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Typical ranks for 3-tensors, nonsingular bilinear maps and determinantal ideals ☆

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Let $m, n \geq 3$, $(m-1)(n-1)+2 \leq p \leq mn$, and $u = mn - p$. The set $\mathbb{R}^{u\times n\times m}$ of all real tensors with size $u \times n \times m$ is one to one corresponding to the set of bilinear maps $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^u$. We show that $\mathbb{R}^{m\times n\times p}$ has plural typical ranks p and $p+1$ if and only if there exists a nonsingular bilinear map $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^u$. We show that there is a dense open subset $\mathscr{O}$ of $\mathbb{R}^{u\times n\times m}$ such that for any $Y \in \mathscr{O}$, the ideal of maximal minors of a matrix defined by Y in a certain way is a prime ideal and the real radical of that is the irrelevant maximal ideal if that is not a real prime ideal. Further, we show that there is a dense open subset $\mathscr{T}$ of $\mathbb{R}^{n\times p\times m}$ and continuous surjective open maps $\nu\colon \mathscr{O} \to \mathbb{R}^{u\times p}$ and $\sigma\colon \mathscr{T} \to \mathbb{R}^{u\times p}$, where $\mathbb{R}^{u\times p}$ is the set of $u \times p$ matrices with entries in $\mathbb{R}$, such that if $\nu(Y) = \sigma(T)$, then $\operatorname{rank} T = p$ if and only if the ideal of maximal minors of the matrix defined by Y is a real prime ideal.

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1. Introduction

For positive integers m, n, and p, we consider an $m\times n\times p$ tensor which is an element of the tensor product of $\mathbb{R}^m$, $\mathbb{R}^n$, and $\mathbb{R}^p$ with standard basis. This tensor can be identified with a 3-way array (a_{ijk}) where $1 \le i \le m$, $1 \le j \le n$ and $1 \le k \le p$. We denote by $\mathbb{R}^{m\times n\times p}$ the set of all $m \times n \times p$ tensors. This set is a topological space with Euclidean topology. Hitchcock [15] defined the rank of a tensor. An integer r is called a typical rank of $\mathbb{R}^{m\times n\times p}$ if the set of tensors with rank r is a semi-algebraic set of dimension mnp. In the other words, r is a typical rank of $\mathbb{R}^{m\times n\times p}$ if the set of tensors with rank r contains a nonempty open set of $\mathbb{R}^{m\times n\times p}$. In this paper we discuss the typical ranks of 3-tensors and connect between plurality of typical ranks and existence of a nonsingular bilinear map.

Let $n \le p$. A typical rank of $\mathbb{R}^{1\times n\times p}$ is equal to an $n\times p$ matrix full rank, that is, n. If $n \ge 2$, then the set of typical ranks of $\mathbb{R}^{2\times n\times p}$ is equal to $\{n, n+1\}$ if $n = p$ and otherwise $\min\{p, 2n\}$ [36]. This is also obtained from the equivalent class: almost all $2\times n\times p$ tensors are equivalent to $((E_n, O_{n\times(p-n)}); (O_{n\times(p-n)}, E_n))$ which has rank $\min\{p, 2n\}$ if $n < p$ (see [18] or [32]), see Section 2 for notation. Suppose that $n \ge m \ge 3$. The set of typical ranks of $\mathbb{R}^{m\times n\times p}$ is equal to $\min\{p, mn\}$ if $(m-1)n < p$ [35]. If $p = (m-1)n$ then the set of typical ranks of $\mathbb{R}^{m\times n\times p}$ depends on the existence of a nonsingular bilinear map $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$: It is equal to $\{p\}$ if there is no nonsingular bilinear map $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ and $\{p, p+1\}$ otherwise [33]. Here, a bilinear map $f\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^r$ is called nonsingular if $f(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0}$ implies $\boldsymbol{x} = \boldsymbol{0}$ or $\boldsymbol{y} = \boldsymbol{0}$.

Suppose that $(m-1)(n-1)+1 \le p \le (m-1)n$. A typical rank of $\mathbb{R}^{m\times n\times p}$ is unknown except a few cases. First, p is a minimal typical rank, since p is a generic rank of $\mathbb{C}^{m\times n\times p}$ [5]. The authors [34] showed that the Hurwitz–Radon function gives a condition that $\mathbb{R}^{m\times n\times (m-1)n}$ has plural typical ranks. We [24] also showed that $\mathbb{R}^{m\times n\times p}$ has plural typical ranks for some (m, n, p) by using the concept of absolutely full column rank tensors. We let $m\#n$ be the minimal integer r such that there is a nonsingular bilinear map $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^r$. Then $m\#n \le m+n-1$ (see Section 2). The set $\mathbb{R}^{r\times m\times n}$ of $r\times m\times n$ tensors is one to one corresponding to the set of bilinear maps $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^r$. By this map the set of absolutely full column rank tensors is one to one corresponding to the set of nonsingular bilinear maps.

Theorem 1.1. *Let $m, n \ge 3$ and $(m-1)(n-1)+1 \le p \le mn$.*

(1) *If there exists a nonsingular bilinear map $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^{mn-p}$, then $\mathbb{R}^{m\times n\times p}$ has plural typical ranks.*
(2) *If $p \ge (m-1)(n-1)+2$ and $\mathbb{R}^{m\times n\times p}$ has plural typical ranks, then there exists a nonsingular bilinear map $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^{mn-p}$.*

(1) of Theorem 1.1 is an extension of one of [24]. Furthermore, we completely determine the set $\operatorname{trank}(m, n, p)$ of typical ranks of $\mathbb{R}^{m\times n\times p}$ for $p\ge(m-1)(n-1)+2$ by the number $m\#n$.

Theorem 1.2. *Let $m, n \geq 3$, $k \geq 2$, and $p = (m-1)(n-1) + k$. The set of typical ranks of $\mathbb{R}^{m\times n\times p}$ is given as follows.*

$$\operatorname{trank}(m,n,p) = \begin{cases} \{p, p+1\}, & 2 \leq k \leq m+n-1-(m\#n) \\ \{p\}, & \max\{2, (m+n)-(m\#n)\} \leq k \leq m+n-2 \\ \{mn\}, & k \geq m+n-1. \end{cases}$$

Consider the case where $p = (m-1)(n-1)+1$, Friedland [12] showed that $\mathbb{R}^{n\times n\times((n-1)^2+1)}$ has plural typical ranks. We extend this result.

Theorem 1.3. *Let $m, n \geq 3$ and $p = (m-1)(n-1)+1$. $\mathbb{R}^{m\times n\times p}$ has plural typical ranks if $m-1$ and $n-1$ are not bit-disjoint.*

This article is organized as follows. Sections 2–7 are preparation to show the above theorems. In Section 2, we set notations and discuss the number $m\#n$. In Section 3, we study absolutely full column rank tensors. Since the set of absolutely full column rank tensors is an open set, there exists a special form of an absolutely full column rank tensor if an absolutely full column rank tensor exists. In Section 4, we state the other notions and deal with ideals of minors of matrices. Theorem 4.31 in Section 4 which corresponds with the real radical ideals is quite interesting in its own right. We show that for integers with $0 < t \leq \min\{u, n\}$ and $m \geq (u-t+1)(n-t+1)+2$, there exist open subsets $\mathscr{O}_1$ and $\mathscr{O}_2$ of $\mathbb{R}^{u\times n\times m}$ such that the union of them is dense, $\mathbb{I}(\mathbb{V}(I_t(M(\boldsymbol{x}, Y)))) = I_t(M(\boldsymbol{x}, Y))$ for $Y \in \mathscr{O}_1$ and $\mathbb{I}(\mathbb{V}(I_t(M(\boldsymbol{x}, Y)))) = (x_1, \ldots, x_m)$ for $Y \in \mathscr{O}_2$, where $I_t(M(\boldsymbol{x}, Y))$ is the ideal generated by all t-minors of the $u \times n$ matrix $M(\boldsymbol{x}, Y) = \sum_{k=1}^{m} x_k Y_k$ given by the indeterminates $x_1, \ldots, x_m$ and $Y = (Y_1; \ldots; Y_m) \in \mathbb{R}^{u\times n\times m}$. From this, we can give a subset of $m \times n \times p$ tensors with rank p for $3 \leq m \leq n$ and $(m-1)(n-1)+2 \leq p \leq (m-1)n$. In Section 5 we discuss a property for the determinantal ideals by using monomial preorder. This property plays an important role for proving Theorem 1.1. We characterize $m \times n \times p$ tensors with rank p in Section 6. In Section 7, we show that the existence of an absolutely full column rank tensor with suitable size implies that $p+1$ is a typical rank of $\mathbb{R}^{m\times n\times p}$. Moreover there exist a nonempty open subset $\mathscr{T}_1$ consisting of tensors with rank p and a possibly empty open subset $\mathscr{T}_2$ consisting of tensors with rank greater than p, corresponding $\mathscr{O}_1$ and $\mathscr{O}_2$ respectively, such that the union of them is a dense subset of $\mathbb{R}^{m\times n\times p}$ (see Theorem 7.14). Finally, in Section 8, we show that $p+2$ is not a typical rank of $\mathbb{R}^{m\times n\times p}$ and complete proofs of the above theorems.

2. Nonsingular bilinear maps

We first recall some basic facts and establish terminology.

Notation.

(1) We denote by $\mathbb{R}^n$ (resp. $\mathbb{R}^{1\times n}$) the set of n-dimensional column (resp. row) real vectors and by E_n the $n\times n$ identity matrix. Let $\boldsymbol{e}_j$ be the j-th column vector of an identity matrix.

(2) For a tensor $x \in \mathbb{R}^n \otimes \mathbb{R}^p \otimes \mathbb{R}^m$ with $x = \sum_{ijk} a_{ijk}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k$, we identify x with $T = (a_{ijk})_{1\le i\le n, 1\le j\le p, 1\le k\le m}$ and denote it by $(A_1; \dots; A_m)$, where $A_k = (a_{ijk})_{1\le i\le n, 1\le j\le p}$ for $k = 1, \dots, m$ is an $n \times p$ matrix, and call $(A_1; \dots; A_m)$ a tensor.

(3) We denote the set of $n\times p\times m$ tensors by $\mathbb{R}^{n\times p\times m}$ and the set of typical ranks by $\operatorname{trank}(n, p, m)$.

(4) For an $n\times p\times m$ tensor $T = (T_1; \dots; T_m)$, an $l\times n$ matrix P and an $k\times p$ matrix Q, we denote by PT the $l\times p\times m$ tensor $(PT_1; \dots; PT_p)$ and by $TQ^\top$ the $n\times k\times m$ tensor $(T_1Q^\top; \dots; T_pQ^\top)$.

(5) For $n\times p$ matrices $A_1, \dots, A_m$, we denote by $(A_1, \dots, A_m)$ the $n\times mp$ matrix obtained by aligning $A_1, \dots, A_m$ horizontally.

(6) We set $\operatorname{Diag}(A_1, A_2, \dots, A_t) = \begin{pmatrix} A_1 & & & O \\ & A_2 & & \\ & & \ddots & \\ O & & & A_t \end{pmatrix}$ for matrices $A_1, A_2, \dots, A_t$.

(7) For an $m\times n$ matrix M, we denote by $M_{\le j}$ (resp. ${}_{j<}M$) the $m\times j$ (resp. $m\times(n-j)$) matrix consisting of the first j (resp. last $n-j$) columns of M. We denote by $M^{\le i}$ (resp. ${}^{i<}M$) the $i\times n$ (resp. $(m-i)\times n$) matrix consisting of the first i (resp. last $m-i$) rows of M. We put $M^{<i} = M^{\le i-1}$, $M_{<i} = M_{\le i-1}$, and $M^{=i} = {}^{i-1<}(M^{\le i})$ which is the i-th row vector of M.

(8) We set $\mathrm{fl}_1(T) = (T_1, \dots, T_m)$ and $\mathrm{fl}_2(T) = \begin{pmatrix} T_1 \\ \vdots \\ T_m \end{pmatrix}$ for a tensor $T = (T_1; \dots; T_m)$.

Definition 2.1. A bilinear map $f\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^l$ is called nonsingular if $f(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0}$ implies $\boldsymbol{x} = \boldsymbol{0}$ or $\boldsymbol{y} = \boldsymbol{0}$. For positive integers m and n, we set

$$m\#n := \min\{l \mid \text{there exists a nonsingular bilinear map } \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^l\}.$$

Let $g\colon \mathbb{R}^{1\times u} \times \mathbb{R}^{1\times v} \to \mathbb{R}^{1\times(u\#v)}$ be a nonsingular bilinear map. For positive integers m and n, let $f\colon \mathbb{R}^{1\times mu} \times \mathbb{R}^{1\times nv} \to \mathbb{R}^{1\times(m+n-1)(u\#v)}$ be a map defined by $f((\boldsymbol{a}_1, \dots, \boldsymbol{a}_m), (\boldsymbol{b}_1, \dots, \boldsymbol{b}_n)) = (g(\boldsymbol{a}_1, \boldsymbol{b}_1), g(\boldsymbol{a}_1, \boldsymbol{b}_2) + g(\boldsymbol{a}_2, \boldsymbol{b}_1), \dots, \sum_{i+j=k} g(\boldsymbol{a}_i, \boldsymbol{b}_j), \dots, g(\boldsymbol{a}_m, \boldsymbol{b}_n))$. It is easily verified that f is a nonsingular bilinear map. Thus we have the following:

Lemma 2.2. *$(mu)\#(nv) \le (m + n - 1)(u\#v)$.*

By applying this lemma to nonsingular bilinear maps obtained by multiplications of $\mathbb{R}$, $\mathbb{C}$, quaternions and octanions respectively, we have the following:

Proposition 2.3 *(cf. [30, Proposition 12.12 (3)]). For $k = 1, 2, 4$ and 8, it holds that $km\#kn \le k(m+n-1)$.*

Let $\mathscr{H}(r,s,n)$ be the condition on the binomial coefficients, called the Stiefel–Hopf criterion, that the binomial coefficient $\binom{n}{k}$ is even whenever $n-s<k<r$. If there exists a continuous, nonsingular, biskew map $\mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$ then the Stiefel–Hopf criterion $\mathscr{H}(r,s,n)$ holds. Put

$$r \circ s = \min\{n \mid \mathscr{H}(r,s,n) \text{ holds}\}.$$

We have

$$\max\{r,s\} \le r \circ s \le r\#s \le r+s-1.$$

Putting $n^* = \lceil \frac{n}{2} \rceil$ for $n \in \mathbb{Z}$, the number $r \circ s$ is easily obtained by the formula

$$r \circ s = \begin{cases} 2(r^* \circ s^*) - 1 & \text{if } r,\ s \text{ are both odd and } r^* \circ s^* = r^* + s^* - 1, \\ 2(r^* \circ s^*) & \text{otherwise} \end{cases}$$

(cf. [30, Proposition 12.9]).

For a positive integer n, we put integers $\alpha_j(n) = 0, 1$, $j \ge 0$ such that $n = \sum_{j=0}^{\infty} \alpha_j(n) 2^j$ is the dyadic expansion of n and let $\alpha(n) := \sum_{j=0}^{\infty} \alpha_j(n)$ be the number of ones in the dyadic expansion of n. Two integers m and n are bit-disjoint if $\{j \mid \alpha_j(m) = 1\}$ and $\{j \mid \alpha_j(n) = 1\}$ are disjoint. For $k > h$, let $\tau(k,h)$ be a nonnegative number defined as

$$\tau(k,h) = \#\{j \ge 0 \mid \alpha_j(k-h) = 0, \alpha_j(k) \ne \alpha_j(h)\}.$$

Proposition 2.4. *$r\#s = r+s-1$ if and only if $r-1$ and $s-1$ are bit-disjoint.*

Proof. If $r-1$ and $s-1$ are bit-disjoint, then $r \circ s = r\#s = r+s-1$ (cf. [30, p. 257]). Moreover, $\tau(k,h) = 0$ if and only if h and $k-h$ are bit-disjoint. There is a nonsingular bilinear map $\mathbb{R}^{h+1} \times \mathbb{R}^{k-h+\tau(k,h)} \to \mathbb{R}^k$ for $k > h \ge 0$ [20] and thus $(h+1)\#(k-h+\tau(k,h)) \le k$. Putting $r = h+1$ and $k = r+s-2$, we have $r\#(s-1+\tau(r+s-2, r-1)) \le r+s-2$. In particular, if $r-1$ and $s-1$ are not bit-disjoint then $r\#s \le r+s-2$. $\square$

Let ρ be the Hurwitz–Radon function defined as $\rho(n) = 2^b + 8c$ for nonnegative integers a, b, c such that $n = (2a+1)2^{b+4c}$ and $0 \le b < 4$. There is a nonsingular bilinear map $\mathbb{R}^n \times \mathbb{R}^{\rho(n)} \to \mathbb{R}^n$ [17,26] and there is no nonsingular bilinear map $\mathbb{R}^n \times \mathbb{R}^{\rho(n)+1} \to \mathbb{R}^n$ for any $n \ge 1$ [1]. Therefore, $n\#\rho(n) \le n$ and $n\#(\rho(n)+1) > n$.

Corollary 2.5. *$n\#n \le 2n-2$. In particular, the equality $n\#n = 2n-2$ holds for $n = 2^a+1$.*

Proof. The inequality $n\#n \le 2n-2$ is clear by Proposition 2.4 since $n-1$ and $n-1$ are not bit-disjoint.

There is an immersion $\mathbb{RP}^n \to \mathbb{R}^{n+k}$ if and only if there is a nonsingular biskew map $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1+k}$ (cf. [2,30]). Note that $\rho(2) = 2$ and $\rho(4) = 4$. Then $2\#2 = 2$ and $3\#3 = 4$ which follows from $4\#4 = 4$. Suppose that $a \ge 2$. Put $m = 2^{a-1}$. Since there is no immersion $\mathbb{RP}^{2m} \to \mathbb{R}^{4m-2}$ (cf. [21]), we have $4m = (2m+1)\#(2m+1)$. □

Many estimations for $m\#n$ are known from immersion problem for manifolds, as projective spaces. For example, the existence of a nonsingular bilinear map $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1+k}$ implies that $\mathbb{RP}^n$ immerses in $\mathbb{R}^{n+k}$ [13].

Proposition 2.6.

(1) $(n+1)\#(n+1) \le 2n - \alpha(n) + 1$ [7].
(2) $(2n+\alpha(n))\#(2n+\alpha(n)) \ge 4n - 2\alpha(n) + 2$ [8].
(3) $(8n+9)\#(8n+9) \ge 16n+6$ *and* $(16n+12)\#(16n+12) \ge 32n+14$ *if* $\alpha(n) = 2$ [10,31].
(4) $(8n+10)\#(8n+10) \ge 16n+1$ *and* $(8n+11)\#(8n+11) \ge 16n+4$ *if* $\alpha(n) = 3$ [9,10].
(5) $(n+1)\#(m+1) \le n+m+1-(\alpha(n)+\alpha(n-m)+\min\{k(n),k(m)\})$ *if m, n are odd and $n \ge m$, where $k(n)$ is a nonnegative function depending only in the mod 8 residue class of n with $k(8a+1) = 0$, $k(8a+3) = k(8a+5) = 1$ and $k(8a+7) = 4$* [23].
(6) $d(h+1)\#(d(k-h)+\tau(k,h)) \le dk$ *for $k > h \ge 0$ and $d = 1, 2, 4, 8$* [20].
(7) $(n+1)\#(n+\tau(2n,n)) \le 2n$.

3. Absolutely full column rank tensors

For a tensor T of $\mathbb{R}^n \otimes \mathbb{R}^p \otimes \mathbb{R}^m$, we define the *rank* of T, denoted by $\operatorname{rank} T$, the minimal number r so that there exist $\boldsymbol{a}_i \in \mathbb{R}^n$, $\boldsymbol{b}_i \in \mathbb{R}^p$, and $\boldsymbol{c}_i \in \mathbb{R}^m$ for $i = 1, \dots, r$ such that

$$T = \sum_{i=1}^{r} \boldsymbol{a}_i \otimes \boldsymbol{b}_i \otimes \boldsymbol{c}_i.$$

The set $\mathbb{R}^{n\times p\times m}$ has an action of $\mathrm{GL}(m) \times \mathrm{GL}(p) \times \mathrm{GL}(n)$ as

$$(A, B, C) \cdot \sum_{i=1}^{r} \boldsymbol{a}_i \otimes \boldsymbol{b}_i \otimes \boldsymbol{c}_i = \sum_{i=1}^{r} A\boldsymbol{a}_i \otimes B\boldsymbol{b}_i \otimes C\boldsymbol{c}_i.$$

For tensors $T_1, T_2 \in \mathbb{R}^{m\times n\times p}$, T_1 and T_2 are said to be *equivalent* if $T_1 = (A, B, C) \cdot T_2$ for some $(A, B, C) \in \mathrm{GL}(n) \times \mathrm{GL}(p) \times \mathrm{GL}(m)$. The equivalence relation preserves the rank. For a subset $\mathscr{U}$ and an open semi-algebraic subset $\mathscr{S}$ of $\mathbb{R}^{m\times n\times p}$, we say that almost all tensors in $\mathscr{S}$ are equivalent to tensors in $\mathscr{U}$ if there exists a semi-algebraic subset $\mathscr{S}_0$ of $\mathscr{S}$ with $\dim \mathscr{S}_0 < mnp$ such that any tensor of $\mathscr{S} \setminus \mathscr{S}_0$ is equivalent to a tensor of $\mathscr{U}$. In particular, for a given tensor T_0, if almost all tensors in $\mathbb{R}^{m\times n\times p}$ are equivalent to $\{T_0\}$, then we say that any tensor is *generically equivalent* to T_0.

