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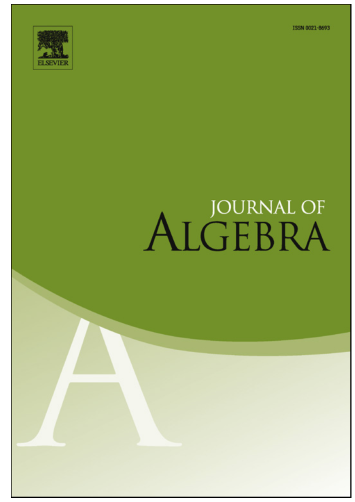

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Locally Nilpotent Skew Derivations with Central Invariants

Dedicated to the memory of Jim Osterburg

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Abstract

Let δ be a locally nilpotent q-skew derivation of an algebra R such that the invariants are central. With some natural assumptions on the q-characteristic, we show that if R is semiprime then R is commutative. We also examine other conditions which imply, even when R is not commutative, that the commutator ideal is contained in the prime radical. These results extend previous work of the authors and of Osterburg and may shed some light on a conjecture of Herstein.

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1. Introduction

In [1] it is shown that if δ is an algebraic skew derivation of a semiprime algebra R, such that the invariants are central, then R must be commutative. This generalized a result in [7] on automorphisms of prime order.

In this paper we turn our attention to locally nilpotent q-skew derivations with central invariants. When contrasting the structure of an algebra to the invariants of a transformation such as an automorphism, derivation, or skew derivation, one typically assumes that the transformation is algebraic. One of the surprising aspects of this paper is that we only need assume that our q-skew derivations are locally nilpotent. Along these lines, the main result of this paper is

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Theorem 3. *Let δ be a locally nilpotent q-skew derivation of a semiprime algebra R such that the invariants of δ are central. If $\delta^N = 0$, assume $char_q(R) \geq N$ or $char_q(R) = 0$, whereas if δ is not nilpotent, assume $char_q(R) = 0$. Then R is commutative.*

In Theorems 4 and 6, we also consider the situation where R need not be semiprime. Starting in the 1970's, a great deal of machinery, such as the Bergman-Isaacs Theorem [4] and Kharchenko's work [6] on inner and outer actions, was developed and used to contrast the structure of algebras to the invariants under the actions of groups and Lie algebras. A second surprising aspect of this paper is that our arguments are self-contained and do not require any previous results on invariants of automorphisms, derivations, or skew derivations.

In [2] a strong connection is shown between skew polynomial rings and rings with locally nilpotent skew derivations. Motivated by the ideas in that paper, we show that, in many cases, if an algebra R has a locally nilpotent q-skew derivation with central invariants, then there is a large subset C of the center such that if $a, b \in R$, then there exists $c \in C$ such that $c[a, b] = 0$. Therefore, the proofs in this paper deal primarily with proving the existence of such central elements c and then examining their annihilators.

We will now introduce the terminology that will be used throughout this paper. R will be an algebra over a field K and q will be a nonzero element of K. If σ is a K-linear automorphism of R, we say that a K-linear map δ is a q-skew derivation if

$$\delta(rs) = \delta(r)s + \sigma(r)\delta(s) \quad \text{and} \quad \delta(\sigma(r)) = q\sigma(\delta(r)),$$

for all $r, s \in R$. Observe that if $\sigma = 1$, then δ is an ordinary derivation, whereas if $\delta = \sigma - 1$, then δ is q-skew with $q = 1$.

We say that δ is locally nilpotent, if for each $r \in R$, there exists $n = n(r) \geq 1$ such that $\delta^n(r) = 0$. It then follows that if we let $R_n = \{r \in R \mid \delta^{n+1}(r) = 0\}$, then

$$R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots,$$

$R = \bigcup_{n \geq 0} R_n$, R_0 is the invariants of δ, and R_1 is the kernel of δ^2.

Since $\delta\sigma = q\sigma\delta$, σ restricts to an automorphism of the K-subalgebra R_0. We let $\delta(R) = \{\delta(r) \mid r \in R\}$ and will let A denote the set $\delta(R) \cap R_0$. Observe that A is an ideal of R_0 and we let $A^\sigma = \{a \in A \mid \sigma(a) = a\}$.

