} = \bigcap\{\mathcal{Z} g \mid g \in \mathfrak{j}\}$. In the particular case when $\mathfrak{j}$ is maximal, $\mathcal{Z}\mathfrak{j}$ is a singleton (because the functions in $\mathcal{M}_n$ separate points [13, 4.17]), and we write $\check{\mathcal{Z}}\mathfrak{j}$ for the unique element of $\mathcal{Z}\mathfrak{j}$.

For later use we record here a classical result, whose proof follows from the Hion–Hölder theorem [6, pp. 45–47], [3, 2.6]:

Lemma 2.3. *For every unital ℓ-group (G, u) and ideal $\mathfrak{m} \in \operatorname{MaxSpec} G$ there is exactly one pair $(\iota_{\mathfrak{m}}, R_{\mathfrak{m}})$ where $R_{\mathfrak{m}}$ is a unital ℓ-subgroup of $(\mathbb{R}, 1)$, and $\iota_{\mathfrak{m}}$ is a unital ℓ-isomorphism of the quotient $(G, u)/\mathfrak{m}$ onto $R_{\mathfrak{m}}$. Upon identifying $(G, u)/\mathfrak{m}$ with $R_{\mathfrak{m}}$ every element $g/\mathfrak{m} \in (G, u)/\mathfrak{m}$ becomes a real number, and we can unambiguously write $g/\mathfrak{m} \in \mathbb{R}$.*

Corollary 2.4. *For any ideal $\mathfrak{i}$ of $\mathcal{M}_n$ let $\operatorname{MaxSpec}_{\supseteq \mathfrak{i}} \mathcal{M}_n$ denote the compact set of all maximal ideals of $\operatorname{MaxSpec} \mathcal{M}_n$ containing $\mathfrak{i}$. Then the map $\mathfrak{n} \mapsto \check{\mathcal{Z}}\mathfrak{n}$ yields a homeomorphism of $\operatorname{MaxSpec}_{\supseteq \mathfrak{i}} \mathcal{M}_n$ onto the compact set $\mathcal{Z}\mathfrak{i} \subseteq [0,1]^n$. The inverse of $\check{\mathcal{Z}}$ is the map $x \in \mathcal{Z}\mathfrak{i} \mapsto \mathfrak{m}_x = \{f \in \mathcal{M}_n \mid f(x) = 0\}$. Further, $f/\mathfrak{m} = f(\check{\mathcal{Z}}(\mathfrak{m}))$ for all $f \in \mathcal{M}_n$ and $\mathfrak{m} \in \operatorname{MaxSpec}_{\supseteq \mathfrak{i}} \in \mathcal{M}_n$.*

Proof. For each $x \in \mathcal{Z}\mathfrak{i}$, $\mathfrak{m}_x$ is a maximal ideal of $\mathcal{M}_n$. Further, for each $f \in \mathfrak{i}$, from $f(x) = 0$ we get $f \in \mathfrak{m}_x$, whence $\mathfrak{m}_x \supseteq \mathfrak{i}$ and $\check{\mathcal{Z}}\mathfrak{m}_x = x$. Let $\mathfrak{p} \in \operatorname{MaxSpec}_{\supseteq \mathfrak{i}} \mathcal{M}_n$. Then $\mathcal{Z}\mathfrak{p} \subseteq \mathcal{Z}\mathfrak{i}$ and for every $f \in \mathfrak{p}$ with $f(\check{\mathcal{Z}}\mathfrak{p}) = 0$ we have $\mathfrak{p} \subseteq \mathfrak{m}_{\check{\mathcal{Z}}(\mathfrak{p})}$ and $\check{\mathcal{Z}}\mathfrak{p} \in \mathcal{Z}\mathfrak{i}$. The assumed maximality of $\mathfrak{p}$ is to the effect that $\mathfrak{p} = \mathfrak{m}_{\check{\mathcal{Z}}(\mathfrak{p})}$, whence $\check{\mathcal{Z}}$ is a one–one map from $\operatorname{MaxSpec}_{\supseteq \mathfrak{i}} \mathcal{M}_n$ onto $\mathcal{Z}\mathfrak{i}$. By definition of spectral topology, $\check{\mathcal{Z}}$ is a homeomorphism. An application of Lemma 2.3 completes the proof. □

Corollary 2.5. *The quotient map $\kappa : \mathcal{M}_n \to \mathcal{M}_n/\mathfrak{i}$ determines the homeomorphism $\mathfrak{m} \mapsto \mathfrak{m}/\mathfrak{i}$ of $\operatorname{MaxSpec}_{\supseteq \mathfrak{i}} \mathcal{M}_n$ onto $\operatorname{MaxSpec} \mathcal{M}_n/\mathfrak{i}$. The inverse map is given by $\kappa^{-1}(\mathfrak{n}) = \{f \in \mathcal{M}_n \mid f/\mathfrak{i} \in \mathfrak{n}\}$ for each $\mathfrak{n} \in \operatorname{MaxSpec} \mathcal{M}_n/\mathfrak{i}$.*

Proof. The routine proof follows by combining Lemma 2.3 with [3, 2.3.8]. □

2.1. Rational polyhedra and unimodular triangulations

We refer to the first few chapters of [5] for background in elementary polyhedral topology. By a *rational polyhedron* P in $\mathbb{R}^n$ we understand a finite union of simplexes $P = S_1 \cup \cdots \cup S_t$ in $\mathbb{R}^n$ such that the coordinates of the vertices of every simplex S_i are rational numbers. For every simplicial complex Σ the point-set union of the simplexes of Σ is called the *support* of Σ and is denoted $|\Sigma|$; Σ is said to be a *triangulation* of $|\Sigma|$.