An integer r is called a *typical rank* of $n \times p \times m$-tensors if there is a nonempty open subset $\mathscr{O}$ of $\mathbb{R}^{n\times p\times m}$ such that $\operatorname{rank} X = r$ for $X \in \mathscr{O}$. Over the complex number field $\mathbb{C}$, it is known that there is a unique typical rank, called the generic rank, of $n \times p \times m$-tensors for any n, p and m. The set of typical ranks of $n \times p \times m$-tensors over $\mathbb{R}$ is denoted by $\operatorname{trank}(n, p, m)$ and the generic rank of $n \times p \times m$-tensors over $\mathbb{C}$ is denoted by $\operatorname{grank}(n, p, m)$.

We recall the following facts.

Theorem 3.1 *([12, Theorem 7.1]). The space $\mathbb{R}^{m_1\times m_2\times m_3}$, m_1, m_2, $m_3 \in \mathbb{N}$, contains a finite number of open connected disjoint semi-algebraic sets $\mathscr{O}_1, \dots, \mathscr{O}_M$ satisfying the following properties.*

(1) *$\mathbb{R}^{m_1\times m_2\times m_3} \setminus \bigcup_{i=1}^{M} \mathscr{O}_i$ is a closed semi-algebraic set $\mathbb{R}^{m_1\times m_2\times m_3}$ of dimension strictly less than $m_1m_2m_3$.*
(2) *Each $T \in \mathscr{O}_i$ has rank r_i for $i = 1, \dots, M$.*
(3) *$\min\{r_1, \dots, r_M\} = \operatorname{grank}(m_1, m_2, m_3)$.*
(4) *$\operatorname{trank}(m_1, m_2, m_3) = \{r \in \mathbb{Z} \mid \min\{r_1, \dots, r_M\} \le r \le \max\{r_1, \dots, r_M\}\}$.*

Let $T = (A_1; \dots; A_p)$ be an $m \times n \times p$ tensor over $\mathbb{R}$. The tensor T is called an *absolutely full column rank* tensor if

$$\operatorname{rank}(\sum_{j=1}^{p} y_j A_j) = n$$

for any $(y_1, \dots, y_p)^\top \in \mathbb{R}^p \setminus \{\mathbf{0}\}$.

From the definition of the absolutely full column rank property, we see the following fact.

Lemma 3.2. *Let T be an $m\times n\times p$ tensor over $\mathbb{R}$ and $P \in \mathrm{GL}(m, \mathbb{R})$. Then T is absolutely full column rank if and only if so is PT.*

Lemma 3.3 *(see Corollary 4.20 or [24, Theorem 3.6]). The set of $m \times n \times p$ absolutely full column rank tensors is an open subset of $\mathbb{R}^{m\times n\times p}$.*

Let $T = (A_1; \dots; A_p)$ be an $m \times n \times p$-tensor. We define $f_T \colon \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$ as

$$f_T(\boldsymbol{x}, \boldsymbol{y}) = \sum_{j=1}^{p} y_j A_j \boldsymbol{x},$$

where $\boldsymbol{y} = (y_1, \dots, y_p)^\top$. Then f_T is a bilinear map. This assignment $T \mapsto f_T$ induces a bijection from $\mathbb{R}^{m\times n\times p}$ to the set of all bilinear maps $\mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$. It is easily verified that $f_T\colon \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$ is nonsingular if and only if T is absolutely full column rank. Therefore

Corollary 3.4. *There is an $m \times n \times p$ absolutely full column rank tensor if and only if there is a nonsingular bilinear map $\mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$, i.e., $n\#p \leq m$.*

Lemma 3.5. *Let n, m, and u be positive integers with $u \leq mn$. Set $p = mn - u$. Then the following conditions are equivalent.*

(1) *$n\#m \leq u$.*
(2) *There is a $u \times n \times m$ absolutely full column rank tensor.*
(3) *There is a $u\times n\times m$ absolutely full column rank tensor Y such that ${}_{p<}\mathrm{fl}_1(Y) = -E_u$.*

Proof. (1) $\Leftrightarrow$ (2) follows from Corollary 3.4.

It is clear that (3) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3): Let $X = (X_1; \dots; X_m)$ be a $u \times n \times m$ absolutely full column rank tensor. By Lemma 3.3, we may assume that ${}_{p<}\mathrm{fl}_1(X)$ is nonsingular. Set $Y = -{}_{p<}\mathrm{fl}_1(X)X$. Then Y satisfies the required conditions. □

4. Ideals of minors

In this section, we state some results on ideals of minors, which we use in the following of this paper and interesting in its own right.

First we recall the definition of normality of a ring.

Definition 4.1 *(see [22, Section 9]).* Let R be a commutative ring. We say that R is normal if R_P is an integrally closed integral domain for any prime ideal P of R.

Remark 4.2.

(1) A Noetherian integral domain is normal if and only if it is integrally closed.
(2) If R is a Noetherian normal ring, then $R \simeq R/P_1 \times \cdots \times R/P_r$, where $P_1, \dots, P_r$ are associated prime ideals of R.

We recall a criterion of normality in terms of Serre's condition.

Definition 4.3 *([22, page 183]).* Let R be a Noetherian ring and i a nonnegative integer.

(1) We say that R satisfies (R_i) if R_P is regular for any prime ideal P of R with $\mathrm{ht}P \leq i$.
(2) We say that R satisfies (S_i) if $\operatorname{depth} R_P \geq \min\{i, \mathrm{ht}P\}$ for any prime ideal P of R.

Lemma 4.4 *([22, Theorem 23.8]). Let R be a Noetherian ring. Then R is normal if and only if R satisfies $(R_1) + (S_2)$.*

The condition $(R_1) + (S_2)$ is restated as follows.

Lemma 4.5. *Let R be a Noetherian ring. Then R satisfies $(R_1) + (S_2)$ if and only if the following condition is satisfied: if P is a prime ideal of R with $\operatorname{depth} R_P \leq 1$, then R_P is regular.*

Proof. First assume that R satisfies $(R_1) + (S_2)$. Let P be a prime ideal of R with $\operatorname{depth} R_P \leq 1$. Since R satisfies (S_2), we see that $\operatorname{depth} R_P \geq \min\{\mathrm{ht} P, 2\}$. Therefore, $\mathrm{ht} P \leq 1$. Thus by (R_1), we see that R_P is regular.

Conversely, assume that R_P is regular for any prime ideal P of R with $\operatorname{depth} R_P \leq 1$. First we show that R satisfies (R_1). If P is a prime ideal with $\mathrm{ht} P \leq 1$, then $\operatorname{depth} R_P \leq \mathrm{ht} P \leq 1$. Thus by assumption, we see that R_P is regular. Next we show that R satisfies (S_2). Let P be an arbitrary prime ideal of R. If $\operatorname{depth} R_P \leq 1$, then by assumption, R_P is regular. Thus $\operatorname{depth} R_P = \mathrm{ht} P = \min\{\mathrm{ht} P, 2\}$. If $\operatorname{depth} R_P \geq 2$, then $\operatorname{depth} R_P \geq \min\{\mathrm{ht} P, 2\}$ holds trivially. $\square$

Next we state notations and definitions used in this section.

Definition 4.6. We denote by u, n, m, and t positive integers with $t \leq \min\{u, n\}$ and set $v = (u - t + 1)(n - t + 1)$. Let $M = (m_{ij})$ be a $u \times n$ matrix with entries in a commutative ring A. We denote by $I_t(M)A$, or simply $I_t(M)$, the ideal of A generated by t-minors of M. For $\alpha(1), \ldots, \alpha(t) \in \{1, \ldots, u\}$ and $\beta(1), \ldots, \beta(t) \in \{1, \ldots, n\}$, we set $[\alpha(1), \ldots, \alpha(t) \mid \beta(1), \ldots, \beta(t)]_M := \det(m_{\alpha(i)\beta(j)})$, and if $u \geq n$ and $\alpha(1), \ldots, \alpha(n) \in \{1, \ldots, u\}$, we set $[\alpha(1), \ldots, \alpha(n)]_M := \det(m_{\alpha(i)j})$. For a tensor $T = (T_1; \ldots; T_m)$ and $\boldsymbol{a} = (a_1, \ldots, a_m)$ we set $M(\boldsymbol{a}, T) := \sum_{i=1}^{m} a_i T_i$ and we define $\Gamma(u \times n) = \{[a_1, \ldots, a_n] \mid 1 \leq a_1 < \cdots < a_n \leq u,\ a_i \in \mathbb{Z}\}$. For $\gamma = [a_1, \ldots, a_n] \in \Gamma(u \times n)$, we set $\operatorname{supp}\gamma = \{a_1, \ldots, a_n\}$. If B is a ring, A is a subring of B and T is a tensor (resp. matrix, vector) with entries in B, we denote by $A[T]$ the subring of B generated by the entries of T over A. If moreover, B is a field, we denote by $A(T)$ the subfield of B generated by the entries of T over A. If the entries of a tensor (resp. matrix, vector) T are independent indeterminates, we say that T is a tensor (resp. matrix, vector) of indeterminates.

Here we note the following fact, which is verified by using [3, Chapter 1 Exercise 2] or [25, (6.13)].

Lemma 4.7. *Let A be a commutative ring, X a square matrix of indeterminates. Then $\det X$ is a non-zerodivisor of $A[X]$.*

Next we recall the following fact.

Lemma 4.8 *([16, Theorem 1 and Corollaries 3 and 4]).* (see also [4, (6.3) Theorem]). *Let A be a Noetherian ring and X a $u \times n$ matrix of indeterminates.*

(1) $\mathrm{ht}(I_t(X)A[X]) = \mathrm{grade}(I_t(X)A[X]) = v$.
(2) *If A is a domain, then $I_t(X)A[X]$ is a prime ideal of $A[X]$.*
(3) *If A is a normal domain, then so is $A[X]/I_t(X)A[X]$.*

We also recall the following fact.

Lemma 4.9 *([16, Theorem 1 and Corollaries 2 and 4]).* (see also [4, (2.1) Theorem]). *Let A be a Noetherian commutative ring and M a $u \times n$ matrix with entries in A. If $I_t(M) \neq A$, then $\mathrm{ht} I_t(M) \leq v$. Moreover, if A is Cohen–Macaulay and $\mathrm{ht} I_t(M) = v$, then $I_t(M)$ is height unmixed.*

The following Lemma is a generalization of [4, (12.4) Lemma].

Lemma 4.10. *Let u, n, m, t and v be as in Definition 4.6, A a commutative Noetherian ring, $T = (t_{ijk})$ a $u \times n \times m$ tensor of indeterminates and $f_1, \dots, f_m$ elements of A. Suppose that $(f_1, \dots, f_m) \neq A$. Set $g = \mathrm{grade}(f_1, \dots, f_m)A$, $\boldsymbol{f} = (f_1, \dots, f_m)$ and $M = M(\boldsymbol{f}, T) = (m_{ij})$.*

(1) $\mathrm{grade}\ I_t(M)A[T] = \min\{g, v\}$.
(2) *If $g \geq v + 1$ and A is a domain, then $I_t(M)A[T]$ is a prime ideal.*
(3) *If $g \geq v + 2$ and A is a Cohen–Macaulay normal domain, then $A[T]/I_t(M)A[T]$ is a normal domain.*

Remark 4.11. If $g \geq v$, then $\mathrm{grade}\ I_t(M) = \mathrm{ht}\, I_t(M) = v$ by Lemma 4.10 (1) and [16, Theorem 1 and Corollary 4].

Proof of Lemma 4.10. Set $R = A[T]$.

First we prove (1). Set $v' = \min\{g, v\}$. Since $I_t(M) \subset (f_1, \dots, f_m)R$, we see that $\mathrm{grade}\, I_t(M)R \leq \mathrm{grade}(f_1, \dots, f_m)R = g$. Thus we see by Lemma 4.9, $\mathrm{grade}\, I_t(M)R \leq v'$.

To prove the converse inequality, it is enough to show that if P be a prime ideal of R with $P \supset I_t(M)$, then $\mathrm{depth}\, R_P \geq v'$. Since if $P \supset (f_1, \dots, f_m)R$, then $\mathrm{depth}\, R_P \geq g \geq v'$, we may assume that $P \not\supset (f_1, \dots, f_m)R$. Take l with $f_l \notin P$. Then M is essentially a matrix of indeterminates over $A[f_l^{-1}][t_{ijk} \mid k \neq l]$. Thus $\mathrm{grade}(I_t(M)R[f_l^{-1}]) = v$ by Lemma 4.8. Since R_P is a localization of $R[f_l^{-1}]$, we see that $\mathrm{depth}\, R_P \geq v \geq v'$.

Next we prove (2). We may assume $f_1, \dots, f_m \neq 0$. Set $B = R/I_t(M)R$. Since $I_t(M)R$ is grade unmixed by (1) and [16, Theorem 1 Corollaries 2 and 4] (see also [27, Corollary of Theorem 1.2] or [22, Exercise 16.3]), we see that every associated prime ideal of $I_t(M)R$ is of grade v. In particular any associated prime ideal of $I_t(M)R$ does not contain $(f_1, \dots, f_m)R$, since $g > v$ by assumption. Thus $(\bar{f}_1, \dots, \bar{f}_m)B$ has grade at least 1, where $\bar{f}_k$ denote the natural image of f_k in B for $1 \le k \le m$.

Since $A[f_l^{-1}][t_{ijk} \mid k \ne l]$ is an integral domain and M is essentially a matrix of indeterminates over $A[f_l^{-1}][t_{ijk} \mid k \ne l]$, we see that $B[\bar{f}_l^{-1}] = R[f_l^{-1}]/I_t(M)R[f_l^{-1}]$ is an integral domain for any l. Thus we see that $\bar{f}_l$ is contained in all associated prime ideals of B but one. We denote this prime ideal by P_l. Since $B[(\bar{f}_l\bar{f}_{l'})^{-1}] = R[(f_l f_{l'})^{-1}]/I_t(M)R[(f_l f_{l'})^{-1}]$ is not a zero ring by the same reason as above, we see that $P_l = P_{l'}$ for any l and l'. In particular, $P_l = P_1$ for any l with $1 \le l \le m$. Since $\operatorname{grade}(\bar{f}_1, \dots, \bar{f}_m)B \ge 1$ and any associated prime of B other than P_1 contains $(\bar{f}_1, \dots, \bar{f}_m)B$, we see that P_1 is the only associated prime ideal of B.

Therefore, $B \subset B[\bar{f}_1^{-1}]$ and we see that B is a domain.

Finally we prove (3). Assume that P is a prime ideal of B with $\operatorname{depth} B_P \le 1$. Since B is Cohen–Macaulay by (1) and [16, Theorem 1 and Corollary 4] and $\operatorname{ht}(\bar{f}_1, \dots, \bar{f}_m)B \ge 2$, we see that $P \not\supset (\bar{f}_1, \dots, \bar{f}_m)B$.

Take l with $\bar{f}_l \notin P$. Then $B[\bar{f}_l^{-1}]$ is a normal domain by Lemma 4.8 and the same argument as above. Since B_P is a localization of $B[\bar{f}_l^{-1}]$, we see that B_P is regular by Lemmas 4.4 and 4.5. Thus B is normal by Lemmas 4.4 and 4.5. $\square$

Here we note the following fact, which can be verified by considering the associated prime ideals of I and using [22, Theorems 15.5, 15.6].

Lemma 4.12. *Let $\mathbb{K}$ be a field, $\boldsymbol{x} = (x_1, \dots, x_m)$ a vector of indeterminates and I a proper ideal of $\mathbb{K}[\boldsymbol{x}]$. Then*

$$\dim \mathbb{K}[\boldsymbol{x}]/I = \max \left\{ r \,\middle|\, \begin{array}{l} \exists i_1, \dots, i_r; \bar{x}_{i_1}, \dots, \bar{x}_{i_r} \textit{ are algebraically} \\ \textit{independent over } \mathbb{K} \end{array} \right\},$$

where $\bar{x}_i$ denote the natural image of x_i in $\mathbb{K}[\boldsymbol{x}]/I$.

Lemma 4.13. *Let $\mathbb{K}$ be a field, $T = (t_{ijk})$ a $u \times n \times m$-tensor of indeterminates, and $\boldsymbol{x} = (x_1, \dots, x_m)$ a vector of indeterminates. Set $R = \mathbb{K}[T]$, $\mathbb{L} = \mathbb{K}(T)$, $M = M(\boldsymbol{x}, T)$ and $v' = \min\{m, v\}$. Then*

$$\mathbb{L}[x_1, \dots, x_{m-v'}] \cap I_t(M)\mathbb{L}[\boldsymbol{x}] = (0), \quad R[x_1, \dots, x_{m-v'}] \cap I_t(M)R[\boldsymbol{x}] = (0),$$

$\mathbb{L}[\boldsymbol{x}]/I_t(M)\mathbb{L}[\boldsymbol{x}]$ is algebraic over the natural image of $\mathbb{L}[x_1, \dots, x_{m-v'}]$ in $\mathbb{L}[\boldsymbol{x}]/I_t(M)\mathbb{L}[\boldsymbol{x}]$ and $R[\boldsymbol{x}]/I_t(M)R[\boldsymbol{x}]$ is algebraic over the natural image of $R[x_1, \dots, x_{m-v'}]$ in $R[\boldsymbol{x}]/I_t(M)R[\boldsymbol{x}]$.

Proof. Since $I_t(M)\mathbb{L}[\boldsymbol{x}]$ is generated by homogeneous polynomials of positive degree with respect to $x_1, \dots, x_m$, we see that $\mathbb{L} \cap I_t(M)\mathbb{L}[\boldsymbol{x}] = (0)$.

By Lemma 4.10, we see that $I_t(M)$ is an ideal of height v'. Thus by Lemma 4.12, we see

$$\mathrm{tr.deg}_{\mathbb{L}}\, \mathbb{L}[\boldsymbol{x}]/I_t(M)\mathbb{L}[\boldsymbol{x}] = m - v'.$$

Thus there is a permutation $i_1, \dots, i_n$ of $1, \dots, n$ such that $\bar{x}_{i_1}, \dots, \bar{x}_{i_{m-v'}}$ are algebraically independent over $\mathbb{L}$ and $\mathbb{L}[\boldsymbol{x}]/I_t(M)\mathbb{L}[\boldsymbol{x}]$ is algebraic over $\mathbb{L}[\bar{x}_{i_1}, \dots, \bar{x}_{i_{m-v'}}]$, where $\bar{x}_i$ denote the natural image of x_i in $\mathbb{L}[\boldsymbol{x}]/I_t(M)\mathbb{L}[\boldsymbol{x}]$. By symmetry, we see that $\bar{x}_1, \dots, \bar{x}_{m-v'}$ are algebraically independent over $\mathbb{L}$ and $\mathbb{L}[\boldsymbol{x}]/I_t(M)\mathbb{L}[\boldsymbol{x}]$ is algebraic over $\mathbb{L}[\bar{x}_1, \dots, \bar{x}_{m-v'}]$. We also see that $R[\boldsymbol{x}]/I_t(M)R[\boldsymbol{x}]$ is algebraic over $R[\bar{x}_1, \dots, \bar{x}_{m-v'}]$.

Since $\bar{x}_1, \dots, \bar{x}_{m-v'}$ are algebraically independent over $\mathbb{L}$, we see that

$$\mathbb{L}[x_1, \dots, x_{m-v'}] \cap I_t(M)\mathbb{L}[\boldsymbol{x}] = (0)$$

and therefore $R[x_1, \dots, x_{m-v'}] \cap I_t(M)R[\boldsymbol{x}] = (0)$. □

Lemma 4.14. *Let $\mathbb{L}/\mathbb{K}$ be a field extension with* $\mathrm{char}\mathbb{K} = 0$, $\boldsymbol{x} = (x_1, \dots, x_m)$ *a vector of indeterminates and Y a $u \times n \times m$ tensor with entries in $\mathbb{L}$. Set $M = M(\boldsymbol{x}, Y)$. Suppose that the entries of Y are algebraically independent over $\mathbb{K}$. Then the following hold.*

(1) *If $m \geq v+1$, then $\mathbb{L}[x_1, \dots, x_{m-v}] \cap I_t(M)\mathbb{L}[\boldsymbol{x}] = (0)$ and $\mathbb{L}[\boldsymbol{x}]/I_t(M)\mathbb{L}[\boldsymbol{x}]$ is algebraic over the natural image of $\mathbb{L}[x_1, \dots, x_{m-v}]$.*
(2) *If $m \geq v+2$, then $\mathbb{L}[\boldsymbol{x}]/I_t(M)\mathbb{L}[\boldsymbol{x}]$ is a normal domain. In particular, $I_t(M)\mathbb{L}[\boldsymbol{x}]$ is a prime ideal of $\mathbb{L}[\boldsymbol{x}]$ of height v.*

Proof. Since the entries of Y are algebraically independent over $\mathbb{K}$, we see by Lemma 4.13 that $\mathbb{K}(Y)[\boldsymbol{x}]/I_t(M)\mathbb{K}(Y)[\boldsymbol{x}]$ is algebraic over $\mathbb{K}(Y)[x_1, \dots, x_{m-v}]$. Thus $\mathbb{L}[\boldsymbol{x}]/I_t(M)\mathbb{L}[\boldsymbol{x}]$ is algebraic over $\mathbb{L}[x_1, \dots, x_{m-v}]$ since $\mathbb{L}[\boldsymbol{x}]/I_t(M)\mathbb{L}[\boldsymbol{x}] = (\mathbb{K}(Y)[\boldsymbol{x}]/I_t(M)\mathbb{K}(Y)[\boldsymbol{x}]) \otimes_{\mathbb{K}(Y)} \mathbb{L}$. On the other hand, since $\mathrm{tr.deg}_{\mathbb{L}}\mathbb{L}[\boldsymbol{x}]/I_t(M)\mathbb{L}[\boldsymbol{x}] = \dim \mathbb{L}[\boldsymbol{x}]/I_t(M)\mathbb{L}[\boldsymbol{x}] \geq m - v$, by Lemmas 4.9 and 4.12, we see that $\bar{x}_1, \dots, \bar{x}_{m-v}$ are algebraically independent over $\mathbb{L}$, where $\bar{x}_i$ denote the natural image of x_i in $\mathbb{L}[\boldsymbol{x}]/I_t(M)\mathbb{L}[\boldsymbol{x}]$. Thus $\mathbb{L}[x_1, \dots, x_{m-v}] \cap I_t(M)\mathbb{L}[\boldsymbol{x}] = (0)$. This proves (1).

