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Ramification theory for Artin–Schreier extensions of valuation rings

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A B S T R A C T

The goal of this paper is to generalize and refine the classical ramification theory of complete discrete valuation rings to more general valuation rings, in the case of Artin–Schreier extensions. We define refined versions of invariants of ramification in the classical ramification theory and compare them. Furthermore, we can treat the defect case.

0. Introduction

We present a generalization and refinement of the classical ramification theory of complete discrete valuation rings to valuation rings satisfying either (I) or (II) (as explained in 0.2), in the case of Artin–Schreier extensions. The classical theory considers the case of complete discrete valued field extension $L|K$ where the residue field k of K is perfect. In his papers [5,6], Kato gives a natural definition of the Swan conductor for complete

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discrete valuation rings with arbitrary (possibly imperfect) residue fields. He also defines the refined Swan conductor rsw in this case using differential 1-forms and powers of the maximal ideal $\mathfrak{m}_L$. The generalization we present is a further refinement of this definition. Moreover, we can deal with the extensions with defect, a case which was not treated previously [7,8].

0.1. Invariants of ramification theory

Let K be a valued field of characteristic $p > 0$ with henselian valuation ring A, valuation v_K and residue field k. Let $L = K(\alpha)$ be the Artin–Schreier extension defined by $\alpha^p - \alpha = f$ for some $f \in K^\times$. Assume that $L|K$ is non-trivial, that is, $[L : K] = p$. Let B be the integral closure of A in L. Since A is henselian, it follows that B is a valuation ring. Let v_L be the valuation on L that extends v_K and let l denote the residue field of L. Let $\Gamma := v_K(K^\times)$ denote the value group of K. The Galois group $\mathrm{Gal}(L|K) = G$ is cyclic of order p, generated by $\sigma : \alpha \mapsto \alpha + 1$.

Let $\mathfrak{A} = \{f \in K^\times \mid \text{the solutions of the equation } \alpha^p - \alpha = f \text{ generate } L \text{ over } K\}$. Consider the ideals $\mathcal{J}_\sigma$ and H, of B and A respectively, defined as below:

$$\mathcal{J}_\sigma = \left(\left\{\frac{\sigma(b)}{b} - 1 \mid b \in L^\times\right\}\right) \subset B \tag{0.1}$$

$$H = \left(\left\{\frac{1}{f} \mid f \in \mathfrak{A}\right\}\right) \subset A \tag{0.2}$$

Our first result compares these two invariants via the norm map $N_{L|K} = N$, by considering the ideal $\mathcal{N}_\sigma$ of A generated by the elements of $N(\mathcal{J}_\sigma)$. We also consider the ideal $\mathcal{I}_\sigma = (\{\sigma(b) - b \mid b \in B\})$ of B. The ideals $\mathcal{I}_\sigma$ and $\mathcal{J}_\sigma$ play the roles of $i(\sigma)$ and $j(\sigma)$ (the Lefschetz numbers in the classical case, as explained in 2.1), respectively, in the generalization.

0.2. Main results

We did not make any assumptions regarding the rank or defect in these definitions. Now consider two special cases of the scenario described above:

(I) **(Defectless)** In this case, we assume that $L|K$ is defectless. For Artin–Schreier extensions $L|K$ considered in this paper, it means that either $v_L(L^\times)/v_L(K^\times)$ has order p and the residue extension $l|k$ is trivial or the residue extension $l|k$ is of degree p and L has the same value group Γ as K.

(II) **(Rank 1)** The value group Γ of K is isomorphic to a subgroup of $\mathbb{R}$ as an ordered group.

We will prove the following results:

Theorem 0.3. *If $L|K$ satisfies (I) or (II), we have the following equality of ideals of A:*

$$H = \mathcal{N}_\sigma \tag{0.4}$$

Theorem 0.5. *If $L|K$ satisfies (I) or* (II)*, we consider the A-module ω_A^1 of logarithmic differential* 1*-forms and the B-module $\omega_{B|A}^1$ of logarithmic differential* 1*-forms over A (as defined in section 1.1). Then*

(i) *There exists a unique homomorphism of A-modules* $\mathrm{rsw} : H/H^2 \to \omega_A^1/(\mathcal{I}_\sigma \cap A)\omega_A^1$ *such that* $\frac{1}{f} \mapsto \mathrm{dlog}\, f$*; for all $f \in \mathfrak{A}$.*

(*ii*) *There is a B-module isomorphism $\varphi_\sigma : \omega_{B|A}^1/\mathcal{J}_\sigma\omega_{B|A}^1 \xrightarrow{\cong} \mathcal{J}_\sigma/\mathcal{J}_\sigma^2$ such that* $\mathrm{dlog}\, x \mapsto \frac{\sigma(x)}{x} - 1$*, for all $x \in L^\times$.*

(*iii*) *Furthermore, these maps induce the following commutative diagram:*

$$\begin{tikzcd} \omega_{B|A}^1/\mathcal{J}_\sigma\omega_{B|A}^1 \arrow[r, "\varphi_\sigma", "\cong"'] \arrow[d, "\overline{\Delta_N}"'] & \mathcal{J}_\sigma/\mathcal{J}_\sigma^2 \arrow[d, hook, "\overline{N}"] \\ \omega_A^1/(\mathcal{I}_\sigma \cap A)\omega_A^1 & H/H^2 \arrow[l, "\mathrm{rsw}"] \end{tikzcd}$$

The maps $\overline{\Delta_N}, \overline{N}$ are induced by the norm map N, as described in section 6.

The map rsw in (i) is a refined generalization of the refined Swan conductor of Kato for complete discrete valuation rings [6].

Remark 0.6. It is worth noting that if $p = 2$, both the results are true without any assumptions regarding defect or rank, as seen in later sections.

Remark 0.7. If $L|K$ is unramified ($e_{L|K} = 1, l|k$ separable of degree p), then we have $i(\sigma) = j(\sigma) = 0$, $\mathcal{I}_\sigma = \mathcal{J}_\sigma = B$ and $H = A$. Consequently, our main results are trivially true. From now on, we assume that $L|K$ is either wild ($e_{L|K} = p, l|k$ trivial), ferocious ($l|k$ purely inseparable of degree p) or with defect.

0.3. Outline of the contents

- **Review, small results, examples:** In sections 1, 2 we present some preliminaries and the discrete valuation ring case. Section 3 contains some elementary results that help us understand the cases I and II.
- **Proofs of main results:** We prove Theorem 0.3 in section 4. In section 5, we analyze the defect case. We use Theorem 0.3 to prove Theorem 5.1, which allows us to express the ring B as a filtered union of rings $A[x]|A$, where elements $x \in L^\times$ are chosen very carefully. We prove Theorem 0.5 for both cases I and II in section 6.

- **The different ideal and further results:** Section 7 presents the description of the different ideal $\mathcal{D}_{B|A}$ when $L|K$ satisfies (I) or (II). This ideal equals the annihilator of the relative Kähler differential module $\Omega^1_{L|K}$ in the classical case [2]. However, this is not true in the case of arbitrary valuations.
- **Appendix:** In Appendix A, we present a non-trivial example of a defect extension. It shows us the difficulties that rise from the defect. We also verify the main results for this example.

1. Preliminaries: differential forms, defect, cyclic extensions, trace

1.1. Definitions: differential forms and different ideal $\mathcal{D}_{B|A}$

Definition 1.1. Differential 1-forms

(i) Let R be a commutative ring. The R-module Ω^1_R of *differential* 1*-forms over* R is defined as follows: Ω^1_R is generated by
  - The set $\{db \mid b \in R\}$ of generators.
  - The relations being the usual rules of differentiation: For all $b, c \in R$,
    (a) (Additivity) $d(b+c) = db + dc$,
    (b) (Leibniz rule) $d(bc) = cdb + bdc$.

(ii) For a commutative ring A and a commutative A-algebra B, the B-module $\Omega^1_{B|A}$ of *relative differential* 1*-forms over* A is defined to be the cokernel of the map $B \otimes_A \Omega^1_A \to \Omega^1_B$.

Definition 1.2. Logarithmic differential 1-forms

(i) For a valuation ring A with the field of fractions K, we define the *A-module* ω^1_A *of logarithmic differential* 1*-forms* as follows: ω^1_A is generated by
  - The set $\{db \mid b \in A\} \cup \{\operatorname{dlog} x \mid x \in K^\times\}$ of generators.
  - The relations being the usual rules of differentiation and an additional rule: For all $b, c \in A$ and for all $x, y \in K^\times$,
    (a) (Additivity) $d(b+c) = db + dc$,
    (b) (Leibniz rule) $d(bc) = cdb + bdc$,
    (c) (Log 1) $\operatorname{dlog}(xy) = \operatorname{dlog} x + \operatorname{dlog} y$,
    (d) (Log 2) $b \operatorname{dlog} b = db$ for all $0 \neq b \in A$.

(ii) Let $L|K$ be an extension of henselian valued fields, B the integral closure of A in L and hence, a valuation ring. We define the *B-module* $\omega^1_{B|A}$ *of logarithmic relative differential* 1*-forms over* A to be the cokernel of the map $B \otimes_A \omega^1_A \to \omega^1_B$.

Definition 1.3. The different ideal $\mathcal{D}_{B|A}$. Let A be a valuation ring with the field of fractions K. Let $L|K$ be a separable extension of fields, B the integral closure of A

in L. As in the classical case, we define the *inverse different* $\mathcal{D}_{B|A}^{-1}$ by $\mathcal{D}_{B|A}^{-1} := \{x \in L \mid \operatorname{Tr}_{L|K}(xB) \subset A\}$.

This is a fractional ideal of L. The *different* $\mathcal{D}_{B|A}$ *of* B *with respect to* A is defined as the inverse ideal of $\mathcal{D}_{B|A}^{-1}$.

1.2. Valuation rings and differential 1*-forms*

Definition 1.4. Let A be a valuation ring with fraction field K and valuation v. For any $x \in K^\times$, we define an A-module homomorphism $dx : M_x \to \omega_A^1$ by $h \mapsto hx\ \operatorname{dlog} x$ where $M_x := \left(\frac{1}{x}\right)$.
For $x = 0$, we define $d0$ to be the zero map: $M_0 \to \omega_A^1$ by $h \mapsto 0$ where $M_0 := K$.

Lemma 1.5. *Let A, K, v be as above and $x, y \in K$. Then we have the following properties.*

- (i) *(Additivity) The A-module homomorphisms $dx, dy, d(x+y) : M \to \omega_A^1$ satisfy $d(x+y) = dx + dy$. Here, $M = M_x \cap M_y \cap M_{x+y}$.*
- (ii) *(Leibniz rule) The A-module homomorphisms $dx, dy, d(xy) : M \to \omega_A^1$ satisfy $d(xy) = ydx + xdy$. Here, $M = M_x \cap M_y \cap M_{xy}$.*

Proof.

- (i) We may assume that $v(x) \le v(y)$ and write $y = ax;\ a \in A$. Note that in $\omega_A^1, da = a \operatorname{dlog} a$ and $d1 = \operatorname{dlog} 1 = 0$. Hence, $(a+1)\operatorname{dlog}(a+1) = d(a+1) = da = a \operatorname{dlog} a$. For all $h \in M$,

$$\begin{aligned} d(x+y)(h) &= h(x+y)\operatorname{dlog}(x+y) \\ &= hx(a+1)\operatorname{dlog}[x(a+1)] \\ &= hx(a+1)[\operatorname{dlog} x + \operatorname{dlog}(a+1)] \\ &= hx(a+1)\ \operatorname{dlog} x + hx(a+1)\ \operatorname{dlog}(a+1) \\ &= hx\ \operatorname{dlog} x + hxa\ \operatorname{dlog} x + hxa\ \operatorname{dlog} a \\ &= hx\ \operatorname{dlog} x + hxa\ \operatorname{dlog} xa \\ &= dx(h) + dy(h) \end{aligned}$$

- (ii) For all $h \in M$,

$$\begin{aligned} d(xy)(h) &= hxy\ \operatorname{dlog}(xy) \\ &= hxy\ \operatorname{dlog} x + hxy\ \operatorname{dlog} y \\ &= ydx(h) + xdy(h) \qquad \square \end{aligned}$$

Lemma 1.6. *Let $L|K$ be as in 0.1. Then we have*

(1) *A surjective B-module homomorphism* $\Phi_\sigma : \Omega^1_{B|A}/\mathcal{I}_\sigma\Omega^1_{B|A} \to \mathcal{I}_\sigma/\mathcal{I}_\sigma^2$ *such that* $\Phi_\sigma(db) = \sigma(b) - b$ *for all* $b \in B$.
(2) *A surjective B-module homomorphism* $\varphi_\sigma : \omega^1_{B|A}/\mathcal{J}_\sigma\omega^1_{B|A} \to \mathcal{J}_\sigma/\mathcal{J}_\sigma^2$ *such that* $\varphi_\sigma(\operatorname{dlog} x) = \frac{\sigma(x)}{x} - 1$ *for all* $x \in L^\times$.

Proof. Since σ fixes K, $\sigma(a) - a = 0$ for all $a \in A$ and $\frac{\sigma(x)}{x} - 1 = 0$ for all $x \in K^\times$. Let $b, c \in B$. The first part follows from $\sigma(b+c) - (b+c) = \sigma(b) - b + \sigma(c) - c$ and

$$\begin{aligned}\sigma(bc) - bc &= (\sigma(b) - b)(\sigma(c) - c) + c(\sigma(b) - b) + b(\sigma(c) - c)\\ &\equiv c(\sigma(b) - b) + b(\sigma(c) - c) \mod \mathcal{I}_\sigma^2.\end{aligned}$$

Let $x, y \in L^\times$. The second assertion follows from

$$\begin{aligned}\frac{\sigma(xy)}{xy} - 1 &= \left(\frac{\sigma(x)}{x} - 1\right)\left(\frac{\sigma(y)}{y} - 1\right) + \frac{\sigma(x)}{x} - 1 + \frac{\sigma(y)}{y} - 1\\ &\equiv \frac{\sigma(x)}{x} - 1 + \frac{\sigma(y)}{y} - 1 \mod \mathcal{J}_\sigma^2. \qquad \square\end{aligned}$$

1.3. Defect: introduction [1]

Definition 1.7. Let $E|F$ be a finite algebraic extension of fields of degree $[E:F] = n$ and v a non-trivial valuation on F. Denote the extensions of v from F to E by $v_1, ...v_g$. Let F_v be the residue field and $v(F^\times)$ the value group for the valued field (F, v). Similarly, define E_{v_i} and $v_i(E^\times)$. For each $1 \le i \le g$, define:

- The ramification index $e_i = (v_i(E^\times) : v(F^\times))$.
- The inertia degree $f_i = [E_{v_i} : F_v]$.