For any rational point $v \in \mathbb{R}^n$ the least common denominator of the coordinates of v is called the *denominator* of v, denoted $\mathrm{den}(v)$. The integer vector $\tilde{v} = \mathrm{den}(v)(v, 1) \in \mathbb{Z}^{n+1}$ is called the *homogeneous correspondent* of v. An m-simplex $U = \mathrm{conv}(w_0, \ldots, w_m) \subseteq [0, 1]^n$ is said to be *unimodular* if it is rational and the set of integer vectors $\{\tilde{w}_0, \ldots, \tilde{w}_m\}$ can be extended to a basis of the free abelian group $\mathbb{Z}^{n+1}$. A simplicial complex is said to be a *unimodular triangulation* (of its support) if all its simplexes are unimodular. The homogeneous correspondent of a unimodular triangulation is known as a regular (or, nonsingular) fan [5].

Proposition 2.6. *(See [12, 4.1, 5.1].) For all $P \subseteq [0, 1]^n$ the following are equivalent*:

(i) *P is a rational polyhedron.*
(ii) *$P = |\Delta|$ for some unimodular triangulation Δ.*
(iii) *For some unimodular triangulation ∇ of $[0, 1]^n$, $P = \bigcup\{S \in \nabla \mid S \subseteq P\}$.*
(iv) *For some $f \in \mathcal{M}_n$, $P = \mathcal{Z} f$.*

Lemma 2.7. *Let $\mathfrak{i}$ be an ideal of $\mathcal{M}_n$. Then the following are equivalent*:

(i) *$\mathfrak{i}$ is principal.*
(ii) *There exists $f \in \mathfrak{i}$ such that $\mathcal{Z}\mathfrak{i} = \mathcal{Z} f$.*

Proof. For the nontrivial direction, let $0 \leqslant f \in \mathfrak{i}$ satisfy $\mathcal{Z}\mathfrak{i} = \mathcal{Z} f$. We must check that, for all $0 \leqslant g \in \mathcal{M}_n$, $g \in \mathfrak{i} \Leftrightarrow \exists k = 1, 2, \ldots$ with $g \leqslant kf$. The $(\Leftarrow)$-direction follows from $f \in \mathfrak{i}$. For the $(\Rightarrow)$-direction, let Λ be a rational triangulation of $[0, 1]^n$ such that both f and g are linear over each simplex of Λ. Let $\{v_1, \ldots, v_s\}$ be the vertices of Λ. Since $\mathcal{Z} f = \mathcal{Z}\mathfrak{i} \subseteq \mathcal{Z} g$, $f(v_i) = 0$ implies $g(v_i) = 0$. For each $i = 1, \ldots, s$ there is $\mathbb{Z} \ni m_i > 0$ such that $m_i f(v_i) \geqslant f(v_i)$. Letting $k = \max(m_1, \ldots, m_s)$, the desired result follows from the assumed linearity properties of f and g. $\square$

3. Finitely presented unital ℓ-groups and bases

Theorem 3.1. *A unital ℓ-group (G, u) is finitely presented iff it has a basis.*

Proof. The $(\Rightarrow)$-direction is proved in [12, 5.2]. For the $(\Leftarrow)$-direction, let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a basis of (G, u), with multiplicities $m_1, \ldots, m_n$. Let $\kappa : \mathcal{M}_n \to (G, u)$ be the unique unital ℓ-homomorphism extending the map $\pi_i \mapsto b_i$, given by Proposition 2.2. Let $\mathfrak{i} = \ker(\kappa)$. By Definition 2.1(i), κ is onto G, whence $(G, u) \cong \mathcal{M}_n/\mathfrak{i}$. Let the function $a \in \mathcal{M}_n$ be defined by $a = |1 - \sum_i m_i \pi_i|$. Then, recalling the notation of (1),

$$0 \leqslant a \in \mathfrak{i}, \quad \text{and} \quad \mathcal{Z} a = \mathcal{T}_{\mathcal{B}}. \tag{4}$$

As a matter of fact, from Definition 2.1(ii) we have $\kappa(\sum_i m_i \pi_i) = \sum_i m_i \kappa(\pi_i) = \sum_i m_i b_i = u$. Recalling (2), we next observe

$$\bigwedge_{C \in \mathcal{B}^{\bowtie}} a_C \in \mathfrak{i}. \tag{5}$$

The result is trivial if $\mathcal{B}$ itself is a cluster in $\mathcal{B}^{\bowtie}$. If this is not the case, for each $C \in \mathcal{B}^{\bowtie}$ let b_{i_C} be an arbitrarily chosen element of $\mathcal{B} \setminus C$. Let $D = \{b_{i_C} \mid C \in \mathcal{B}^{\bowtie}\}$. Then $D \notin \mathcal{B}^{\bowtie}$, for otherwise, $b_{i_D} \in D$, which is impossible. Therefore, $\kappa(\bigwedge_{C\in\mathcal{B}^{\bowtie}} \pi_{i_C}) = \bigwedge_{C\in\mathcal{B}^{\bowtie}} b_{i_C} = 0$, i.e., $\bigwedge_{C\in\mathcal{B}^{\bowtie}} \pi_{i_C} \in \mathfrak{i}$. Since each $b_{i_C} \in \mathcal{B} \setminus C$ is arbitrary, the desired result (5) now follows from the distributivity of the underlying lattice of (G, u).