We let $char(R)$ denote the ordinary characteristic of the field K. However, we define the q-characteristic of R, denoted as $char_q(R)$, as the smallest $n \geq 1$ such that $1+q+\cdots+q^{n-1} = 0$. If no such n exists, we say $char_q(R) = 0$. The center of R will be denoted as $Z(R)$. If $B \subseteq Z(R)$ and $S \subseteq R$, we let $ann_S(B) = \{s \in S \mid Bs = 0\}$.

2. The results

We begin our work with

Lemma 1. *Let $x \in R_1$ such that $0 \neq \delta(x) \in A^\sigma$ and then let $\alpha = \delta(x)$. If $m \geq 1$ and either $char_q(R) = 0$ or $m < char_q(R)$, then*

$$\alpha R_m \subseteq xR_{m-1} + R_{m-1}.$$

and

$$\alpha^m R_m \subseteq R_0 + R_0 x + \cdots + R_0 x^m.$$

Proof. Let $\gamma = 1 + q + \cdots + q^{m-1}$; using our assumption on $char_q(R)$, we know that $\gamma \neq 0$. For the first set containment, it suffices to show that if $w \in R_m$, then $\alpha w \in xR_{m-1} + R_{m-1}$. Observe that $\delta(w) \in R_{m-1}$ and $\delta^{m+1}(w) = 0$. Therefore

$$\delta^m(\gamma\alpha w) = \gamma\alpha\delta^m(w)$$

and

$$\delta^m(x\delta(w)) = \gamma\sigma^{m-1}(\delta(x))\delta^m(w) = \gamma\alpha\delta^m(w).$$

Hence $\delta^m(\gamma\alpha w - x\delta(w)) = 0$ and $\gamma\alpha w - x\delta(w) \in R_{m-1}$. Since γ is an invertible element of the ground field, this immediately implies that $\alpha w \in xR_{m-1} + R_{m-1}$, as needed.

We will prove the second set inclusion using Mathematical Induction. Observe that the first part of this lemma handles the $m = 1$ case. Therefore it suffices to assume the result for m and then prove it for $m + 1$. By the first part of this lemma,

$$\alpha R_{m+1} \subseteq xR_m + R_m.$$

Multiplying the above by α^m and applying the induction hypothesis, we obtain

$$\alpha^{m+1} R_{m+1} \subseteq \alpha^m(xR_m + R_m) \subseteq x(R_0 + R_0 x + \cdots + R_0 x^m) + (R_0 + R_0 x + \cdots + R_0 x^m) \subseteq$$

$$R_0 + R_0 x + \cdots + R_0 x^m + R_0 x^{m+1},$$

concluding the proof. □

Next we prove a simple but extremely useful result about A^σ in the semiprime case.

Lemma 2. *If R is semiprime, then $A = A^\sigma$.*

Proof. It suffices to show that if $a \in A$, then $\sigma(a) = a$. If $r \in R$, since a is central and $\delta(a) = 0$, we have

$$0 = \delta(ar - ra) = \delta(a)r + \sigma(a)\delta(r) - \delta(r)a - \sigma(r)\delta(a) = \sigma(a)\delta(r) - \delta(r)a.$$

As a result, $(\sigma(a) - a)\delta(R) = 0$.

Since $a \in \delta(R)$, let $x \in R$ such that $\delta(x) = a$. Therefore

$$\sigma(a) - a = \sigma(\delta(x)) - \delta(x) = q^{-1}\delta(\sigma(x)) - \delta(x) = \delta(q^{-1}\sigma(x) - x).$$

As a result, $\sigma(a) - a \in \delta(R)$. This tells us that

$$(\sigma(a) - a)^2 \in (\sigma(a) - a)\delta(R) = 0.$$

Since R is semiprime, the center of R does not contain any nonzero nilpotent elements. However $\sigma(a) - a$ belongs to the center of R and has square zero, thus $\sigma(a) - a = 0$. Therefore $\sigma(a) = a$, concluding the proof. $\square$

We can now prove the first main result of this paper. The special case of Theorem 3 where δ is nilpotent was proved in [1]. However, the argument here is more elementary.