Next we prove (2). Take a transcendence basis S of $\mathbb{L}/\mathbb{K}(Y)$ and put

$$\begin{aligned} A &= \mathbb{K}(Y)[\boldsymbol{x}]/I_t(M)\mathbb{K}(Y)[\boldsymbol{x}], \\ C &= \mathbb{K}(Y)(S)[\boldsymbol{x}]/I_t(M)\mathbb{K}(Y)(S)[\boldsymbol{x}] \text{ and} \\ B &= \mathbb{L}[\boldsymbol{x}]/I_t(M)\mathbb{L}[\boldsymbol{x}]. \end{aligned}$$

By Lemma 4.10 (3), we see that A is a normal domain.

Since

$$\mathbb{K}(Y)[S][\boldsymbol{x}]/I_t(M)\mathbb{K}(Y)[S][\boldsymbol{x}] = (\mathbb{K}(Y)[\boldsymbol{x}]/I_t(M)\mathbb{K}(Y)[\boldsymbol{x}]) \otimes_{\mathbb{K}(Y)} \mathbb{K}(Y)[S] = A[S]$$

is a polynomial ring (with possibly infinitely many variables) over A, it is an integrally closed integral domain. Since C is a localization of the above ring, C is a normal domain.

Now let P be a prime ideal of B with $\operatorname{depth} B_P \le 1$. By Lemmas 4.4 and 4.5, it is enough to show that B_P is regular. Set $Q = C \cap P$. Then since $B = C \otimes_{\mathbb{K}(Y)(S)} \mathbb{L}$ is flat over C, we see that $\operatorname{depth} C_Q \le 1$ by [22, Theorem 23.3 Corollary]. Thus C_Q is regular since C is normal. The fiber ring $C_Q/QC_Q \otimes_C B_P$ is a localization of

$$C_Q/QC_Q \otimes_C B = C_Q/QC_Q \otimes_{\mathbb{K}(Y)(S)} \mathbb{L}$$

which is a 0-dimensional reduced ring, thus regular, since $\mathbb{L}$ is separably algebraic over $\mathbb{K}(Y)(S)$. (Note that $\mathbb{L}$ is an inductive limit of finitely generated algebraic extension fields of $\mathbb{K}(Y)(S)$. Or see [25, Theorem 3.2.6 and Theorem 3.2.8 (i)] and note the assumption of the existence of a field containing M and N is not used in the proof of [25, Theorem 3.2.8 (i)].) Thus by [22, Theorem 23.7], we see that B_P is regular.

Thus, B is a normal ring. Since B is a nonnegatively graded ring whose degree 0 component is a field, B is not a direct product of 2 or more rings. Therefore, B is a domain by Remark 4.2. Moreover, we see by (1), that $\operatorname{ht} I_t(M)\mathbb{L}[\boldsymbol{x}] = v$. $\square$

Definition 4.15. Let u, n, m, t and v be as in Definition 4.6. We set

$$\begin{aligned}
&\mathscr{A}_t^{u\times n\times m} := \{Y \in \mathbb{R}^{u\times n\times m} \mid I_t(M(\boldsymbol{a}, Y)) \neq (0) \text{ for any } \boldsymbol{a} \in \mathbb{R}^{1\times m} \setminus \{\boldsymbol{0}\}\},\\
&\mathscr{C}_t^{u\times n\times m} := \mathbb{R}^{u\times n\times m} \setminus \mathscr{A}_t^{u\times n\times m},\\
&\mathscr{I} := \{Y \in \mathbb{R}^{u\times n\times m} \mid \text{the entries of } Y \text{ are algebraically independent over } \mathbb{Q}\},
\end{aligned}$$

and for $Y = (Y_1; \dots ; Y_m) \in \mathbb{R}^{u\times n\times m}$ and for integers k, k' with $t \le k \le u$ and $t \le k' \le n$, we set

$$\mu_{k,k'}(\boldsymbol{x}, Y) := [1, \dots, t-1, k \mid 1, \dots, t-1, k']_{M(\boldsymbol{x},Y)},$$

where $\boldsymbol{x}$ is a vector of indeterminates. We also define

$$\begin{aligned}
&J_t(\boldsymbol{x}, Y) = \frac{\partial(\mu_{tt}, \mu_{t,t+1}, \dots, \mu_{tn}, \mu_{t+1,t}, \dots, \mu_{t+1,n}, \dots, \mu_{u,t}, \dots, \mu_{un})}{\partial(x_{m-v+1}, \dots, x_m)}(\boldsymbol{x}, Y),\\
&S_t(Y) := \left\{ \boldsymbol{a} \in \mathbb{R}^{1\times m} \;\middle|\; \begin{array}{l} \det M(\boldsymbol{a}, Y)^{<t}_{<t} \neq 0, J_t(\boldsymbol{a}, Y) \neq 0 \text{ and} \\ I_t(M(\boldsymbol{a}, Y)) = (0) \end{array} \right\},\\
&\mathscr{P}_t := \{Y \in \mathbb{R}^{u\times n\times m} \mid S_t(Y) \neq \emptyset\}.
\end{aligned}$$

Remark 4.16.

(1) $\mathscr{A}_1^{u\times n\times m} \supset \mathscr{A}_2^{u\times n\times m} \supset \cdots \supset \mathscr{A}_{\min\{u,n\}}^{u\times n\times m}$.
(2) $\mathscr{A}_t^{u\times n\times m}$ is stable under the action of $\mathrm{GL}(u, \mathbb{R})$ for any t.
(3) $\mathscr{C}_t^{u\times n\times m} \neq \emptyset$ for any t.
(4) $\mathscr{P}_t$ is a subset of $\mathscr{C}_t^{u\times n\times m}$ and

$$S_t(Y) = \left\{ \boldsymbol{a} \in \mathbb{R}^{1\times m} \;\middle|\; \begin{array}{l} \det M(\boldsymbol{a}, Y)^{<t}_{<t} \neq 0, J_t(\boldsymbol{a}, Y) \neq 0 \text{ and there exist} \\ \text{linearly independent vectors } \boldsymbol{b}_t, \dots, \boldsymbol{b}_n \in \mathbb{R}^n \\ \text{such that } M(\boldsymbol{a}, Y)\boldsymbol{b}_j = \boldsymbol{0} \text{ for any } t \leq j \leq n \end{array} \right\}$$

Lemma 4.17. *Let u, n and t be as in Definition 4.6, A an integral domain and M a $u \times n$ matrix with entries in A. Suppose that $\det M^{<t}_{<t} \neq 0$ and $[1, \dots, t-1, k \mid 1, \dots, t-1, k']_M = 0$ for any integer with $t \leq k \leq u$ and $t \leq k' \leq n$. Then $I_t(M) = (0)$. In particular,*

$$S_t(Y) = \left\{ \boldsymbol{a} \in \mathbb{R}^{1\times m} \;\middle|\; \begin{array}{l} \det M(\boldsymbol{a}, Y)^{<t}_{<t} \neq 0, J_t(\boldsymbol{a}, Y) \neq 0 \textit{ and} \\ \qquad\qquad [1, \dots, t-1, k \mid 1, \dots, t-1, k']_M = 0 \\ \textit{for any integer with } t \leq k \leq u \textit{ and } t \leq k' \leq n. \end{array} \right\}$$

Proof. Set

$$\xi_l := \begin{pmatrix} (-1)^{t+1}[1, \dots, t-1 \mid 2, \dots, t-1, l]_M \\ (-1)^{t+2}[1, \dots, t-1 \mid 1, 3, \dots, t-1, l]_M \\ \vdots \\ (-1)^{2t-1}[1, \dots, t-1 \mid 1, \dots, t-2, l]_M \\ \boldsymbol{0} \\ (-1)^{2t}[1, \dots, t-1 \mid 1, \dots, t-2, t-1]_M \\ \boldsymbol{0} \end{pmatrix} \in A^n$$

for each l with $t \leq l \leq n$ ($(-1)^{2t}[1, \dots, t-1 \mid 1, \dots, t-2, t-1]_M$ in the l-th position). Then, since $[1, \dots, t-1 \mid 1 \dots, t-1]_M = \det M^{<t}_{<t} \neq 0$, we see that $\xi_t, \dots, \xi_n$ are linearly independent over A. Since the k-th entry of $M\xi_l$ is $[1, \dots, t-1, k \mid 1, \dots, t-1, l]_M$, we see, by assumption, that $M\xi_l = \boldsymbol{0}$ for $t \leq l \leq n$. Thus, $\operatorname{rank} M < t$ and we see that $I_t(M) = (0)$. □

It is verified the following fact, since $\mathbb{Q}$ is a countable field.

Lemma 4.18. *$\mathscr{I}$ is a dense subset of $\mathbb{R}^{u\times n\times m}$.*

We also see that $\mathscr{A}_t^{u\times n\times m}$ is an open subset of $\mathbb{R}^{u\times n\times m}$. First note the following fact, which is easily verified.

Lemma 4.19. *Let X and Y be topological spaces with X compact and $f\colon X \times Y \to \mathbb{R}$ is a continuous map. Set $g\colon Y \to \mathbb{R}$ by $g(y) := \min_{x\in X} f(x, y)$. Then, g is a continuous map.*

Corollary 4.20. *$\mathscr{A}_t^{u\times n\times m}$ is an open subset of $\mathbb{R}^{u\times n\times m}$.*

Proof. Since $\mathscr{A}_t^{u\times n\times m}$ is the set consisting of $Y \in \mathbb{R}^{u\times n\times m}$ such that

$$\min_{\boldsymbol{a}\in S^{m-1}}(\text{the maximum of the absolute values of } t\text{-minors of } M(a,Y)) > 0,$$

we see the result by the previous lemma. □

Lemma 4.21. *If $v < m$, then $\mathscr{P}_t$ is a dense subset of $\mathscr{C}_t^{u\times n\times m}$. In particular, $\mathscr{P}_t \neq \emptyset$.*

Proof. Let $Y \in \mathscr{C}_t^{u\times n\times m}$ and $\mathscr{U}$ an open neighborhood of Y in $\mathbb{R}^{u\times n\times m}$. In order to prove the first assertion, it suffices to show that $\mathscr{P}_t \cap \mathscr{U} \neq \emptyset$.

There exist a nonzero vector $\boldsymbol{a} \in \mathbb{R}^{1\times m}$ and linearly independent vectors $\boldsymbol{b}_t, \dots, \boldsymbol{b}_n \in \mathbb{R}^n$ such that $M(\boldsymbol{a}, Y)\boldsymbol{b}_j = \boldsymbol{0}$ for $t \le j \le n$. Let $g_3 \in \mathrm{GL}(m)$ and $g_2 \in \mathrm{GL}(n)$ such that the first entry of $g_3^\top \boldsymbol{a}$ is nonzero, ${}^{t\le}(g_2^\top \boldsymbol{b}_t, \dots, g_2^\top \boldsymbol{b}_n)$ is nonsingular and sufficiently close to E_m and E_n respectively so that $(1, g_2^{-1}, g_3^{-1}) \cdot Y \in \mathscr{U}$. By replacing Y, $\boldsymbol{a}$ and $\boldsymbol{b}_t, \dots, \boldsymbol{b}_n$ by $(1, g_2^{-1}, g_3^{-1}) \cdot Y$, $g_3^\top \boldsymbol{a}$ and $g_2^\top \boldsymbol{b}_t, \dots, g_2^\top \boldsymbol{b}_n$ respectively, we may assume that the first entry of $\boldsymbol{a}$ is nonzero and ${}^{t\le}(\boldsymbol{b}_t, \dots, \boldsymbol{b}_n)$ is nonsingular.

Let $e \in \mathbb{R}$. We take a tensor $P(e) = (p_{ijk}) \in \mathbb{R}^{u\times n\times m}$ as follows. For any i, j, k with $j < t$ or $k \neq 1$, we put

$$p_{ijk} = \begin{cases} e^{ij} & (k = 1, i < t, j < t), \\ e & ((i,j,k) = (t+l, t+l', m-v+1+l+l'(u-t+1)), \\ & \qquad 0 \le l \le u-t, 0 \le l' \le n-t), \\ 0 & (\text{otherwise}) \end{cases}$$

and take p_{ij1} for $1 \le i \le u$ and $t \le j \le n$ so that $M(\boldsymbol{a}, P(e))\boldsymbol{b}_j = \boldsymbol{0}$ for $t \le j \le n$. Note that we can take such p_{ij1} since the first entry of $\boldsymbol{a}$ is nonzero and ${}^{t\le}(\boldsymbol{b}_t, \dots, \boldsymbol{b}_n)$ is nonsingular.

Then we have

$$\det M(\boldsymbol{a}, Y + P(e))_{<t}^{<t} \neq 0 \quad \text{and} \quad J_t(\boldsymbol{a}, Y + P(e)) \neq 0$$

for $e \gg 0$.

Therefore, since the entries of $P(e)$ are polynomials of e, we see that for a real number $e_0 \neq 0$ which is sufficiently closed to 0,

$$\det M(\boldsymbol{a}, Y + P(e_0))_{<n}^{<n} \neq 0, \tag{4.1}$$

$$J_t(\boldsymbol{a}, Y + P(e_0)) \neq 0, \tag{4.2}$$

$$Y + P(e_0) \in \mathscr{U}. \tag{4.3}$$

(4.1), (4.2) and the fact $M(\boldsymbol{a}, Y + P(e_0))\boldsymbol{b}_j = M(\boldsymbol{a}, Y)\boldsymbol{b}_j + M(\boldsymbol{a}, P(e_0))\boldsymbol{b}_j = \boldsymbol{0}$ for $t \leq j \leq n$ imply that $\boldsymbol{a} \in S(Y + P(e_0))$. Thus we have $Y + P(e_0) \in \mathscr{P}_t$ and we see that $\mathscr{P}_t \cap \mathscr{U} \neq \emptyset$.

The latter assertion follows from Remark 4.16. $\square$

Lemma 4.22. *Suppose that $v < m$. Then the set $\mathscr{P}_t$ is an open subset of $\mathbb{R}^{u\times n\times m}$ and for any $Y \in \mathscr{P}_t$ and $\boldsymbol{a} = (\boldsymbol{a}_1, \boldsymbol{a}_2) \in S_t(Y)$, where $\boldsymbol{a}_1 \in \mathbb{R}^{1\times(m-v)}$ and $\boldsymbol{a}_2 \in \mathbb{R}^{1\times v}$, there exists an open neighborhood $O(\boldsymbol{a}, Y)$ of $\boldsymbol{a}_1 \in \mathbb{R}^{1\times(m-v)}$ such that for any $\boldsymbol{b}_1 \in O(\boldsymbol{a}, Y)$, there exists $\boldsymbol{b}_2 \in \mathbb{R}^{1\times v}$ such that $(\boldsymbol{b}_1, \boldsymbol{b}_2) \in S_t(Y)$.*

Proof. Assume that $Y \in \mathscr{P}_t$ and $\boldsymbol{a} = (\boldsymbol{a}_1, \boldsymbol{a}_2) \in S_t(Y)$. Then $\mu_{kk'}(\boldsymbol{a}, Y) = 0$ for any $t \leq k \leq u$ and $t \leq k' \leq n$. Thus by implicit function theorem, we see that there is an open neighborhood U of $(\boldsymbol{a}_1, Y)$ in $\mathbb{R}^{1\times(m-v)} \times \mathbb{R}^{u\times n\times m}$ and a continuous map $\nu\colon U \to \mathbb{R}^{1\times v}$ such that $\nu(\boldsymbol{a}_1, Y) = \boldsymbol{a}_2$, and $\mu_{kk'}(\boldsymbol{b}, Z) = 0$ for any $(\boldsymbol{b}_1, Z) \in U$ and any k, k' with $t \leq k \leq u$ and $t \leq k' \leq n$, where $\boldsymbol{b} := (\boldsymbol{b}_1, \nu(\boldsymbol{b}_1, Z))$. By replacing U by a smaller neighborhood if necessary, we may assume that $\det M(\boldsymbol{b}, Z)^{<t}_{<t} \neq 0$ and $J_t(\boldsymbol{b}, Z) \neq 0$ for any $(\boldsymbol{b}_1, Z) \in U$.

Assume $(\boldsymbol{b}_1, Z) \in U$. Put $\boldsymbol{b} = (\boldsymbol{b}_1, \nu(\boldsymbol{b}_1, Z))$. By Lemma 4.17, we see that $\boldsymbol{b} \in S_t(Z)$. Thus it suffices to set $O(\boldsymbol{a}, Y) := \{\boldsymbol{b}_1 \in \mathbb{R}^{1\times(m-v)} \mid (\boldsymbol{b}_1, Y) \in U\}$. Moreover, since $\{Z \in \mathbb{R}^{u\times n\times m} \mid (\boldsymbol{a}_1, Z) \in U\}$ is an open subset of $\mathscr{P}_t$ containing Y, we see that $\mathscr{P}_t$ is an open subset of $\mathbb{R}^{u\times n\times m}$. $\square$

By Corollary 4.20 and Lemmas 4.21 and 4.22, we see the following:

Corollary 4.23. *If $v < m$, then $\mathscr{P}_t \subset \operatorname{int} \mathscr{C}_t^{u\times n\times m}$ and $\overline{\mathscr{P}_t} = \overline{\operatorname{int} \mathscr{C}_t^{u\times n\times m}} = \mathscr{C}_t^{u\times n\times m}$.*

Definition 4.24. We set $\tilde{\mathscr{P}}_t := \{PY \mid P \in \mathrm{GL}(u, \mathbb{R}), Y \in \mathscr{P}_t\}$.

Lemma 4.25. *$\tilde{\mathscr{P}}_t$ is an open subset of $\mathbb{R}^{u\times n\times m}$, stable under the action of $\mathrm{GL}(u, \mathbb{R})$ and $\overline{\tilde{\mathscr{P}}_t} = \mathscr{C}_t^{u\times n\times m}$.*

Proof. Since $\tilde{\mathscr{P}}_t = \bigcup_{P\in\mathrm{GL}(u,\mathbb{R})} P\mathscr{P}_t$ and $P\mathscr{P}_t$ is an open subset of $\mathbb{R}^{u\times n\times m}$ for any $P \in \mathrm{GL}(u, \mathbb{R})$ by Lemma 4.22 and the fact that multiplication of a nonsingular matrix is a homeomorphism on $\mathbb{R}^{u\times n\times m}$. Therefore, $\tilde{\mathscr{P}}_t$ is an open subset of $\mathbb{R}^{u\times n\times m}$. The fact that $\tilde{\mathscr{P}}_t$ is stable under the action of $\mathrm{GL}(u, \mathbb{R})$ is clear from the definition of $\tilde{\mathscr{P}}_t$. Finally, since $\mathscr{A}_t^{u\times n\times m}$ is stable under the action of $\mathrm{GL}(u, \mathbb{R})$, we see, by Remark 4.16, that $\tilde{\mathscr{P}}_t \subset \mathscr{C}_t^{u\times n\times m}$. Therefore, we see that $\overline{\tilde{\mathscr{P}}_t} = \mathscr{C}_t^{u\times n\times m}$ by Lemma 4.21. $\square$

Lemma 4.26. *Let $\mathbb{L}$ be an infinite field and $\boldsymbol{x} = (x_1, \ldots, x_m)$ a vector of indeterminates. Set $v' = \min\{m, v\}$ and $v'' = \min\{m, (u - t + 2)(n - t + 2)\}$. Then there is a Zariski dense open subset $\mathscr{Q}_1$ of $\mathbb{L}^{u\times n\times m}$ such that if $Y \in \mathscr{Q}_1$, then $\mathbb{L}[\boldsymbol{x}]/I_t(M(\boldsymbol{x}, Y))\mathbb{L}[\boldsymbol{x}]$ is algebraic over the natural image of $\mathbb{L}[x_1, \ldots, x_{m-v'}]$, $\mathbb{L}[x_1, \ldots, x_{m-v'}] \cap I_t(M(\boldsymbol{x}, Y))\mathbb{L}[\boldsymbol{x}] = (0)$, $\mathrm{ht} I_t(M)\mathbb{L}[\boldsymbol{x}] = v'$, $\mathbb{L}[\boldsymbol{x}]/I_{t-1}(M(\boldsymbol{x}, Y))\mathbb{L}[\boldsymbol{x}]$ is algebraic over* *the natural image of* $\mathbb{L}[x_1, \dots, x_{m-v''}]$, $\mathbb{L}[x_1, \dots, x_{m-v''}] \cap I_{t-1}(M(\boldsymbol{x}, Y))\mathbb{L}[\boldsymbol{x}] = (0)$ *and* $\mathrm{ht} I_{t-1}(M(\boldsymbol{x}, Y))\mathbb{L}[\boldsymbol{x}] = v''$.

Proof. Let $T = (t_{ijk})$ be the $u \times n \times m$ tensor of indeterminates. Then by Lemma 4.13, we see that $\mathbb{L}[T][\boldsymbol{x}]/I_t(M(\boldsymbol{x}, T))\mathbb{L}[T][\boldsymbol{x}]$ is algebraic over the natural image of $\mathbb{L}[T][x_1, \dots, x_{m-v'}]$. Denote the natural image of x_l in $\mathbb{L}[T][\boldsymbol{x}]/I_t(M(\boldsymbol{x}, T))\mathbb{L}[T][\boldsymbol{x}]$ by $\bar{x}_l$. Take a nonzero polynomial $f_l(t)$ with coefficient in $\mathbb{L}[T][x_1, \dots, x_{m-v'}]$ such that $f_l(\bar{x}_l) = 0$ for each l with $m - v' + 1 \le l \le m$. Let g be the product of all nonzero elements of $\mathbb{L}[T]$ appearing as the coefficient of at least one of f_l and set $\mathscr{Q}'_1 = \mathbb{L}^{u \times n \times m} \setminus \mathbb{V}(g)$. Then $\mathscr{Q}'_1$ is a Zariski dense open subset of $\mathbb{L}^{u \times n \times m}$.