Let $f^* \in \mathcal{M}_n$ be defined by $f^* = a \vee \bigwedge_{C\in\mathcal{B}^{\bowtie}} a_C$. From (3)–(5) it follows that

$$0 \leqslant f^* \in \mathfrak{i} \quad \text{and} \quad \mathcal{Z} f^* = \mathcal{Z} a \cap \bigcup_{C\in\mathcal{B}^{\bowtie}} \mathcal{Z} a_C = \bigcup_{C\in\mathcal{B}^{\bowtie}} \mathcal{T}_C, \tag{6}$$

whence $\mathcal{Z} f^* \supseteq \mathcal{Z}\mathfrak{i}$. To prove the converse inclusion, for each cluster K of $\mathcal{B}$ we set

$$\operatorname{apogee}(K) = \{\mathfrak{n} \in \operatorname{MaxSpec} \mathcal{M}_n/\mathfrak{i} \mid \mathfrak{n} \supseteq \mathcal{B} \setminus K\}.$$

For each $\mathfrak{n} \in \operatorname{MaxSpec} \mathcal{M}_n/\mathfrak{i}$, letting $C_{\mathfrak{n}}$ be the cluster of all $b \in \mathcal{B}$ such that $b \notin \mathfrak{n}$, it follows that $\mathcal{B} \setminus C_{\mathfrak{n}} \subseteq \mathfrak{n}$, whence $\mathfrak{n} \in \operatorname{apogee}(C_{\mathfrak{n}})$. Thus, $\bigcup_{C\in\mathcal{B}^{\bowtie}} \operatorname{apogee}(C) \supseteq \operatorname{MaxSpec} \mathcal{M}_n/\mathfrak{i}$. Since the converse inclusion holds by definition, we have

$$\operatorname{MaxSpec} \mathcal{M}_n/\mathfrak{i} = \bigcup_{C\in\mathcal{B}^{\bowtie}} \operatorname{apogee}(C). \tag{7}$$

For each $K \in \mathcal{B}^{\bowtie}$ we let $\operatorname{apogee}_{\mathbb{R}}(K)$ denote the inverse image of $\operatorname{apogee}(K)$ under the composition of the homeomorphisms $x \mapsto \mathfrak{m}_x \mapsto \mathfrak{m}_x/\mathfrak{i}$ of Corollaries 2.4 and 2.5, where $\mathfrak{m}_x = \{f \in \mathcal{M}_n \mid f(x) = 0\}$. In other words,

$$\operatorname{apogee}_{\mathbb{R}}(K) = \bigl\{x \in \mathcal{Z}\mathfrak{i} \bigm| \mathfrak{m}_x/\mathfrak{i} \in \operatorname{apogee}(K)\bigr\}. \tag{8}$$

From (6)–(7) we get

$$\bigcup_{C\in\mathcal{B}^{\bowtie}} \operatorname{apogee}_{\mathbb{R}}(C) = \mathcal{Z}\mathfrak{i} \subseteq \mathcal{Z} f^* = \bigcup_{C\in\mathcal{B}^{\bowtie}} \mathcal{T}_C. \tag{9}$$

Claim 1. *For each $C = \{b_{i_1}, \ldots, b_{i_t}\} \in \mathcal{B}^{\bowtie}$, $\operatorname{apogee}_{\mathbb{R}}(C) \subseteq \mathcal{T}_C$.*

Indeed, by Definition 2.1(iii) we have

$$\begin{aligned} \operatorname{apogee}(C) &= \{\mathfrak{n} \in \operatorname{MaxSpec} \mathcal{M}_n/\mathfrak{i} \mid b/\mathfrak{n} = 0 \text{ for all } b \in \mathcal{B} \setminus C\} \\ &= \left\{\mathfrak{n} \in \operatorname{MaxSpec} \mathcal{M}_n/\mathfrak{i} \;\middle|\; \frac{m_{i_1} b_{i_1} + \cdots + m_{i_t} b_{i_t}}{\mathfrak{n}} = 1\right\}. \end{aligned} \tag{10}$$

By Lemma 2.3, for each $\mathfrak{m} \in \operatorname{MaxSpec}_{\supseteq\mathfrak{i}} \mathcal{M}_n$, both $\frac{\mathcal{M}_n}{\mathfrak{m}}$ and its isomorphic copy $\frac{\mathcal{M}_n/\mathfrak{i}}{\mathfrak{m}/\mathfrak{i}}$ are canonically isomorphic to the same unital ℓ-subgroup of $(\mathbb{R}, 1)$. Thus for each $f \in \mathcal{M}_n$ the real numbers $\frac{f/\mathfrak{i}}{\mathfrak{m}/\mathfrak{i}}$ and $\frac{f}{\mathfrak{m}}$ are identical. By Corollaries 2.4 and 2.5, for each $\mathfrak{n} \in \operatorname{MaxSpec} \mathcal{M}_n/\mathfrak{i}$ we have $\frac{f/\mathfrak{i}}{\mathfrak{n}/\mathfrak{i}} = f(\check{\mathcal{Z}}(\kappa^{-1}(\mathfrak{n})))$, or equivalently,

$$f(x) = \frac{f/\mathfrak{i}}{\mathfrak{m}_x/\mathfrak{i}} \quad \text{for all } x \in \mathcal{Z}\mathfrak{i}. \tag{11}$$