Theorem 3. *Let δ be a locally nilpotent q-skew derivation of a semiprime algebra R such that the invariants of δ are central. If $\delta^N = 0$, assume $char_q(R) \geq N$ or $char_q(R) = 0$, whereas if δ is not nilpotent, assume $char_q(R) = 0$. Then R is commutative.*

Proof. If $a, b \in R$, let $i, j \geq 0$ be the smallest integers such that $a \in R_i$ and $b \in R_j$. Observe that if $\delta^N = 0$, then $i, j \leq N - 1$. Therefore, regardless of whether δ is nilpotent, Lemma 1 asserts that if $\alpha \in A^\sigma$ and $\alpha = \delta(x)$, then

$$\alpha^i a \in R_0 + R_0 x + \cdots + R_0 x^i$$

and

$$\alpha^j b \in R_0 + R_0 x + \cdots + R_0 x^j.$$

Since R_0 is central, the set $R_0 + R_0 x + R_0 x^2 + \cdots$ is a commutative subring of R. Therefore $\alpha^i a$ and $\alpha^j b$ belong to the same commutative subring of R. Hence

$$0 = [\alpha^i a, \alpha^j b] = \alpha^{i+j}[a, b] = 0.$$

Since R is semiprime, α^{i+j} and α have the same annihilator. As a result, $\alpha[a, b] = 0$. The previous equation holds for all $\alpha \in A^\sigma$ and $a, b \in R$, therefore

$$A^\sigma [R, R] = 0.$$

In addition, since R is semiprime, Lemma 2 asserts that $A = A^\sigma$. Therefore $A[R, R] = 0$.

Next, let $B = ann_R(A)$; we claim that B is stable under δ. Note that since A is central, B is both the left and right annihilator of A. Observe that if $a \in A, b \in B$, we have

$$0 = \delta(ba) = \delta(b)a + \sigma(b)\delta(a) = \delta(b)a,$$

hence $\delta(B)$ annihilates A and $\delta(B) \subseteq B$.

We now consider the case where $\delta(B) \neq 0$. Since δ is locally nilpotent, the set $\delta(B) \cap R_0$ is nonzero. Furthermore, $\delta(B) \cap R_0 \subseteq \delta(R) \cap R_0 = A$. Therefore

$$(\delta(B) \cap R_0)^2 \subseteq AB = 0.$$

However, this is a contradiction as $\delta(B) \cap R_0$ is contained in the center of a semiprime ring and cannot contain nonzero nilpotent elements.

In light of the above, it must be the case that $\delta(B) = 0$, hence B is contained in the center of R. Recall that $A[R, R] = 0$, hence B contains the set $[R, R]$. Therefore

$$[[R, R], R] \subseteq [B, R] = 0.$$

As a result, all commutators in R are central. Therefore if $r, s \in R$, then $[r, s]$ and $[r, rs] = r[r, s]$ must be central and we have

$$0 = [[r, rs], s] = [r[r, s], s] = [r, s]^2.$$

Since $[r, s]$ is both central and nilpotent, the ideal of R generated by $[r, s]$ is nilpotent. However R is semiprime, therefore $[r, s] = 0$ and we see that R is commutative. □

The reason we can use self-contained arguments to obtain results for q-skew derivations which are not algebraic is due to the strength of the assumption that R_0 is central. In [3], it is shown that if R is a domain and d is an algebraic skew derivation (not necessarily q-skew) then R satisfies a polynomial identity if and only if the invariants of d satisfy a polynomial identity.

We can contrast this to the case for locally nilpotent derivations where we only assume the invariants are commutative. Let $R = K[x, y \mid [x, y] = 1]$ be the first Weyl algebra in characteristic 0. In this case, the derivation d defined as $d(r) = [r, y]$, for all $r \in R$, is locally nilpotent. The invariants of d are the commutative polynomial ring $K[y]$ but R does not satisfy a polynomial identity. Therefore, even for domains, the commutativity of the invariants of a locally nilpotent derivation is not sufficient to force the ring to satisfy a polynomial identity.

In light of this, our results for locally nilpotent q-skew derivations cannot be extended to the case where the invariants are commutative. We now show that there are assumptions, other than R being semiprime, strong enough to force R to be commutative. The assumption that $char(R) \neq 2$ may seem surprising but, after the proof of Corollary 5, we will see that it is necessary.