Fact I: For each $1 \le i \le g$, e_i and f_i are finite. Moreover, we have the **fundamental inequality**:

$$[E:F] = n \ge \sum_{i=1}^{g} e_i f_i \tag{1.8}$$

If the equality holds, it is called the **fundamental equality**.
Fact II: When (F, v) is henselian, $g = 1$ and we deal with a single ramification index $e_{E|F} = e$ and a single inertia degree $f_{E|F} = f$. Furthermore, in this case, n is divisible by the product ef and we can write

$$n = d_{E|F}e_{E|F}f_{E|F} \tag{1.9}$$

for some positive integer $d_{E|F}$.

Definition 1.10. The integer $d_{E|F}$ above is called the *defect* of the extension $(E|F, v)$. It is known that $d_{E|F}$ is a power of q; where $q = \max\{\text{char}(F_v), 1\}$.

1.4. Cyclic extensions of prime degree

Let $E|F$ be a cyclic degree p Galois extension of henselian valued fields, where $p = \text{char}\, F > 0$. Let $\mathcal{O}_E$ and $\mathcal{O}_F$ denote the valuation rings of E and F respectively. Let $\overline{E}$ and $\overline{F}$ be the respective residue fields.

Lemma 1.11. *If $E|F$ is ramified and defectless, then we have two cases:*

(*a*) *Order of $v(E^\times)/v(F^\times)$ is p and it is generated by $v_E(\mu)$ for some $\mu \in E^\times$.*
(*b*) *There is some $\mu \in E^\times$ such that the residue extension $\overline{E}|\overline{F}$ is purely inseparable of degree p, generated by the residue class of μ.*

Lemma 1.12. *Let $E|F, \mu$ be as in Lemma 1.11 and $x_i \in F$ for all $0 \le i \le p-1$. Then $\sum_{i=0}^{p-1} x_i\mu^i \in \mathcal{O}_E$ if and only if $x_i\mu^i \in \mathcal{O}_E$ for all i.*

Proof. If $x_i\mu^i \in \mathcal{O}_E$ for all i, then clearly, $\sum_{i=0}^{p-1} x_i\mu^i \in \mathcal{O}_E$. For the converse, we observe that if $v_E(x_i\mu^i)$ are all distinct for $0 \le i \le p-1$, then $0 \le v_E\left(\sum_{i=0}^{p-1} x_i\mu^i\right) = \min_{0\le i\le p-1} v_E(x_i\mu^i)$. Hence, the converse is true in this case. Now let us break down the rest into two cases (a) and (b) as described in the lemma above:

(a) We claim that in this case, $v_E(x_i\mu^i); 0 \le i \le p-1,\ x_i \neq 0$ all have to be distinct. Assume to the contrary. Let $0 \le i < j \le p-1$ be such that $v_E(x_i\mu^i) = v_E(x_j\mu^j)$; x_i, x_j are non-zero. Then $v_E(\mu^{j-i}) = (j-i)v_E(\mu) = v_E\left(\frac{x_i}{x_j}\right) \in v(F^\times)$. This is impossible, since the order of $v_E(\mu)$ in $v(E^\times)/v(F^\times)$ is p and $p \nmid j-i$.
(b) We observe that $v(\mu) = 0$. The only case we need to consider is when $\min_{0\le i\le p-1} v(x_i\mu^i) = v < 0$ and the minimum is achieved by more than one $x_i\mu^i$. Let $0 \le i_1 < \dots < i_r \le p-1; r \ge 2$ integer such that $v(x_{i_s}\mu^{i_s}) = v$ for all $1 \le s \le r$. Clearly, $v(x_{i_s}) = v$ for all $1 \le s \le r$. In particular, $x_{i_1} \neq 0$. Since $v\left(\sum_{s=1}^{r} x_{i_s}\mu^{i_s}\right) > v$, we see that $v\left(\sum_{s=1}^{r} \frac{x_{i_s}}{x_{i_1}}\mu^{i_s}\right) > 0$.

Equivalently, $z = \sum_{s=1}^{r} \frac{x_{i_s}}{x_{i_1}} \mu^{i_s} \in \mathfrak{m}_E$; where $\mathfrak{m}_E$ is the maximal ideal of $\mathcal{O}_E$.

Since $\overline{\mu}^i$'s are $\overline{F}$-linearly independent; $0 \le i \le p-1$, $z \in \mathfrak{m}_E \Leftrightarrow \overline{z} = 0 \in \overline{E}$ $\Leftrightarrow \overline{\left(\frac{x_{i_s}}{x_{i_1}}\right)} = 0 \in \overline{F}$ for all s. However, this is impossible since $v(x_{i_s}) = v$ for all $1 \le s \le r$. □

Lemma 1.13. *Let μ be as in Lemma 1.11. Then* $\operatorname{dlog}\mu$ *generates the $\mathcal{O}_E$-module $\omega^1_{\mathcal{O}_E|\mathcal{O}_F}$.*

Proof. It is enough to consider the elements $\operatorname{dlog}(x\mu^i); 0 \le i \le p-1,\ x \in K^\times$. $\operatorname{dlog}(x\mu^i) = \operatorname{dlog} x + i\ \operatorname{dlog}\mu$. The rest follows from the fact that $\operatorname{dlog} x = 0$ in $\omega^1_{\mathcal{O}_E|\mathcal{O}_F}$. □

1.5. Trace

Lemma 1.14. *Let R be an integrally closed integral domain with the field of fractions F. Let $E|F$ be a separable extension of fields of degree n. Suppose that $\beta \in E$ is such that $E = F(\beta)$. Let $g(T) = \min_F(\beta)$, the minimal polynomial of β over F. Then*

(1) $\operatorname{Tr}_{E|F}\left(\frac{\beta^m}{g'(\beta)}\right)$ *is zero for all* $1 \le m \le n-2$ *and* $\operatorname{Tr}_{E|F}\left(\frac{\beta^{n-1}}{g'(\beta)}\right) = 1$.

(2) *Assume, in addition, that β is integral over R. Then* $\{x \in E \mid \operatorname{Tr}_{E|F}(xR[\beta]) \subset R\} = \frac{1}{g'(\beta)} R[\beta]$.

Details can be found in section 6.3 of [4].

2. Discrete valuation rings

2.1. Classical theory: complete discrete valuation rings with perfect residue fields

Let K be a complete discrete valued field of residue characteristic $p > 0$ with normalized valuation v_K, valuation ring A and perfect residue field k. Consider $L|K$, a finite Galois extension of K. Let $e_{L|K}$ be the ramification index of $L|K$ and $G = \operatorname{Gal}(L|K)$. Let v_L be the valuation on L that extends v_K, B the integral closure of A in L and l the residue field of L. In this case, we have the following invariants of ramification theory:

- The Lefschetz number $i(\sigma)$ and the logarithmic Lefschetz number $j(\sigma)$ for $\sigma \in G\backslash\{1\}$ are defined as

$$i(\sigma) = \min\{v_L(\sigma(a) - a) \mid a \in B\} \tag{2.1}$$

$$j(\sigma) = \min\{v_L(\frac{\sigma(a)}{a} - 1) \mid a \in L^\times\} \tag{2.2}$$

Both the numbers are non-negative integers.

- For a finite dimensional representation ρ of G over a field of characteristic zero, the Artin conductor $\mathrm{Art}(\rho)$ and the Swan conductor $\mathrm{Sw}(\rho)$ are defined as

$$\mathrm{Art}(\rho) = \frac{1}{e_{L|K}} \sum_{\sigma \in G \setminus \{1\}} i(\sigma)(\dim(\rho) - \mathrm{Tr}(\rho(\sigma))) \tag{2.3}$$

$$\mathrm{Sw}(\rho) = \frac{1}{e_{L|K}} \sum_{\sigma \in G \setminus \{1\}} j(\sigma)(\dim(\rho) - \mathrm{Tr}(\rho(\sigma))) \tag{2.4}$$

  Both these conductors are integers. This is a consequence of the Hasse–Arf Theorem (see [3]).

The invariants $j(\sigma)$ and $\mathrm{Sw}(\rho)$ are the parts of $i(\sigma)$ and $\mathrm{Art}(\rho)$, respectively, which handle the wild ramification. We wish to generalize these to all valuation rings considered in this paper. Namely, the case where L is a non-trivial Artin–Schreier extension of K, a valued field with henselian valuation ring, defined by $\alpha^p - \alpha = f$, where $f \in K$. Let us begin with the case of discrete valuation rings, possibly with imperfect residue fields.

2.2. Best f and Swan conductor

Let K be a complete discrete valued field of residue characteristic $p > 0$ with normalized valuation v_K, valuation ring A and residue field k. We do not assume that k is perfect. Let $L = K(\alpha)$ be the (non-trivial) Artin–Schreier extension defined by $\alpha^p - \alpha = f$, where $f \in K$. Let v_L, B and l denote the valuation, valuation ring and the residue field of L, respectively. We define the Swan conductor of this extension as described below.

Definition 2.5. Let $\mathfrak{P} : K \to K$ denote the additive homomorphism $x \mapsto x^p - x$. Note that the extension L does not change when f is replaced by any element $g \in K$ such that $g \equiv f \mod \mathfrak{P}(K)$. Because, if $g = f + h^p - h$ for some $h \in K$, then the corresponding Artin–Schreier extension is generated by $\alpha + h$ over K.

(1) If there is such $g \in A$, L is unramified over K and the Swan conductor is defined to be 0.
(2) If there is no such $g \in A$, the Swan conductor is defined to be $\min\{-v_K(g) \mid g \equiv f \mod \mathfrak{P}(K)\}$. An element f of K which attains this minimum will be referred to as "best f" throughout this paper. It is well-defined modulo $\mathfrak{P}(K)$.

This definition coincides with the classical definition of the Swan conductor when k is perfect.

Existence of best f relies on the existence of $\min\{-v_K(g) \mid g \equiv f \mod \mathfrak{P}(K)\}$. This is guaranteed in the case of discrete valuation rings, but not in the case of general valuation rings.

Example 2.6. Let $K = k((t))$ where k is of characteristic $p > 0$. t is a prime element of K. Let n be a positive integer coprime to p. In this case, the Swan conductor of the extension given by $\alpha^p - \alpha = \frac{1}{t^n}$ is n.

More generally, let $m \geq 0$ be an integer and n as above. Then the Swan conductor of the extension given by $\alpha^p - \alpha = \frac{1}{t^{np^m}}$ is also n. This follows from $\frac{1}{t^{np^m}} \equiv \frac{1}{t^n} \bmod \mathfrak{P}(K)$.

A concrete description of the Swan conductor is given by the following lemma:

Lemma 2.7. *By replacing f with an element of $\{g \in K \mid g \equiv f \mod \mathfrak{P}(K)\}$, we have best f which satisfies exactly one of the following properties:*

(*i*) $f \in A$.
(*ii*) $v_K(f) = -n$ *where n is a positive integer relatively prime to p.*
(*iii*) $f = at^{-n}$ *where $n > 0$, $p|n$, t is a prime element of K and $a \in A^\times$ such that the residue class of a in k does not belong to $k^p = \{x^p \mid x \in k\}$.*

In the case (i), the Swan conductor is 0. In the cases (ii) and (iii), the Swan conductor is n.

2.3. Refined Swan conductor rsw

Definition 2.8. Let K be a discrete valued field of residue characteristic $p > 0$ with normalized valuation v_K, valuation ring A and residue field k (possibly imperfect). Let $L = K(\alpha)$ be the Artin–Schreier extension defined by $\alpha^p - \alpha = f$ where f is best. The *refined Swan conductor (*rsw*)* of this extension is defined to be the A-homomorphism $df : \left(\frac{1}{f}\right) \to \omega_A^1$ given by $h \mapsto (hf)\ \mathrm{dlog}\, f$. We note that for $h \in \left(\frac{1}{f}\right)$, $hf \in A$ and hence, $(hf)\ \mathrm{dlog}\, f$ is indeed an element of ω_A^1.

The A-homomorphism rsw is well-defined up to certain relations, as discussed below.

Lemma 2.9. *Let $L|K$ be as above, given by best f, $H = \left(\frac{1}{f}\right)$. Then* rsw *is well-defined as the A-homomorphism : $H \to \omega_A^1/\mathbb{I}\omega_A^1$; where $\mathbb{I}$ is the ideal $\{x \in K \mid v_K(x) \geq \left(\frac{p-1}{p}\right) v_K\left(\frac{1}{f}\right)\}$ of A.*

Proof. Let g be best as well. Hence, there exists $a \in K$ such that $g = f + a^p - a$ and $v_K(f + a^p - a) = v_K(f)$. Since $v_K(a) \geq v_K(f), H \cap M_a = H$. By Lemma 1.5, $dg - df = -da$ on H.

For $h = \frac{b}{f} \in H; b \in A, da(h) = ha\ \ \mathrm{dlog}\, a = b\left(\frac{a}{f}\right)\mathrm{dlog}\, a \in \left(\frac{a}{f}\right)\omega_A^1$. It is enough to show that $v_K\left(\frac{a}{f}\right) \geq \left(\frac{p-1}{p}\right) v_K\left(\frac{1}{f}\right)$. This is clear in the case $a \in A$.

If $a \in K \backslash A$, then $v_K(a^p - a) = pv_K(a) \geq v_K(f) = v_K(f + a^p - a)$. Hence, proved. □

Remark 2.10. We note that $\mathbb{I} = \{x \in K \mid v_K(x) \geq \left(\frac{p-1}{p}\right) v_K\left(\frac{1}{f}\right)\} = \{x \in K \mid v_L(x) \geq \left(\frac{p-1}{p}\right) v_L\left(\frac{1}{f}\right)\}$.

3. Small results

In this section, we present some small results that help us understand the two special cases I and II better. First we extend the notion of "best f" to the general case.

3.1. Best f

Definition 3.1. Let K be as in 0.1, $\mathfrak{P} : K \to K$ as before. We say that $f \in K^\times$ is best if either $f \in A^\times$ or if f satisfies $-v(f) = \inf\{-v(g) \mid g \equiv f \mod \mathfrak{P}(K)\}$.

Since we cannot guarantee the existence of best f in general, as seen in the example below, we will reinterpret the notion of the refined Swan conductor using the logarithmic differential 1-forms over A, as stated in Theorem 0.5.

Example 3.2. (Non-DVR)
Consider the extension $L|K$ as described in Appendix A. The value group Γ is isomorphic to $\mathbb{Z}[\frac{1}{p}]$. We have a sequence of elements $f_i \in \mathfrak{A}$ for all integers $i \geq 0$, each better than the previous one, such that

(1) $-v(f_i) = n - \displaystyle\sum_{j=1}^{i} \frac{1}{p^j}$
(2) The ideal H of A is generated by $\{\frac{1}{f_i} \mid i \geq 0\}$.

Since $\displaystyle\inf_{i \geq 0} -v(f_i) = n - \frac{1}{p-1} = c \in \mathbb{R}\backslash\Gamma$, there is no best f.