From (8) and (10) it follows that $(y_1,\dots,y_n)\in \mathrm{apogee}_{\mathbb{R}}(C)$ if and only if

$$\frac{m_{i_1}b_{i_1}+\cdots+m_{i_t}b_{i_t}}{\mathfrak{m}_x/\mathfrak{i}}=\frac{(m_{i_1}y_{i_1}+\cdots+m_{i_t}y_{i_t})/\mathfrak{i}}{\mathfrak{m}_x/\mathfrak{i}}=1.$$

Recalling (1) and (11), and writing $\mathrm{apogee}_{\mathbb{R}}(C)=\{(y_1,\dots,y_n)\in \mathcal{Z}\mathfrak{i}\mid m_{i_1}y_{i_1}+\cdots+m_{i_t}y_{i_t}=1\}\subseteq \mathcal{T}_C$, Claim 1 is settled.

Claim 2. *For every $C\in\mathcal{B}^{\bowtie}$, $\mathrm{apogee}_{\mathbb{R}}(C)=\mathcal{T}_C$.*

The proof, by induction on the number $l=1,2,\dots$ of elements of C, closely follows the algebraic topological argument of [12, Theorem 4.5, Claim 3].

Basis. For a unique $j\in\{1,\dots,n\}$ we have $C=\{b_j\}=\{\pi_j/\mathfrak{i}\}$. By Definition 2.1(ii), $\mathrm{apogee}(C)$ contains exactly one element $\mathfrak{n}$. By Lemma 2.3, $\mathfrak{n}$ is the only maximal ideal of $\mathcal{M}_n/\mathfrak{i}$ such that $0=b/\mathfrak{n}$ for all $b\neq b_j$. By (10), $\mathfrak{n}$ is uniquely determined by the condition $1=m_jb_j/\mathfrak{n}=(m_j\pi_j/\mathfrak{i})/\mathfrak{n}$. Letting $z\in\mathcal{Z}\mathfrak{i}$ be the image of $\mathfrak{n}$ in $\mathrm{apogee}_{\mathbb{R}}(C)$, by (1) and Claim 1 we have $z=e_j/m_j$. We conclude that $\mathrm{apogee}_{\mathbb{R}}(C)=\{e_j/m_j\}=\mathrm{conv}\{e_j/m_j\}=\mathcal{T}_C$.

Induction step. Assuming C has $l+1$ elements, let us write $C=\{b_{i_0},\dots,b_{i_l}\}$. Since every l-element subset C' of C is a cluster of $\mathcal{B}$, by induction hypothesis $\mathrm{apogee}_{\mathbb{R}}(C')=\mathcal{T}_{C'}$. $\mathcal{T}_{C'}$ is known as a *facet* of $\mathcal{T}_C$. By Claim 1, $\mathrm{apogee}_{\mathbb{R}}(C)$ is a nonempty subset of $\mathcal{T}_C$ containing all facets of $\mathcal{T}_C$. Further, $\mathrm{apogee}_{\mathbb{R}}(C)$ is homeomorphic to an l-simplex, because so is its homeomorphic copy $\mathrm{apogee}(C)$, by Definition 2.1(ii). Observe that $\mathcal{T}_C$ is *contractible* (i.e., $\mathcal{T}_C$ is continuously shrinkable to a point). By way of contradiction, suppose $\mathrm{apogee}_{\mathbb{R}}(C)$ is a proper subset of $\mathcal{T}_C$. Then a routine exercise in algebraic topology shows that $\mathrm{apogee}_{\mathbb{R}}(C)$ is not contractible. Thus $\mathrm{apogee}_{\mathbb{R}}(C)$ is not homeomorphic to any l-simplex, a contradiction showing that $\mathrm{apogee}_{\mathbb{R}}(C)=\mathcal{T}_C$. Claim 2 is settled.

By Claim 2 and (9) we can write $\mathcal{Z}f^*=\bigcup_{C\in\mathcal{B}^{\bowtie}}\mathcal{T}_C=\mathcal{Z}\mathfrak{i}$. By Lemma 2.7, $\mathfrak{i}$ is the ideal generated by f^*, and $(G,u)\cong\mathcal{M}_n/\mathfrak{i}$ is finitely presented. □

4. A class of projective unital ℓ-groups

Lemma 4.1. *Let $S=\mathrm{conv}(x_1,\dots,x_k)\subseteq[0,1]^n$ be a unimodular $(k-1)$-simplex and $v\in\{0,1\}^n$ a vertex of $[0,1]^n$. Then for every $Y\subseteq\{x_1,\dots,x_k\}$ there is a matrix $M\in\mathbb{Z}^{n\times n}$ and a vector $b\in\mathbb{Z}^n$ such that*

$$Mx_i+b_i=\begin{cases} v & \text{if } x_i\in Y,\\ x_i & \text{otherwise}.\end{cases}$$