Theorem 4. *Let δ be a locally nilpotent q-skew derivation of an algebra R such that the invariants of δ are central and $char(R) \neq 2$. If $\delta^N = 0$, assume $char(R), char_q(R) \geq N$ or $char(R) = char_q(R) = 0$, whereas if δ is not nilpotent, assume $char(R) = char_q(R) = 0$. If the set $\delta(R) \cap R_0 \cap R^\sigma$ has zero annihilator, then R is commutative.*

Proof. The beginning of the proof is quite similar to the proof of Theorem 3. If $a, b \in R$, let $i, j \geq 0$ be the smallest integers such that $a \in R_i$ and $b \in R_j$. Observe that if $\delta^N = 0$, then $i, j \leq N - 1$. Therefore, regardless of whether δ is nilpotent, Lemma 1 asserts that if $\alpha, \beta \in A^\sigma$ and $\alpha = \delta(x), \beta = \delta(y)$, then

$$\alpha^i a \in R_0 + R_0 x + \cdots + R_0 x^i$$

and

$$\beta^j b \in R_0 + R_0 y + \cdots + R_0 y^j$$

Next, suppose $\gamma \in A^\sigma$ and $\gamma = \delta(z)$. By Lemma 1,

$$\gamma x, \gamma y \in R_0 + R_0 z.$$

Therefore

$$0 = [\gamma x, \gamma y] = \gamma^2 [x, y].$$

Observe that, for any $s, t \geq 1$, $[x^s, y^t]$ is a sum of terms of the form $c[x, y]e$, where $c, d \in R$. Since γ is central, it now follows that

$$\gamma^2 [x^s, y^t] = 0.$$

The previous equation asserts that

$$[\gamma \alpha^i a, \gamma \beta^j b] = \gamma^2 [\alpha^i a, \beta^j b] \in \gamma^2 [R_0 + R_0 x + \cdots + R_0 x^i, R_0 + R_0 y + \cdots + R_0 y^j] = 0.$$

Hence

$$\alpha^i \beta^j \gamma^2 [a, b] = 0.$$

Let $B_1 = \{r \in A^\sigma \mid r\beta^j \gamma^2 [a, b] = 0\}$. Observe that, for every $\alpha \in A^\sigma$, $\alpha^i \in B_1$. Since the characteristic of R is either 0 or exceeds i, we can linearize the expression r^i (or apply the commutative version of the Nagata Higman theorem), to show that $(A^\sigma)^i \subseteq B_1$. Therefore

$$(A^\sigma)^i \beta^j \gamma^2 [a, b] = 0.$$

If we now let $B_2 = \{r \in A^\sigma \mid (A^\sigma)^i r \gamma^2 [a, b] = 0\}$, reasoning as above shows that $(A^\sigma)^j \subseteq B_2$, hence

$$(A^\sigma)^i (A^\sigma)^j \gamma^2 [a, b] = 0.$$

If $char R \neq 2$, we can let $B_3 = \{r \in R \mid (A^\sigma)^i (A^\sigma)^j r [a, b] = 0\}$. Reasoning as above shows that $(A^\sigma)^2 \subseteq B_3$, therefore

$$(A^\sigma)^{i+j+2} [a, b] = (A^\sigma)^i (A^\sigma)^j (A^\sigma)^2 [a, b] = 0.$$

However we are assuming that $A^\sigma = \delta(R) \cap R_0 \cap R^\sigma$ has zero annihilator, therefore $[a, b] = 0$. Therefore $[R, R] = 0$ and R is commutative

□

In [5], Herstein conjectured if σ is an automorphism of prime order p of a ring R such that $pR = 0$ and the invariants of R are central, must the commutator ideal of R be nil. Specializing Theorem 4 to the case where $\delta = \sigma - 1$ and R is an algebra over $\mathbb{Z}_p$, we have

Corollary 5. *Let R be a ring with an automorphism σ of prime order $p > 2$ such that $pR = 0$. If the invariants R^σ are central and the set $R^\sigma \cap \{\sigma(r) - r \mid r \in R\}$ has zero annihilator in R, then R is commutative.*

When $\sigma^p = 1$, $\delta = \sigma - 1$, and $pR = 0$, then the functions δ^{p-1} is the same as the trace map $\sigma^{p-1} + \cdots + \sigma + 1$. In [7], a corollary of the main result on central invariants shows that if there exists $a \in R$ such that $\delta^{p-1}(a)$ is regular in R, then the commutator ideal is contained in the prime radical of R. Observe that Corollary 5 obtains a stronger conclusion with a weaker assumption than the corollary in [7].

It is interesting to note that Herstein's conjecture predated the enormous progress made in the 1970's and 1980's in the study of finite groups acting on associative rings. Yet the problem remains unsolved. We hope that the techniques in Theorems 3 and 4 might result in additional progress towards solving this problem. We now give an example which illustrates that the assumption that $char(R) \neq 2$ in Theorem 4 is necessary.