Corollary 3.3. *Let K be as in 0.1 and $L|K$ satisfy (I), given by $\alpha^p - \alpha = f$ where f is best. Then*

(*i*) *B is described as follows:*
  (a) *If $e_{L|K} = p, B = \displaystyle\sum_{i=0}^{p-1} A_i\alpha^i$ where $A_0 := A$ and for all $1 \leq i \leq p-1$,*
  $A_i := \{x \in K \mid v_L(x) \geq -iv_L(\alpha)\} = \{x \in A \mid v_L(x) > -iv_L(\alpha)\}$.
  (b) *If $e_{L|K} = 1, B = A[\alpha\gamma]$ where $\gamma \in A$ such that $\alpha\gamma \in B^\times$.*
(*ii*) $\operatorname{dlog}\alpha$ *generates the B-module $\omega^1_{B|A}$.*

Proof.

(i) We apply Lemma 1.12

(a) $-v_0 := v_L(\alpha)$ generates the group $v_L(L^\times)/v_L(K^\times)$ of order p. In particular, for all $1 \leq i \leq p-1$, $iv_0 \notin v_L(K^\times)$. For $x \in K^\times$, $x\alpha^i \in B$ if and only if $v_L(x) \geq iv_0$ if and only if $v_L(x) > iv_0$.

(b) Since $e = 1$, there exists $\gamma \in A$ such that $\alpha\gamma \in B^\times$. We just take $\mu = \alpha\gamma$.

(ii) This is a direct consequence of (i) and Lemma 1.13. □

3.2. Fractional ideals in a valued field

Let F be a valued field with valuation v, value group Γ, valuation ring $\mathcal{O} := \mathcal{O}_F$ and residue field $\overline{F}$. A subset S of F is a *fractional ideal* of F if there exists $0 \neq b \in \mathcal{O}$ such that bS is an (integral) ideal of $\mathcal{O}$.

We note that in such a case, $S = \{x \in F \mid v(x) \geq v(s) \text{ for some } s \in S\} = \cup_{s\in S}\, s\mathcal{O}$.

Definition 3.4. Consider the case (II), we can regard Γ as an ordered subgroup of $\mathbb{R}$. Let S be a fractional ideal of F and $\inf_{s\in S} v(s) = t \in \mathbb{R}$. We define *$F$-valuation of S* as follows:

(i) If $t \in \Gamma \subset \mathbb{R}, v(S) := t$.

(ii) If $t \in \mathbb{R}\backslash\Gamma, v(S) := t^+$.

We can define the *F-valuation of S* by (i) when S is generated by a single element $s \in F$, even if Γ is not isomorphic to an ordered subgroup of $\mathbb{R}$. In that case, $v(S) := v(s)$ and $S = s'\mathcal{O}$ for any $s' \in F$ such that $v(s') = v(s)$.

3.3. Defect and $\mathcal{J}_\sigma$

Lemma 3.5. *The fractional ideals $\mathcal{J}_\sigma$ and H are integral ideals of L and K respectively, that is,*

(i) $\mathcal{J}_\sigma = \left(\{\frac{\sigma(b)}{b} - 1 \mid b \in L^\times\}\right) \subset B.$

(ii) $H = \left(\{\frac{1}{f} \mid f \in \mathfrak{A}\}\right) \subset A.$

Proof.

(i) For $b \in L^\times$, $v_L(\sigma(b) - b) \geq \min\{v_L(\sigma(b)), v_L(b)\} = v_L(\sigma(b)) = v_L(b)$. Hence, $\frac{\sigma(b)}{b} - 1 \in B$.

(ii) We need to show that for each $f \in \mathfrak{A}, \dfrac{1}{f} \in A$. Assume to the contrary that there is some $f \in \mathfrak{m}_K \cap \mathfrak{A}$. Since K is henselian, roots of $\alpha^p - \alpha = f$ are already in K, contradicting our assumption that $L|K$ is non-trivial. □

In Lemma 3.6, we define the A-linear maps D_i which will be used in the proof of Proposition 3.10.

Lemma 3.6. *Let $L|K$ be as in 0.1 and $b \in B$ such that $\sigma(b) - b$ generates $\mathcal{I}_\sigma$. Define A-linear maps $D_i : L \to L$ inductively for $0 \le i \le p-1$ by*

$$D_0 := id_L : L \to L,\ D_i(x) := \frac{(\sigma-1)(D_{i-1}(x))}{(\sigma-1)(D_{i-1}(b^i))}\ ; 1 \le i \le p-1 \tag{3.7}$$

These maps have the following properties:

(1) $D_i(b^i) = 1;\ 0 \le i \le p-1.$
(2) $D_i(b^j) = 0;\ 0 \le j \le i-1,\ 1 \le i \le p-1.$
(3) *For $x \in B$, $D_i(xb) = \sigma^i(b)D_i(x) + D_{i-1}(x);\ 0 \le i \le p-1$. (If $i = 0$, we set $D_{i-1}(x) = 0$.)*
(4) $D_i(b^{i+1}) = \sum_{j=0}^{i} \sigma^j(b);\quad 0 \le i \le p-1.$
(5) *For each $0 \le i \le p-2$, $(\sigma-1)(D_i(b^{i+1})) = \sigma^{i+1}(b) - b$ and hence, is a generator of $\mathcal{I}_\sigma$.*
In particular, it is non-zero.

Proof. First we note that $(\sigma-1)(D_0(b^1)) = (\sigma-1)(b) \neq 0$ and hence, the definition of D_1 is valid. As we prove (1)–(5) by induction on i, validity of the definition of D_i for $1 \le i \le p-1$ will become clear.

(1) follows directly from the definition. (2) is clearly true for $i = 1$, since $D_1(1) = 0$. If $0 \le j \le i-1 \le p-2$, $(\sigma-1)(D_{i-1}(b^j)) = (\sigma-1)(1)$ or $(\sigma-1)(0)$ and hence, $D_i(b^j) = 0$. The $i = 0$ case of (3)–(5) follows directly from the definition.
For $i = 1$, (3)–(5) follow from

$$\begin{aligned} D_1(xb) &= \frac{(\sigma-1)(D_0(xb))}{(\sigma-1)(D_0(b))} = \frac{(\sigma-1)(xb)}{(\sigma-1)(b)} \\ &= \frac{(\sigma-1)(x)\sigma(b) + x(\sigma-1)(b)}{(\sigma-1)(b)} \\ &= \sigma(b)D_1(x) + x = \sigma(b)D_1(x) + D_0(x) \end{aligned}$$

Let $2 \le i \le p-1$ and assume that (3)–(5) are true for $0, ..., i-2, i-1$. Then we have:

$$\begin{aligned} D_i(xb) &= \frac{(\sigma-1)(D_{i-1}(xb))}{(\sigma-1)(D_{i-1}(b^i))} \\ &= \frac{(\sigma-1)(\sigma^{i-1}(b)D_{i-1}(x) + D_{i-2}(x))}{\sigma^i(b) - b} \qquad \text{by (3).} \\ &= \frac{(\sigma-1)(D_{i-1}(x)).\sigma^i(b) + (\sigma-1)(\sigma^{i-1}(b)).D_{i-1}(x) + (\sigma-1)(D_{i-2}(x))}{\sigma^i(b) - b} \\ &= \sigma^i(b)D_i(x) + \frac{(\sigma-1)(\sigma^{i-1}(b)).D_{i-1}(x) + (\sigma-1)(D_{i-2}(x))}{\sigma^i(b) - b} \end{aligned}$$

by (3.7) and (5).

$$= \sigma^i(b)D_i(x) + \frac{(\sigma-1)(\sigma^{i-1}(b)).D_{i-1}(x) + (\sigma-1)(D_{i-2}(b^{i-1})).D_{i-1}(x)}{\sigma^i(b)-b}$$

by (3.7) for $i-1$.

$$= \sigma^i(b)D_i(x) + D_{i-1}(x)\frac{(\sigma-1)(\sigma^{i-1}(b)) + \sigma^{i-1}(b) - b}{\sigma^i(b)-b} \text{ by (5) for } i-2.$$

$$= \sigma^i(b)D_i(x) + D_{i-1}(x)$$

This proves (3) for i. (4) follows from (3). For any fixed $0 \le i \le p-2, (\sigma-1)(D_i(b^{i+1}))$ has the same valuation as $(\sigma-1)(b)$ and hence generates $\mathcal{I}_\sigma$. □

Corollary 3.8. *For all $0 \le i \le p-1, D_i(B)$ is a subset of B.*

Proof. This is clearly true for $i = 0$. We proceed by induction. Fix some $1 \le i \le p-1$ and assume that the statement is true for $i-1$. Hence, for all $x \in B, D_{i-1}(x) \in B \Rightarrow (\sigma-1)(D_{i-1}(x)) \in \mathcal{I}_\sigma$. By Lemma 3.6(5), $D_i(x) = \dfrac{(\sigma-1)(D_{i-1}(x))}{(\sigma-1)(D_{i-1}(b^i))} \in B$. □

Lemma 3.9. *If $L|K$ is as in 0.1 and has defect, then*

(*i*) $\left(\sigma(b)-b \mid b \in B^\times\right) = \mathcal{I}_\sigma = \mathcal{J}_\sigma = \left(\dfrac{\sigma(b)}{b} - 1 \mid b \in B^\times\right)$.

(*ii*) $\Omega^1_{B|A} = \omega^1_{B|A}$.

Proof. Given any $b \in L$, there are elements $a \in K,\ b' \in B^\times$ such that $b = ab'$.

(i) For $b \in B, a \in A$ and $\sigma(b) - b = a(\sigma(b') - b')$.
For $b \in L^\times, a \in K^\times$ and $\frac{\sigma(b)}{b} - 1 = \frac{a\sigma(b')}{ab'} - 1 = \frac{\sigma(b')}{b'} - 1$.
Furthermore, $b' \in B^\times \Rightarrow \frac{\sigma(b')}{b'} - 1 = \frac{1}{b'}(\sigma(b') - b') \in \mathcal{I}_\sigma$.

(ii) Let $b \in L^\times$. $\operatorname{dlog} b = \operatorname{dlog} a + \operatorname{dlog} b' = \operatorname{dlog} b'$ since $\operatorname{dlog} a = 0$ in $\omega^1_{B|A}$.
$b' \in B^\times \Rightarrow \operatorname{dlog} b' = \frac{1}{b'}db' \in \Omega^1_{B|A}$. □

Proposition 3.10. *Let $L|K$ be as in 0.1. $\mathcal{J}_\sigma$ is principal if and only if $L|K$ is defectless.*

Proof. If the extension is defectless, by Lemma 1.11, Lemma 1.13 and Lemma 1.6(a) $\mathcal{J}_\sigma$ is principal. Now suppose that the extension is with defect and that $\mathcal{J}_\sigma$ is principal. Hence, by Lemma 3.9 $\mathcal{J}_\sigma = \mathcal{I}_\sigma$. Let $b \in B$ such that $\sigma(b) - b$ generates $\mathcal{I}_\sigma$.
We claim that $B = A[b]$.

Consider $x_i \in K; 0 \le i \le p-1$ such that $y = \displaystyle\sum_{i=0}^{p-1} x_i b^i \in B$. We must show that $x_i \in A$; for all i.

Define $y_i := \sum_{j=0}^{p-i} x_j b^j; 1 \le i \le p$. We show that $y_i \in B$ and consequently, $D_{p-i}(y_i) = x_{p-i} \in B$ by Corollary 3.8. Clearly, $y_1 = y \in B$. Assume that $y_i \in B$ for some $1 \le i \le p$.

Since $x_{p-i}, b \in B$, $y_{i+1} = y_i - x_{p-i}b^{p-i} \in B$. Thus, $x_i \in B \cap K = A$; for all i and hence, $B = A[b]$.

Since the extension is with defect, $f = 1$ and $b = a + b'$ for some $a \in A$ and for some $b' \in m_L$. Therefore, we may assume $b \in m_L$. Also, due to the defect, $e = 1$ and $b = ab'$ for some $a \in m_K$ and for some unit b' of B. $\sigma(b) - b = a(\sigma(b') - b')$. $\sigma(b') - b' = c(\sigma(b) - b)$ for some $c \in B$. Hence, $ac = 1$. This is impossible since $a \in m_K$. Thus, the extension must be defectless if $\mathcal{J}_\sigma$ is principal. □

4. Proof of Theorem 0.3

We prove that $H = \mathcal{N}_\sigma$.

Let $f \in \mathfrak{A}$. Then $(-1)^p N(\alpha) = -f$. Equivalently, $\frac{1}{f} = N(\frac{1}{\alpha}) = N(\frac{\sigma(\alpha)}{\alpha} - 1)$. From this, it follows that H is a subset of $\mathcal{N}_\sigma$, without any assumptions regarding defect or the value group Γ_K. Next, we prove the reverse inclusion $\mathcal{N}_\sigma \subset H$. If $L|K$ is defectless, this follows directly from results in section 3 (see Proposition 3.10 and Corollary 3.3(i)). Because, H is generated by $\frac{1}{f}$, where f is best. Since $\mathcal{J}_\sigma = \left(\frac{1}{\alpha}\right)$, $\mathcal{N}_\sigma = \left(N(\frac{1}{\alpha})\right) = H$. Proof in the defect case, however, requires some work.

Let $L|K$ satisfy (II) and have defect. The value group $\Gamma = \Gamma_K$ can be regarded as an ordered subgroup of $\mathbb{R}$. Let v denote the valuation on L and also on K. We analyze a special case first.

4.1. Case $p = 2$

For any $x \in L$, $\sigma(\sigma(x) - x) = x - \sigma(x) = \sigma(x) - x$, since the characteristic is 2. Hence, $\sigma(x) - x = \sigma(x) + x = \mathrm{Tr}_{L|K}(x) \in K$; for all $x \in L$. For any fixed $x \in L$, let $y = \sigma(x) - x$. $\sigma(\frac{x}{y}) - \frac{x}{y} = 1$ if y is non-zero, that is, if x does not belong to K. Let $z = \frac{x}{y}$. $z + \sigma(z) = 2z + 1 = 1$ and $N(z) = z(z+1) = z^2 + z = z^2 - z = f \in K$. Thus, $\frac{x}{y}$ is a solution of an Artin–Schreier extension $\alpha^2 - \alpha = f; f \in K$. All Artin–Schreier extensions over K having solution in L are obtained in this way.

Any generator of $\mathcal{J}_\sigma$ has the form $\frac{\sigma(x)-x}{x}$. Letting $\frac{1}{f} = N(\frac{\sigma(x)-x}{x})$ we get the corresponding Artin–Schreier extension.

Remark 4.1. We don't need Γ to be an ordered subgroup of $\mathbb{R}$ for this case, the argument is true for any value group.