Proof. Since S is unimodular, the set $\{\tilde{x}_1,\dots,\tilde{x}_k\}$ of homogeneous correspondents of $x_1,\dots,x_k$ can be extended to a basis $\{\tilde{x}_1,\dots,\tilde{x}_k,q_{k+1},\dots,q_{n+1}\}$ of the free abelian group $\mathbb{Z}^{n+1}$. The $(n+1)\times(n+1)$ matrix D with column vectors $\tilde{x}_1,\dots,\tilde{x}_k,q_{k+1},\dots,q_{n+1}$ is invertible and $D^{-1}\in\mathbb{Z}^{(n+1)\times(n+1)}$. For each $i=1,\dots,t$ let $c_i\in\mathbb{Z}^{n+1}$ be defined by $c_i=\mathrm{den}(x_i)(v,1)$ if $x_i\in Y$, and otherwise $c_i=\tilde{x}_i$. Let $C\in\mathbb{Z}^{(n+1)\times(n+1)}$ be the matrix whose columns are given by the vectors $c_1,\dots,c_k,q_{k+1},\dots,q_{n+1}$. Since D and C have the same $(n+1)$th row,

$$CD^{-1}=\left(\begin{array}{c|c} M & d\\ \hline 0,\dots,0 & 1\end{array}\right)$$

for some $n\times n$ integer matrix M and $d\in\mathbb{Z}^n$. For each $i=1,\dots,k$, $(CD^{-1})\tilde{x}_i=(CD^{-1})\,\mathrm{den}(x_i)(x_i,1)=\mathrm{den}(x_i)(Mx_i+d,1)$. In conclusion, $(CD^{-1})\tilde{x}_i=c_i=\mathrm{den}(x_i)(v,1)$ if $x_i\in Y$ and $(CD^{-1})\tilde{x}_i=\tilde{x}_k=\mathrm{den}(x_i)(x_i,1)$ otherwise. □

Theorem 4.2. *Suppose the unital ℓ-group (G,u) has a basis $\mathcal{B}$ with $\bigwedge \mathcal{B} \neq 0$. Suppose at least one of the multiplicities of $\mathcal{B}$ is equal to 1. Then (G,u) is projective.*

Proof. Let $1 = m_1 \leqslant m_2 \leqslant \cdots \leqslant m_n$ be the multiplicities of $\mathcal{B}$. Proposition 2.6 yields a unimodular triangulation Δ of $[0,1]^n$ such that the simplex $\mathcal{T}_{\mathcal{B}}$ in (1) is a union of simplexes of Δ, and all vertices of (every simplex of) Δ have rational coordinates. We next define the function $\mathbf{f}\colon [0,1]^n \to [0,1]^n$ by stipulating that, for each vertex v of Δ,

$$\mathbf{f}(v) = \begin{cases} v & \text{if } v \in \mathcal{T}_{\mathcal{B}}, \\ e_1 & \text{if } v \notin \mathcal{T}_{\mathcal{B}} \end{cases} \tag{12}$$

and $\mathbf{f}$ is linear over each simplex of Δ. Then $\mathbf{f}$ is a continuous map and $\mathbf{f} \restriction \mathcal{T}_{\mathcal{B}}$ is the identity map on $\mathcal{T}_{\mathcal{B}}$. For any simplex S of Δ, let ∂S denote the set of extremal points of S. Since $\mathbf{f}$ is linear over S and $\mathbf{f}(v) \in \mathcal{T}_{\mathcal{B}}$ for each $v \in \partial S$, we have $\mathbf{f}(S) = \mathbf{f}(\mathrm{conv}(\partial S)) = \mathrm{conv}(\mathbf{f}(\partial S)) \subseteq \mathrm{conv}(\mathcal{T}_{\mathcal{B}}) = \mathcal{T}_{\mathcal{B}}$, whence

$$\mathbf{f}\big([0,1]^n\big) = \mathcal{T}_{\mathcal{B}}. \tag{13}$$

We have thus shown that $\mathbf{f} \circ \mathbf{f} = \mathbf{f}$ and $\mathbf{f}$ is a continuous retraction of $[0,1]^n$ onto $\mathcal{T}_{\mathcal{B}}$ which is linear on each simplex of Δ.

By Lemma 4.1, the coefficients of each linear piece of $\mathbf{f}$ are integers. Therefore, the map $\varphi : \mathcal{M}_n \to \mathcal{M}_n$ given by

$$\varphi(g) = g \circ \mathbf{f} \tag{14}$$

is well defined. It follows straightforwardly that φ is a unital ℓ-homomorphism. Since $\mathbf{f} \circ \mathbf{f} = \mathbf{f}$ then $\varphi \circ \varphi = \varphi$. In other words, φ is an idempotent endomorphism of $\mathcal{M}_n$. Stated otherwise, the unital ℓ-subgroup $\varphi(\mathcal{M}_n)$ of $\mathcal{M}_n$ is a retraction of $\mathcal{M}_n$. Applying now the universal property of $\mathcal{M}_n$ (Proposition 2.2) one sees that $\mathcal{M}_n$ is projective. A routine exercise using the fact that $\varphi(\mathcal{M}_n)$ is a retraction of $\mathcal{M}_n$ shows that $\varphi(\mathcal{M}_n)$ is projective.