Example. Let K be a field of characteristic 2 and let R be the K-algebra generated by the set $\{x_i\}_{i\geq 1} \cup \{z_i\}_{i\geq 1}$ subject to the relations

1. ${x_i}^2 = {z_i}^2 = 0$, for all $i \geq 1$,
2. each z_i is central,
3. $[x_i, x_j] = z_i z_j$, for all $i, j \geq 1$.

We then define the derivation d on R as $d(x_i) = z_i$ and $d(z_i) = 0$, for all $i \geq 1$. Since $2R = 0$, d preserves the defining relations for R, therefore d is a derivation of R such that d^2 is also a derivation. Observe that R is generated by R_1, therefore d^2 vanishes on all the generators of R, hence $d^2 = 0$.

Every element of R can be written uniquely as a linear combination of monomials of the form $z_{i_1} \cdots z_{i_n} x_{j_i} \cdots x_{j_m}$, where the subscripts of the z_i's and x_j's are both strictly increasing and the number of $z_i's$ and $x_j's$ occurring can be any integer greater than or equal to 0. In this example, $\sigma = 1$ and $\delta(R) \subseteq R_0$, therefore $A^\sigma = \delta(R)$, hence each $z_i \in A^\sigma$, for $i \geq 1$. Thus the set $\delta(R) \cap R_0 \cap R^\sigma$ does indeed have zero annihilator in R. Since R is certainly not commutative, to conclude the example, it suffices to show that $R_0 \subseteq Z(R)$.

Suppose for the moment, that $d(R) \subseteq Z(R)$. Then, if $b \in R_0$, we have

$$d(Rb) = d(R)b.$$

Since $d(R) \subseteq Z(R)$ taking the commutator of both sides of the previous equation with R, we obtain

$$0 = [d(R)b, R] = d(R)[b, R].$$

However, $d(R)$ has zero annihilator, therefore $[b, R] = 0$, hence $b \in Z(R)$. Therefore $R_0 \subseteq Z(R)$ and it now suffices to show that $d(R) \subseteq Z(R)$.

By re-ordering the x_i's, it suffices to show that if $x_1, r \in R$, then $[d(r), x_1] = 0$. Since $d(x_1)$ is central, this is equivalent to showing $d([r, x_1]) = 0$. We only need to consider the case where r is a monomial in the z_i's and x_j's. But since each z_i is a central invariant, we only need to consider the case where r is a monomial in the x_j's. One possibility is that $r = x_1\Delta$, where x_1 does not appear in Δ. Then

$$d([x_1\Delta, x_1]) = d(x_1[\Delta, x_1]) = z_1[\Delta, x_1] + x_1 d([\Delta, x_1]).$$

When expanding $[\Delta, x_1]$, each term contains a factor of z_1. Therefore $z_1[\Delta, x_1] = 0$. As a result, it now suffices to show that $d([\Delta, x_1]) = 0$.

In light of our reductions, we may assume that $r = x_2x_3\cdots x_n$. For $2 \leq i \leq n$, let Δ_i be r with x_i removed. Then

$$d(r) = \sum_{2\leq i\leq n} z_i\Delta_i.$$

Next, if $2 \leq i \neq j \leq n$, let $\Delta_{i,j}$ be r with both x_i and x_j removed. For fixed (but arbitrary) i, we have

$$[\Delta_i, x_1] = \sum_{2\leq i\neq j\leq n} z_1z_j\Delta_{i,j}.$$

Now, summing over all i, we obtain

$$[d(r), x_1] = [\sum_{2\leq i\leq n} z_i\Delta_i, x_1] = \sum_{2\leq i\neq j\leq n} z_iz_1z_j\Delta_{i,j}.$$

Observe that since $\Delta_{i,j} = \Delta_{j,i}$, terms with $\Delta_{i,j}$ occur twice, once preceded by $z_iz_1z_j$ and once preceded by $z_jz_1z_i$. Therefore

$$0 = [d(r), x_1],$$

hence $d(R) \subseteq Z(R)$, concluding the example. □

In light of the above, when $char(R) = 2$, the other hypotheses in Theorem 5 are not sufficient to force R to be commutative, However, if δ is a derivation (as in the example above) or of the form $\sigma - 1$, it can be shown that $[R, R] \subseteq Z(R)$. Furthermore, if δ is neither a derivation nor of the form $\sigma - 1$, it can be shown that $[[R, R], [R, R]] \subseteq Z(R)$.