4.2. Case $p > 2$

We wish to show $\mathcal{N}_\sigma \subset H$, equivalently, for each $\beta \in L^\times \backslash K^\times$ the ideal of A generated by $N(\frac{\sigma(\beta)}{\beta} - 1)$ is a subset of H.
Let us begin with some elementary observations:

(O1) **We may assume** $\beta \in B \backslash A$**:** $\frac{\sigma(1/\beta)}{1/\beta} - 1 = (\frac{\sigma(\beta)}{\beta} - 1)(-\frac{\beta}{\sigma(\beta)})$. Since $(-\frac{\beta}{\sigma(\beta)}) \in B^\times$, norms of elements $\frac{\sigma(1/\beta)}{1/\beta} - 1$ and $\frac{\sigma(\beta)}{\beta} - 1$ generate the same ideal of A.

(O2) **Trace and** $(\sigma - 1)$**:**
We have the formal expression $(\sigma-1)^{p-1} = \frac{(\sigma-1)^p}{\sigma-1} = \frac{\sigma^p-1}{\sigma-1} = \sigma^{p-1} + \sigma^{p-2} + ... + \sigma + 1$.
Thus, for any $x \in L$, $(\sigma-1)^{p-1}(x) = \mathrm{Tr}_{L|K}(x)$.

(O3) **Reduction:** If we can find an element $x_\beta = x \in L\backslash K$ satisfying an Artin–Schreier equation over K and such that $v(\frac{(\sigma-1)(x)}{x}) \leq v(\frac{\sigma(\beta)}{\beta} - 1)$, then we have:
$0 \leq v(N(\frac{(\sigma-1)(x)}{x})) = t_1 \leq v(N(\frac{\sigma(\beta)}{\beta} - 1)) = t_2$
After this, it is sufficient to show that the ideal of A generated by $N(\frac{\sigma(x)}{x} - 1)$ is a subset of H.

(O4) $\sigma - 1$ **and changes in valuation:** Let $b \in L^\times$.
- $\frac{\sigma(b)-b}{b} \in B \Rightarrow v(\sigma(b) - b) = v(b) + s_b$ for some $s_b \geq 0$.
- For $1 \leq i \leq p-1$,
$v(\sigma^i(b) - b) = v(\sum_{1\leq j\leq i} \sigma^j(b) - \sigma^{j-1}(b)) \geq \min_{1\leq j\leq i}\{v(\sigma^j(b) - \sigma^{j-1}(b))\} = v(\sigma(b) - b)$
- By the same argument, applied to $\tau = \sigma^i$, ($\sigma = \tau^m$ for some $1 \leq m \leq p-1$) we have
$v(\sigma(b) - b) \geq v(\sigma^i(b) - b)$ and thus, the following equality:
For all $1 \leq i \leq p-1$,
$v(\sigma^i(b) - b) = v(b) + s_b$.

Proof of Theorem 0.3. For given β as above, we will now construct the special element x_β [see (O3)] and prove that the ideal of A generated by $N(\frac{\sigma(\beta)}{\beta} - 1)$ is indeed a subset of H. Let $g(T) = \min_K(\beta)$ and $x_\beta = x := (\sigma-1)^{p-2}(\frac{\beta^{p-1}}{g'(\beta)})$.

Put $y = \sigma(x) - x = (\sigma-1)(x)$. By (O2) and Lemma 1.14, $y = \mathrm{Tr}_{L|K}(\frac{\beta^{p-1}}{g'(\beta)}) = 1$. As in the case $p = 2$, $y \neq 0$ and we have $\sigma(\frac{x}{y}) - \frac{x}{y} = (\sigma-1)(x) = 1$.

Observe that $x = (\sigma-1)^{p-2}\left(\frac{\beta^{p-1}}{g'(\beta)}\right) \in L\backslash K$ satisfies $\sigma(x) = x + 1$ and hence, the Artin–Schreier equation $\alpha^p - \alpha = N(x)$. Thus, we have

$$\frac{1}{N(x)} = N\left(1/(\sigma-1)^{p-2}(\frac{\beta^{p-1}}{g'(\beta)})\right) \in H. \tag{4.2}$$

Now we need to relate the principal ideals generated by $N(\frac{(\sigma-1)(x)}{x})$ and $N(\frac{\sigma(\beta)}{\beta} - 1)$. For this, we look at the L-valuation of these elements. Let $v(\frac{(\sigma-1)(x)}{x}) = s' \geq 0$ and $v(\frac{(\sigma-1)(\beta)}{\beta}) = s \geq 0$.

If $s' \leq s$, then $N(\frac{\sigma(\beta)}{\beta} - 1) \in H$ and hence, $\left(N(\frac{\sigma(\beta)}{\beta} - 1)\right) = N(\frac{\sigma(\beta)}{\beta} - 1)A \subset H$.

Now suppose that $s' > s$. Put $r = \frac{\beta^{p-1}}{g'(\beta)}$. Then $g'(\beta) = \prod_{1\leq i\leq p-1}(\beta - \sigma^i(\beta))$. Hence, by (O4),

$$v(r) = -(p-1)s \tag{4.3}$$

For $1 \leq i \leq p-1$, let $v((\sigma-1)^i(r)) = v((\sigma-1)^{i-1}(r))+c_i$; $c_i \geq 0$. $c_{p-1} = s'$ by definition. Since $v((\sigma-1)^{p-1}(r)) = v(1) = 0$, from (4.3), we see that

$$\sum_{i=1}^{p-1} c_i = -v(r) = (p-1)s \tag{4.4}$$

Let $c := \inf\{v(\frac{\sigma(b)}{b} - 1) \mid b \in L^\times\} = \inf\{s_b \mid b \in L^\times\} \in \mathbb{R}$, where s_b is as described in (O4).

We observe $(p-1)s = \sum_{i=1}^{p-2} c_i + s' \geq (p-2)c + s' > (p-2)c + s \geq (p-1)c \geq 0$.

In particular, $(p-2)(s-c) \geq s' - s > 0$. By the definition of c, we can take s very close to c such that $s' \leq s$ for this new s.

This concludes the proof. $\square$

Corollary 4.5. *Under the assumptions of Theorem 0.3, the following statements are equivalent:*

(1) *Best f exists.*
(2) *H is a principal ideal of A.*
(3) *$\mathcal{J}_\sigma$ is a principal ideal of B.*
(4) *$L|K$ is defectless.*

5. Filtered union in the defect case

To generalize the results to the defect case, we write the ring B as a filtered union of rings $A[x]$, where the elements x are chosen very carefully. Although these are not valuation rings, each ring is generated by a single element (over A). This makes the extensions $K(x)|K$ and the corresponding differential modules, special ideals easier to understand. We will use Theorem 0.3 to prove these results.

Theorem 5.1. *Consider $\mathscr{S} = \{\alpha \in L \mid \alpha^p - \alpha = f; f \in K$, and α generates $L|K\}$. For each $\alpha \in \mathscr{S}$, we can find $\alpha' \in B^\times \cap \alpha K^\times$ such that $B = \cup_{\alpha\in\mathscr{S}} A[\alpha']$ is a filtered union, that is, the following are true:*

(i) *For any $\alpha_1, \alpha_2 \in \mathscr{S}$, either $A[\alpha_1'] \subset A[\alpha_2']$ or $A[\alpha_2'] \subset A[\alpha_1']$.*
(ii) *Given any $\beta \in B$, there exists $\alpha \in \mathscr{S}$ such that $\beta \in A[\alpha']$.*

5.1. $p = 2$

First we consider the filtered union in the $p = 2$ case, as given by the result below.

Proposition 5.2. *For $p = 2$, $B = \cup_{\alpha\in L\setminus K} A[\frac{\mathrm{Tr}(\alpha)}{\alpha}]$ is a filtered union.*

Proof. We are dealing with the defect case, so $v_L = v_K = v$. Let $\alpha_1, \alpha_2 \in L \backslash K$. $\frac{\mathrm{Tr}(\alpha_i)}{\alpha_i} = \beta_i \in B$. $\sigma(\frac{\alpha_i}{\mathrm{Tr}(\alpha_i)}) = \frac{\sigma(\alpha_i)}{\mathrm{Tr}(\alpha_i)} = \frac{\alpha_i}{\mathrm{Tr}(\alpha_i)} + 1$ since $p = 2$. We have $(\frac{\alpha_i}{\mathrm{Tr}(\alpha_i)})^2 - \frac{\alpha_i}{\mathrm{Tr}(\alpha_i)} = \frac{1}{c_i}$; $c_i \in A$.
$\sigma(\frac{\alpha_1}{\mathrm{Tr}(\alpha_1)} - \frac{\alpha_2}{\mathrm{Tr}(\alpha_2)}) = \frac{\alpha_1}{\mathrm{Tr}(\alpha_1)} + 1 - \frac{\alpha_2}{\mathrm{Tr}(\alpha_2)} - 1 = \frac{\alpha_1}{\mathrm{Tr}(\alpha_1)} - \frac{\alpha_2}{\mathrm{Tr}(\alpha_2)}$
Therefore, $\frac{\alpha_1}{\mathrm{Tr}(\alpha_1)} - \frac{\alpha_2}{\mathrm{Tr}(\alpha_2)} = \frac{1}{\beta_1} - \frac{1}{\beta_2} = g \in K$ and $\frac{1}{c_1} = \frac{1}{c_2} + g^2 - g$. We note that $(\frac{1}{\beta_i})^2 - \frac{1}{\beta_i} = \frac{1}{c_i}$, that is, $\beta_i = \frac{c_i}{\beta_i} - c_i$. We will prove the following two statements:

1. If $v(c_1) > v(c_2)$, then $A[\beta_1]$ is a subset of $A[\beta_2]$.
2. If $v(c_1) = v(c_2)$, then $A[\beta_1] = A[\beta_2]$.

In (1), it is enough to show that $\beta_1 \in A[\beta_2]$. Since $\frac{c_2}{\beta_2} = \beta_2 + c_2$, it is an element of $A[\beta_2]$. Consequently, $\frac{c_1}{\beta_2} = \frac{c_2}{\beta_2}\frac{c_1}{c_2} \in A[\beta_2]$.
Claim. $\beta_1 \in A[\beta_2] \Leftrightarrow \dfrac{c_1}{\beta_1} \in A[\beta_2] \Leftrightarrow \dfrac{c_1}{\beta_2} \in A[\beta_2]$.

Proof of Claim. This can be shown by following steps:

- $\beta_1 = \frac{c_1}{\beta_1} - c_1$, $c_1 \in A$.
- $\frac{c_1}{\beta_1} = c_1(\frac{1}{\beta_2} + g) = \frac{c_1}{\beta_2} + c_1 g$.
  Now $-v(c_1) = v(\frac{1}{c_1}) = v(\frac{1}{c_2} + g^2 - g) < -v(c_2) = v(\frac{1}{c_2}) \leq 0$
  $\Rightarrow -v(c_1) = v(\frac{1}{c_2} + g^2 - g) = v(g^2 - g) = 2v(g)$ (the last equality follows from $0 > -v(c_1)$)
  $\Rightarrow v(c_1 g) = -2v(g) + v(g) = -v(g) > 0$
  $\Rightarrow c_1 g \in A$ (since $c_1 g$ is already in K).

The proof of (2) is very similar to the proof of (1). We just need to show that $v(c_1 g) \geq 0$. If $v(g) \geq 0$, this is clearly true. Let $v(g) < 0, v(c_1) = v(c_2) = v \geq 0$. Since $v(\frac{1}{c_2} + g^2 - g) = v(\frac{1}{c_2}), 2v(g) = v(g^2 - g) \geq v(\frac{1}{c_2}) = -v$. Hence, $v(c_1 g) \geq v - \frac{v}{2} = \frac{v}{2} \geq 0$. □

Remark 5.3. This particular construction in the case $p = 2$ doesn't appear to have an easy generalization to the case $p > 2$. We use a different approach.

5.2. Some elementary results for $p > 2$

Due to the defect, given any $\alpha \in \mathscr{S}$ there exists $\gamma_\alpha = \gamma \in A$ such that $v(\gamma) = -v(\alpha) = -\frac{1}{p}v(f)$. Define $\alpha' = \alpha\gamma \in B^\times$. We claim that this choice of α' satisfies the conditions of Theorem 5.1. We note that the ring $A[\alpha']$ does not depend on the choice of γ.

Lemma 5.4. *If $\alpha_1, \alpha_2 \in \mathscr{S}$ such that $v(\alpha_1) \leq v(\alpha_2)$, then $A[\alpha_1'] \subset A[\alpha_2']$.*

Proof. We have (by choosing appropriate conjugates) $\sigma(\alpha_2-\alpha_1) = (\alpha_2+1)-(\alpha_1+1) = \alpha_2 - \alpha_1$. Hence, $\alpha_2 - \alpha_1 =: h \in K$.

$v(\alpha_1) \leq v(\alpha_2) \Rightarrow v(\gamma_1) \geq v(\gamma_2)$ and $v(h) \geq v(\alpha_1) = -v(\gamma_1)$. Therefore, $\frac{\gamma_1}{\gamma_2}, \gamma_1 h \in A$. Consequently, $\alpha'_1 = \gamma_1(\alpha_2 - h) = \frac{\gamma_1}{\gamma_2}\alpha'_2 - \gamma_1 h \in A[\alpha'_2]$. $\square$

Lemma 5.5. *Given any $\beta \in B$, there exists $\alpha \in \mathscr{S}$ such that $(\sigma(\beta) - \beta) \subset (\sigma(\alpha') - \alpha')$.*

Proof. Let $v := v(\sigma(\beta) - \beta)$, $v_0 := \inf_{b \in B^\times} v(\sigma(b) - b) \in \mathbb{R}$. Hence, $\mathcal{I}_\sigma = \mathcal{J}_\sigma = \{x \in L^\times \mid v(x) > v_0\}$ and $\mathcal{N}_\sigma = \{x \in K^\times \mid v(x) > pv_0\}$. Since this is the defect case, by Proposition 3.10, $\mathcal{I}_\sigma$ is not a principal ideal. We need to show that $v > c$, where $c \in \mathbb{R}$ is defined by
$c := \inf_{\alpha \in \mathscr{S}} v(\sigma(\alpha') - \alpha') = \inf_{\alpha \in \mathscr{S}} v(\gamma_\alpha) = \inf_{\alpha \in \mathscr{S}} -v(\alpha) = \inf_{f \in \mathfrak{A}} -\frac{1}{p} v(f)$.
Note that $H = \{x \in K^\times \mid v(x) > pc\}$. By Theorem 0.3, $H = \mathcal{N}_\sigma$ and hence, $c = v_0$. To conclude the proof, we observe that $\sigma(\beta) - \beta \in \mathcal{I}_\sigma \Rightarrow v > v_0 = c$. $\square$

Lemma 5.6. *For $x, y \in L$, we have $(\sigma-1)^n(xy) = \sum_{k=0}^{n} \binom{n}{k} (\sigma-1)^{n-k}(x)(\sigma-1)^k(\sigma^{n-k}(y))$ In particular, for $n = 1$, $(\sigma - 1)(xy) = (\sigma - 1)(x)\sigma(y) + x(\sigma - 1)(y)$.*

Proof. This can be proved by using induction on n and the binomial identity $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$. $\square$

5.3. Filtered union for $p > 2$

Proposition 5.7. *Given any $\beta \in B^\times$, there exists $\alpha \in \mathscr{S}$ such that $(\sigma - 1)^{p-1}(\frac{1}{F'(\alpha')} A[\alpha', \beta]) \subset B$. Here, F denotes the minimal polynomial of α' over K.*

Proof. We compute valuation of these elements and show that it is non-negative.