Suppose that $Y \in \mathscr{Q}'_1$. And let $\tilde{x}_i$ be the natural image of x_i in $\mathbb{L}[\boldsymbol{x}]/I_t(M(\boldsymbol{x}, Y))\mathbb{L}[\boldsymbol{x}]$ and $\tilde{f}_l$ be an element of $\mathbb{L}[x_1, \dots, x_{m-v'}]$ obtained by substituting Y to T. Then $\tilde{f}_l$ is a nonzero element of $\mathbb{L}[x_1, \dots, x_{m-v'}]$ and $\tilde{f}_l(\tilde{x}_l) = 0$ for $m - v' + 1 \le l \le m$. Therefore, $\mathbb{L}[\boldsymbol{x}]/I_t(M(\boldsymbol{x}, Y))\mathbb{L}[\boldsymbol{x}]$ is algebraic over the natural image of $\mathbb{L}[x_1, \dots, x_{m-v'}]$. Thus $\mathrm{ht} I_t(M(\boldsymbol{x}, Y))\mathbb{L}[\boldsymbol{x}] = m - \dim \mathbb{L}[\boldsymbol{x}]/I_t(M)\mathbb{L}[\boldsymbol{x}] = m - \mathrm{tr.deg}_{\mathbb{L}} \mathbb{L}[\boldsymbol{x}]/I_t(M)\mathbb{L}[\boldsymbol{x}] \ge v'$. Thus, $\mathrm{ht} I_t(M(\boldsymbol{x}, Y))\mathbb{L}[\boldsymbol{x}] = v'$ by Lemma 4.9 and we see that $\mathrm{tr.deg}_{\mathbb{L}} \mathbb{L}[\boldsymbol{x}]/I_t(M(\boldsymbol{x}, Y))\mathbb{L}[\boldsymbol{x}] = m - v'$. Therefore $\tilde{x}_1, \dots, \tilde{x}_{m-v'}$ are algebraically independent over $\mathbb{L}$, that is, $\mathbb{L}[x_1, \dots, x_{m-v'}] \cap I_t(M(\boldsymbol{x}, Y))\mathbb{L}[\boldsymbol{x}] = (0)$.

We see by the same way that there is a Zariski dense open subset $\mathscr{Q}''_1$ of $\mathbb{L}^{u \times n \times m}$ such that if $Y \in \mathscr{Q}''_1$, then $\mathbb{L}[\boldsymbol{x}]/I_{t-1}(M(\boldsymbol{x}, Y))\mathbb{L}[\boldsymbol{x}]$ is algebraic over the natural image of $\mathbb{L}[x_1, \dots, x_{m-v''}]$, $\mathbb{L}[x_1, \dots, x_{m-v''}] \cap I_{t-1}(M(\boldsymbol{x}, Y))\mathbb{L}[\boldsymbol{x}] = (0)$ and $\mathrm{ht} I_{t-1}(M(\boldsymbol{x}, Y))\mathbb{L}[\boldsymbol{x}] = v''$. Thus it is enough to set $\mathscr{Q}_1 = \mathscr{Q}'_1 \cap \mathscr{Q}''_1$. $\square$

Let $\mathbb{L}$ be a field, $\boldsymbol{x} = (x_1, \dots, x_m)$ a vector of indeterminates and M a $u \times n$ matrix with entries in $\mathbb{L}[\boldsymbol{x}]$. Suppose that $\mathrm{ht} I_t(M) = v$ and $\det(M^{<t}_{<t}) \notin \sqrt{I_t(M)}$. Then $I_t(M)\mathbb{L}[\boldsymbol{x}][\det(M^{<t}_{<t})^{-1}]$ is a proper ideal of $\mathbb{L}[\boldsymbol{x}][\det(M^{<t}_{<t})^{-1}]$ and M is equivalent to the matrix of the following form in $\mathbb{L}[\boldsymbol{x}][(\det(M^{<t}_{<t})^{-1}]$.

$$\begin{pmatrix} E_{t-1} & O \\ O & * \end{pmatrix}$$

In particular, $I_t(M)$ is a complete intersection ideal in $\mathbb{L}[\boldsymbol{x}][(\det(M^{<t}_{<t})^{-1}]$. By symmetry, we see that if $\mathrm{ht} I_{t-1}(M) > v$, then $I_t(M)$ is a generically complete intersection ideal.

We use the notation of [11, p. 219]. Let $\mathbb{L}$ be a field of characteristic 0, T a $u \times n \times m$ tensor of indeterminates and $\boldsymbol{x} = (x_1, \dots, x_m)$ a vector of indeterminates. Set $\mathbb{M} = \mathbb{L}(T)$. Suppose that $m > v$. Then $I_t(M(\boldsymbol{x}, T))\mathbb{M}[\boldsymbol{x}]$ is a prime ideal and $\mathrm{ht} I_{t-1}(M(\boldsymbol{x}, T))\mathbb{M}[\boldsymbol{x}] > v$ by Lemma 4.10. Thus

$$I_t(M(\boldsymbol{x}, T)) : \mathscr{J}_{m-v}(I_t(M(\boldsymbol{x}, T))) = I_t(M(\boldsymbol{x}, T))$$

by [11, Theorem 2.1] and the argument above. Thus if we set $I' = I_t(M(\boldsymbol{x}, T)) + \mathscr{J}_{m-v}(I_t(M(\boldsymbol{x}, T)))$, then $\mathrm{ht} I' > v$. Therefore the natural images of $x_1, \dots, x_{m-v}$ in $\mathbb{M}[\boldsymbol{x}]/I'$ are algebraically dependent over $\mathbb{M}$ by Lemma 4.12. Take a transcendence basis

$x_{i_1}, \dots x_{i_d}$ of $\mathbb{M}[\boldsymbol{x}]/I'$ over $\mathbb{M}$. By symmetry, we may assume that $i_k = k$ for $1 \le k \le d$. Since $\mathrm{ht} I' > v$, we see that $d < m - v$. Take a nonzero polynomial $f(t)$ with coefficients in $\mathbb{L}[T][x_1, \dots, x_d]$ such that $f(x_{m-v}) \in I'$ and let g be the product of all nonzero elements of $\mathbb{L}[T]$ which appear in some nonzero coefficient of f. Set $\mathscr{Q}_2 = \mathbb{L}^{u\times n\times m} \setminus \mathbb{V}(g)$. Then $\mathscr{Q}_2$ is a Zariski dense open subset of $\mathbb{L}^{u\times n\times m}$ and if $Y \in \mathscr{Q}_1 \cap \mathscr{Q}_2$, where $\mathscr{Q}_1$ is the one in Lemma 4.26, then $\mathrm{ht}(I_t(M(\boldsymbol{x}, Y)) + \mathscr{J}_{m-v}(I_t(M(\boldsymbol{x}, Y)))) > v$ since $\mathrm{tr.deg}_{\mathbb{L}} \mathbb{L}[\boldsymbol{x}]/(I_t(M(\boldsymbol{x}, Y)) + \mathscr{J}_{m-v}(I_t(M(\boldsymbol{x}, Y)))) < m - v$.

Until the end of this section, assume that $m \ge v + 2$ and let U be the $m \times m$ matrix of indeterminates, T the $u \times n \times m$ tensor of indeterminates, $\boldsymbol{x} = (x_1, \dots, x_m)$ the vector of indeterminates and $\mathbb{L}$ the algebraic closure of $\mathbb{R}(U)$.

Set

$$\begin{pmatrix} x'_1 \\ \vdots \\ x'_m \end{pmatrix} = U \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}.$$

Then $\mathbb{L}(T)[x'_1, \dots, x'_m] = \mathbb{L}(T)[\boldsymbol{x}]$ and $\mathbb{L}(T)[x'_1, \dots, x'_{m-v+1}] \cap I_t(M(\boldsymbol{x}, T))$ is a principal ideal generated by a polynomial F called the ground form of $I_t(M(\boldsymbol{x}, T))$, since $I_t(M(\boldsymbol{x}, T))$ is a prime ideal therefore is unmixed of height v. See [29, parts 28 and 29].

Since $\mathbb{L}(T)[x'_1, \dots, x'_{m-v+1}] \cap I_t(M(\boldsymbol{x}, T))$ is the elimination ideal, F is obtained by the Buchberger's algorithm. Let g_3 be the products of all elements of $\mathbb{L}[T]$ which appear as a numerator or a denominator of a nonzero coefficient of at least one polynomial in the process of Buchberger's algorithm to obtain the reduced Gröbner basis of $I_t(M(\boldsymbol{x}, T))$ in $\mathbb{L}[T][\boldsymbol{x}]$. Set $\mathscr{Q}_3 = \mathbb{L}^{u\times n\times m} \setminus \mathbb{V}(g_3)$. Then $\mathscr{Q}_3$ is a Zariski dense open subset of $\mathbb{L}^{u\times n\times m}$ and if $Y \in \mathscr{Q}_3$, then the Buchberger's algorithm to obtain the reduced Gröbner basis of $I_t(M(\boldsymbol{x}, Y))\mathbb{L}[\boldsymbol{x}]$ in $\mathbb{L}[\boldsymbol{x}]$ is identical with that of $I_t(M(\boldsymbol{x}, T))\mathbb{L}(T)[\boldsymbol{x}]$ in $\mathbb{L}(T)[\boldsymbol{x}]$. In particular, $\mathbb{L}[x'_1, \dots, x'_{m-v+1}] \cap I_t(M(\boldsymbol{x}, Y))$ is a principal ideal generated by F_Y, the polynomial obtained by substituting Y in T in the coefficients of F.

Let $d = \deg F$ and let $P_{\mathbb{L}}(d, m{-}v{+}1)$ (resp. $P_{\mathbb{R}}(d, m{-}v{+}1)$) be the set of homogeneous polynomials with coefficients in $\mathbb{L}$ (resp. $\mathbb{R}$) with variables $x'_1, \dots, x'_{m-v+1}$ and degree d. Since $m - v + 1 \ge 3$ and $\mathbb{L}$ is an algebraically closed field containing $\mathbb{R}$, we see by [14] that

$$\begin{aligned} &\{G \in P_{\mathbb{R}}(d, m - v + 1) \mid G \text{ is absolutely irreducible}\} \\ &= P_{\mathbb{R}}(d, m - v + 1) \cap \{G \in P_{\mathbb{L}}(d, m - v + 1) \mid G \text{ is irreducible}\} \end{aligned}$$

is a Zariski dense open subset of $P_{\mathbb{R}}(d, m - v + 1)$.

Definition 4.27. Set

$$\mathscr{Q} = \{Y \in \mathscr{Q}_1 \cap \mathscr{Q}_2 \cap \mathscr{Q}_3 \cap \mathbb{R}^{u\times n\times m} \mid F_Y \text{ is absolutely irreducible}\},$$

where $\mathscr{Q}_1$ is the one in Lemma 4.26 and $\mathscr{Q}_2$, $\mathscr{Q}_3$ and F_Y are the ones defined after the proof of Lemma 4.26.

Remark 4.28. $\mathscr{Q}$ is a Zariski open subset of $\mathbb{R}^{u\times n\times m}$, since the correspondence Y to F_Y is a rational map whose domain contains $\mathscr{Q}_3$.

Moreover, we see the following fact.

Lemma 4.29. *$\mathscr{Q} \supset \mathscr{I} \cap \mathscr{Q}_1 \cap \mathscr{Q}_2 \cap \mathscr{Q}_3$. In particular, $\mathscr{Q}$ is not an empty set.*

Proof. Suppose that $Y \in \mathscr{I} \cap \mathscr{Q}_1 \cap \mathscr{Q}_2 \cap \mathscr{Q}_3$. Then we see, by applying Lemma 4.14 to $\mathbb{L}/\mathbb{Q}$, that $I_t(M(\boldsymbol{x}, Y))\mathbb{L}[\boldsymbol{x}]$ is a prime ideal. Thus the elimination ideal is also prime and therefore the generator F_Y of the elimination ideal is an irreducible polynomial in $\mathbb{L}[x'_1, \dots, x'_{m-v+1}]$. Therefore, $Y \in \mathscr{Q}$. Since $\mathscr{I}$ is a dense subset of $\mathbb{R}^{u\times n\times m}$ by Lemma 4.18, we see that $\mathscr{I} \cap \mathscr{Q}_1 \cap \mathscr{Q}_2 \cap \mathscr{Q}_3 \neq \emptyset$. Thus, $\mathscr{Q} \neq \emptyset$. $\square$

Thus we see that $\mathscr{Q}$ is a non-empty Zariski open subset of $\mathbb{R}^{u\times n\times m}$. In particular, $\mathscr{Q}$ is dense.

Lemma 4.30. *If $Y \in \mathscr{Q}$, then $I_t(M(\boldsymbol{x}, Y))\mathbb{R}[\boldsymbol{x}]$ is a prime ideal of height v.*

Proof. Since $Y \in \mathscr{Q}$, $\mathrm{ht} I_t(M(\boldsymbol{x}, Y))\mathbb{R}[\boldsymbol{x}] = v$ and $\mathbb{L}[x'_1, \dots, x'_{m-v+1}] \cap I_t(M(\boldsymbol{x}, Y))\mathbb{L}[\boldsymbol{x}] = (F_Y)\mathbb{L}[x'_1, \dots, x'_{m-v+1}]$. Thus

$$\mathbb{R}(U)[x'_1, \dots, x'_{m-v+1}] \cap I_t(M(\boldsymbol{x}, Y))\mathbb{R}(U)[\boldsymbol{x}] = (F_Y)\mathbb{R}(U)[x'_1, \dots, x'_{m-v+1}]$$

since $\mathbb{L}$ is faithfully flat over $\mathbb{R}(U)$. Thus we see that F_Y is the ground form of $I_t(M(\boldsymbol{x}, Y))\mathbb{R}[\boldsymbol{x}]$ [29, part 28]. Since F_Y is an irreducible polynomial in $\mathbb{L}[x'_1, \dots, x'_{m-v+1}]$ and therefore in $\mathbb{R}(U)[x'_1, \dots, x'_{m-v+1}]$, we see by [29, part 31], that $I_t(M(\boldsymbol{x}, Y))\mathbb{R}[\boldsymbol{x}]$ is a primary ideal.

On the other hand, since $Y \in \mathscr{Q}_1 \cap \mathscr{Q}_2$, we see that

$$\mathrm{ht}(I_t(M(\boldsymbol{x}, Y)) + \mathscr{J}_{m-v}(I_t(M(\boldsymbol{x}, Y)))) > v.$$

Since $I_t(M(\boldsymbol{x}, Y))$ is a primary ideal of height v, we see that

$$I_t(M(\boldsymbol{x}, Y)) : \mathscr{J}_{m-v}(I_t(M(\boldsymbol{x}, Y))) = I_t(M(\boldsymbol{x}, Y)).$$

Therefore, by [11, Theorem 2.1], we see that $I_t(M(\boldsymbol{x}, Y))$ is a radical ideal. Thus $I_t(M(\boldsymbol{x}, Y))$ is a prime ideal. $\square$

Now we show the following result.

Theorem 4.31. *Suppose that $m \geq v+2$. Set $\mathscr{O}_1 = \mathscr{Q} \cap \tilde{\mathscr{P}}_t$ and $\mathscr{O}_2 = \mathscr{Q} \cap \mathscr{A}_t^{u\times n\times m}$. Then the following hold.*

(1) *$\mathscr{O}_1$ and $\mathscr{O}_2$ are disjoint open subsets of $\mathbb{R}^{u\times n\times m}$ and $\mathscr{O}_1 \neq \emptyset$.*
(2) *$\mathscr{O}_1 \cup \mathscr{O}_2$ is a dense subset of $\mathbb{R}^{u\times n\times m}$.*
(3) *$\overline{\mathscr{O}_1} = \mathscr{C}_t^{u\times n\times m} = \mathbb{R}^{u\times n\times m} \setminus \mathscr{A}_t^{u\times n\times m}$.*
(4) *If $Y \in \mathscr{O}_1 \cup \mathscr{O}_2$, then $I_t(M(\boldsymbol{x}, Y))\mathbb{R}[\boldsymbol{x}]$ is a prime ideal of height v.*
(5) *If $Y \in \mathscr{O}_1$, then $\mathbb{I}(\mathbb{V}(I_t(M(\boldsymbol{x}, Y)))) = I_t(M(\boldsymbol{x}, Y))$.*
(6) *If $Y \in \mathscr{O}_2$, then $\mathbb{I}(\mathbb{V}(I_t(M(\boldsymbol{x}, Y)))) = (x_1, \ldots, x_m)$.*

Proof. The set $\mathscr{A}_t^{u\times n\times m}$ is an open subset of $\mathbb{R}^{u\times n\times m}$ by Corollary 4.20 and $\tilde{\mathscr{P}}_t$ is a nonempty open subset of $\mathbb{R}^{u\times n\times m}$ with $\mathscr{A}_t^{u\times n\times m} \cap \tilde{\mathscr{P}}_t = \emptyset$ by Lemmas 4.21 and 4.25. Further $\mathscr{Q}$ is a Zariski open subset. Thus (1) holds.

(2): Since $\mathscr{Q}$ and $\mathscr{A}_t^{u\times n\times m} \cup \tilde{\mathscr{P}}_t$ are dense open subsets of $\mathbb{R}^{u\times n\times m}$ by Corollary 4.20 and Lemma 4.25, we see that $\mathscr{O}_1 \cup \mathscr{O}_2 \supset \mathscr{Q} \cap (\mathscr{A}_t^{u\times n\times m} \cup \mathscr{P}_t)$ is also a dense open subset of $\mathbb{R}^{u\times n\times m}$.

(3): Since $\mathscr{Q}$ is a dense subset of $\mathbb{R}^{u\times n\times m}$, and $\tilde{\mathscr{P}}_t$ is an open set, we see that $\overline{\mathscr{O}_1} = \overline{\mathscr{Q} \cap \tilde{\mathscr{P}}_t} = \overline{\tilde{\mathscr{P}}_t} = \mathscr{C}_t^{u\times n\times m}$ by Lemma 4.25.

(4) follows from Lemma 4.30.

(5): Assume the contrary and take $g \in \mathbb{I}(\mathbb{V}(I_t(M(\boldsymbol{x}, Y))))$ with $g \notin I_t(M(\boldsymbol{x}, Y))$. Set $J = (g)\mathbb{R}[\boldsymbol{x}] + I_t(M(\boldsymbol{x}, Y))$. Then $J \supsetneq I_t(M(\boldsymbol{x}, Y))\mathbb{R}[\boldsymbol{x}]$. Since $I_t(M(\boldsymbol{x}, Y))\mathbb{R}[\boldsymbol{x}]$ is a prime ideal of height v by Lemma 4.30, we see that $\mathrm{ht} J > v$ and therefore $\mathbb{R}[x_1, \ldots, x_{m-v}] \cap J \neq (0)$.

Take $0 \neq f \in J \cap \mathbb{R}[x_1, \ldots, x_{m-v}]$. Since $Y \in \tilde{\mathscr{P}}_t$, we can take $P \in \mathrm{GL}(u, \mathbb{R})$ such that $PY \in \mathscr{P}_t$. Take $\boldsymbol{b} \in S_t(PY)$. Since $O(\boldsymbol{b}, PY)$ defined in Lemma 4.22 is an open set and f is a non-zero polynomial, we can take $(a_1, \ldots, a_{m-v}) \in O(\boldsymbol{b}, PY)$ with $f(a_1, \ldots, a_{m-v}) \neq 0$. On the other hand, we see that there are $a_{m-v+1}, \ldots, a_m \in \mathbb{R}$ such that $I_t(M(\boldsymbol{a}, Y)) = I_t(PM(\boldsymbol{a}, Y)) = I_t(M(\boldsymbol{a}, PY)) = (0)$ by Lemma 4.22, where $\boldsymbol{a} = (a_1, \ldots, a_m)$. Thus by assumption, we see that $g(\boldsymbol{a}) = 0$. This contradicts to the fact that $f \in J = (g)\mathbb{R}[\boldsymbol{x}] + I_t(M(\boldsymbol{x}, Y))\mathbb{R}[\boldsymbol{x}]$ and $f(a_1, \ldots, a_{m-v}) \neq 0$.

Finally, (6) is clear from the definition of $\mathscr{A}_t^{u\times n\times m}$. $\square$

5. Monomial preorder

In this section, we introduce the notion of monomial preorder and prove a result about ideals of minors by using it.

First we recall the notion of preorder.

Definition 5.1. Let S be a nonempty set and $\preceq$ a binary relation on S. We say that $\preceq$ is a *preorder* on S or $(S, \preceq)$ is a *preordered set* if the following two conditions are satisfied.

(1) $a \preceq a$ for any $a \in S$.
(2) $a \preceq b$, $b \preceq c \Rightarrow a \preceq c$.

If moreover,

(3) $a \preceq b$ or $b \preceq a$ for any a, $b \in S$.

is satisfied, then we say that $(S, \preceq)$ is a *totally preordered set* or $\preceq$ is a *total preorder*.

Notation. Let $(S, \preceq)$ be a preordered set. We denote by $b \succeq a$ the fact $a \preceq b$. We denote by $a \prec b$ or by $b \succ a$ the fact that $a \preceq b$ and $b \npreceq a$. We also denote by $a \sim b$ the fact that $a \preceq b$ and $b \preceq a$.

Remark 5.2. The binary relation $\sim$ defined above is an equivalence relation and if $a \sim a'$ and $b \sim b'$, then

$$a \preceq b \iff a' \preceq b'.$$

In particular, we can define a binary relation $\leq$ on the quotient set $P = S/\sim$ by $\bar{a} \leq \bar{b} \overset{\text{def}}{\iff} a \preceq b$, where $\bar{a}$ is the equivalence class which a belongs to. It is easily verified that $(P, \leq)$ is a partially ordered set and $(S, \preceq)$ is a totally preordered set if and only if $(P, \leq)$ is a totally ordered set. As usual, we denote $\bar{a} > \bar{b}$ the fact $\bar{a} \geq \bar{b}$ and $\bar{a} \neq \bar{b}$.