In Lemma 1, we see that the subalgebra of R generated by R_1 represents a large piece of R. In our final result, we consider the case where R is generated by R_1. Recall that the prime radical is the intersection of all the prime ideals of R, the nil radical is a nil ideal containing all the nil ideals of R, and the prime radical is always contained in the nil radical.

Theorem 6. *Let R be an algebra with a locally nilpotent q-skew derivation δ such that $char(R)$, $char_q(R) \neq 2$. If the invariants of δ are central and R is generated by $R_1 = \{r \in R \mid \delta^2(r) = 0\}$, then the commutator ideal $R[R, R]R$ is contained in the prime radical.*

Proof. Suppose $\beta \in A$ and $\beta = \delta(y)$. If $z \in R_1$, since $\beta, \delta(z) \in R_0$ and R_0 is central, we have

$$\delta(\beta z - \delta(z)y) = \delta(z\beta - y\delta(z)) = \delta(z)\beta - \delta(y)\delta(z) = \beta\delta(z) - \beta\delta(z) = 0.$$

Therefore $\beta z - \delta(z)y \in R_0$, hence $\beta z \in \delta(z)y + R_0 \subseteq R_0 + R_0 y$.

As a result,

$$\beta R_1 \subseteq R_0 + R_0 y$$

and

$$\beta^2[R_1, R_1] = [\beta R_1, \beta R_1] \subset [R_0 + R_0 y, R_0 + R_0 y] = 0.$$

Since R is generated by R_1, it follows that if $r, s \in R$, then $[r, s]$ is a sum of terms of the form $a[b, c]d$, where $b, c \in R_1$ and $a, d \in R$. Since β is central, we see that $\beta^2[r, s] = 0$. As a result,

$$\beta^2[R, R] = 0.$$

Since the characteristic of R is not 2, we can once again linearize $\beta^2[R, R] = 0$ to obtain

$$A^2[R, R] = 0.$$

Let $B = ann_R(A^2)$; observe that B contains the commutator ideal $R[R, R]R$. Since δ is q-skew, both $\delta(R)$ and R_0 are σ-stable, hence $\sigma(A^2) = A^2$. It immediately follows that $\sigma(B) = B$. Furthermore, if $b \in B, a \in A^2$, we have

$$0 = \delta(ba) = \delta(b)a + \sigma(b)\delta(a) = \delta(b)a.$$

Therefore $\delta(B) \subseteq ann_R(A^2) = B$ and B is δ-stable.

In light of the above, δ restricts to B as a locally nilpotent q-skew derivation with invariants which are central in R. Since $\delta(R_1) \subseteq A \subseteq Z(R)$ and $\delta^2(R_1) = 0$, it follows that if $r_1, r_2, \ldots, r_n \in R_1$, with $n \geq 3$, then

$$\delta^2(r_1 r_2 \cdots r_n) \in A^2 R.$$

Therefore

$$\delta^2(B^3) \subseteq A^2 B = 0.$$

As a result, $\delta(B^3)$ is contained in the invariants, hence $\delta(B^3) \subseteq Z(R)$

If $a, b \in B^3$, we have

$$0 = \delta^2(ab) = \delta^2(a)b + (1+q)\sigma(\delta(a))\delta(b) + \sigma^2(a)b = (1+q)\sigma(\delta(a))\delta(b).$$

Since $1 + q \neq 0$, it follows from the previous equation that

$$\delta(B^3)\delta(B^3) = \sigma(\delta(B^3))\delta(B^3) = 0.$$

Furthermore, if $a, b \in B^3$, it follows that

$$\delta(a\delta(b)) = \delta(a)\delta(b) + \sigma(a)\delta^2(b) = 0,$$

hence $a\delta(b)$ is also contained in the invariants of δ. As a result $B^3\delta(B^3) \subseteq R_0 \subseteq Z(R)$.

Since $\delta(B^3)$ and $B^3\delta(B^3)$ are both central,

$$0 = [B^3\delta(B^3), B^3] = [B^3, B^3]\delta(B^3).$$

We now let $C = ann_{B^3}(\delta(B^3))$; observe that C contains the ideal $R[B^3, B^3]R$. Since $\sigma(C) = C$, we have

$$\delta(C^2) \subseteq \delta(C)C + \sigma(C)\delta(C) \subseteq \delta(B^3)C + C\delta(B^3) = 0.$$

Therefore C^2 is also in the invariants of δ, hence $C^2 \subseteq Z(R)$.