For all $\alpha \in \mathscr{S}, 1 \leq k \leq p-1$, $\sigma^k(\alpha') - \alpha' = (\sigma^k(\alpha) - \alpha)\gamma = k\gamma$. Therefore, $F'(\alpha') = -\gamma^{p-1}$. In particular, it is an element of K and hence, fixed by σ.

We wish to select α such that for all $i, j \geq 0$,

$$v((\sigma - 1)^{p-1}(\alpha'^{\,i}\beta^j)) \geq v(F'(\alpha')) = (p-1)v(\gamma) \tag{5.8}$$

(Step 1) **Construction of the special α'**

We begin with an α_0 satisfying $(\sigma(\beta) - \beta) \subset (\sigma(\alpha'_0) - \alpha'_0)$. Let $(\sigma - 1)(\beta) = b_1\gamma_0; b_1 \in B$. Therefore, $(\sigma - 1)^2(\beta) = (\sigma - 1)(b_1)\gamma_0$. We don't know much about the valuation of $(\sigma-1)(b_1)$, however. Let α_1 be such that $((\sigma - 1)(b_1)) \subset ((\sigma - 1)(\alpha'_1))$. Write $(\sigma-1)(b_1) = b_2\gamma_1; b_2 \in B$. Now we can write $(\sigma-1)^2(\beta) = b_2\gamma_1\gamma_0$. Using this process, we can find b_i's and α_i's such that $(\sigma - 1)^i(\beta) = b_i\gamma_{i-1}...\gamma_1\gamma_0$; where $b_i \in B$.

Let γ be the γ_j with smallest valuation involved in the expression for $i = p-1$. Let α denote the corresponding α_j. We will show that this α satisfies the required property (5.8).

(Step 2) **Proof for** β

$(\sigma(\beta) - \beta) \subset (\sigma(\alpha'_0) - \alpha'_0) \subset (\sigma(\alpha') - \alpha') = (\gamma)$, since $v(\gamma) \leq v(\gamma_0)$. Due to the choice of γ, we also have $v((\sigma - 1)^t(\beta)) \geq tv(\gamma)$ for all $1 \leq t \leq p-1$. In particular, this is true for $t = p-1$, proving the statement (5.8) for the case $i = 0, j = 1$.

(Step 3) **Terms** $\alpha'^{\,i}\beta^j$

For the terms of the form β^j, we use induction on j and Lemma 5.6. Valuation of each term in the expansion is at least $(p-1)v(\gamma)$. In fact, by a similar argument, $v((\sigma - 1)^k(\beta^j)) \geq kv(\gamma)$ for all $1 \leq k \leq p-1$.

For the general terms $\alpha'^{\,i}\beta^j$, first note that $(\sigma - 1)^k(\alpha') = (\sigma - 1)^{k-1}(\gamma) = 0$ for all $k > 1$. Therefore, (again using the identity) we have

$(\sigma - 1)^{p-1}(\alpha'^{\,i}\beta^j) = \alpha'^{\,i}(\sigma - 1)^{p-1}(\beta^j) + (p-1)(\sigma - 1)(\alpha'^{\,i})(\sigma - 1)^{p-2}(\sigma(\beta^j))$.

Once again, both these terms have valuation $\geq (p-1)v(\gamma)$.

This concludes the proof of the proposition. □

5.4. Proof of Theorem 5.1

Let β and corresponding special α' be as described above in (Step 1). We recall that for an A-module $R \subset L$, $R^* := \{x \in L \mid \mathrm{Tr}_{L|K}(xR) \subset A\}$.

(1) $A[\alpha', \beta]^* = A[\alpha']^*$

Proof. Clearly, $A[\alpha', \beta]^* \subset A[\alpha']^* = \frac{1}{F'(\alpha')}A[\alpha']$. We proved that $(\sigma - 1)^{p-1}(\frac{1}{F'(\alpha')}A[\alpha', \beta]) \subset B$. Since $(\sigma - 1)^{p-1} = \mathrm{Tr}_{L|K}$ has image in K, $\mathrm{Tr}_{L|K}(\frac{1}{F'(\alpha')}A[\alpha', \beta]) \subset B \cap K = A$ and we have the reverse inclusion. □

(2) $R := A[\alpha', \beta], S := A[\alpha']$ are finitely generated free A-modules.

Proof. Since β, α' are integral over A, R and S are finitely generated A-modules. A is a valuation ring and R, S are finitely generated torsion-free A-modules. Therefore, R, S are free A-modules (of finite ranks). □

(3) $A[\alpha', \beta] = A[\alpha']$

Proof. R is a free A-module of finite rank. Hence, $R^{**} = (R^*)^* = R$. Similarly, $S^{**} = S$. By (1), $R^* = S^*$ and hence, $R = S$. □

These statements, in combination with Proposition 5.7 prove part (ii) of Theorem 5.1. Part (i) was already proved in Lemma 5.4. This concludes the proof.

6. Proof of Theorem 0.5

Lemma 6.1. $N_{L|K} = N : B \to A/(\mathcal{I}_\sigma \cap A)$ *is a surjective ring homomorphism.*

Proof. We just need to check the additive property of $N : B \to A/(\mathcal{I}_\sigma \cap A)$ in order to prove that it is a ring homomorphism. For $x \in B, N(x) = x \prod_{i=1}^{p-1} \sigma^i(x)$.

For each $1 \le i \le p-1, \sigma^i(x) \equiv x \mod \mathcal{I}_\sigma$.

Thus, $N : B \to B/\mathcal{I}_\sigma$ is just the p-power map, that is, $x \mapsto x^p \mod \mathcal{I}_\sigma$ and hence, additive. This makes $N : B \to A/(\mathcal{I}_\sigma \cap A)$ additive as well. $\square$

Remark 6.2. We don't need any assumptions regarding defect or rank here.

6.1. Case I: relation between the ideals $H, \mathbb{I}, \mathcal{I}_\sigma, \mathcal{J}_\sigma$

Notation 6.3. Case I is the defectless case, so best f exists and we can define the ideal $\mathbb{I}$ of A by

$\mathbb{I} := \left(\{\frac{a}{f} \in K \mid v_K(f + a^p - a) = v_K(f)\}\right)$. It is worth noting that this definition coincides with the one in Lemma 2.9. Let $v_L(\alpha) = -v_0 \le 0$. Hence, $v_L(f) = -pv_0$, $H = \{x \in K \mid v_L(x) \ge pv_0\}$, $\mathbb{I} = \{x \in K \mid v_L(x) \ge (p-1)v_0\}$ and $\mathcal{J}_\sigma = \{x \in L \mid v_L(x) \ge v_0\}$.

Proposition 6.4. *$H \subset \mathbb{I} \subset \mathcal{I}_\sigma \cap A$.*

Proof. Comparing valuations mentioned above, it is clear that $H \subset \mathbb{I}$.

We break down the rest of the argument into several cases:

- If $e_{L|K} = 1, \mathcal{I}_\sigma = \mathcal{J}_\sigma = \{x \in L \mid v_L(x) \ge v_0\}$ and the result follows.
- Let $e_{L|K} = p$.
  - (i) $p > 2$
    $\frac{1}{\alpha} \in B \Rightarrow \sigma\left(\frac{1}{\alpha}\right) - \frac{1}{\alpha} = \frac{-1}{\alpha(\alpha+1)} \in \mathcal{I}_\sigma$.
    Hence, $\{x \in L \mid v_L(x) \ge 2v_0\} \subset \mathcal{I}_\sigma$. Since $p > 2,\ p-1 \ge 2$ and hence, $\mathbb{I} \subset \mathcal{I}_\sigma \cap A$.
    We cannot use this argument for $p = 2$, since in that case, $p - 1 = 1 < 2$.
  - (ii) $p = 2$
    Let $\frac{a}{f} \in K$ such that $v_K(f + a^2 - a) = v_K(f)$. Consider $b = \frac{a\alpha}{f}$. Then
    $v_L(b) = v_L(\alpha) + v_L(\frac{a}{f}) \ge -v_0 + v_0 = 0 \Rightarrow b \in B \Rightarrow \sigma(b) - b \in \mathcal{I}_\sigma$.
    $\sigma(b) - b = \sigma(b) + b = \mathrm{Tr}(b) = \frac{a}{f}$
    $\mathrm{Tr}(\alpha) = \frac{a}{f}$ since
    $\mathrm{Tr}(\alpha) = 1$.
    Hence, $\frac{a}{f} \in \mathcal{I}_\sigma \cap K = \mathcal{I}_\sigma \cap A$. This concludes the proof. $\square$

6.2. Case I

Let f be best, $b \in B$. We prove that the following diagram commutes:

$$\begin{array}{ccc} \omega^1_{B|A}/\mathcal{J}_\sigma\omega^1_{B|A} & \xrightarrow[\cong]{\varphi_\sigma} & \mathcal{J}_\sigma/\mathcal{J}_\sigma^2 \\ \overline{\Delta_N}\big\downarrow & & \big\downarrow\overline{N} \\ \omega^1_A/(\mathcal{I}_\sigma \cap A)\omega^1_A & \xleftarrow[\text{rsw}]{} & H/H^2 \end{array} \quad \text{where the maps are given by} \quad \begin{array}{ccc} b\,\mathrm{dlog}\,\alpha & \xrightarrow{\varphi_\sigma} & b\frac{1}{\alpha} \\ \overline{\Delta_N}\big\downarrow & & \big\downarrow\overline{N} \\ N(b)\,\mathrm{dlog}\,f & \xleftarrow[\text{rsw}]{} & N(b)\frac{1}{f} \end{array}.$$

Proof. Consider the map $\varphi_\sigma : \omega^1_{B|A}/\mathcal{J}_\sigma\omega^1_{B|A} \to \mathcal{J}_\sigma/\mathcal{J}_\sigma^2$. By Lemma 1.6, we know that φ_σ is a surjective B-module homomorphism. We prove that it is injective.

Since $\omega^1_{B|A}$ is generated by $\mathrm{dlog}\,\alpha$, it is enough to consider elements of the form $b\,\mathrm{dlog}\,\alpha$; where $b \in B$. $b\,\mathrm{dlog}\,\alpha \in \mathit{Ker}(\varphi_\sigma) \Leftrightarrow b(\frac{\sigma(\alpha)}{\alpha} - 1) = b\frac{1}{\alpha} \in \mathcal{J}_\sigma^2 \Leftrightarrow b \in \mathcal{J}_\sigma \Leftrightarrow b\,\mathrm{dlog}\,\alpha \in \mathcal{J}_\sigma\omega^1_{B|A}$. Therefore, φ is a B-module isomorphism.

Next, we note that H is generated by $\frac{1}{f}$ and $N(\alpha) = f$. By Lemma 6.1, we have additivity of the two vertical maps. Since $H \subset \mathbb{I} \subset \mathcal{I}_\sigma \cap A$, the map rsw is independent of the choice of best f. $\square$

6.3. Preparation for Case II

6.3.1. Valuation on A and B

Fix some $\alpha_0 \in \mathscr{S}$ as our starting point. We may only consider $\alpha \in \mathscr{S}$ such that $v(\alpha_0) < v(\alpha)$. Consider the subset $\mathscr{S}_0$ of $\mathscr{S}$ consisting of such α's. Let $v(\alpha_0) = -\mu < 0, \gamma_0 \in A$ such that $v(\gamma_0) = \mu$. For each $\alpha \in \mathscr{S}_0$, we have corresponding $\gamma_\alpha \in A$ with $v(\gamma_\alpha) = -v(\alpha) < v(\gamma_0) = \mu$ and $\alpha' = \alpha\gamma_\alpha \in B^\times$. Let F_α denote the minimal polynomial of α' over K. We recall that $F'_\alpha(\alpha') = -\gamma_\alpha^{p-1}$ and hence, we have the isomorphism $\Omega^1_{A[\alpha']|A} \cong A[\alpha']/(\gamma_\alpha^{p-1})$ described in 6.3.3.
Let $f_\alpha := \alpha^p - \alpha = N(\alpha) \in K$.

6.3.2. Special ideals

Due to the defect, we have $\mathcal{I}_\sigma = \mathcal{J}_\sigma$ by Lemma 3.9.
Let $v_0 := \inf\{v(\frac{\sigma(b)}{b} - 1) \mid b \in B^\times\} \in \mathbb{R}$. Then

(a) $\mathcal{I}_\sigma = \mathcal{J}_\sigma = \{b \in B \mid v(b) > v_0\}$, and consequently, by Theorem 0.3,
(b) $\mathcal{N}_\sigma = \{a \in A \mid v(a) > pv_0\} = H$.