Definition 5.3. Let $x_1, \ldots, x_r$ be indeterminates. We denote the set of monomials or power products of $x_1, \ldots, x_r$ by $PP(x_1, \ldots, x_r)$. A *monomial preorder* on $x_1, \ldots, x_r$ is a total preorder $\preceq$ on $PP(x_1, \ldots, x_r)$ satisfying the following conditions.

(1) $1 \preceq m$ for any $m \in PP(x_1, \ldots, x_r)$.
(2) For m_1, m_2, $m \in PP(x_1, \ldots, x_r)$,

$$m_1 \preceq m_2 \iff m_1 m \preceq m_2 m.$$

Let $\sim$ be the equivalence relation on $PP(x_1, \ldots, x_r)$ defined by the monomial preorder $\preceq$. We denote by $P(x_1, \ldots, x_r)$ the quotient set $PP(x_1, \ldots, x_r)/\sim$ and by $\operatorname{qdeg} m$ the class of m in $P(x_1, \ldots, x_r)$ and call it the *quasi-degree* of m, where $m \in PP(x_1, \ldots, x_r)$.

Remark 5.4. Our definition of monomial preorder may seem to be different from that of [19], but it is identical except we allow $m \sim 1$ for a monomial $m \neq 1$.

Example 5.5 *(cf. [19, Example 3.1]).* Let $x_1, \ldots, x_r$ be indeterminates, $W = (\boldsymbol{w}_1, \ldots, \boldsymbol{w}_s)$ an $r \times s$ matrix whose entries are real numbers such that the first nonzero entry of each row is positive. If one defines

$$x^{\boldsymbol{a}} \preceq x^{\boldsymbol{b}} \overset{\text{def}}{\iff} (\boldsymbol{a} \cdot \boldsymbol{w}_1, \ldots, \boldsymbol{a} \cdot \boldsymbol{w}_s) \leq_{\text{lex}} (\boldsymbol{b} \cdot \boldsymbol{w}_1, \ldots, \boldsymbol{b} \cdot \boldsymbol{w}_s),$$

where $\cdot$ denotes the inner product and $\leq_{\text{lex}}$ denotes the lexicographic order, then $\preceq$ is a monomial preorder. In fact, one can prove by the same way as [28] that every monomial preorder is of this type.

Definition 5.6. Let K be a field and $x_1, \ldots, x_r$ indeterminates. If a monomial preorder on $x_1, \ldots, x_r$ is defined, we say that $K[x_1, \ldots, x_r]$ is a polynomial ring with monomial preorder. Let f be a nonzero element of $K[x_1, \ldots, x_r]$. We say that f is a *form* if all the monomials appearing in f have the same quasi-degree. We denote by $\operatorname{qdeg} f$ the quasi-degree of the monomials appearing in f. Let g be a nonzero element of $K[x_1, \ldots, x_r]$. Then there is a unique expression

$$g = g_1 + g_2 + \cdots + g_t$$

of g, where g_i is a form for $1 \leq i \leq t$ and $\operatorname{qdeg} g_1 > \operatorname{qdeg} g_2 > \cdots > \operatorname{qdeg} g_t$. We define the leading form of g, denoted $\operatorname{lf}(g)$ as g_1.

Remark 5.7. Let $K[x_1, \ldots, x_r]$ be a polynomial ring with monomial preorder and f, g nonzero elements of $K[x_1, \ldots, x_r]$. Then $\operatorname{lf}(fg) = \operatorname{lf}(f)\operatorname{lf}(g)$.

Remark 5.8. It is essential to assume both implications in (2) of Definition 5.3. For example, let x and y be indeterminates. We define total preorder on $PP(x, y)$ by $1 \prec y \prec x$ and $m_1 \prec m_2$ if the total degree of m_1 is less than that of m_2. Then it is easily verified that

(1) $1 \prec m$ for any $m \in PP(x, y) \setminus \{1\}$.
(2) $m_1 \preceq m_2 \Rightarrow m_1 m \preceq m_2 m$.

Let $f = x + y$. Then $\operatorname{lf}(f) = x$ while $\operatorname{lf}(f^2) = x^2 + 2xy + y^2 \neq x^2 = (\operatorname{lf}(f))^2$.

Definition 5.9. Let $x_1, \ldots, x_r$ be indeterminates. Suppose that a total preorder on $\{x_1, \ldots, x_r\}$ is defined. Rewrite the set $\{x_1, \ldots, x_r\}$ as follows. $\{x_1, \ldots, x_r\} = \{y_{11}, \ldots, y_{1s_1}, y_{21}, \ldots, y_{2s_s}, \ldots, y_{t1}, \ldots, y_{ts_t}\}$, $s_1 + \cdots + s_t = r$, $y_{11} \sim \cdots \sim y_{1s_1} \succ y_{21} \sim \cdots \sim y_{2s_2} \succ \cdots \succ y_{t1} \sim \cdots \sim y_{ts_t}$.

The lexicographic monomial preorder on $PP(x_1, \ldots, x_r)$ is defined as follows. $\prod_{i=1}^{t} \prod_{j=1}^{s_i} y_{ij}^{a_{ij}} \preceq \prod_{i=1}^{t} \prod_{j=1}^{s_i} y_{ij}^{b_{ij}}$ if and only if one of the following conditions is satisfied.

- $\sum_{j=1}^{s_1} a_{1j} < \sum_{j=1}^{s_1} b_{1j}$.
- $\sum_{j=1}^{s_1} a_{1j} = \sum_{j=1}^{s_1} b_{1j}$ and $\sum_{j=1}^{s_2} a_{2j} < \sum_{j=1}^{s_2} b_{2j}$.
- $\sum_{j=1}^{s_1} a_{1j} = \sum_{j=1}^{s_1} b_{1j}$, $\sum_{j=1}^{s_2} a_{2j} = \sum_{j=1}^{s_2} b_{2j}$ and $\sum_{j=1}^{s_3} a_{3j} < \sum_{j=1}^{s_3} b_{3j}$.

  $\vdots$

- $\sum_{j=1}^{s_1} a_{1j} = \sum_{j=1}^{s_1} b_{1j}$, $\sum_{j=1}^{s_2} a_{2j} = \sum_{j=1}^{s_2} b_{2j}$, …, $\sum_{j=1}^{s_{t-2}} a_{t-2,j} = \sum_{j=1}^{s_{t-2}} b_{t-2,j}$ and $\sum_{j=1}^{s_{t-1}} a_{t-1,j} < \sum_{j=1}^{s_{t-1}} b_{t-1,j}$.
- $\sum_{j=1}^{s_1} a_{1j} = \sum_{j=1}^{s_1} b_{1j}$, $\sum_{j=1}^{s_2} a_{2j} = \sum_{j=1}^{s_2} b_{2j}$, …, $\sum_{j=1}^{s_{t-1}} a_{t-1,j} = \sum_{j=1}^{s_{t-1}} b_{t-1,j}$ and $\sum_{j=1}^{s_t} a_{t,j} \leq \sum_{j=1}^{s_t} b_{t,j}$.

Remark 5.10. Suppose that $x_1, \dots, x_r$ are indeterminates and total preorder $\preceq$ on $\{x_1, \dots, x_r\}$ is defined. Suppose also that

$$x_1 \sim \cdots \sim x_{m_1} \succ x_{m_1+1} \sim \cdots \sim x_{m_2} \succ \cdots \succ x_{m_{t-1}+1} \sim \cdots \sim x_{m_t},$$

$m_t = r$. Then the lexicographic monomial preorder induced by this preorder on $\{x_1, \dots, x_r\}$ is the one defined as in Example 5.5 by the $r \times t$ matrix whose j-th column has 1 in $m_{j-1}+1$ through m_j-th position and 0 in others, where we set $m_0 = 0$.

Definition 5.11. We set

$$[a_1, \dots, \overset{k}{a_i}, \dots, a_n] := [a_1, \dots, a_{i-1}, k, a_{i+1}, \dots, a_n]$$

and

$$[a_1, \dots, \overset{k}{a_i}, \dots, \overset{l}{a_j}, \dots, a_n] := [a_1, \dots, a_{i-1}, k, a_{i+1}, \dots, a_{j-1}, l, a_{j+1}, \dots, a_n].$$

Lemma 5.12. *Let K be a field, $K[x_1, \dots, x_r]$ a polynomial ring with monomial preorder, S a subset of $\{x_1, \dots, x_r\}$ and $g_1, \dots, g_t \in K[x_1, \dots, x_r]$. Set $L = K[S]$. If $\mathrm{lf}(g_1), \dots, \mathrm{lf}(g_t)$ are linearly independent over L, then $g_1, \dots, g_t$ are linearly independent over L.*

Proof. Assume the contrary and suppose that

$$\sum_i c_i g_i = 0$$

is a non-trivial relation where $c_i \in L$ and $c_i \neq 0$ for any i which appears in the above sum. Then

$$\sum{}' \mathrm{lf}(c_i)\mathrm{lf}(g_i) = 0,$$

where $\sum'$ runs through i's with $\mathrm{qdeg}\,\mathrm{lf}(c_i g_i)$ are maximal. Since $\mathrm{lf}(c_i) \in L$ for any i, it contradicts the assumption. $\square$

Lemma 5.13 *(Plücker relations, see e.g. [4, (4,4) Lemma]). For every $u \times n$-matrix M, $u \geq n$, with entries in a commutative ring and all indices $a_1, \dots, a_k, b_l, \dots, b_n, c_1, \dots, c_s \in \{1, \dots, u\}$ such that $s = n - k + l - 1 > n$, $t = n - k > 0$ one has*

$$\sum_{\substack{i_1<\cdots<i_t \\ i_{t+1}<\cdots<i_s \\ \{1,\ldots,s\}=\{i_1,\ldots,i_s\}}} \operatorname{sgn}(i_1,\ldots,i_s)[a_1,\ldots,a_k,c_{i_1},\ldots,c_{i_t}]_M[c_{i_{t+1}},\ldots,c_{i_s},b_l,\ldots,b_n]_M = 0,$$

where $\operatorname{sgn}(\sigma)$ *is the signature of a permutation* σ *and the notations are defined in Definition 4.6.*

An element a is called a *non-zerodivisor* if $ab = 0$ implies $b = 0$.

Lemma 5.14. *Let* $A = A_0 \oplus A_1 \oplus \cdots$ *be a graded ring,* $X = (x_{ij})$ *a* $u \times n$ *matrix with* $u > n$ *and entries in* A_1 *and* $\boldsymbol{y} = (y_1,\ldots,y_n) \in A_1^{1\times n}$*. Set* $\tilde{X} = \begin{pmatrix} X \\ \boldsymbol{y} \end{pmatrix}$ *and* $\Gamma = \Gamma(u \times n)$*. Suppose that*

- $\delta = [1,2,\ldots,n]_X$ *is a non-zerodivisor of* A*,*
- *for any* k_1 *and* k_2 *with* $1 \le k_1 < k_2 \le n$*,*

$$\begin{aligned}
&\delta_X\gamma_X \quad (\gamma\in\Gamma),\\
&[1,\ldots,\overset{n+1}{k_2},\ldots,n]_X\gamma_X \quad (\gamma\in\Gamma\setminus\{\delta\}) \text{ and}\\
&[1,\ldots,\overset{n+1}{k_1},\ldots,n]_X\gamma_X \quad (\gamma\in\Gamma,\ \operatorname{supp}\gamma \not\supset \{1,\ldots,\hat{k}_2,\ldots,n\})
\end{aligned}$$

  are linearly independent over A_0 *and*
- $I_n(\tilde{X}) = I_n(X)$*,*

where the notations are defined in Definition 4.6. Then, $\boldsymbol{y}$ *is an* A_0*-linear combination of rows of* X*.*

Proof. We denote $\gamma_{\tilde{X}}$ as γ and $\sum_{\gamma\in\Gamma}$ as $\sum_\gamma$ for simplicity. Set

$$[1,\ldots,\overset{u+1}{k},\ldots,n] = \sum_\gamma a_\gamma^{(k)}\gamma$$

and

$$[1,\ldots,\overset{n+1}{k_1},\ldots,\overset{u+1}{k_2},\ldots,n] = \sum_\gamma a_\gamma^{(k_1,k_2)}\gamma$$

where $a_\gamma^{(k)}$, $a_\gamma^{(k_1,k_2)} \in A$. By considering the degree, we may assume that $a_\gamma^{(k)}$, $a_\gamma^{(k_1,k_2)} \in A_0$.

$$[1,\ldots,\overset{n+1}{k_1},\ldots,\overset{u+1}{k_2},\ldots,n]_{\tilde{X}\operatorname{Cof}(X^{\le n})}$$

$$
\begin{aligned}
&= \det(\mathrm{Cof}(X^{\leq n}))[1,\dots,\overset{n+1}{k_1},\dots,\overset{u+1}{k_2},\dots,n] \\
&= \delta^{n-1}\sum_{\gamma} a_\gamma^{(k_1,k_2)}\gamma,
\end{aligned}
$$

where $\mathrm{Cof}(X^{\leq n})$ denotes the matrix of cofactors of $X^{\leq n}$. On the other hand, since

$$
\tilde{X}\mathrm{Cof}(X^{\leq n}) = \begin{pmatrix}
\delta & & & \\
 & \delta & & \\
 & & \ddots & \\
 & & & \delta \\
[\overset{n+1}{1},2\dots,n] & [1,\overset{n+1}{2},\dots,n] & \cdots & [1,2,\dots,\overset{n+1}{n}] \\
[\overset{n+2}{1},2\dots,n] & [1,\overset{n+2}{2},\dots,n] & \cdots & [1,2,\dots,\overset{n+2}{n}] \\
\vdots & \vdots & \vdots & \vdots \\
[\overset{u+1}{1},2\dots,n] & [1,\overset{u+1}{2},\dots,n] & \cdots & [1,2,\dots,\overset{u+1}{n}]
\end{pmatrix},
$$

we see that

$$
\begin{aligned}
&[1,\dots,\overset{n+1}{k_1},\dots,\overset{u+1}{k_2},\dots,n]_{\tilde{X}\mathrm{Cof}(X^{\leq n})} \\
&= \delta^{n-2}\det\begin{pmatrix}
[1,\dots,\overset{n+1}{k_1},\dots,n] & [1,\dots,\overset{n+1}{k_2},\dots,n] \\
[1,\dots,\overset{u+1}{k_1},\dots,n] & [1,\dots,\overset{u+1}{k_2},\dots,n]
\end{pmatrix}.
\end{aligned}
$$

Since δ is a non-zerodivisor, we see that

$$
\begin{aligned}
&\sum_{\gamma} a_\gamma^{(k_1,k_2)}\delta\gamma \\
&= [1,\dots,\overset{n+1}{k_1},\dots,n][1,\dots,\overset{u+1}{k_2},\dots,n] - [1,\dots,\overset{n+1}{k_2},\dots,n][1,\dots,\overset{u+1}{k_1},\dots,n] \\
&= \sum_{\gamma} a_\gamma^{(k_2)}[1,\dots,\overset{n+1}{k_1},\dots,n]\gamma - \sum_{\gamma} a_\gamma^{(k_1)}[1,\dots,\overset{n+1}{k_2},\dots,n]\gamma \\
&= -a_\delta^{(k_1)}\delta[1,\dots,\overset{n+1}{k_2},\dots,n] - \sum_{\gamma\in\Gamma\setminus\{\delta\}} a_\gamma^{(k_1)}[1,\dots,\overset{n+1}{k_2},\dots,n]\gamma \\
&\quad + a_\delta^{(k_2)}\delta[1,\dots,\overset{n+1}{k_1},\dots,n] + \sum_{\substack{\mathrm{supp}\,\gamma\cap\{1,\dots,n\} \\ =\{1,\dots,\hat{k}_2,\dots,n\}}} a_\gamma^{(k_2)}[1,\dots,\overset{n+1}{k_1},\dots,n]\gamma \\
&\quad + \sum_{\mathrm{supp}\,\gamma\not\supset\{1,\dots,\hat{k}_2,\dots,n\}} a_\gamma^{(k_2)}[1,\dots,\overset{n+1}{k_1},\dots,n]\gamma
\end{aligned}
$$

Suppose that $\mathrm{supp}\gamma \cap \{1,\dots,n\} = \{1,\dots,\hat{k}_2,\dots,n\}$. Then $\gamma = [1,\dots,\hat{k}_2,\dots,n,l]$ for some l with $n+1 \le l \le u$. Thus $\gamma = (-1)^{n-k_2}[1,\dots,\overset{l}{k_2},\dots,n]$. By applying Lemma 5.13 to

$$[1,\dots,\overset{l}{k_2},\dots,n][1,\dots,\overset{n+1}{k_1},\dots,n],$$

by substituting u, n, k, s, l by u, n, k_1-1, $n+1$, k_1+1 respectively, we see that

$$\begin{aligned}
&[1,\dots,\overset{l}{k_2},\dots,n][1,\dots,\overset{n+1}{k_1},\dots,n] \\
&= \delta[1,\dots,\overset{n+1}{k_1}\dots,\overset{l}{k_2},\dots,n] + [1,\dots,\overset{n+1}{k_2},\dots,n][1,\dots,\overset{l}{k_1},\dots,n] \\
&= \delta[1,\dots,\overset{n+1}{k_1}\dots,\overset{l}{k_2},\dots,n] + (-1)^{n-k_1}[1,\dots,\overset{n+1}{k_2},\dots,n][1,\dots,\hat{k}_1,\dots,n,l]
\end{aligned}$$

Therefore, if we set

$$b_\gamma^{(k_1)} = \begin{cases} a_\gamma^{(k_1)} - (-1)^{k_2-k_1} a^{(k_2)}_{[1,\dots,\hat{k}_2,\dots,n,l]} & \text{if } \gamma = [1,\dots,\hat{k}_1\dots,n,l] \\ a_\gamma^{(k_1)} & \text{otherwise} \end{cases}$$

for $\gamma \in \Gamma \setminus \{\delta\}$, we see that there are $b_\gamma \in A_0$ such that

$$\sum_\gamma b_\gamma \delta\gamma - \sum_{\gamma\in\Gamma\setminus\{\delta\}} b_\gamma^{(k_1)}[1,\dots,\overset{n+1}{k_2},\dots,n]\gamma \\
+ \sum_{\mathrm{supp}\gamma \not\supset \{1,\dots,\hat{k}_2,\dots,n\}} a_\gamma^{(k_2)}[1,\dots,\overset{n+1}{k_1},\dots,n]\gamma = 0.$$

Thus we see, by the assumption, that

$$b_\gamma^{(k_1)} = 0 \quad \text{if } \gamma \in \Gamma \setminus \{\delta\}$$

and

$$a_\gamma^{(k_2)} = 0 \quad \text{if } \mathrm{supp}\gamma \not\supset \{1,\dots,\hat{k}_2,\dots,n\}.$$

Therefore, by the definition of $b_\gamma^{(k_1)}$, we see that

$$(-1)^{n-k_1} a^{(k_1)}_{[1,\dots,\hat{k}_1,\dots,n,l]} = (-1)^{n-k_2} a^{(k_2)}_{[1,\dots,\hat{k}_2,\dots,n,l]}$$

and

$$a_\gamma^{(k_1)} = 0 \quad \text{if } \mathrm{supp}\gamma \not\supset \{1,\dots,\hat{k}_1,\dots,n\}.$$

Since these hold for any k_1, k_2 with $1 \le k_1 < k_2 \le n$, we see that if we set

$$c_l = a^{(n)}_{[1,\dots,n-1,l]}$$

for l with $n+1 \le l \le u$,

$$\begin{aligned}[1,\dots,\overset{u+1}{k},\dots,n] &= a^{(k)}_\delta \delta + (-1)^{n-k}\sum_{l=n+1}^{u} c_l[1,\dots,\hat{k},\dots,n,l] \\ &= a^{(k)}_\delta \delta + \sum_{l=n+1}^{u} c_l[1,\dots,\overset{l}{k},\dots,n]\end{aligned}$$

for any k with $1 \le k \le n$.