To conclude the proof it is enough to show that $\varphi(\mathcal{M}_n)$ is unitally ℓ-isomorphic to (G,u). In proving the $(\Leftarrow)$-direction of Theorem 3.1 we have seen that (G,u) is unitally ℓ-isomorphic to $\mathcal{M}_n/\mathfrak{i}$, for some ideal $\mathfrak{i}$ having following characterization:

$$\mathfrak{i} = \left\{ g \in \mathcal{M}_n \;\middle|\; \mathcal{Z}g \supseteq \bigcup_{C \in \mathcal{B}^{\bowtie}} \mathcal{T}_C \right\} = \{ g \in \mathcal{M}_n \mid \mathcal{Z}g \supseteq \mathcal{T}_{\mathcal{B}} \}.$$

By (13) and (14),

$$\begin{aligned} g \in \ker(\varphi) \quad &\Leftrightarrow \quad g \circ \mathbf{f} = 0 \quad \Leftrightarrow \quad g\big(\mathbf{f}\big([0,1]^n\big)\big) = \{0\} \\ &\Leftrightarrow \quad g(\mathcal{T}_{\mathcal{B}}) = \{0\} \quad \Leftrightarrow \quad \mathcal{Z}g \supseteq \mathcal{T}_{\mathcal{B}} \quad \Leftrightarrow \quad g \in \mathfrak{i}. \end{aligned}$$

Therefore, $(G,u) \cong \mathcal{M}_n/\mathfrak{i} = \mathcal{M}_n/\ker(\varphi) \cong \varphi(\mathcal{M}_n)$, and the proof is complete. $\square$

5. The underlying dimension group of a unital ℓ-group with a basis

In the category $\mathcal{P}$ of *partially ordered abelian groups with order-unit* [8, p. 12] objects are pairs (G,u), where G is a partially ordered abelian group and u is an order-unit of G. A morphism

$\phi : (G, u) \to (H, v)$ of $\mathcal{P}$ is a *unital* (i.e., unit-preserving) *positive* (in the sense that $\phi(G^+) \subseteq H^+$) homomorphism.

Following [8, p. 47], by a *unital simplicial* group we understand an object (G, u) of $\mathcal{P}$ that is isomorphic (in $\mathcal{P}$) to the free abelian group $\mathbb{Z}^n$ for some integer $n > 0$ equipped with the product ordering: $(x_1, \ldots, x_n) \geqslant 0$ iff $x_i \geqslant 0$, $\forall i = 1, \ldots, n$. The order-unit u has the form $u = (u_1, \ldots, u_n)$ where each u_i is an integer > 0.

A *unital dimension group* (G, u) is an object of $\mathcal{P}$ such that $G = G^+ - G^+$, sums of intervals are intervals, and for any $g \in G$ if $kg \in G^+$ for some $0 < k \in \mathbb{Z}$, then already $g \in G^+$. For short, G is directed, Riesz, and unperforated [8, p. 44]. By Elliott classification theory [4], countable unital dimension groups are complete classifiers of AF C*-algebras, i.e., the norm limits of ascending sequences of finite-dimensional C*-algebras, all with the same unit.

Given a unital ℓ-group (G, u) let $(G, u)_{\mathrm{dim}}$ denote the underlying group of (G, u) equipped with the same positive cone G^+ and order-unit u, but forgetting the lattice structure of (G, u). Then $(G, u)_{\mathrm{dim}}$ is a unital dimension group. Thus in particular, every unital simplicial group is a unital dimension group. Since the properties of directedness, Riesz, and unperforatedness are preserved by direct limits, then direct limits of unital simplicial groups are unital dimension groups.

The Effros–Handelman–Shen theorem [4], [8, 3.21] (also see Grillet's theorem [10, 2.1] jointly with [9, Remark 3.2]) states the converse: for every unital dimension group (G, u) we can write

$$(G, u) \cong \lim\bigl\{\phi_{ij}\colon \bigl(\mathbb{Z}^{n_i}, u_i\bigr) \to \bigl(\mathbb{Z}^{n_j}, u_j\bigr) \bigm| i, j \in I\bigr\}$$

for some direct system of unital simplicial groups and unital positive homomorphisms in $\mathcal{P}$. For dimension groups of the form $(G, u)_{\mathrm{dim}}$, with (G, u) a unital ℓ-group, Marra [11] proved that the maps ϕ_{ij} can be assumed to be 1–1.

A further simplification occurs when (G, u) has a basis: as a matter of fact, in Theorem 5.3 below we will prove that the set of bases of (G, u) is rich enough to provide a direct system of unital simplicial groups and 1–1 unital homomorphisms such that $(G, u)_{\mathrm{dim}}$ is the limit of this system in the category $\mathcal{P}$. To this purpose, given a basis $\mathcal{B} = \{b_1, \ldots, b_n\}$ of a unital ℓ-group (G, u), we let $\operatorname{grp}\mathcal{B} = \mathbb{Z}b_1 + \cdots + \mathbb{Z}b_n$ denote the group generated by $\mathcal{B}$ in (the underlying group of) G. Similarly, $\operatorname{sgr}\mathcal{B} = \mathbb{Z}_{\geqslant 0}b_1 + \cdots + \mathbb{Z}_{\geqslant 0}b_n$ will denote the semigroup generated by $\mathcal{B}$ together with the zero element.