We have $\inf\{v(\frac{\sigma(b)}{b} - 1) \mid b \in B^\times\} = \inf\{v(\sigma(b) - b) \mid b \in B^\times\} = \inf\{v(\sigma(\alpha') - \alpha') \mid \alpha \in \mathscr{S}_0\} \in \mathbb{R}$. The last equality follows from Lemma 5.5. Therefore,

$$v_0 = \inf\{v(\gamma_\alpha) \mid \alpha \in \mathscr{S}_0\} \in \mathbb{R} \tag{6.5}$$

6.3.3. Differential modules $\Omega^1_{A[\alpha']|A}$'s

We compare $\Omega^1_{A[\alpha_0']|A}$ and $\Omega^1_{A[\alpha']|A}$. Let $c_\alpha := \gamma_\alpha^{p-1}$, $c_0 := \gamma_0^{p-1}$ and the ratio $\gamma_0/\gamma_\alpha =: a_\alpha \in A$. Then we have the following commutative diagram:

$$\begin{tikzcd}
\Omega^1_{A[\alpha_0']|A} \arrow[r, "\cong"] \arrow[d, hook, "\rho_\alpha"] & A[\alpha_0']/(c_0) \arrow[r, "\cong"] \arrow[d, hook, "\iota_\alpha"] & (\frac{1}{a_0})A[\alpha_0']/(\frac{c_0}{a_0})A[\alpha_0'] \arrow[d, hook, "j_\alpha"] \\
\Omega^1_{A[\alpha']|A} \arrow[r, "\cong"] & A[\alpha']/(c_\alpha) \arrow[r, "\cong"] & (\frac{1}{a_\alpha})A[\alpha']/(\frac{c_\alpha}{a_\alpha})A[\alpha']
\end{tikzcd}$$

Here, $a_0 = \gamma_0/\gamma_0 = 1 \in A$ and the isomorphisms are given by $b_0 d\alpha_0' \mapsto b_0 \mapsto \dfrac{b_0}{a_0}$; for all $b_0 \in A[\alpha_0']$ and $bd\alpha' \mapsto b \mapsto \frac{b}{a_\alpha}$; for all $b \in A[\alpha']$. The vertical maps are described as follows. We look at the relationship between the generators α_0', α' and similarly, between $d\alpha_0'$, $d\alpha'$. Since α and α_0 give rise to the same extension $L|K$, $\alpha_0 - \alpha =: h \in K$. Comparing the valuations, we see that $v(\alpha_0) = v(h) < v(\alpha)$ and hence, $u = h\gamma_0 \in A^\times$.

$$\alpha_0' = (\alpha + h)\gamma_0 = (\alpha + h)\gamma_\alpha \cdot\ a_\alpha = a_\alpha\alpha' + u \tag{6.6}$$

Since $\alpha' \in B$ and $a_\alpha, u \in A$, $\alpha' da_\alpha = 0 = du$ in the differential module $\Omega^1_{A[\alpha']|A}$. Therefore, we have

$$d\alpha_0' = a_\alpha d\alpha' + \alpha' da_\alpha + du = a_\alpha d\alpha' \tag{6.7}$$

Thus, ρ_α, ι_α are given by multiplication by a_α. The map j_α is also multiplication by a_α and rises from the inclusions

$$(\frac{1}{a_0})A[\alpha_0'] \subset (\frac{1}{a_\alpha})A[\alpha'];\ \frac{1}{a_0} \mapsto \frac{1}{a_0}a_\alpha = \frac{1}{a_\alpha}a_\alpha^2 \tag{6.8}$$

and

$$(\frac{c_0}{a_0})A[\alpha_0'] \subset (\frac{c_\alpha}{a_\alpha})A[\alpha'];\ \frac{c_0}{a_0} \mapsto \frac{c_0}{a_0}a_\alpha = \frac{c_\alpha}{a_\alpha}a_\alpha^p \tag{6.9}$$

Lemma 6.10. *Consider the fractional ideals Θ and Θ' of L given by $\Theta = \{x \in L \mid v(x) > v_0 - \mu\}$ and $\Theta' = \{x \in L \mid v(x) > pv_0 - \mu\}$. Then we have:*

(a) $\Omega^1_{B|A} \cong \Theta/\Theta'$,
(b) $\Theta/\mathcal{J}_\sigma\Theta \cong \mathcal{J}_\sigma/\mathcal{J}_\sigma^2$.

Proof.

(a) Let I be the fractional ideal of L generated by the elements $(\frac{1}{a_\alpha})$. Let I' be the fractional ideal of L generated by the elements $(\frac{c_\alpha}{a_\alpha})$. Under the isomorphisms described

in the preceding discussion, we can identify each $\Omega^1_{A[\alpha']|A}$ with $(\frac{1}{a_\alpha})A[\alpha']/(\frac{c_\alpha}{a_\alpha})A[\alpha']$. Taking limit over α's, we can identify $\Omega^1_{B|A}$ with I/I'.
Since $-v(a_\alpha) = v(\gamma_\alpha) - v(\gamma_0) = v(\gamma_\alpha) - \mu$, $I = \{x \in L \mid v(x) > \inf_\alpha v(\gamma_\alpha) - \mu\} = \Theta$.
Similarly, $v(c_\alpha) = (p-1)v(\gamma_\alpha) \Rightarrow v(\frac{c_\alpha}{a_\alpha}) = pv(\gamma_\alpha) - \mu \Rightarrow I' = \Theta'$.

(b) This follows from the fact that $\Theta \cong \mathcal{J}_\sigma$ as B-modules, via the map $\times\gamma_0 : x \mapsto x\gamma_0$. □

6.4. Proof of Theorem 0.5 in Case II

Due to the defect, we consider $\Omega^1_{B|A}$ and Ω^1_A instead:

$$\begin{tikzcd} \Omega^1_{B|A}/\mathcal{J}_\sigma\Omega^1_{B|A} \arrow[r, "\varphi_\sigma", "\cong"'] \arrow[d, "\overline{\Delta_N}"'] & \mathcal{J}_\sigma/\mathcal{J}_\sigma^2 \arrow[d, "\int \overline{N}"] \\ \Omega^1_A/(\mathcal{I}_\sigma \cap A)\Omega^1_A & H/H^2 \arrow[l, "\mathrm{rsw}"'] \end{tikzcd}$$

As discussed in Lemma 6.10, we can write $\Omega^1_{B|A} = \varinjlim_{\alpha\in\mathscr{S}_0} \Omega^1_{A[\alpha']|A}$ and it is enough to consider the diagram for each $\alpha \in \mathscr{S}_0$:

$$\begin{tikzcd} \Omega^1_{A[\alpha']|A}/(\frac{1}{\alpha})A[\alpha']\Omega^1_{A[\alpha']|A} \arrow[r, "\varphi_\sigma", "\cong"'] \arrow[d, "\overline{\Delta_N}"'] & (\frac{1}{\alpha})A[\alpha']/(\frac{1}{\alpha})^2A[\alpha'] \arrow[d, "\int \overline{N}"] \\ \Omega^1_A/(\mathcal{I}_\sigma \cap A)\Omega^1_A & (\frac{1}{f_\alpha})A/(\frac{1}{f_\alpha})^2A \arrow[l, "\mathrm{rsw}"'] \end{tikzcd} \tag{6.11}$$

where the maps are given by

$$\begin{tikzcd} bd\alpha' \arrow[r, "\varphi_\sigma"] \arrow[d, "\overline{\Delta_N}"'] & b\alpha'\frac{1}{\alpha} \arrow[d, "\overline{N}"] \\ N(b\alpha')\,\mathrm{dlog}\, f_\alpha & N(b\alpha')\frac{1}{f_\alpha} \arrow[l, "\mathrm{rsw}"'] \end{tikzcd}$$

We note that in $\omega^1_{B|A}$, $\mathrm{dlog}\,\alpha = \mathrm{dlog}\,\alpha' + \mathrm{dlog}\,\gamma_\alpha = \mathrm{dlog}\,\alpha' = \frac{d\alpha'}{\alpha'}$ and $\frac{\sigma(\alpha')}{\alpha'} - 1 = \frac{1}{\alpha}$. At each α-level, we observe the following:

(i) The map $\varphi_\sigma : \Omega^1_{A[\alpha']|A}/(\frac{1}{\alpha})\Omega^1_{A[\alpha']|A} \to (\frac{1}{\alpha})/(\frac{1}{\alpha})^2$ is same as the one obtained from Lemma 6.10.

Proof. By Lemma 6.10, $\Omega^1_{A[\alpha']|A}/(\frac{1}{\alpha})\Omega^1_{A[\alpha']|A} \cong (\frac{1}{a_\alpha})/(\frac{1}{\alpha})(\frac{1}{a_\alpha}) \cong (\frac{1}{\alpha})/(\frac{1}{\alpha})^2$ under the composition $d\alpha' \mapsto \frac{1}{a_\alpha} \mapsto \gamma_0\frac{1}{a_\alpha} = \gamma_\alpha = \frac{\alpha'}{\alpha}$.
On the other hand, $\varphi_\sigma(d\alpha') = \alpha'\left(\frac{\sigma(\alpha')}{\alpha'} - 1\right) = \frac{\alpha'}{\alpha}$. □

(ii) The map rsw is well-defined.

Proof. Define the ideal $\mathbb{I}_\alpha$ of A by $\mathbb{I}_\alpha := \Big(\{\frac{a}{f_\alpha} \in K \mid v_K(f_\alpha + a^p - a) = v_K(f_\alpha)\}\Big)$. As in case (I), we have $(\frac{1}{f_\alpha})A \subset \mathbb{I}_\alpha \subset (\frac{1}{\alpha})A[\alpha'] \cap A$. Since $(\frac{1}{\alpha})A[\alpha'] \cap A \subset \mathcal{J}_\sigma \cap A = \mathcal{I}_\sigma \cap A$, the map rsw is well-defined. □

7. The different ideal $\mathcal{D}_{B|A}$

7.1. Basic properties

We recall that $\mathcal{D}_{B|A}^{-1} := \{x \in L \mid \mathrm{Tr}_{L|K}(xB) \subset A\} = B^*$ and the different ideal $\mathcal{D}_{B|A}$ is defined to be its inverse ideal.

Lemma 7.1. *Let $\mu \in B \backslash A, L = K(\mu)$, and $F(T) \in K[T]$ the minimal polynomial of μ over K, then $A[\mu]^* = \dfrac{1}{F'(\mu)} A[\mu]$.*

Proof. See Lemma 6.76 of [4]. □

Now we describe the different ideal $\mathcal{D}_{B|A}$ in the cases I and II. We will assume that the extension $L|K$ is ramified. Consider the following three sub-cases:

- **Case (i):** $e_{L|K} = 1, f_{L|K} = p$.
  Let v denote both v_L and v_K. Assume that $L|K$ is generated by $\alpha^p - \alpha = f$ where f is best. There exists $\gamma \in A$ such that $\alpha' := \alpha\gamma \in B^\times$ and $l|k$ is purely inseparable, generated by the residue class of α'. Let $v(\alpha) = -v_0$. Hence, $v(f) = -pv_0, v(\gamma) = v_0$. Since $f\gamma^p \in A^\times, F(T) = T^p - T\gamma^{p-1} - f\gamma^p$ is the minimal polynomial of α' over A. Therefore, $F'(T) = pT^{p-1} - (p-1)\gamma^{p-1} = \gamma^{p-1}$. By Lemma 1.11, Lemma 1.13, $B = A[\alpha']$ and hence, $\mathcal{D}_{B|A}^{-1} = B^* = A[\alpha']^* = \frac{1}{F'(\alpha')} A[\alpha']$ is clearly a fractional ideal of L, generated by a single element $\frac{1}{F'(\alpha')}$.
- **Case (ii):** $e_{L|K} = p, f_{L|K} = 1$.
  Let f be best, $v_L(\alpha) = -v_0$. Recall that $B = \sum_{i=0}^{p-1} A_i\alpha^i; A_0 := A$, for all $1 \le i \le p-1$,
  $A_i := \{x \in K \mid v(x) \ge iv_0\} = \{x \in A \mid v(x) > iv_0\}$. Let $y \in L$. Then for all $0 \le i \le p-1$,

$$y = \sum_{j=0}^{p-1} y_j\alpha^j \in \mathcal{D}_{B|A}^{-1}; y_j \in K \Leftrightarrow \mathrm{Tr}_{L|K}(y\alpha^i A_i) \subset A \tag{7.2}$$

α has the minimal polynomial $F(T) = T^p - T - f$. Hence, $F'(\alpha) = -1$.
For $1 \le i \le p-1, \alpha^{i+(p-1)} = \alpha^i + f\alpha^{i-1}$. By Lemma 1.14, we have

$$\mathrm{Tr}_{L|K}(\alpha^i) = \begin{cases} 0; & 0 \le i \le p-2 \\ -1; & i = p-1, 2(p-1) \\ 0; & p \le i \le 2(p-1)-1 \end{cases}$$

Let $x_i \in A_i$. Then

$$\mathrm{Tr}(x_i y \alpha^i) = \mathrm{Tr}(\sum_{j=0}^{p-1} x_i y_j \alpha^{i+j}) = \begin{cases} -x_0 y_{p-1}; & i = 0 \\ -x_i y_{p-1-i}; & 1 \le i \le p-2 \\ -x_{p-1} y_0 - x_{p-1} y_{p-1}; & i = p-1 \end{cases}$$

Hence, $y \in \mathcal{D}^{-1}_{B|A}$ if and only if $A_0 y_{p-1}, A_{p-1}(y_0 + y_{p-1}), A_i y_{p-1-i} \subset A$ (for all $1 \le i \le p-2$).

- **Case (iii):** Rank 1 and $e_{L|K} = 1, f_{L|K} = 1$
  Let $\Gamma \subset \mathbb{R}$ and let v denote both v_L, v_K. By Theorem 5.1, we can write $B = \cup_{\alpha \in \mathscr{S}} A[\alpha']$, where $\alpha' = \alpha\gamma_\alpha \in B^\times$, $\gamma_\alpha \in A$. Recall that $v_0 := \inf_{\alpha \in \mathscr{S}} v(\gamma_\alpha) \in \mathbb{R}\backslash\Gamma$. By an argument similar to Case (i) above, we have $\mathcal{D}^{-1}_{A[\alpha']|A} = \{x \in L \mid v(x) \ge (p-1)v(\alpha) = -(p-1)v(\gamma_\alpha)\}$.
  Since all the $A[\alpha']$'s and B have the same fraction field $L, \mathcal{D}^{-1}_{B|A} \subset \mathcal{D}^{-1}_{A[\alpha']|A}$ for all $\alpha \in \mathscr{S}$. Hence, $\gamma_\alpha^{p-1}\mathcal{D}^{-1}_{B|A} \subset \gamma_\alpha^{p-1}\mathcal{D}^{-1}_{A[\alpha']|A} \subset A[\alpha'] \subset B$ and $\mathcal{D}^{-1}_{B|A}$ is a fractional ideal of L described by

$$\begin{aligned} \mathcal{D}^{-1}_{B|A} &= \cap_{\alpha \in \mathscr{S}} \mathcal{D}^{-1}_{A[\alpha']|A} \\ &= \cap_{\alpha \in \mathscr{S}} \{x \in L \mid v(x) \ge (p-1)v(\alpha)\} \\ &= \{x \in L \mid v(x) \ge (p-1)v(\alpha)\ \forall\ \alpha \in \mathscr{S}\} \\ &= \{x \in L \mid v(x) \ge -(p-1)v_0\} \end{aligned}$$

7.2. Results in the case $e_{L|K} = 1$

Let $L|K$ satisfy (I) or (II) and assume further that $e_{L|K} = 1$.

Lemma 7.3. $\{x \in L \mid \mathrm{Tr}_{L|K}(xB) \subset H\} = \mathcal{J}_\sigma$.

Proof. Since $e_{L|K} = 1$, given any $x \in L$, there are elements $x' \in B^\times, a \in K$ such that $x = x'a$. Hence, $\mathrm{Tr}(xB) = a\,\mathrm{Tr}(x'B) = a\,\mathrm{Tr}(B)$.