Set

$$\boldsymbol{z} = \boldsymbol{y} - \sum_{s=1}^{n} a^{(s)}_\delta X^{=s} - \sum_{l=n+1}^{u} c_l X^{=l}$$

and

$$Z = \begin{pmatrix} X \\ \boldsymbol{z} \end{pmatrix}.$$

Then

$$\begin{aligned}&[1,\dots,\overset{u+1}{k},\dots,n]_Z \\ &= [1,\dots,\overset{u+1}{k},\dots,n]_{\tilde{X}} - \sum_{s=1}^{n} a^{(s)}_\delta [1,\dots\overset{s}{k},\dots n]_X - \sum_{l=n+1}^{u} c_l[1,\dots\overset{l}{k},\dots n]_X \\ &= 0\end{aligned}$$

for any k with $1 \le k \le n$. Thus $\boldsymbol{z}\mathrm{Cof}(X^{\le n}) = \mathbf{0}$ and we see that $\boldsymbol{z} = \mathbf{0}$ since $\det \mathrm{Cof}(X^{\le n}) = \delta^{n-1}$ is a non-zerodivisor. $\square$

Lemma 5.15. *Let L be a field, n, m integers with $n \ge m \ge 3$, $x_1,\dots,x_m$ indeterminates and α, $\beta \in L$ with $0 \ne \alpha \ne \beta \ne 0$. Then the following polynomials are linearly independent over L.*

$$\begin{aligned}&x_1^{2n} \\ &x_1^{2n-1}(x_2 - \beta x_m) \\ &x_1^{2n-1}x_s \quad (3 \le s \le m-1) \\ &x_1^{n}(x_2 - \alpha x_m)(x_2 - \beta x_m)x_{b_1}\cdots x_{b_{n-2}} \quad (1 \le b_1 \le \cdots \le b_{n-2} \le 2) \\ &x_1^{n}(x_2 - \alpha x_m)x_{b_1}\cdots x_{b_{n-2}}x_s \quad (1 \le b_1 \le \cdots \le b_{n-2} \le 2,\ 3 \le s \le m-1)\end{aligned}$$

$$
\begin{aligned}
&x_1^n x_{b_1} \cdots x_{b_n} \quad (1 \le b_1 \le \cdots \le b_n \le m-1,\ b_{n-1} \ge 3)\\
&x_1^{2n-2}(x_2 - \beta x_m)^2\\
&x_1^{2n-2}(x_2 - \beta x_m)x_s \quad (3 \le s \le m-1)\\
&x_1^{n-1}(x_2 - \alpha x_m)(x_2 - \beta x_m)^2 x_{b_1} \cdots x_{b_{n-2}} \quad (1 \le b_1 \le \cdots \le b_{n-2} \le 2)\\
&x_1^{n-1}(x_2 - \alpha x_m)(x_2 - \beta x_m) x_{b_1} \cdots x_{b_{n-2}} x_s\\
&\qquad\qquad (1 \le b_1 \le \cdots \le b_{n-2} \le 2,\ 3 \le s \le m-1)\\
&x_1^{n-1}(x_2 - \beta x_m) x_{b_1} \cdots x_{b_n} \quad (1 \le b_1 \le \cdots \le b_n \le m-1,\ b_{n-1} \ge 3)\\
&x_1^{n-2}(x_2 - \alpha x_m)^2(x_2 - \beta x_m)^2 x_{b_1} \cdots x_{b_{n-2}} \quad (1 \le b_1 \le \cdots \le b_{n-2} \le 2)\\
&x_1^{n-2}(x_2 - \alpha x_m)^2(x_2 - \beta x_m) x_{b_1} \cdots x_{b_{n-2}} x_s\\
&\qquad\qquad (1 \le b_1 \le \cdots \le b_{n-2} \le 2,\ 3 \le s \le m-1)\\
&x_1^{n-2}(x_2 - \alpha x_m)(x_2 - \beta x_m) x_{b_1} \cdots x_{b_n}\\
&\qquad\qquad (1 \le b_1 \le \cdots \le b_n \le m-1,\ b_{n-1} \ge 3)
\end{aligned}
$$

Proof. Set $\deg x_1 = \deg x_2 = \deg x_m = 0$ and $\deg x_3 = \cdots = \deg x_{m-1} = 1$. Then the polynomials under consideration are homogeneous. Thus it is enough to show that for each integer d, the polynomials of degree d in the above list are linearly independent over L.

First consider the polynomials with degree more than 1. They are

$$
\begin{aligned}
&x_1^n x_{b_1} \cdots x_{b_n} \quad (1 \le b_1 \le \cdots \le b_n \le m-1,\ b_{n-1} \ge 3)\\
&x_1^{n-1}(x_2 - \beta x_m) x_{b_1} \cdots x_{b_n} \quad (1 \le b_1 \le \cdots \le b_n \le m-1,\ b_{n-1} \ge 3)\\
&x_1^{n-2}(x_2 - \alpha x_m)(x_2 - \beta x_m) x_{b_1} \cdots x_{b_n}\\
&\qquad\qquad (1 \le b_1 \le \cdots \le b_n \le m-1,\ b_{n-1} \ge 3).
\end{aligned}
$$

By first substituting $\beta^{-1}x_2$ to x_m and next by substituting $\alpha^{-1}x_2$ to x_m one sees that these polynomials are linearly independent.

Next consider the polynomials with degree 1. They are

$$
\begin{aligned}
&x_1^{2n-1} x_s \quad (3 \le s \le m-1)\\
&x_1^n(x_2 - \alpha x_m) x_{b_1} \cdots x_{b_{n-2}} x_s \quad (1 \le b_1 \le \cdots \le b_{n-2} \le 2,\ 3 \le s \le m-1)\\
&x_1^{2n-2}(x_2 - \beta x_m)x_s \quad (3 \le s \le m-1)\\
&x_1^{n-1}(x_2 - \alpha x_m)(x_2 - \beta x_m) x_{b_1} \cdots x_{b_{n-2}} x_s\\
&\qquad\qquad (1 \le b_1 \le \cdots \le b_{n-2} \le 2,\ 3 \le s \le m-1)\\
&x_1^{n-2}(x_2 - \alpha x_m)^2(x_2 - \beta x_m) x_{b_1} \cdots x_{b_{n-2}} x_s\\
&\qquad\qquad (1 \le b_1 \le \cdots \le b_{n-2} \le 2,\ 3 \le s \le m-1).
\end{aligned}
$$

By a similar but more subtle argument as above, one sees that these polynomials are linearly independent.

Finally consider the polynomials with degree 0. They are

$$\begin{aligned}
&x_1^{2n}\\
&x_1^{2n-1}(x_2-\beta x_m)\\
&x_1^{n}(x_2-\alpha x_m)(x_2-\beta x_m)x_{b_1}\cdots x_{b_{n-2}} \quad (1\le b_1\le\cdots\le b_{n-2}\le 2)\\
&x_1^{2n-2}(x_2-\beta x_m)^2\\
&x_1^{n-1}(x_2-\alpha x_m)(x_2-\beta x_m)^2x_{b_1}\cdots x_{b_{n-2}} \quad (1\le b_1\le\cdots\le b_{n-2}\le 2)\\
&x_1^{n-2}(x_2-\alpha x_m)^2(x_2-\beta x_m)^2x_{b_1}\cdots x_{b_{n-2}} \quad (1\le b_1\le\cdots\le b_{n-2}\le 2)
\end{aligned}$$

By a similar but more subtle argument, one sees that these polynomials are linearly independent. □

Lemma 5.16. *Let K be a field, $T=(T_1;\cdots;T_m)=(t_{ijk})$ a $u\times n\times m$-tensor of indeterminates with $u>n\ge m\ge u-n+2$, $\boldsymbol{x}=(x_1,\ldots,x_m)$ a vector of indeterminates. Set $X=M(\boldsymbol{x},T)$ (cf. Definition 4.6), $\Gamma=\Gamma(u\times n)$, $\delta=\delta_0=[1,\ldots,n]$ and $A=K[T][\boldsymbol{x}]$. Then δ_X is a non-zerodivisor of A and for any k_1, k_2 with $1\le k_1,k_2\le n$, $k_1\ne k_2$,*

$$\begin{aligned}
&\delta_X\gamma_X \quad (\gamma\in\Gamma)\\
&[1,\ldots,\overset{n+1}{k_2},\ldots,n]_X\gamma_X \quad (\gamma\in\Gamma\setminus\{\delta\})\\
&[1,\ldots,\overset{n+1}{k_1},\ldots,n]_X\gamma_X \quad (\gamma\in\Gamma,\operatorname{supp}\gamma\not\supset\{1,\ldots,\hat{k}_2,\ldots,n\})
\end{aligned}$$

are linearly independent over $K[T]$.

Proof. The first assertion is clear since A is an integral domain and $\delta_X\ne 0$. Next we prove the second assertion. By symmetry, we may assume that $k_1=n-1$ and $k_2=n$. Set $\delta_1=[1,2,\ldots,n-1,n+1]$ and $\delta_2=[1,2,\ldots,n-2,n,n+1]$. We introduce the lexicographic monomial preorder induced by the preorder on the indeterminates defined as follows.

If one of the following is satisfied, we define $t_{ijk}\succ t_{i'j'k'}$.

- $i<i'$.
- $i=i'$ and $j<j'$.
- $i=i'$, $j=j'$, $k<k'$ and "$i<j$ or $i>j+u-n$".

In case $0\le i-j\le u-n$ and $(i,j)\ne(n,n-1),(n+1,n)$, we define

$$t_{i,j,i-j+1}\succ t_{i,j,i-j+2}\succ\cdots\succ t_{i,j,m}\succ t_{i,j,1}\succ\cdots\succ t_{i,j,i-j}.$$

In case $(i,j) = (n, n-1)$ or $(n+1, n)$, we define

$$t_{i,j,2} \sim t_{i,j,m} \succ t_{i,j,1} \succ t_{i,j,3} \succ \cdots \succ t_{i,j,m-1}.$$

And

$$t_{ijk} \succ x_1 \sim x_2 \sim \cdots \sim x_m$$

for any i, j and k.

Set

$$\begin{aligned}
\Gamma_0 &= \{[a_1, \ldots, a_n] \in \Gamma \mid a_{n-1} = n-1\}, \\
\Gamma_1 &= \{[a_1, \ldots, a_n] \in \Gamma \mid a_{n-1} = n\}, \\
\Gamma_2 &= \{[a_1, \ldots, a_n] \in \Gamma \mid a_{n-1} \geq n+1\}, \\
\Gamma_{00} &= \{[a_1, \ldots, a_n] \in \Gamma_0 \mid a_n = n\} = \{\delta_0\}, \\
\Gamma_{01} &= \{[a_1, \ldots, a_n] \in \Gamma_0 \mid a_n = n+1\} = \{\delta_1\}, \\
\Gamma_{02} &= \{[a_1, \ldots, a_n] \in \Gamma_0 \mid a_n \geq n+2\}, \\
\Gamma_{11} &= \{[a_1, \ldots, a_n] \in \Gamma_1 \mid a_n = n+1\} = \{\delta_2\}, \\
\Gamma_{12} &= \{[a_1, \ldots, a_n] \in \Gamma_1 \mid a_n \geq n+2\}.
\end{aligned}$$

Then

$$\begin{aligned}
\Gamma &= \Gamma_0 \sqcup \Gamma_1 \sqcup \Gamma_2 \\
\Gamma_0 &= \Gamma_{00} \sqcup \Gamma_{01} \sqcup \Gamma_{02} \\
\Gamma_1 &= \Gamma_{11} \sqcup \Gamma_{12}.
\end{aligned}$$

Set $\alpha_1 = t_{n,n-1,2}$, $\alpha_2 = t_{n,n-1,m}$, $\beta_1 = t_{n+1,n,2}$ and $\beta_2 = t_{n+1,n,m}$. Then for $\gamma = [a_1, \ldots, a_n] \in \Gamma$, $\operatorname{lf}(\gamma_X)$ is, up to multiplication of nonzero element of $K[T]$, as follows.

$$\begin{aligned}
&x_1^n \quad \text{if } \gamma \in \Gamma_{00} \\
&x_1^{n-1}(\beta_1 x_2 + \beta_2 x_m) \quad \text{if } \gamma \in \Gamma_{01} \\
&x_1^{n-1} x_{a_n - n+1} \quad \text{if } \gamma \in \Gamma_{02} \\
&x_{a_1} x_{a_2-1} \cdots x_{a_{n-2}-n+3}(\alpha_1 x_2 + \alpha_2 x_m)(\beta_1 x_2 + \beta_2 x_m) \quad \text{if } \gamma \in \Gamma_{11} \\
&x_{a_1} x_{a_2-1} \cdots x_{a_{n-2}-n+3}(\alpha_1 x_2 + \alpha_2 x_m) x_{a_n-n+1} \quad \text{if } \gamma \in \Gamma_{12} \\
&x_{a_1} x_{a_2-1} \cdots x_{a_n-n+1} \quad \text{if } \gamma \in \Gamma_2.
\end{aligned}$$

Therefore, for $\gamma = [a_1, \ldots, a_n] \in \Gamma$, $\operatorname{lf}(\delta_X \gamma_X)$ is, up to multiplication of nonzero element of $K[T]$,

$$
\begin{aligned}
&x_1^{2n} \quad \text{if } \gamma \in \Gamma_{00} \\
&x_1^{2n-1}(\beta_1 x_2 + \beta_2 x_m) \quad \text{if } \gamma \in \Gamma_{01} \\
&x_1^{2n-1} x_{a_n-n+1} \quad \text{if } \gamma \in \Gamma_{02} \\
&x_1^n x_{a_1} x_{a_2-1} \cdots x_{a_{n-2}-n+3}(\alpha_1 x_2 + \alpha_2 x_m)(\beta_1 x_2 + \beta_2 x_m) \quad \text{if } \gamma \in \Gamma_{11} \\
&x_1^n x_{a_1} x_{a_2-1} \cdots x_{a_{n-2}-n+3}(\alpha_1 x_2 + \alpha_2 x_m) x_{a_n-n+1} \quad \text{if } \gamma \in \Gamma_{12} \\
&x_1^n x_{a_1} x_{a_2-1} \cdots x_{a_n-n+1} \quad \text{if } \gamma \in \Gamma_2.
\end{aligned}
$$

For $\gamma = [a_1, \dots, a_n] \in \Gamma \setminus \{\delta\}$, $\mathrm{lf}((\delta_1)_X \gamma_X)$ is, up to multiplication of nonzero element of $K[T]$,

$$
\begin{aligned}
&x_1^{2n-2}(\beta_1 x_2 + \beta_2 x_m)^2 \quad \text{if } \gamma \in \Gamma_{01} \\
&x_1^{2n-2}(\beta_1 x_2 + \beta_2 x_m) x_{a_n-n+1} \quad \text{if } \gamma \in \Gamma_{02} \\
&x_1^{n-1} x_{a_1} x_{a_2-1} \cdots x_{a_{n-2}-n+3}(\alpha_1 x_2 + \alpha_2 x_m)(\beta_1 x_2 + \beta_2 x_m)^2 \quad \text{if } \gamma \in \Gamma_{11} \\
&x_1^{n-1} x_{a_1} x_{a_2-1} \cdots x_{a_{n-2}-n+3}(\alpha_1 x_2 + \alpha_2 x_m)(\beta_1 x_2 + \beta_2 x_m) x_{a_n-n+1} \quad \text{if } \gamma \in \Gamma_{12} \\
&x_1^{n-1}(\beta_1 x_2 + \beta_2 x_m) x_{a_1} x_{a_2-1} \cdots x_{a_n-n+1} \quad \text{if } \gamma \in \Gamma_2.
\end{aligned}
$$

Finally, consider the leading form of $(\delta_2)_X \gamma_X$, where $\gamma = [a_1, \dots, a_n] \in \Gamma$ and $\mathrm{supp}\gamma \not\supset \{1, \dots, n-1\}$. It is easily verified that $\mathrm{supp}\gamma \not\supset \{1, \dots, n-1\}$ if and only if $\gamma \in \Gamma_1 \sqcup \Gamma_2$. Thus the leading form of $(\delta_2)_X \gamma_X$ is, up to multiplication of nonzero element of $K[T]$,

$$
\begin{aligned}
&x_1^{n-2} x_{a_1} x_{a_2-1} \cdots x_{a_{n-2}-n+3}(\alpha_1 x_2 + \alpha_2 x_m)^2(\beta_1 x_2 + \beta_2 x_m)^2 \quad \text{if} \gamma \in \Gamma_{11} \\
&x_1^{n-2} x_{a_1} x_{a_2-1} \cdots x_{a_{n-2}-n+3}(\alpha_1 x_2 + \alpha_2 x_m)^2(\beta_1 x_2 + \beta_2 x_m) x_{a_n-n+1} \quad \text{if } \gamma \in \Gamma_{12} \\
&x_1^{n-2}(\alpha_1 x_2 + \alpha_2 x_m)(\beta_1 x_2 + \beta_2 x_m) x_{a_1} x_{a_2-1} \cdots x_{a_n-n+1} \quad \text{if } \gamma \in \Gamma_2.
\end{aligned}
$$

Since

$$
\begin{aligned}
&1 \le a_1 \le a_2 - 1 \le \cdots \le a_n - n + 1 \le u - n + 1 \le m - 1 \\
&a_{n-2} - n + 3 \le 2 \quad \text{if } \gamma \in \Gamma_1 \\
&a_{n-1} - n + 2 \ge 3 \quad \text{if } \gamma \in \Gamma_2
\end{aligned}
$$

and

$$
a_n - n + 1 \ge 3 \quad \text{if } \gamma \in \Gamma_{02} \sqcup \Gamma_{12},
$$

we see by Lemma 5.15 that

$$
\begin{aligned}
&\mathrm{lf}(\delta_X \gamma_X) \quad (\gamma \in \Gamma) \\
&\mathrm{lf}((\delta_1)_X \gamma_X) \quad (\gamma \in \Gamma \setminus \{\delta\})
\end{aligned}
$$

$$\mathrm{lf}((\delta_2)_X\gamma_X) \quad (\gamma \in \Gamma,\ \mathrm{supp}\gamma \not\supset \{1,\dots,n-1\})$$

are linearly independent over $K[T]$. The assertion follows by Lemma 5.12. $\square$

Corollary 5.17. *Let $\mathbb{K}$ be a field, T a $u \times n \times m$ tensor of indeterminates with $u > n \geq m \geq u - n + 2$, R a commutative ring containing $\mathbb{K}(T)$, $\boldsymbol{x} = (x_1, \dots, x_m)$ a vector of indeterminates. Set $M = M(\boldsymbol{x}, T)$ (cf. Definition 4.6) and $B = R[\boldsymbol{x}]$. Then δ_M is a non-zerodivisor of B and for any k_1, k_2 with $1 \leq k_1, k_2 \leq n$, $k_1 \neq k_2$,*

$$\begin{aligned}
&\delta_M\gamma_M \quad (\gamma \in \Gamma)\\
&[1,\dots,\overset{n+1}{k_2},\dots,n]_M\gamma_M \quad (\gamma \in \Gamma \setminus \{\delta\})\\
&[1,\dots,\overset{n+1}{k_1},\dots,n]_M\gamma_M \quad (\gamma \in \Gamma,\ \mathrm{supp}\gamma \not\supset \{1,\dots,\hat{k}_2,\dots,n\})
\end{aligned}$$

are linearly independent over R.

Proof. Set $A = \mathbb{K}[T][\boldsymbol{x}]$. Then $B = A \otimes_{\mathbb{K}[T]} R$.

Since R is flat over $\mathbb{K}[T]$, we see that B is flat over A. By Lemma 5.16, we see that δ_M is a non-zerodivisor of A and for any k_1, k_2 with $1 \leq k_1, k_2 \leq n$, $k_1 \neq k_2$,

$$\begin{aligned}
&\delta_M\gamma_M \quad (\gamma \in \Gamma)\\
&[1,\dots,\overset{n+1}{k_2},\dots,n]_M\gamma_M \quad (\gamma \in \Gamma \setminus \{\delta\})\\
&[1,\dots,\overset{n+1}{k_1},\dots,n]_M\gamma_M \quad (\gamma \in \Gamma,\ \mathrm{supp}\gamma \not\supset \{1,\dots,\hat{k}_2,\dots,n\})
\end{aligned}$$

are linearly independent over $\mathbb{K}[T]$. Since R (resp. B) is flat over $\mathbb{K}[T]$ (resp. A), we see that δ_M is a non-zerodivisor of B and for any k_1, k_2 with $1 \leq k_1, k_2 \leq n$, $k_1 \neq k_2$,

$$\begin{aligned}
&\delta_M\gamma_M \quad (\gamma \in \Gamma)\\
&[1,\dots,\overset{n+1}{k_2},\dots,n]_M\gamma_M \quad (\gamma \in \Gamma \setminus \{\delta\})\\
&[1,\dots,\overset{n+1}{k_1},\dots,n]_M\gamma_M \quad (\gamma \in \Gamma, \mathrm{supp}\gamma \not\supset \{1,\dots,\hat{k}_2,\dots,n\})
\end{aligned}$$

are linearly independent over R. $\square$

Corollary 5.18. *Let $\boldsymbol{x} = (x_1, \dots, x_m)$ be a vector of indeterminates. Suppose that $u > n$ and $Y \in \mathscr{I}$ (cf. Definition 4.15) and set $M = M(\boldsymbol{x}, Y)$. Then δ_M is a non-zerodivisor of $\mathbb{R}[\boldsymbol{x}]$ and for any k_1, k_2 with $1 \leq k_1, k_2 \leq n$, $k_1 \neq k_2$,*

$$\begin{aligned}
&\delta_M\gamma_M \quad (\gamma \in \Gamma)\\
&[1,\dots,\overset{n+1}{k_2},\dots,n]_M\gamma_M \quad (\gamma \in \Gamma \setminus \{\delta\})
\end{aligned}$$

$$[1,\ldots,\overset{n+1}{k_1},\ldots,n]_M\gamma_M \quad (\gamma\in\Gamma,\ \mathrm{supp}\gamma\not\supset\{1,\ldots,\hat{k}_2,\ldots,n\})$$

are linearly independent over $\mathbb{R}$*.*

Definition 5.19. Let $\boldsymbol{x}=(x_1,\ldots,x_m)$ be a vector of indeterminates. We set $\mathscr{Q}':=\{Y\in\mathbb{R}^{u\times n\times m}\mid \delta_M\neq 0$ and $\delta_M\gamma_M\quad(\gamma\in\Gamma)$, $[1,\ldots,\overset{n+1}{k_2},\ldots,n]_M\gamma_M\quad(\gamma\in\Gamma\setminus\{\delta\})$, $[1,\ldots,\overset{n+1}{k_1},\ldots,n]_M\gamma_M\quad(\gamma\in\Gamma,\ \mathrm{supp}\gamma\not\supset\{1,\ldots,\hat{k}_2,\ldots,n\})$ are linearly independent over $\mathbb{R}$ for any k_1, k_2 with $1\le k_1<k_2\le n$, where $M=M(\boldsymbol{x},Y)\}$.

Remark 5.20. $\mathscr{Q}'$ is a Zariski open set of $\mathbb{R}^{u\times n\times m}$ and by Corollary 5.18, we see that $\mathscr{Q}'\supset\mathscr{I}$. In particular, $\mathscr{Q}'$ is a Zariski dense open subset of $\mathbb{R}^{u\times n\times m}$.

By Lemma 5.14, we see the following fact.

Proposition 5.21. *Let u, n and m be integers with $u>n\ge m\ge u-n+2$ and let $\boldsymbol{x}=(x_1,\ldots,x_m)$ be a vector of indeterminates. Suppose that $Y\in\mathscr{Q}'$ and $\boldsymbol{y}\in\mathbb{R}^{1\times n\times m}$. Set*

$$\tilde{Y}=\begin{pmatrix}Y\\ \boldsymbol{y}\end{pmatrix}.$$

If $I_n(M(\boldsymbol{x},\tilde{Y}))=I_n(M(\boldsymbol{x},Y))$, then $\mathrm{fl}_1(\boldsymbol{y})$ is an $\mathbb{R}$-linear combination of rows of $\mathrm{fl}_1(Y)$.