Assuming, as we are doing throughout the rest of this paper, that the elements of $\mathcal{B}$ are listed in some prescribed order, by definition of $\mathcal{B}$ the n-tuple of multiplicities $m_{\mathcal{B}} = (m_1, \ldots, m_n)$ is uniquely determined by the n-tuple $(b_1, \ldots, b_n)$.

Proposition 5.1. *Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a basis of a unital ℓ-group (G, u). Let $G_{\mathcal{B}} = (\operatorname{grp}\mathcal{B}, \operatorname{sgr}\mathcal{B}, u)$ denote the group $\operatorname{grp}\mathcal{B}$ equipped with the positive cone $\operatorname{sgr}\mathcal{B}$ and with the distinguished order-unit $u = \sum m_i b_i$. Let the simplicial group $\mathbb{Z}_{\mathcal{B}}$ be defined by $\mathbb{Z}_{\mathcal{B}} = (\mathbb{Z}^n, (\mathbb{Z}^+)^n, m_{\mathcal{B}})$, with the n-tuple $m_{\mathcal{B}}$ as the order-unit. Then*:

- (I) *$\mathcal{B}$ is a free generating set of the free abelian group $\operatorname{grp}\mathcal{B}$ of rank n.*
- (II) $G^+ \cap \operatorname{grp}\mathcal{B} = \operatorname{sgr}\mathcal{B}$.
- (III) *The map $b_i \mapsto e_i$ uniquely extends to an isomorphism $\psi_{\mathcal{B}} : \operatorname{grp}_{\mathcal{B}} \cong \mathbb{Z}^n$.*
- (IV) *$\psi_{\mathcal{B}}$ is in fact an isomorphism (in the category $\mathcal{P}$) of $G_{\mathcal{B}}$ onto $\mathbb{Z}_{\mathcal{B}}$, whence $G_{\mathcal{B}}$ is a unital simplicial group, called the* basic group of *$\mathcal{B}$; further, $\mathcal{B}$ is the set of atoms (= minimal positive nonzero elements) of $G_{\mathcal{B}}$; thus if $\mathcal{B}' \neq \mathcal{B}$ is another basis of (G, u) then $G_{\mathcal{B}} \neq G_{\mathcal{B}'}$.*

Proof. (I) By condition (ii) in the definition of $\mathcal{B}$, no nonzero linear combination of the elements of $\mathcal{B}$ is zero in (the $\mathbb{Z}$-module) G. Since G is torsion-free, $\mathcal{B}$ is a free generating set in $\operatorname{grp}\mathcal{B}$, and $\operatorname{grp}\mathcal{B}$ is free abelian of rank n.

(II) Suppose $g \in G^+ \cap \operatorname{grp}\mathcal{B}$, and write $g = \sum_{i=1}^n l_i b_i$ for suitable integers $l_1, \ldots, l_n$. Fix now $j \in \{1, \ldots, n\}$ and let $\mathfrak{n}_j$ be the only maximal ideal of G such that $b_k \in \mathfrak{n}_j$ for all $k \neq j$, as given by condition (ii) in the definition of $\mathcal{B}$. By condition (iii) we have

$$0 \leqslant \sum_{i=1}^{n} l_i b_i \quad \Rightarrow \quad 0 \leqslant \frac{\sum_{i=1}^{n} l_i b_i}{\mathfrak{n}_j} = \frac{l_j b_j}{\mathfrak{n}_j} = \frac{l_j}{m_j},$$

whence $0 \leqslant l_j$ for all j, and $g \in \operatorname{sgr} \mathcal{B}$. The converse inclusion is trivial.

(III) The map $b_i \mapsto e_i$ is a one–one correspondence between the free generating set $\mathcal{B}$ of $\operatorname{grp} \mathcal{B}$ and the free generating set $\{e_1, \ldots, e_n\}$ of $\mathbb{Z}^n$.

(IV) It is easy to see that $\mathcal{B}$ is the set of atoms of $G_{\mathcal{B}}$, and $\{e_1, \ldots, e_n\}$ is the set of atoms of the simplicial group $(\mathbb{Z}^n, (\mathbb{Z}^+)^n)$. Thus $\psi_{\mathcal{B}}$ is an isomorphism of $G_{\mathcal{B}}$ onto $(\mathbb{Z}^n, \mathbb{Z}^{+n})$, and $G_{\mathcal{B}}$ is simplicial. Trivially, $\psi_{\mathcal{B}}$ preserves order-units. So $G_{\mathcal{B}}$ is a unital simplicial group which is isomorphic (in $\mathcal{P}$) to $\mathbb{Z}_{\mathcal{B}}$. The rest is clear. □

Given two bases $\mathcal{B}'$ and $\mathcal{B}$ of a unital ℓ-group (G, u) we say that $\mathcal{B}'$ *refines* $\mathcal{B}$ if $\mathcal{B} \subseteq \operatorname{sgr} \mathcal{B}'$. Then from the above proposition we immediately obtain