- **Case (i):** We note that $\mathrm{Tr}(\frac{1}{\alpha}) = \frac{-1}{f}$. Hence, $\mathcal{J}_\sigma = \left(\frac{1}{\alpha}\right) B \subset \{x \in L \mid \mathrm{Tr}(xB) \subset H\}$. Conversely, suppose that $\mathrm{Tr}(xB) \subset H = \left(\frac{1}{f}\right) A$. In particular, $a\,\mathrm{Tr}\left(\frac{1}{\alpha'}\right) = a\,\mathrm{Tr}\left(\frac{1}{\alpha\gamma}\right) = \frac{a}{\gamma}\,\mathrm{Tr}\left(\frac{1}{\alpha}\right) = \frac{a}{\gamma}\left(\frac{-1}{f}\right) \in H$. Hence, $\frac{a}{\gamma} \in A \Rightarrow a\alpha \in B \Rightarrow a \in \mathcal{J}_\sigma$.
- **Case (iii):** The argument is very similar to the case (i). Again, $\mathcal{J}_\sigma \subset \{x \in L \mid \mathrm{Tr}(xB) \subset H\}$. Conversely, suppose that $\mathrm{Tr}(xB) \subset H$. Hence, for all $\alpha \in \mathscr{S}$,

$\frac{a}{\gamma_\alpha}\left(\frac{-1}{f_\alpha}\right) \in H$
$\Rightarrow v(a) - v(\gamma_\alpha) - v(f_\alpha) > pv_0$
$\Rightarrow v(a) > (p-1)(v_0 - v(\gamma_\alpha)) + v_0$.
Since this is true for all $\alpha \in \mathscr{S}$, we have $v(a) \geq v_0$.
But $v_0 \notin \Gamma \Rightarrow v(a) > v_0 \Rightarrow a \in \mathcal{J}_\sigma$. $\square$

Lemma 7.4. *Consider the rank 1 case, i.e., case (II). For an ideal I of A and $a \in K$, $aI \subset I$ if and only if $a \in A$.*

Corollary 7.5. *In particular, if $L|K$ satisfies (II) and $e_{L|K} = 1$, then $\{x \in L \mid \mathrm{Tr}(x\mathcal{J}_\sigma) \subset H\} = B$.*

Proof. By Lemma 7.3, $\{x \in L \mid \mathrm{Tr}(x\mathcal{J}_\sigma) \subset H\} = \{x \in L \mid x\mathcal{J}_\sigma \subset \mathcal{J}_\sigma\}$ and hence, clearly contains B. The reverse inclusion follows from Lemma 7.4. $\square$

Proposition 7.6. *In the cases (i) and (iii), $\mathcal{D}_{B|A}^{-1}$ is described by:*

- ***Case (i):*** $\mathcal{D}_{B|A}^{-1} = \mathcal{J}_\sigma^{1-p}$ *and*
- ***Case (iii):*** $\mathcal{D}_{B|A}^{-1} = \{x \in L \mid xBH \subset \mathcal{J}_\sigma\}$.

Proof. Since $e_{L|K} = 1, \mathcal{I}_\sigma = \mathcal{J}_\sigma$.

- **Case (i):** $v(F'(\mu)) = (p-1)v(\gamma) = (p-1)v_0 \Rightarrow \mathcal{D}_{B|A}^{-1} = \{x \in L \mid v(x) \geq -(p-1)v_0\}$. The rest follows from $\mathcal{J}_\sigma = \mathcal{I}_\sigma = \left(\frac{1}{\alpha}\right) B$.
- **Case (iii):** By Lemma 7.4, $\mathrm{Tr}(xB) \subset A$ if and only if $\mathrm{Tr}(xB)H \subset H$. By Lemma 7.3, $\mathrm{Tr}(xB)H \subset H$ if and only $xBH \subset \mathcal{J}_\sigma$. $\square$

7.3. Results in the case $e_{L|K} = p$

We study the case (ii) in this section.

7.3.1. Preparation

Lemma 7.7. *Let S be a fractional ideal of L and $\alpha \in L^\times$ such that $v_L(\alpha)$ generates $v_L(L^\times)/v_L(K^\times)$. Then for $y = \sum_{j=0}^{p-1} y_j\alpha^j;\ y_j \in K, y \in S$ if and only if $y_i\alpha^i \in S$ for all $0 \leq i \leq p-1$.*

Proof. Since $e_{L|K} = p, v_L(y_i\alpha^i); y_i \neq 0$ are all distinct. If $y \in S$, then for some $s \in S$, we have $v_L(y) = \min\limits_{0 \leq i \leq p-1} v_L(y_i\alpha^i) \geq v_L(s)$. Thus, $v_L(y_i\alpha^i) \geq v_L(s)$ for all $0 \leq i \leq p-1$ and hence, $y_i\alpha^i \in S$ for all $0 \leq i \leq p-1$. The converse is clearly true. $\square$

Two important applications of the lemma are below.

- Consider $S = \mathcal{D}_{B|A}^{-1}, y \in L$. $y \in \mathcal{D}_{B|A}^{-1} \Leftrightarrow \mathrm{Tr}(y_i\alpha^i b) \in A$ for all $b \in B$ for all $0 \le i \le p-1$.
Hence, $\mathcal{D}_{B|A}^{-1} = \cup_{0\le i\le p-1}\mathcal{D}_i B$ where $\mathcal{D}_i := \{y\alpha^i \mid y \in K, y\alpha^i \in \mathcal{D}_{B|A}^{-1}\}$.
Fix some i, let $y \in K$. Write $b = \sum_{j=0}^{p-1} x_j\alpha^j$; $x_j \in A_j$. $\mathrm{Tr}(y\alpha^i b) \in A \Leftrightarrow \sum_{j=0}^{p-1} yx_j \,\mathrm{Tr}(\alpha^{i+j}) \in A$.
Thus, if $i = p-1$, then
$y\alpha^{p-1} \in \mathcal{D}_{B|A}^{-1} \Leftrightarrow v_L(y) + v_L(x_0 + x_{p-1}) \ge 0$ for all $x_0 \in A$, for all $x_{p-1} \in A_{p-1}$.
$\Leftrightarrow v_L(y) \ge 0$ and hence, $\mathcal{D}_{p-1}B = A\alpha^{p-1}B = \alpha^{p-1}B = \mathcal{J}_\sigma^{-(p-1)}$.
If $0 \le i \le p-2$,
$y\alpha^i \in \mathcal{D}_{B|A}^{-1} \Leftrightarrow v_L(y) + v_L(x_{p-1-i}) \ge 0$ for all $x_{p-1-i} \in A_{p-1-i}$
$\Leftrightarrow y\alpha^i.x_{p-1-i}\alpha^{p-1-i} \in \alpha^{p-1}B$
$\Leftrightarrow y\alpha^i A_{p-1-i}\alpha^{p-1-i} \subset \alpha^{p-1}B$.
- Consider $S = \mathcal{I}_\sigma$.
$\mathcal{I}_\sigma$ is generated by $\{(\sigma-1)(x_i\alpha^i) \mid x_i \in A_i, 1 \le i \le p-1\}$. For a fixed i, $(\sigma-1)(A_i\alpha^i)B = A_i\alpha^i[(1+\frac{1}{\alpha})^i - 1]B = A_i\alpha^i\frac{1}{\alpha}B = A_i\alpha^i\mathcal{J}_\sigma$. Thus,
$$\mathcal{I}_\sigma = [\cup_{1\le i\le p-1} A_i\alpha^i B]\mathcal{J}_\sigma. \tag{7.8}$$

Definition 7.9. We consider the B-sub-module ${\Omega_{B|A}^1}'$ of $\Omega_{B|A}^1$ generated by the set $\{db \mid b \in \mathfrak{m}_L\}$ of generators (and the relations described for $\Omega_{B|A}^1$).

Lemma 7.10. *${\Omega_{B|A}^1}' \cong \Omega_{B|A}^1$ as B-modules.*

Proof. ${\Omega_{B|A}^1}' \to \Omega_{B|A}^1$ is the map $db \mapsto db$. Consider the map $\pi : \Omega_{B|A}^1 \to {\Omega_{B|A}^1}'$ described below.

For $b \in B$, there exists $x \in A$ such that $b - x \in \mathfrak{m}_L$. We define $\pi(db) = d(b-x)$. Note that this definition is independent of the choice of x. It is enough to show that π preserves the relations.

Let $b, c \in B, x, y \in A$ such that $b - x, c - y \in \mathfrak{m}_K$.
Additivity is preserved, since $\pi(d(b+c)) = d(b+c-x-y) = d(b-x) + d(c-y) = \pi(db) + \pi(dc)$.
Since $dx = 0, dy = 0$ and $bc - xy = c(b-x) + x(c-y) \in \mathfrak{m}_L$,

$$\begin{aligned} cd(b-x) + bd(c-y) &= cd(b-x) + (b-x)dc - (b-x)dc + (b-x)d(c-y) \\ &\quad + xd(c-y) + (c-y)dx \\ &= d(c(b-x)) + d(x(c-y)) + (b-x)d(c-y) - (b-x)dc \\ &= d(bc - xc + xc - xy) + (b-x)[d(c-y) - dc] \\ &= d(bc-xy) - (b-x)dy = d(bc-xy) \end{aligned}$$

Hence, $\pi(d(bc)) = c\pi(db) + b\pi(dc)$. $\square$

We do not have a good description, as in Proposition 7.6, of the different ideal in this case. However, with further assumptions on the value group Γ_K, we obtain similar results.

7.3.2. Some results in a special case

Notation 7.11. Let $L|K$ satisfy (II). Assume further that $e_{L|K} = p$ and the value group Γ_K of K (as an ordered subgroup of $\mathbb{R}$) is not isomorphic to $\mathbb{Z}$. Thus, $L|K$ is a defectless Artin–Schreier extension and Γ_K is a dense ordered subgroup of $\mathbb{R}$.

Lemma 7.12. *Under the assumptions above (Notation 7.11),*

(a) *For* $1 \le i \le p-1, A_iB = \mathcal{J}_\sigma^i \mathfrak{m}_L$.
(b) $\mathcal{I}_\sigma = \mathcal{J}_\sigma \mathfrak{m}_L$.
(c) $\mathfrak{m}_L^n = \mathfrak{m}_L$ *for all integers* $n \ge 1$*, and consequently,* $\mathcal{I}_\sigma^n = \mathcal{J}_\sigma^n \mathfrak{m}_L$.

Proof.

(a) For $1 \le i \le p-1, A_iB = \{x \in K \mid v_L(x) > iv_0\}B = \{x \in L \mid v_L(x) > iv_0\}$. Hence, $A_iB = \frac{1}{\alpha^i}\mathfrak{m}_L = \mathcal{J}_\sigma^i \mathfrak{m}_L$.
(b) By (a), for $1 \le i \le p-1, A_i\alpha^iB = \frac{1}{\alpha^i}\alpha^i\mathfrak{m}_L = \mathfrak{m}_L$. Hence, by Equation (7.8), $\mathcal{I}_\sigma = [\cup_{1\le i\le p-1} A_i\alpha^i B]\mathcal{J}_\sigma = \mathcal{J}_\sigma\mathfrak{m}_L$.
(c) Let $x \in \mathfrak{m}_L, v_L(x) > 0$. Since the value group is dense in $\mathbb{R}$, there exists an element y of $\mathfrak{m}_L$ satisfying $0 < v_L(y) < v_L(x)/n$. Therefore, $(x) \subset (y^n) \subset \mathfrak{m}_L^n$ and we can conclude that $\mathfrak{m}_L = \mathfrak{m}_L^n$. The rest follows from (b). $\square$

Remark 7.13. In the general case when $e_{L|K} = p, 1 \le i \le p-1$, we have $A_iB \subset \mathcal{J}_\sigma^i \mathfrak{m}_L$ and $\mathcal{I}_\sigma \subset \mathcal{J}_\sigma \mathfrak{m}_L$.

Proposition 7.14. *Under the assumptions above (Notation 7.11),*

(a) $\mathcal{D}_{B|A}^{-1} = \mathcal{J}_\sigma^{-(p-1)}$.
(b) $\Omega^1_{B|A} \cong \omega^1_{B|A} \otimes_B \mathfrak{m}_L \cong \dfrac{\mathcal{I}_\sigma}{\mathcal{I}_\sigma^p}$.

Proof.

(a) We recall that $\mathcal{D}_{p-1} = \mathcal{J}_\sigma^{-(p-1)}$ and hence, $\mathcal{J}_\sigma^{-(p-1)} \subset \mathcal{D}_{B|A}^{-1}$. If $0 \le i \le p-2$,
$y\alpha^i \in \mathcal{D}_{B|A}^{-1}$
$\Leftrightarrow v_L(y) + v_L(x_{p-1-i}) \ge 0$ for all $x_{p-1-i} \in A_{p-1-i}$
$\Leftrightarrow v_L(y) + (p-1-i)v_0 \ge 0$ (since Γ_K is dense)

$\Leftrightarrow v_L(y\alpha^i) \geq -(p-1)v_0$
$\Leftrightarrow y\alpha^i \in \mathcal{J}_\sigma^{-(p-1)}$.
Hence, $\mathcal{D}_{B|A}^{-1} \subset \mathcal{J}_\sigma^{-(p-1)}$ and we have the equality $\mathcal{D}_{B|A}^{-1} = \mathcal{J}_\sigma^{-(p-1)}$.

(b) We defined a map $\pi : \Omega^1_{B|A} \to {\Omega^1_{B|A}}'$ in Lemma 7.10. Let $\Omega := \Omega^1_{B|A}, \Omega' := {\Omega^1_{B|A}}'$, for convenience. Consider the following maps:

$$\xi : \Omega' \to \omega^1_{B|A} \otimes_B \mathfrak{m}_L;\ \xi(db) = \operatorname{dlog} b \otimes b \tag{7.15}$$

where $0 \neq b \in \mathfrak{m}_L$ and

$$\psi : \omega^1_{B|A} \otimes_B \mathfrak{m}_L \to \Omega;\ \psi(\operatorname{dlog} b \otimes c) = \frac{c}{ab} d(ab) \tag{7.16}$$

where $b \in L^\times, c \in \mathfrak{m}_L, a \in K^\times; 0 \leq v_L(ab) \leq v_L(c)$. Such an a exists since Γ_K is dense in $\mathbb{R}$.

We verify that these maps are well-defined. Furthermore, $\xi \circ \pi \circ \psi : \omega^1_{B|A} \otimes_B \mathfrak{m}_L \to \omega^1_{B|A} \otimes_B \mathfrak{m}_L$ and $\psi \circ \xi \circ \pi : \Omega \to \Omega$ are isomorphisms.