Proof. Set $M=M(\boldsymbol{x},Y)$. Since $\mathbb{R}[\boldsymbol{x}]$ is a domain and $\delta_M\neq 0$ by the definition of $\mathscr{Q}'$, we see that δ_M is a non-zerodivisor of $\mathbb{R}[\boldsymbol{x}]$. Moreover,

$$\begin{aligned}&\delta_M\gamma_M \quad (\gamma\in\Gamma)\\ &[1,\ldots,\overset{n+1}{k_2},\ldots,n]_M\gamma_M \quad (\gamma\in\Gamma\setminus\{\delta\})\\ &[1,\ldots,\overset{n+1}{k_1},\ldots,n]_M\gamma_M \quad (\gamma\in\Gamma,\ \mathrm{supp}\gamma\not\supset\{1,\ldots,\hat{k}_2,\ldots,n\})\end{aligned}$$

are linearly independent over $\mathbb{R}$ for any k_1, k_2 with $1\le k_1<k_2\le n$ by the definition of $\mathscr{Q}'$. Thus by Lemma 5.14, we see that $M(\boldsymbol{x},\boldsymbol{y})$ is an $\mathbb{R}$-linear combination of rows of $M=M(\boldsymbol{x},Y)$. Since $x_1,\ldots,x_m$ are indeterminates, we see that $\mathrm{fl}_1(\boldsymbol{y})$ is an $\mathbb{R}$-linear combination of rows of $\mathrm{fl}_1(Y)$. $\square$

6. Tensor with rank p

Let $3\le m\le n$, $(m-1)(n-1)+1\le p\le(m-1)n$ and set $l=(m-1)n-p$ and $u=n+l$. In the following of this paper, we use the results of the previous sections by setting $t=n$. See Definition 4.6. Then $v=l+1$ and it follows that $v<m$ since $l\le m-2$. Note also $u+p=nm$. We make bunch of definitions used in the sequel of this paper.

Definition 6.1. We put

$$\mathscr{V} = \mathscr{V}^{n\times p\times m} := \{T \in \mathbb{R}^{n\times p\times m} \mid \mathrm{fl}_2(T)^{\le p} \text{ is nonsingular}\}$$

and define $\sigma\colon \mathscr{V} \to \mathbb{R}^{u\times p}$ be a map defined as

$$\sigma(T) = ({}^{p<}\mathrm{fl}_2(T))(\mathrm{fl}_2(T)^{\le p})^{-1}.$$

We denote by $\mathscr{A}^{u\times n\times m}$ the set of all $u\times n\times m$ absolutely full column rank tensors and put $\mathscr{C}^{u\times n\times m} = \mathbb{R}^{u\times n\times m} \setminus \mathscr{A}^{u\times n\times m}$. Note that $\mathscr{A}^{u\times n\times m} = \mathscr{A}_n^{u\times n\times m}$ and $\mathscr{C}^{u\times n\times m} = \mathscr{C}_n^{u\times n\times m}$ in the notation of Definition 4.15.

Let $\mathcal{M}$ be the subset of $\mathbb{R}^{u\times nm}$ consisting of all matrices $W = (W_1, \dots, W_m)$ satisfying that there are $A = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_p) \in \mathbb{R}^{n\times p}$ and $p \times p$ diagonal matrices $D_1, \dots, D_m$ such that $D_k = \mathrm{Diag}(d_{1k}, \dots, d_{pk})$ for $1 \le k \le m$, $(d_{j1}W_1 + \cdots + d_{jm}W_m)\boldsymbol{a}_j = \boldsymbol{0}$ for $1 \le j \le p$, and

$$\begin{pmatrix} AD_1 \\ \vdots \\ AD_{m-2} \\ A^{\le n-l}D_{m-1} \end{pmatrix} \tag{6.1}$$

is nonsingular. Let $\iota\colon \mathbb{R}^{u\times p} \to \mathbb{R}^{u\times nm}$ be a map which sends A to $(A, -E_u)$. Moreover put

$$\mathcal{C} := \{W \in \mathbb{R}^{u\times nm} \mid W \notin \mathrm{fl}_1(\mathscr{A}^{u\times n\times m})\} = \{W \in \mathbb{R}^{u\times n\times m} \mid W \in \mathrm{fl}_1(\mathscr{C}^{u\times n\times m})\}.$$

We define $\phi\colon \mathbb{R}^{1\times m} \times \mathbb{R}^n \to \mathbb{R}^p$ a map defined as

$$\phi(\boldsymbol{a}, \boldsymbol{b}) = \begin{pmatrix} a_1\boldsymbol{b} \\ a_2\boldsymbol{b} \\ \vdots \\ a_{m-1}\boldsymbol{b}^{\le n-l} \end{pmatrix} \in \mathbb{R}^p, \text{ where } \boldsymbol{a} = (a_1, \dots, a_{m-1}, a_m).$$

Recall that the set $\mathscr{A}^{u\times n\times m}$ is an open subset of $\mathbb{R}^{u\times n\times m}$ by Lemma 3.3 or Corollary 4.20.

Proposition 6.2. *σ is an open, surjective and continuous map.*

Proof. Clearly σ is surjective and continuous. Let $\mathscr{O}$ be an open subset of $\mathscr{V}$ and let $h\colon \mathbb{R}^{nm\times p} \to \mathbb{R}^{p\times p} \times \mathbb{R}^{u\times p}$ be a homeomorphism defined as $h(M) = (M^{\le p}, {}^{p<}M)$. Then $h(\mathrm{fl}_2(\mathscr{O}))$ can be written by

$$\bigcup_\lambda O_{1,\lambda} \times O_{2,\lambda}$$

for some open subsets $O_{1,\lambda} \subset \mathbb{R}^{p\times p}$ and $O_{2,\lambda} \subset \mathbb{R}^{u\times p}$ and thus

$$\sigma(\mathscr{O}) = \bigcup_{\lambda} \bigcup_{A\in O_{1,\lambda}} O_{2,\lambda}A^{-1}.$$

The set $O_{2,\lambda}A^{-1}$ is open and then $\sigma(\mathscr{O})$ is open. □

The following fact follows from the definition.

Lemma 6.3. *$\mathcal{M}$ is stable under the action of* $\mathrm{GL}(u,\mathbb{R})$.

Lemma 6.4. *$\mathcal{M} \subset \mathcal{C}$.*

Proof. By observing the first column of (6.1), we see that $\boldsymbol{a}_1 \neq \mathbf{0}$ and at least one of d_{11}, $d_{12}, \ldots, d_{1,m-1}$ is nonzero, since $\begin{pmatrix} AD_1 \\ \vdots \\ AD_{m-2} \\ A^{\leq n-l}D_{m-1} \end{pmatrix}$ is nonsingular, where $D_k = \mathrm{Diag}(d_{1k}, \ldots, d_{pk})$ for $1 \leq k \leq m-1$. Since $d_{11}W_1 + \cdots + d_{1m}W_m$ is singular, $(W_1; \ldots; W_m)$ is not absolutely full column rank, i.e., $\mathcal{M} \subset \mathcal{C}$. □

Theorem 6.5. *Let $X \in \mathscr{V} \subset \mathbb{R}^{n\times p\times m}$. Put $(W_1, \ldots, W_{m-1}, W_m) = \iota(\sigma(X))$, where $W_1, \ldots, W_m \in \mathbb{R}^{u\times n}$. The following four statements are equivalent.*

(1) $\operatorname{rank} X = p$.
(2) *There are an $n \times p$ matrix A, and diagonal $p \times p$ matrices $D_1, \ldots, D_m$ such that*

$$W_1AD_1 + W_2AD_2 + \cdots + W_{m-1}AD_{m-1} + W_mAD_m = O \tag{6.2}$$

and

$$N = \begin{pmatrix} AD_1 \\ \vdots \\ AD_{m-2} \\ A^{\leq n-l}D_{m-1} \end{pmatrix} \tag{6.3}$$

is nonsingular.
(3) $\iota(\sigma(X)) \in \mathcal{M}$.

Proof. It holds that $\operatorname{rank} X \geq p$, since $\mathrm{fl}_2(X)^{\leq p}$ has rank p. Put $(S_1; \ldots; S_m) = X(\mathrm{fl}_2(X)^{\leq p})^{-1}$ Then $\operatorname{rank} X = \operatorname{rank}(S_1; \ldots; S_m)$ and

$$\begin{pmatrix} {}^{n-l<}S_{m-1} \\ S_m \end{pmatrix} = (W_1, W_2, \ldots, W_{m-2}, (W_{m-1})_{\leq n-l}).$$

(1) ⇒ (2): Since $\operatorname{rank}(S_1;\dots;S_m)=p$, there are an $n\times p$-matrix A, a $p\times p$-matrix Q, and $p\times p$ diagonal matrices $D_1,\dots,D_m$ such that

$$AD_kQ=S_k \quad \text{for } k=1,\dots,m.$$

Since

$$NQ=\begin{pmatrix} AD_1\\ \vdots \\ AD_{m-2}\\ A^{\le n-l}D_{m-1}\end{pmatrix}Q=\begin{pmatrix} S_1\\ \vdots \\ S_{m-2}\\ (S_{m-1})^{\le n-l}\end{pmatrix}=E_p,$$

we see that N and Q are nonsingular and $Q^{-1}=N$. Since

$$\begin{pmatrix} {}^{n-l<}S_{m-1}\\ S_m\end{pmatrix}=\sigma(X)=(W_1,W_2,\dots,W_{m-2},(W_{m-1})_{\le n-l}),$$

and $AD_k=S_kN$ for $k=m-1$, m, we see that

$$\begin{aligned}
&O_{u\times p}\\
&=\begin{pmatrix} {}^{n-l<}S_{m-1}\\ S_m\end{pmatrix}N-\begin{pmatrix} {}^{n-l<}AD_{m-1}\\ AD_m\end{pmatrix}\\
&=W_1AD_1+\cdots+W_{m-2}AD_{m-2}\\
&\qquad +(W_{m-1})_{\le n-l}A^{\le n-l}D_{m-1}+\begin{pmatrix} -E_l\\ O\end{pmatrix}{}^{n-l<}AD_{m-1}+\begin{pmatrix} O\\ -E_n\end{pmatrix}AD_m\\
&=W_1AD_1+\cdots+W_{m-2}AD_{m-2}+W_{m-1}AD_{m-1}+W_mAD_m.
\end{aligned}$$

Therefore the equation (6.2) holds.

(2) ⇒ (1): Set $Q=N^{-1}$. Then, since $NQ=E_p$, we see that

$$AD_kQ=S_k \quad 1\le k\le m-2$$

and

$$A^{\le n-l}D_{m-1}Q=S_{m-1}^{\le n-l}.$$

Furthermore, since

$$\begin{aligned}
&W_1AD_1+\cdots+W_{m-2}AD_{m-2}\\
&\qquad +(W_{m-1})_{\le n-l}A^{\le n-l}D_{m-1}+\begin{pmatrix} -E_l\\ O\end{pmatrix}{}^{n-l<}AD_{m-1}+\begin{pmatrix} O\\ -E_n\end{pmatrix}AD_m\\
&=W_1AD_1+\cdots+W_mAD_m\\
&=O_{u\times p},
\end{aligned}$$

we see that

$$\begin{pmatrix} {}^{n-l<}S_{m-1} \\ S_m \end{pmatrix} N = \begin{pmatrix} E_l \\ O \end{pmatrix} {}^{n-l<}AD_{m-1} + \begin{pmatrix} O \\ E_n \end{pmatrix} AD_m.$$

Thus

$$\begin{pmatrix} {}^{n-l<}AD_{m-1} \\ AD_m \end{pmatrix} Q = \begin{pmatrix} {}^{n-l<}S_{m-1} \\ S_m \end{pmatrix}$$

and we see that $AD_kQ = S_k$ for $k = m-1, m$. Therefore, $\operatorname{rank} X = \operatorname{rank}(S_1; \dots; S_m) \le p$ and we see (1).

Finally it is easy to see that (2) $\Leftrightarrow$ (3). □

Since $\mathcal{M} \subset \mathcal{C}$ by Lemma 6.4, we see the following:

Proposition 6.6. *For $X \in \mathscr{V}$, if $\operatorname{rank} X = p$, then $\iota(\sigma(X)) \notin \mathrm{fl}_1(\mathscr{A}^{u\times n\times m})$.*

7. Contribution of absolutely full column rank property

Let m, n and p be integers with $3 \le m \le n$ and $(m-1)(n-1)+1 \le p \le (m-1)n$. We set $u = nm - p$ and $t = n$ and use the results of Sections 4 and 5. Note $v = u - n + 1 = (m-1)n - p + 1$ in the notation of Definition 4.6.

It is known that the generic rank $\operatorname{grank}(n, p, m)$ of $n \times p \times m$ tensors over $\mathbb{C}$ is equal to p ([5, Theorem 3.1] or [6, Theorem 2.4 and Remark 2.5]) and it is also equal to the minimal typical rank of $n \times p \times m$ tensors over $\mathbb{R}$. Thus if we discuss the plurality of typical ranks, it is enough to consider whether there exists a typical rank that is greater than p or not.

Definition 7.1. We set $\mathcal{A} := \iota^{-1}(\mathrm{fl}_1(\mathscr{A}^{u\times n\times m})) \subset \mathbb{R}^{u\times p}$, where ι and $\mathscr{A}^{u\times n\times m}$ are defined in Definition 6.1.

Lemma 7.2. *If $Y \in \mathscr{V}^{n\times p\times m}$ and $\sigma(Y) \in \mathcal{A}$, then $\operatorname{rank} Y > p$.*

Proof. This follows from the fact that $\operatorname{rank} Y \ge p$ if $Y \in \mathscr{V}^{n\times p\times m}$ and Proposition 6.6. □

Theorem 7.3. *If $\mathscr{A}^{u\times n\times m} \ne \emptyset$, then there are plural typical ranks of $n \times p \times m$ tensors over $\mathbb{R}$.*

Proof. By Lemma 3.5, we see that $\mathcal{A} \ne \emptyset$. Since $\mathscr{A}^{u\times n\times m}$ is an open subset of $\mathbb{R}^{u\times n\times m}$, we see that $\mathcal{A}$ is an open subset of $\mathbb{R}^{u\times p}$. Moreover, since $\sigma\colon \mathscr{V}^{n\times p\times m} \to \mathbb{R}^{u\times p}$ is a surjective continuous map, we see that $\sigma^{-1}(\mathcal{A})$ is a nonempty open subset of $\mathscr{V}^{n\times p\times m}$. Thus, there is a typical rank greater than p by Lemma 7.2.

Since p is a typical rank of $n \times p \times m$ tensors over $\mathbb{R}$, we see that there are plural typical ranks of $n \times p \times m$ tensors over $\mathbb{R}$. $\square$

From now on until the end of this section, we assume that $p \geq (m-1)(n-1)+2$. Thus, $m \geq v+2$.

Definition 7.4. Let $Y \in \mathbb{R}^{u \times n \times m}$ and let $\boldsymbol{x} = (x_1, \ldots, x_m)$ be a vector of indeterminates. For $i_1, \ldots, i_{n-1} \in \{1, \ldots, u\}$, we set

$$\psi_{i_1,\ldots,i_{n-1}}(\boldsymbol{x}, Y) := \begin{pmatrix} (-1)^{n+1}[i_1, \ldots, i_{n-1} \mid 2, \ldots, n-1, n]_{M(\boldsymbol{x},Y)} \\ (-1)^{n+2}[i_1, \ldots, i_{n-1} \mid 1, 3, \ldots, n-1, n]_{M(\boldsymbol{x},Y)} \\ \vdots \\ (-1)^{2n}[i_1, \ldots, i_{n-1} \mid 1, \ldots, n-2, n-1]_{M(\boldsymbol{x},Y)} \end{pmatrix} \in \mathbb{R}[\boldsymbol{x}]^n.$$

For the definition $[a_1, \ldots, a_t \mid b_1, \ldots, b_t]$, see Definition 4.6. We define $\hat{\psi}_{i_1,\ldots,i_{n-1}} : \mathbb{R}^{1\times m} \times \mathbb{R}^{u\times n\times m} \to \mathbb{R}[\boldsymbol{x}]^p$ by

$$\hat{\psi}_{i_1,\ldots,i_{n-1}}(\boldsymbol{x}, Y) := \begin{pmatrix} x_1\psi_{i_1,\ldots,i_{n-1}}(\boldsymbol{x}, Y) \\ x_2\psi_{i_1,\ldots,i_{n-1}}(\boldsymbol{x}, Y) \\ \vdots \\ x_{m-1}\psi_{i_1,\ldots,i_{n-1}}(\boldsymbol{x}, Y) \end{pmatrix}^{\leq p} \in \mathbb{R}[\boldsymbol{x}]^p.$$

We also define the $\mathbb{R}$-vector space $U(Y)$ by

$$U(Y) := \left\langle \hat{\psi}_{i_1,\ldots,i_{n-1}}(\boldsymbol{u}, Y) \;\middle|\; \begin{matrix} \boldsymbol{u} \in \mathbb{V}(I_n(M(\boldsymbol{x}, Y))), \\ i_1, \ldots, i_{n-1} \in \{1, \ldots, u\} \end{matrix} \right\rangle \subset \mathbb{R}^p.$$

For $\boldsymbol{c} = (c_{11}, \ldots, c_{n1}, c_{12}, \ldots, c_{n2}, \ldots, c_{1m}, \ldots, c_{nm}) \in \mathbb{R}^{1\times nm}$, we set $Z_k = \begin{pmatrix} & Y_k & \\ c_{1k} & \cdots & c_{nk} \end{pmatrix}$ for $1 \leq k \leq m$, $Z = (Z_1; \ldots; Z_m) \in \mathbb{R}^{(u+1)\times n\times m}$ and

$$g_{i_1,\ldots,i_{n-1}}(\boldsymbol{x}, Y, \boldsymbol{c}) = [i_1, \ldots, i_{n-1}, u+1]_{M(\boldsymbol{x},Z)}$$

for any $i_1, \ldots, i_{n-1} \in \{1, \ldots, u\}$. For the definition $[i_1, \ldots, i_{n-1}, u+1]_{M(\boldsymbol{x},Z)}$, see Definition 4.6.

Lemma 7.5. *Suppose that $Y \in \mathbb{R}^{u\times n\times m}$. Then the following claims are equivalent.*

(1) $\dim U(Y) = p$.
(2) *If $\boldsymbol{c} \in \mathbb{R}^{1\times nm}$ satisfies the following conditions, then $\boldsymbol{c} = \boldsymbol{0}$.*
  (∗) ${}_{p<}\boldsymbol{c} = \boldsymbol{0}$ *and*
  (∗∗) $g_{i_1,\ldots i_{n-1}}(\boldsymbol{u}, Y, \boldsymbol{c}) = 0$ *for any* $\boldsymbol{u} \in \mathbb{V}(I_n(M(\boldsymbol{x}, Y)))$ *and any* $i_1, \ldots, i_{n-1} \in \{1, \ldots, u\}$.

Proof. The vector $\boldsymbol{d} \in \mathbb{R}^p$ is perpendicular to $U(Y)$ if and only if $\boldsymbol{d}$ is perpendicular to $\hat{\psi}_{i_1,\dots,i_{n-1}}(\boldsymbol{u}, Y)$ for any $\boldsymbol{u} \in \mathbb{V}(I_n(M(\boldsymbol{x}, Y)))$ and any $i_1, \dots, i_{n-1}$. Since the inner product of $\hat{\psi}_{i_1,\dots,i_n}(\boldsymbol{u}, Y)$ with $\boldsymbol{d}$ is $g_{i_1,\dots,i_{n-1}}(\boldsymbol{u}, Y, (\boldsymbol{d}^\top, \boldsymbol{0}))$, the result follows. □

Next we show the following result. For the definition of $\mathcal{M}$, see Definition 6.1.