Proposition 5.2. *Let $\mathcal{B}' = \{b'_1, \ldots, b'_{n'}\}$ and $\mathcal{B} = \{b_1, \ldots, b_n\}$ be bases of a unital ℓ-group (G, u) such that $\mathcal{B}'$ refines $\mathcal{B}$. Then for each $i = 1, \ldots, n$, b_i is expressible as a linear combination $b_i = m_{1i} b'_1 + \cdots + m_{n'i} b'_{n'}$, for uniquely determined integers $m_{ki} \geqslant 0$ $(k = 1, \ldots, n')$. Further, the rank of the $n' \times n$ matrix $M_{\mathcal{B}\mathcal{B}'}$ whose entries are the m_{ki}, equals n. Finally, the inclusion $G_{\mathcal{B}} \to G_{\mathcal{B}'}$ induces the unital positive 1–1 homomorphism $\phi_{\mathcal{B}\mathcal{B}'} : (y_1, \ldots, y_n) \in \mathbb{Z}^n \mapsto (z_1, \ldots, z_{n'}) = M_{\mathcal{B}\mathcal{B}'}(y_1, \ldots, y_n) \in \mathbb{Z}^{n'}$ of $(\mathbb{Z}_{\mathcal{B}}, m_{\mathcal{B}})$ into $(\mathbb{Z}_{\mathcal{B}'}, m_{\mathcal{B}'})$, and we have a commutative diagram*

$$\begin{tikzcd} G_{\mathcal{B}} \arrow[r, "\text{inclusion}"] \arrow[d, "\psi_{\mathcal{B}}"] & G_{\mathcal{B}'} \arrow[d, "\psi_{\mathcal{B}'}"] \\ (\mathbb{Z}_{\mathcal{B}}, m_{\mathcal{B}}) \arrow[r, "\phi_{\mathcal{B}\mathcal{B}'}"] & (\mathbb{Z}_{\mathcal{B}'}, m_{\mathcal{B}'}). \end{tikzcd} \tag{15}$$

Theorem 5.3. *Suppose the unital ℓ-group (G, u) has a basis.*

(I) *Any two basic groups $G_{\mathcal{B}}, G_{\mathcal{F}}$ of (G, u) are jointly embeddable (by unit preserving, order preserving inclusions) into some basic group $G_{\mathcal{B}'}$ of (G, u).*
(II) *There exists a direct system $\{\phi_{\mathcal{B}\mathcal{B}'} : (\mathbb{Z}_{\mathcal{B}}, m_{\mathcal{B}}) \to (\mathbb{Z}_{\mathcal{B}'}, m_{\mathcal{B}'})\}$ of unital simplicial groups and unital positive 1–1 homomorphisms in $\mathcal{P}$, indexed by all pairs $\mathcal{B}, \mathcal{B}'$ of bases of (G, u) such that $\mathcal{B} \subseteq \operatorname{sgr} \mathcal{B}'$.*
(III) *Further, $\lim\{\phi_{\mathcal{B}\mathcal{B}'} : (\mathbb{Z}_{\mathcal{B}}, m_{\mathcal{B}}) \to (\mathbb{Z}_{\mathcal{B}'}, m_{\mathcal{B}'})\} \cong (G, u)_{\mathrm{dim}}$.*

Proof. (I)–(II) By Theorem 3.1, (G, u) is finitely presented, and for some $n = 1, 2, \ldots,$ and principal ideal $\mathfrak{j}$ of $\mathcal{M}_n$, (G, u) is isomorphic to $\mathcal{M}_n/\mathfrak{j}$. Suppose $\mathfrak{j}$ is generated by $f \in \mathcal{M}_n$. A variant of [7, 5.2] shows that $\mathcal{M}_n/\mathfrak{j} \cong \mathcal{M}_n \upharpoonright \mathcal{Z}f$. A fortiori, (G, u) is archimedean. From [12, 5.4] it follows that $\mathcal{B}$ and $\mathcal{F}$ have a joint refinement $\mathcal{B}'$. Direct inspection of that proof shows that $\mathcal{B}'$ is obtained from $\mathcal{B}$ by finitely many applications of the following operation: Replace a 2-cluster $\{b, c\}$ of a basis $\mathcal{A}$, by the three elements $b \wedge c, b - (b \wedge c), c - (b \wedge c)$. The result is a basis $\mathcal{A}'$ such that $\mathcal{A} \subseteq \operatorname{sgr} \mathcal{A}'$. Thus $\mathcal{B} \subseteq \operatorname{sgr} \mathcal{B}'$. From Proposition 5.2 we now obtain (I) and (II). For (III), in view of (15) it is sufficient to prove $G = \bigcup\{\operatorname{grp} \mathcal{B} \mid \mathcal{B}$ a basis of $(G, u)\}$ and $G^+ = \bigcup\{\operatorname{sgr} \mathcal{B} \mid \mathcal{B}$ a basis of $(G, u)\}$. Since $G = G^+ - G^+$, we need only prove that for every $p \in G^+, (G, u)$ has a basis $\mathcal{B}$ such that $p \in \operatorname{sgr} \mathcal{B}$. Since $(G, u) \cong \mathcal{M}_n \upharpoonright \mathcal{Z}f$ is archimedean, the proof follows from [12, 5.4]. □

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