In order to show that the commutator ideal $R[R, R]R$ is in the prime radical, we need to show that, for every prime ideal P of R, we have $R[R, R]R \subseteq P$. Suppose there exists a prime ideal P such that $R[R, R]R \not\subset P$. Since $R[R, R]R \subseteq B$, it follows that $B \not\subset P$. Furthermore, since P is prime, $B^3 \not\subset P$.

As a result, $B^3 + P/P$ is a nonzero ideal of the prime ring R/P. Since $R[R, R]R \not\subset P$, the prime ring R/P is not commutative. Therefore, every nonzero ideal of R/P is not commutative. Hence $B^3 + P/P$ is not commutative and it follows that $[B^3, B^3] \not\subset P$.

In light of the above, $R[B^3, B^3]R \not\subset P$ and since $R[B^3, B^3]R \subseteq C$, we have $C \not\subset P$. Since P is prime, $C^2 \not\subset P$. Therefore $C^2 + P/P$ is a nonzero ideal of the prime ring R/P. Since R/P is not commutative, $C^2 + P/P$ is also not commutative, therefore $[C^2, C^2] \nsubseteq P$. But this contradicts the fact that $[C^2, R] = 0$.

Thus $R[R, R]R$ is indeed contained in each prime ideal of R and is therefore contained in the prime radical. □

To this point, the only example we have given where central invariants do not imply commutativity is in characteristic 2. In light of this, we conclude this paper with an example of an algebra R in characteristic 0 with a locally nilpotent derivation d such that d is not nilpotent, R_0 is central, but R is not commutative. Observe that R will be a direct sum of noncommutative algebras on which d acts nilpotently with central invariants.

Example. If $n \geq 2$, let K be a field of characteristic 0 and let $R(n)$ be the K-algebra generated by x_1, x_2, y_1, y_2 such that

1. x_1, x_2 are central,
2. $x_{i_1} \cdot x_{i_2} \cdots x_{i_n} = 0$, where each $x_{i_j} \in \{x_1, x_2\}$,
3. $[y_1, y_2] = {x_1}^{n-1}$,
4. the derivation d_n is defined as $d_n(x_1) = d_n(x_2) = 0$, $d_n(y_1) = x_1$, and $d_n(y_2) = x_2$.

Since ${d_n}^{n-1}(y_1) = (n-1)!{x_1}^{n-1}$, we see that ${d_n}^{n-1} \neq 0$. Let $\Delta_1, \ldots, \Delta_m$ be all the products of length n in x_1 and x_2. Observe that $d_n(R(n)) \subseteq x_1R(n) + x_2R(n)$, hence ${d_n}^n(R(n)) \subseteq \sum_{i=1}^m \Delta_i R(n)$. However, in $R(n)$ each $\Delta_i = 0$, hence ${d_n}^n(R(n)) = 0$.

If $r \in R(n)$ is an invariant, we can write $r = \sum_{i,j\geq 0} {y_1}^i {y_2}^j + w$, where every monomial in w has at least one x_1 or x_2. Observe that

$$d_n(\sum_{i,j\geq 0} {y_1}^i {y_2}^j) = \sum_{i\geq 1, j\geq 0} i x_1 {y_1}^{i-1} {y_2}^j + \sum_{i\geq 0, j\geq 1} j x_2 {y_1}^i {y_2}^{j-1}.$$

On the other hand, every monomial in $d_n(w)$ must have degree at least 2 in the x's. Therefore, if $d_n(r) = 0$, then $r = w \in x_1 R(n) + x_2 R(n)$. However, $[R(n), R(n)] \subseteq {x_1}^{n-1} R(n)$, hence $[w, R(n)] = 0$. Therefore w is central and so, d_n has central invariants.

Therefore d_n is nilpotent on $R(n)$ with central invariants, however $R(n)$ is not commutative. Finally let $R = \oplus_{n=2}^{\infty} R(n)$ and we can define the derivation d on R as $d((r_1, r_2, \ldots)) = (d_1(r_1), d_2(r_2), \ldots)$, where each $r_i \in R(i)$. Certainly R is not commutative, d has central invariants, and d is not nilpotent. However, each element of R only has a finite number of nonzero components, therefore d is locally nilpotent. Finally, as in Theorem 6, R is generated by R_1. $\square$

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