Lemma 7.6. *If* $\dim U(Y) = p$*, then* $\mathrm{fl}_1(Y) \in \mathcal{M}$*.*

Proof. Set $Y = (Y_1; \dots; Y_m)$. Suppose that $\dim U(Y) = p$. Then there are $\boldsymbol{u}_1, \dots, \boldsymbol{u}_p \in \mathbb{V}(I_n(M(\boldsymbol{x}, Y)))$ and $t_{11}, \dots, t_{1,n-1}, \dots, t_{p1}, \dots, t_{p,n-1}$ such that

$$\hat{\psi}_{t_{11},\dots,t_{1,n-1}}(\boldsymbol{u}_1, Y), \dots, \hat{\psi}_{t_{p1},\dots,t_{p,n-1}}(\boldsymbol{u}_p, Y)$$

are linearly independent over $\mathbb{R}$. Set $\boldsymbol{u}_j = (u_{j1}, \dots, u_{jm})$ for $1 \le j \le p$, $D_k = \mathrm{Diag}(u_{1k}, \dots, u_{pk})$ for $1 \le k \le m$ and

$$A = (\psi_{t_{11},\dots,t_{1,n-1}}(\boldsymbol{u}_1, Y), \dots, \psi_{t_{p1},\dots,t_{p,n-1}}(\boldsymbol{u}_p, Y)).$$

Then,

$$\begin{aligned}\begin{pmatrix} AD_1 \\ \vdots \\ AD_{m-2} \\ A^{\le n-l} D_{m-1} \end{pmatrix} &= \begin{pmatrix} AD_1 \\ \vdots \\ AD_{m-2} \\ AD_{m-1} \end{pmatrix}^{\le p} \\ &= (\hat{\psi}_{t_{11},\dots,t_{1,n-1}}(\boldsymbol{u}_1, Y), \dots, \hat{\psi}_{t_{p1},\dots,t_{p,n-1}}(\boldsymbol{u}_p, Y))\end{aligned}$$

is a nonsingular matrix and

$$(u_{j1}Y_1 + \cdots + u_{jm}Y_m)\psi_{t_{j1},\dots,t_{j,n-1}}(\boldsymbol{u}_j, Y) = \begin{pmatrix} [1, t_{j1}, \dots, t_{j,n-1}]_{M(\boldsymbol{u}_j, Y)} \\ [2, t_{j1}, \dots, t_{j,n-1}]_{M(\boldsymbol{u}_j, Y)} \\ \vdots \\ [u, t_{j1}, \dots, t_{j,n-1}]_{M(\boldsymbol{u}_j, Y)} \end{pmatrix} = \boldsymbol{0}$$

since $I_n(M(\boldsymbol{u}_j, Y)) = (0)$ for $1 \le j \le p$. □

Definition 7.7. Set $\mathscr{U} := \{Y \in \mathbb{R}^{u \times n \times m} \mid {}_{p<}\mathrm{fl}_1(Y) \text{ is nonsingular}\}$, $\mathscr{O}_3 := \mathscr{U} \cap \mathscr{Q} \cap \mathscr{Q}' \cap \tilde{\mathscr{P}}_n = \mathscr{O}_1 \cap \mathscr{U} \cap \mathscr{Q}'$ and $\mathscr{O}_4 := \mathscr{U} \cap \mathscr{Q} \cap \mathscr{Q}' \cap \mathscr{A}^{u \times n \times m} = \mathscr{O}_2 \cap \mathscr{U} \cap \mathscr{Q}'$, where $\mathscr{Q}$, $\mathscr{Q}'$ and $\tilde{\mathscr{P}}_n$ are the ones defined in Definitions 4.27, 5.19, and 4.24 and $\mathscr{O}_1$ and $\mathscr{O}_2$ are the ones in Theorem 4.31 under $t = n$. Define $\nu \colon \mathscr{U} \to \mathbb{R}^{u \times p}$ as $\nu(Y) := -({}_{p<}\mathrm{fl}_1(Y))^{-1}\mathrm{fl}_1(Y)_{\le p}$ for $i = 1, 2$, where σ is the one defined in Definition 6.1. Set $\mathcal{O}_i = \nu(\mathscr{O}_{i+2}) \subset \mathbb{R}^{u \times p}$ and $\mathscr{T}_i = \sigma^{-1}(\mathcal{O}_i) \subset \mathscr{V}^{n \times p \times m}$ for $i = 1, 2$.

The following fact is immediately verified.

Lemma 7.8. $\iota(\nu(Y)) = \mathrm{fl}_1(-({}_{p<}\mathrm{fl}_1(Y))^{-1}Y)$.

By the same way as Theorem 4.31 (1), (2) and (3), we see the following fact.

Lemma 7.9. *Then the following hold.*

(1) $\mathscr{O}_3$ *and* $\mathscr{O}_4$ *are disjoint open subsets of* $\mathbb{R}^{u\times n\times m}$ *and* $\mathscr{O}_3$ *is nonempty.*
(2) $\mathscr{O}_3 \cup \mathscr{O}_4$ *is a dense subset of* $\mathbb{R}^{u\times n\times m}$.
(3) $\overline{\mathscr{O}_3} = \mathscr{C}^{u\times n\times m} = \mathbb{R}^{u\times n\times m} \setminus \mathscr{A}^{u\times n\times m}$.

Lemma 7.10. $\mathrm{fl}_1(\mathscr{O}_3) \subset \mathcal{M}$.

Proof. Let $Y \in \mathscr{O}_3$. By Lemmas 7.6 and 7.5, it is enough to show that if $\boldsymbol{c} \in \mathbb{R}^{1\times nm}$ satisfies $(*)$ and $(**)$, then $\boldsymbol{c} = \boldsymbol{0}$.

Set $\boldsymbol{c} = (\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m)$, where $\boldsymbol{c}_j \in \mathbb{R}^{1\times n}$, $\boldsymbol{c}' = (\boldsymbol{c}_1; \cdots; \boldsymbol{c}_m) \in \mathbb{R}^{1\times n\times m}$ and $\tilde{Y} = \begin{pmatrix} Y \\ \boldsymbol{c}' \end{pmatrix}$. Then by $(**)$, $g_{i_1,\ldots,i_{n-1}}(\boldsymbol{x}, Y, \boldsymbol{c}) \in \mathbb{I}(\mathbb{V}(I_n(M(\boldsymbol{x}, Y))))$ for any $i_1, \ldots, i_{n-1} \in \{1, \ldots, u\}$. Therefore, by the definition of $\mathscr{O}_3$ and Theorem 4.31 (5), we see that $g_{i_1,\ldots,i_{n-1}}(\boldsymbol{x}, Y, \boldsymbol{c}) \in I_n(M(\boldsymbol{x}, Y))$ for any $i_1, \ldots, i_{n-1} \in \{1, \ldots, u\}$. Thus we see that $I_n(M(\boldsymbol{x}, \tilde{Y})) = I_n(M(\boldsymbol{x}, Y))$. Thus, by Proposition 5.21, we see that $\mathrm{fl}_1(\boldsymbol{c})$ is an $\mathbb{R}$-linear combination of rows of $\mathrm{fl}_1(Y)$. Since ${}_{p<}\boldsymbol{c} = \boldsymbol{0}$ and $Y \in \mathscr{U}$, we see that $\boldsymbol{c} = \boldsymbol{0}$. □

By the same way as Proposition 6.2, we see the following:

Proposition 7.11. ν *is an open, surjective and continuous map.*

We see the following fact.

Lemma 7.12. *Then the following hold.*

(1) $Y \in \mathscr{A}^{u\times n\times m}$ *if and only if* $\nu(Y) \in \mathcal{A}$ *for* $Y \in \mathscr{U}$.
(2) $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *are disjoint open subsets of* $\mathbb{R}^{u\times p}$ *and* $\mathcal{O}_1 \neq \emptyset$.
(3) $\mathcal{O}_1 \cup \mathcal{O}_2$ *is a dense subset of* $\mathbb{R}^{u\times p}$.
(4) $\overline{\mathcal{O}_1} = \mathbb{R}^{u\times p} \setminus \mathcal{A}$.
(5) $\mathcal{O}_2 \subset \mathcal{A}$ *and* $\overline{\mathcal{O}_2} = \overline{\mathcal{A}}$.

Proof. (1): Suppose that $Y \in \mathscr{U}$. Since $\mathscr{A}^{u\times n\times m}$ and $\mathscr{U}$ are stable under the action of $\mathrm{GL}(u, \mathbb{R})$, we see that $Y \in \mathscr{A}^{u\times n\times m}$ if and only if $-({}_{p<}\mathrm{fl}_1(Y))^{-1}Y \in \mathscr{A}^{u\times n\times m}$. Since $\iota(\nu(Y)) = \mathrm{fl}_1(-({}_{p<}\mathrm{fl}_1(Y))^{-1}Y)$ and fl_1 is a bijection, we see (1).

We see (2) by the facts that $\tilde{\mathscr{P}}_n$ and $\mathscr{A}^{u\times n\times m}$ are stable under the action of $\mathrm{GL}(u, \mathbb{R})$, Lemma 7.9, and Proposition 7.11. (3) also follows from Lemma 7.9 and Proposition 7.11. We see by (1) that if $Y \in \mathscr{O}_3$, then $\nu(Y) \notin \mathcal{A}$. Thus $\mathcal{O}_1 = \nu(\mathscr{O}_3) \subset \mathbb{R}^{u\times p} \setminus \mathcal{A}$. Since

$\overline{\mathscr{O}_3} = \mathbb{R}^{u\times n\times m}\setminus\mathscr{A}^{u\times n\times m}$ by Lemmas 7.9 (3), we see that $\overline{\mathcal{O}_1} \supset \nu(\overline{\mathscr{O}_3}\cap\mathscr{U}) = \nu((\mathbb{R}^{u\times n\times m}\setminus \mathscr{A}^{u\times n\times m})\cap\mathscr{U}) = \mathbb{R}^{u\times p}\setminus\mathcal{A}$ by (1) and the surjectivity of ν. Thus we see (4). Therefore $\mathcal{O}_2 \subset \mathcal{A}$ by (2). Further, we see that $\overline{\mathcal{O}_2} \supset \mathcal{A}$ by (3) and (4). Thus we see (5). □

Lemma 7.13. *Let X and Y be topological spaces, $f\colon X\to Y$ a mapping and B a subset of Y.*

(1) *If f is continuous, then $f^{-1}(\overline{B}) \supset \overline{f^{-1}(B)}$.*
(2) *If f is an open map, then $f^{-1}(\overline{B}) \subset \overline{f^{-1}(B)}$.*

Proof. (1): Since $f^{-1}(\overline{B})$ is a closed subset of X containing $f^{-1}(B)$, we see that $f^{-1}(\overline{B}) \supset \overline{f^{-1}(B)}$.

(2): Suppose that $x \in f^{-1}(\overline{B})$ and let U be an open neighborhood of x. We show that $U \cap f^{-1}(B) \neq \emptyset$.

Since $f(x) \in \overline{B}$, $f(x) \in f(U)$ and $f(U)$ is an open subset of Y, we see that $f(U)\cap B \neq \emptyset$. Take $b \in f(U)\cap B$ and $a \in U$ such that $f(a) = b$. Then, since $f(a) \in B$, we see that $a \in f^{-1}(B)$. Thus, $a \in U\cap f^{-1}(B)$ and we see that $U \cap f^{-1}(B) \neq \emptyset$. □

Theorem 7.14. *Let m, n and p be integers with $3 \le m \le n$ and $(m-1)(n-1)+2 \le p \le (m-1)n$. The following hold.*

(1) *$\mathscr{T}_1$ and $\mathscr{T}_2$ are disjoint open subsets of $\mathscr{V}^{n\times p\times m}$ and $\mathscr{T}_1$ is nonempty.*
(2) *$\mathscr{T}_1 \cup \mathscr{T}_2$ is a dense subset of $\mathbb{R}^{n\times p\times m}$.*
(3) *$\overline{\mathscr{T}_1} \cap \mathscr{V}^{n\times p\times m} = \mathscr{V}^{n\times p\times m} \setminus \sigma^{-1}(\mathcal{A})$ and $\overline{\mathscr{T}_2} \cap \mathscr{V}^{n\times p\times m} = \overline{\sigma^{-1}(\mathcal{A})} \cap \mathscr{V}^{n\times p\times m}$.*
(4) *If $T \in \mathscr{T}_1$, then $\operatorname{rank} T = p$.*
(5) *If $T \in \mathscr{T}_2$, then $\operatorname{rank} T > p$.*

Proof. First note that $\overline{\sigma^{-1}(\mathcal{X})} \cap \mathscr{V}^{n\times p\times m} = \sigma^{-1}(\overline{\mathcal{X}})$ for any subset $\mathcal{X}$ of $\mathbb{R}^{u\times p}$ by Lemma 7.13, since σ is an open continuous map.

(1) and (2) follow from Lemma 7.12 and the facts that σ is surjective and $\mathscr{V}^{n\times p\times m}$ is a dense subset of $\mathbb{R}^{n\times p\times m}$.

(3): We see by Lemma 7.12 that $\overline{\mathscr{T}}_1 \cap \mathscr{V}^{n\times p\times m} = \sigma^{-1}(\overline{\mathcal{O}_1}) = \sigma^{-1}(\mathbb{R}^{u\times p}\setminus\mathcal{A}) = \mathscr{V}^{n\times p\times m}\setminus\sigma^{-1}(\mathcal{A})$ and $\overline{\mathscr{T}}_2 \cap \mathscr{V}^{n\times p\times m} = \sigma^{-1}(\overline{\mathcal{O}_2}) = \overline{\sigma^{-1}(\mathcal{A})} \cap \mathscr{V}^{n\times p\times m}$.

(4): Suppose that $T \in \mathscr{T}_1$. Then $\sigma(T) \in \mathcal{O}_1$. Thus there exists $Y \in \mathscr{O}_3$ such that $\nu(Y) = \sigma(T)$. By Lemma 7.10, we see that $\mathrm{fl}_1(Y) \in \mathcal{M}$. Hence $\iota(\sigma(T)) = \iota(\nu(Y)) = -({}_{p<}\mathrm{fl}_1(Y))^{-1}\mathrm{fl}_1(Y) \in \mathcal{M}$, since $\mathcal{M}$ is stable under the action of $\mathrm{GL}(u,\mathbb{R})$. Therefore $\operatorname{rank} T = p$ by Theorem 6.5.

(5): If $T \in \mathscr{T}_2$, then $\sigma(T) \in \mathcal{O}_2 \subset \mathcal{A}$ by Lemma 7.12. Thus $\operatorname{rank} T > p$ by Lemma 7.2. □

8. Upper bound for typical ranks

In Lemma 7.2, we see a class of tensors with rank greater than p. To complete the proof of Theorem 1.2, we give an upper bound of the set of typical ranks of $\mathbb{R}^{n\times p\times m}$:

Theorem 8.1. *Let $3 \le m \le n$ and $(m-1)(n-1)+1 \le p \le (m-1)n$. Any typical rank of $\mathbb{R}^{n\times p\times m}$ is less than or equal to $p+1$.*

We prepare the proof.

Let $3 \le m \le n$, $(m-1)(n-1)+1 \le p < (m-1)n$ and $u = mn-p$. Let $\sigma'\colon \mathscr{V}^{n\times(p+1)\times m} \to \mathbb{R}^{(u-1)\times(p+1)}$ be the counterpart of $\sigma\colon \mathscr{V}^{n\times p\times m} \to \mathbb{R}^{u\times p}$. Also, let $\mathcal{A}' \subset \mathbb{R}^{(u-1)\times(p+1)}$ and $\mathscr{T}_1' \subset \mathscr{V}^{n\times(p+1)\times m}$ be the counterparts of $\mathcal{A} \subset \mathbb{R}^{u\times p}$ and $\mathscr{T}_1 \subset \mathscr{V}^{n\times p\times m}$ respectively. Let $\pi\colon \mathbb{R}^{n\times(p+1)\times m} \to \mathbb{R}^{n\times p\times m}$ be a canonical projection defined as $\pi(Y_1;\dots;Y_m) = ((Y_1)_{\le p};\dots;(Y_m)_{\le p})$. Clearly π is a continuous, surjective and open map.

Lemma 8.2. *$\pi(\mathscr{T}_1')$ is an open dense subset of $\mathbb{R}^{n\times p\times m}$.*

Proof. Since $\mathscr{T}_1'$ is an open set and π is an open map, $\pi(\mathscr{T}_1')$ is an open subset of $\mathbb{R}^{n\times p\times m}$. We show that $\pi(\mathscr{T}_1')$ is dense. Let $X \in \mathscr{V}^{n\times p\times m}$. Consider the map $f\colon \mathscr{V}^{n\times p\times m} \to \mathscr{V}^{n\times(p+1)\times m}$ defined as

$$f(X_1;\dots;X_{m-2};X_{m-1};X_m) = ((X_1,\mathbf{0});\dots;(X_{m-2},\mathbf{0});(X_{m-1},\boldsymbol{e});(X_m,\mathbf{0}))$$

where $\boldsymbol{e}$ is the $(2n-u+1)$th column vector of the identity matrix E_n. Since the $(p+1)$th column vector of the matrix $\sigma'(f(X))$ is zero, $f(X) \notin \sigma'^{-1}(\mathcal{A}')$ holds and by Theorem 7.14 (3), $f(X) \in \overline{\mathscr{T}_1'}$. Since $\pi \circ f$ is the identity map and π is continuous, $X \in \pi(\overline{\mathscr{T}_1'}) \subset \overline{\pi(\mathscr{T}_1')}$ holds. Therefore $\mathscr{V}^{n\times p\times m} \subset \overline{\pi(\mathscr{T}_1')}$ and thus $\mathbb{R}^{n\times p\times m} = \overline{\pi(\mathscr{T}_1')}$. $\square$

By Theorem 7.14 (5), and Lemma 8.2, we have immediately the following corollary.

Corollary 8.3. *Let $3 \le m \le n$ and $(m-1)(n-1)+1 \le p < (m-1)n$. $\mathscr{T}_2 \ne \emptyset$ if and only if $\mathscr{T}_2 \cap \pi(\mathscr{T}_1') \ne \emptyset$, and $\operatorname{rank} T = p+1$ for any $T \in \mathscr{T}_2 \cap \pi(\mathscr{T}_1')$.*

Note that arbitrary tensor of $\pi(\mathscr{T}_1')$ has rank less than or equal to $p+1$ by Theorem 7.14 (4).

Proof of Theorem 8.1. The assertion for $p = (m-1)n$ holds by [33]. Suppose that $(m-1)(n-1)+1 \le p < (m-1)n$. Then $\operatorname{rank}(T) \le p+1$ for $T \in \pi(\mathscr{T}_1')$. Since $\pi(\mathscr{T}_1')$ is dense, arbitrary integer greater than $p+1$ is not a typical rank. $\square$

Recall that $\operatorname{trank}(m,n,p) = \operatorname{trank}(n,p,m)$. We are ready to prove main theorems.

Proof of Theorem 1.1. (1) follows from Theorem 7.3 and Corollary 3.4.

(2): We may assume that $3 \leq m \leq n$ without the loss of generality. Ten Berge [35] showed that $\mathbb{R}^{m\times n\times p}$ has a unique typical rank for $p \geq (m-1)n+1$. Therefore, we see that $p \leq (m-1)n$. Set $u = mn - p$. By Theorem 7.14 (2) and (4), we see that $\mathscr{T}_2 \neq \emptyset$. Furthermore, $\mathscr{T}_2 \neq \emptyset \Rightarrow \mathscr{O}_4 \neq \emptyset$ by definitions and the surjectivity of σ. Since $\mathscr{O}_4 \subset \mathscr{A}^{u\times n\times m}$, we see that there exists an absolutely full column rank $u \times n \times m$ tensor. The result follows from Corollary 3.4. □

Proof of Theorem 1.2. We may assume that $3 \leq m \leq n$. Note that

$$\operatorname{trank}(m,n,p) = \{\min\{p, mn\}\}$$

for $k \geq m$ [35]. Suppose that $2 \leq k \leq m-1$. By Theorem 8.1, the maximal typical rank of $\mathbb{R}^{m\times n\times p}$ is less than or equal to $p+1$. Since p is the minimal typical rank of $\mathbb{R}^{m\times n\times p}$, $\operatorname{trank}(m,n,p)$ is $\{p\}$ or $\{p, p+1\}$. By Theorem 1.1, $\mathbb{R}^{m\times n\times p}$ has a unique typical rank if and only if $m\#n \geq mn - p + 1$, equivalently, $k \geq m + n - (m\#n)$. This completes the proof. □

We immediately have Theorem 1.3 by Proposition 2.4 and Theorem 7.3. In the case where $p = (m-1)(n-1)+1$, we have many examples for having plural typical ranks.

Corollary 8.4. *Let $m, n \geq 3$ and $a \geq 1$. If $m \equiv 2^{a-1} + s \pmod{2^a}$ and $n \equiv 2^{a-1} + t \pmod{2^a}$ for some integers s and t with $1 \leq s, t \leq 2^{a-1}$ then $\mathbb{R}^{m\times n\times((m-1)(n-1)+1)}$ has plural typical ranks.*

Proposition 8.5. *Let $a = 4, 8$. If m and n are divisible by a, then for each $1 \leq k < a$, $\mathbb{R}^{m\times n\times((m-1)(n-1)+k)}$ has plural typical ranks.*

Proof. For $a = 4, 8$, if m and n are divisible by a, then $m\#n \leq m + n - a$ by Proposition 2.3 and thus $m + n - 1 - (m\#n) \geq a - 1$. Then the assertion follows by Theorem 1.2. □

Corollary 8.6.

(1) *$\mathbb{R}^{4\times 4\times k}$ has plural typical rank whenever $10 \leq k \leq 12$.*
(2) *$\mathbb{R}^{8\times 8\times k}$ has plural typical rank whenever $50 \leq k \leq 56$.*

Proposition 8.7. *Let $m, n \geq 3$. If $\mathbb{R}^{m\times n\times((m-1)(n-1)+1)}$ has a unique typical rank, then $\operatorname{trank}(m, n, (m-1)(n-1)-k) = \{(m-1)(n-1)+1\}$ holds whenever $0 \leq k < \frac{(m-1)(n-1)}{m+n-1}$.*

Proof. Let $0 \leq k < \frac{(m-1)(n-1)}{m+n-1}$, $q = (m-1)(n-1) - k$ and $p = (m-1)(n-1)+1$. Suppose that $\mathbb{R}^{m\times n\times p}$ has a unique typical rank. Then $\operatorname{trank}(n, p, m) = \{p\}$. Since the set of all $n \times p \times m$ tensors with rank p is a dense subset of $\mathbb{R}^{n\times p\times m}$, the image of this set by a canonical projection $\mathbb{R}^{n\times p\times m} \to \mathbb{R}^{n\times q\times m}$ is also a dense subset of $\mathbb{R}^{n\times q\times m}$. Thus any typical rank of $\mathbb{R}^{n\times q\times m}$ is less than or equal to p. On the other hand, by elementary calculation, we see that $(m-1)(n-1) < \frac{mnq}{m+n+q-2} \Leftrightarrow k < \frac{(m-1)(n-1)}{m+n-1}$. Thus the minimal typical rank of $\mathbb{R}^{n\times q\times m}$ is greater than or equal to p. Therefore $\mathbb{R}^{n\times q\times m}$ has a unique typical rank p. $\square$

Corollary 8.8. *Let $3 \le m \le n$. Suppose that $\mathbb{R}^{m\times n\times((m-1)(n-1)+1)}$ has a unique typical rank. If $0 \le k \le \lfloor \frac{m}{2} \rfloor - 1$ then* $\operatorname{trank}(m, n, (m-1)(n-1)-k) = \{(m-1)(n-1)+1\}$.

Proof. Let $0 \le k \le \lfloor \frac{m}{2} \rfloor - 1$. Then $(m+n-1)(k+1) \le (m+n-1)\frac{m}{2} \le (n+n-1)\frac{m}{2} < mn$ and thus $(m+n-1)k < (m-1)(n-1)$. Therefore the assertion follows from Proposition 8.7. $\square$

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