- Let $0 \neq b, c \in \mathfrak{m}_L, 0 < v_L(c) \leq v_L(b)$. We can write $b = ch\ ; h \in B$.

$$\begin{aligned}
\operatorname{dlog}(b+c) \otimes (b+c) &= \operatorname{dlog} c(1+h) \otimes c(1+h) \\
&= (1+h) \operatorname{dlog} c \otimes c + (1+h) \operatorname{dlog}(1+h) \otimes c \\
&= \operatorname{dlog} c \otimes c + h \operatorname{dlog} c \otimes c + h \operatorname{dlog} h \otimes c \\
&= \operatorname{dlog} c \otimes c + h \operatorname{dlog} ch \otimes c \\
&= \operatorname{dlog} c \otimes c + \operatorname{dlog} ch \otimes ch \\
&= \operatorname{dlog} c \otimes c + \operatorname{dlog} b \otimes b
\end{aligned}$$

- Let $0 \neq b, c \in \mathfrak{m}_L$

$$\begin{aligned}
\operatorname{dlog}(bc) \otimes (bc) &= \operatorname{dlog} b \otimes bc + \operatorname{dlog} c \otimes bc \\
&= c \operatorname{dlog} b \otimes b + b \operatorname{dlog} c \otimes c
\end{aligned}$$

Thus, ξ is well-defined. Next, we check that ψ is well-defined.

- Let $b \in L^\times, c \in \mathfrak{m}_L, a, a' \in K^\times$ such that $0 \leq v_L(ab), v_L(a'b) \leq v_L(c)$. Since $da = 0 = da'$,
  $\frac{c}{ab} d(ab) = \frac{c}{ab}(adb + bda) = \frac{c}{b} db = \frac{c}{a'b} d(a'b)$.
  Thus, ψ is independent of choice of a.
- Let $0 \neq b \in B, c \in \mathfrak{m}_L, a \in K^\times$ as described in the definition of ψ. Since $da = 0$, we have
  $\psi(db \otimes c) = b\frac{c}{ab} d(ab) = \frac{c}{a}(adb + bda) = cdb$.
  Hence, ψ preserves additivity and Leibniz rule.

- Let $b, b' \in L, c, c' \in \mathfrak{m}_L, a, a' \in K^\times$ such that $0 \leq v_L(ab) \leq v_L(c)$ and $0 \leq v_L(a'b') \leq v_L(c')$.
  Furthermore, since Γ_K is dense in $\mathbb{R}$, we can choose a, a' such that $0 \leq v_L(aa'bb') \leq v_L(c)$.
  $\frac{c}{aa'bb'} d(aa'bb') = \frac{c}{aa'bb'}[a'b'd(ab) + abd(a'b')] = \frac{c}{ab} d(ab) + \frac{c}{a'b'} d(a'b')$

Thus, ψ is well-defined.
Next, we consider the maps $\xi \circ \pi \circ \psi : \omega^1_{B|A} \otimes_B \mathfrak{m}_L \to \omega^1_{B|A} \otimes_B \mathfrak{m}_L$ and $\psi \circ \xi \circ \pi : \Omega \to \Omega$.

- Let $b \in L^\times, c \in \mathfrak{m}_L, a \in K^\times, x \in A$ such that $0 \leq v_L(ab) \leq v_L(c)$ and $ab - x \in \mathfrak{m}_L$.

$$\begin{aligned}
\xi \circ \pi \circ \psi(\operatorname{dlog} b \otimes c) &= \frac{c}{ab} \operatorname{dlog}(ab - x) \otimes (ab - x) \\
&= \frac{ab - x}{ab} \operatorname{dlog}(ab - x) \otimes c \\
&= \frac{ab}{ab} \operatorname{dlog}(ab) \otimes c \\
&= \operatorname{dlog} a \otimes c + \operatorname{dlog} b \otimes c = \operatorname{dlog} b \otimes c
\end{aligned}$$

- Let $0 \neq b \in B, x \in A$ such that $b - x \in \mathfrak{m}_L$.

$$\begin{aligned}
\psi \circ \xi \circ \pi(db) &= \psi(\operatorname{dlog}(b - x) \otimes (b - x)) \\
&= \left(\frac{b - x}{b - x}\right) d(b - x) \\
&= d(b - x) = db
\end{aligned}$$

This proves the first isomorphism.
Next, we prove that

$$\omega^1_{B|A} \cong B/\mathcal{J}_\sigma^{p-1} \tag{7.17}$$

By Corollary 3.3(ii), $\omega^1_{B|A}$ is generated by $\operatorname{dlog} \alpha = -\operatorname{dlog}\left(\frac{1}{\alpha}\right)$. In $\omega^1_{B|A}$, we have

$$\begin{aligned}
0 &= -\left(1 - \frac{1}{\alpha^{p-1}}\right) \operatorname{dlog}(\frac{1}{f}) = \left(1 - \frac{1}{\alpha^{p-1}}\right) \operatorname{dlog} f \\
&= \left(1 - \frac{1}{\alpha^{p-1}}\right) \operatorname{dlog}(\alpha^p) + \left(1 - \frac{1}{\alpha^{p-1}}\right) \operatorname{dlog}\left(1 - \frac{1}{\alpha^{p-1}}\right) \\
&= d\left(1 - \frac{1}{\alpha^{p-1}}\right) = d\left(-\frac{1}{\alpha^{p-1}}\right) \\
&= -d\left(\frac{1}{\alpha^{p-1}}\right) = \left(1 - \frac{1}{\alpha^{p-1}}\right) \\
&= -(p-1)\left(\frac{1}{\alpha^{p-1}}\right) \operatorname{dlog}\left(\frac{1}{\alpha}\right)
\end{aligned}$$

Therefore, $\mathcal{J}_\sigma^{p-1} = \left(\frac{1}{\alpha^{p-1}}\right)$ annihilates $\omega^1_{B|A}$.
Conversely, let $0 \neq b \in B$ such that $b\ \omega^1_{B|A} = 0$. Hence, for all $1 \leq i \leq p-1$, $x_i \in A_i, bd(x_i\alpha^i) = 0$
$\Rightarrow b \in \cap_{i,x_i} G'_{i,x_i}(x_i\alpha^i)B$, where G_{i,x_i} is the minimal polynomial of $x_i\alpha^i$ over K. Let $G := G_{i,x_i}$ for fixed (i, x_i). Then

$$\begin{aligned} G'(x_i\alpha^i) &= \prod_{1\leq j\leq p-1} x_i\alpha^i \left(1 - \left(\frac{\alpha+j}{\alpha}\right)^i\right) \\ &= (x_i\alpha^i)^{p-1} \prod_{1\leq j\leq p-1} \left(1 - \left(\frac{\alpha+j}{\alpha}\right)^i\right) \\ &= (x_i\alpha^i)^{p-1} \prod_{1\leq j\leq p-1} \left(1 - s\left(\frac{\alpha+j}{\alpha}\right)\right) u;\ u \in B^\times \\ &= (x_i\alpha^i)^{p-1} \left(\frac{-1}{\alpha}\right)^{p-1} (p-1)!\ u \end{aligned}$$

Thus, $b \in \cap_{i,x_i} G'_{i,x_i}(x_i\alpha^i)B = \cap_{i,x_i}(x_i\alpha^i)^{p-1}\mathcal{J}_\sigma^{p-1} \Rightarrow b \in \mathcal{J}_\sigma^{p-1}$
By Equation (7.17) and Lemma 7.12,
$\omega^1_{B|A} \otimes \mathfrak{m}_L \cong B/\mathcal{J}_\sigma^{p-1} \otimes \mathfrak{m}_L \cong \mathfrak{m}_L/\mathcal{J}_\sigma^{p-1}\mathfrak{m}_L \cong \mathcal{J}_\sigma\mathfrak{m}_L/\mathcal{J}_\sigma^p\mathfrak{m}_L = \mathcal{I}_\sigma/\mathcal{I}_\sigma^p$. □

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Appendix A. Non-trivial example of defect extension

For some of the well-known examples of defect extensions, our main results are trivially true, since the differential modules are all 0. We construct an example below that exhibits complications created by the defect.

Example A.1. Let k be a perfect field of characteristic $p > 0$ and let A_0 be the local ring of a smooth algebraic surface over k at some closed point, and assume that we are given an Artin–Schreier extension L of the field of fractions K of A_0 given by

$$\alpha^p - \alpha = \frac{a+y}{x^n}$$

where x and y are regular parameters of A_0, $a \in k \setminus \mathbb{F}_p$, and $n \geq 1$ is coprime to p. We assume $n \geq 3$ if $p = 2$. We will construct two dimensional regular local rings $A_i \subset K$

($i \geq 0$) such that

$$A_0 \subset A_1 \subset A_2 \subset \dots$$

as follows, by using successive blow ups. We will have a valuation ring $A := \bigcup_i A_i$ for which this Artin–Schreier extension has defect.

A.1. Construction

Let $u := y + a$. Define $x' \in K$ by $x = x'y^p$ and let A_0' be local ring of $A_0[x'] \subset K$ at the maximal ideal generated by $x' - 1$ and y. Then A_0' is a two dimensional regular local ring with regular parameters $x' - 1$ and y. Since n is coprime to p, $z := (x')^{-n} - 1$ and y are also regular parameters of A_0'. Define $z' \in K$ by $z = z'y$ and let A_1 be the local ring of $A'[z'] \subset K$ at the maximal ideal generated by $z' - 1$ and y.

Then the above Artin–Schreier equation is rewritten as follows. We have

$$\begin{aligned} f_0 &:= \frac{a+y}{x^n} = \frac{a+y}{(x')^n y^{np}} = \frac{(a+y)(1+z'y)}{y^{np}} \\ &= \frac{a}{y^{np}} + \frac{a+1+a(z'-1)+z'y}{y^{np-1}} \\ &= \frac{a+1+y_1}{x_1^{np-1}} + c^p - c \end{aligned} \tag{A.2}$$

with

$$x_1 = y, \quad y_1 = a(z'-1) + z'y + a^{1/p}y^{n(p-1)-1}, \quad c = a^{1/p}y^{-n}.$$

In A_1, x_1 and y_1 are regular parameters, and the same Artin–Schreier extension is obtained by

$$\alpha_1^p - \alpha_1 = \frac{a+1+y_1}{x_1^{np-1}} =: f_1.$$

We can repeat this process and get $A_0 \subset A_1 \subset A_2 \subset \dots$ inductively. To sum up, we have the following for all $i \geq 0$:

In A_i, the regular parameters are x_i and y_i, as described in the construction (and we put $x = x_0, y = y_0, \alpha = \alpha_0, n = n_0$). The same Artin–Schreier extension is give by

$$\alpha_i^p - \alpha_i = \frac{a+i+y_i}{x_i^{n_i}} =: f_i$$

where the integers n_i satisfy the recursive relation $n_{i+1} = pn_i - 1$.

A.2. Valuation on A and B

Let B be the integral closure of A in L. Due to their construction using successive blow ups we note that A and B are valuation rings [9]. Let $v_K = v$ be the valuation on K. We see from the calculations below that the value group of K is $\Gamma \cong \mathbb{Z}[\frac{1}{p}]; v(x_0) \mapsto 1$. For all $i \geq 0$, we have the following:

(1) $n_i = p^i n - (p^{i-1} + \cdots + p^2 + p + 1) = p^i n - \dfrac{p^i - 1}{p-1}$.

(2) $v(x_i) = pv(y_i) = pv(x_{i+1})$

And hence, we get

$$v(x_i) = \frac{1}{p^i}, v(y_i) = \frac{1}{p^{i+1}}$$

(3) $-v(f_i) = n_i v(x_i) = n - \dfrac{1}{p-1} + \dfrac{1}{p^i(p-1)}$.

Since Γ is p-divisible, $L|K$ has defect. We will use v to denote v_L as well. By the computations above, it follows that

$$-v(\alpha_i) = \frac{1}{p}(-v(f_i)) = \frac{n}{p} - \frac{1}{p(p-1)} + \frac{1}{p^{i+1}(p-1)}$$

A.3. Special ideals and differential modules

Due to the defect, we have $\mathcal{I}_\sigma = \mathcal{J}_\sigma$ and it is enough to look at Ω^1's instead of ω^1's (see Lemma 3.9).

The elements $\dfrac{1}{\alpha_i}$ for $i \geq 0$ generate the ideal $\mathcal{J}_\sigma$ of B and the elements $\dfrac{1}{f_i}$ for $i \geq 0$ generate the ideal H of A.

- Since $\inf_{i\geq 0}\left(\dfrac{n}{p} - \dfrac{1}{p(p-1)} + \dfrac{1}{p^{i+1}(p-1)}\right) = \dfrac{n}{p} - \dfrac{1}{p(p-1)}$, we have $\mathcal{I}_\sigma = \mathcal{J}_\sigma = \{b \in B \mid v(b) > \frac{1}{p}(n - \frac{1}{p-1}) =: v_0\}$, and consequently,
- $\mathcal{N}_\sigma = \{a \in A \mid v(a) > (n - \frac{1}{p-1}) = pv_0\}$.
- Since $\inf_{i\geq 0} -v(f_i) = \inf_{i\geq 0}\left(n - \dfrac{1}{p-1} + \dfrac{1}{p^i(p-1)}\right) = n - \dfrac{1}{p-1}$, there is no best f and furthermore,
  $H = \{a \in A \mid v(a) > (n - \frac{1}{p-1})\}$

Thus, Theorem 0.3 is clearly true in this case.

Next, use the notation from the proof of Theorem 0.5 and consider the differential modules $\Omega^1_{B|A}, \Omega^1_{B_i|A_i}$'s.
Let $\beta_i := \alpha_i y_i^{n_i}$. Then the integral closure of A_i in L is given by $B_i = A_i[\beta_i]$. Let $F_i(T)$ be the minimal polynomial of β_i over A_i. Then $F_i'(T) = -y_i^{n_i(p-1)}$.

We have an isomorphism: $A_i[\beta_i]/F_i'(\beta_i) \to \Omega^1_{B_i|A_i}$ of B_i-modules via the A_i-linear map $a \mapsto ad\beta_i$; for all $a \in A_i$.

We use α_0 as our starting point. The valuation of α_0 is $-n/p$ and $v_0 = \dfrac{1}{p}\left(n - \dfrac{1}{p-1}\right)$. The fractional ideals Θ and Θ' of L are described by $\Theta = \{x \in L \mid v(x) > -\dfrac{1}{p(p-1)} =: v_1\}$ and $\Theta' = \{x \in L \mid v(x) > \left(\dfrac{n(p-1)-1}{p}\right) + v_1 =: v_2\}$. Then we have:

- $\Omega^1_{B|A} \cong \Theta/\Theta'$,
- $\Theta/\mathcal{J}_\sigma\Theta \cong \mathcal{J}_\sigma/\mathcal{J}_\sigma^2$.

From this, Theorem 0.5 will follow.

We can also verify that

- $\mathcal{D}^{-1}_{B|A} = \cap_{i\geq 0}\mathcal{D}^{-1}_{B_i|A_i}$,
- $\mathcal{D}^{-1}_{B|A} = \{x \in L \mid v(x) > -(p-1)v_0\}$,
- $\mathcal{D}_{B|A} = \mathcal{J}_\sigma^{p-1}$ is the annihilator of $\Omega^1_{B|A}$.

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