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POINCARÉ SERIES OF FIBER PRODUCTS AND WEAK COMPLETE INTERSECTION IDEALS

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Abstract. We study the Poincaré series of modules over a fiber product of commutative local rings. We introduce the notion of a weak complete intersection ideal; these are the ideals with the property that every differential in their minimal free resolutions can be represented by a matrix whose entries are in the ideal itself. We show that many properties of the Poincaré series that are known to hold for a fiber product over the maximal ideal also hold for those over weak intersection ideals.

1. Introduction

Given a commutative local ring R, with residue field k, the growth of a free resolution of an R-module M is encoded in the formal power series

$$P_M^R = \sum_{i=0}^{\infty} \operatorname{rank}_k \operatorname{Tor}_i^R(M,k) z^i \in \mathbb{Z}[\![z]\!],$$

known as the *Poincaré series of* M. Formulas involving Poincaré series and change of rings have been studied extensively. For example, suppose that Q and R are local rings with the same residue field k and $\varphi\colon Q \to R$ is a local homomorphism, Serre shows that for every finite R-module M, there is a coefficientwise inequality of formal power series

$$P_M^R \preccurlyeq \frac{P_M^Q}{1+z-zP_R^Q}, \tag{1.0.1}$$

see [2, 3.3.2] for a proof. Specialization of the ring homomorphism yields more accurate formulas. In [5], Dress and Krämer study the Poincaré series of modules over the fiber product $R \times_k S$ of two commutative local rings R and S over the common residue field k. They give a formula that allows one to compute the Poincaré series of an S-module M when considered as a $R \times_k S$-module:

$$P_M^{R\times_k S} = \frac{P_M^S P_k^R}{P_k^R + P_k^S - P_k^R P_k^S}.$$

Our goal, in this paper, is to investigate the Poincaré series of modules over a fiber product in a more general setting. In Section 3, we prove that similar formulas hold if the fiber product is over a weak complete intersection ideal; see Section 2 for the definition. In particular, we prove the following; see Theorem 3.12.

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Theorem A. *Let R, S and T be commutative local rings such that there are surjective ring homomorphisms $S \to R \to T$. If the kernel of $R \to T$ is a weak complete intersection ideal in R, then the equality*

$$P_M^{R \times_T S} = \frac{P_M^S P_T^R}{P_T^R + P_T^S - P_T^R P_T^S}$$

holds for every finitely generated S-module M.

We introduce the notion of a weak complete intersection ideal in Section 2. These are the ideals with the property that the R/I-modules $\operatorname{Tor}_i^R(R/I, R/I)$ are free for every i. Clearly the maximal ideal satisfies this property. Also, if I is a complete intersection ideal in R, then R/I is resolved by the Koszul complex and therefore I is a weak complete intersection ideal. The following theorem provides a family of weak complete intersection ideals; see Theorem 2.5.

Theorem B. *Let S be a commutative local ring, let I be a complete intersection ideal in S and let $s \geq 2$ be an integer. If $R = S/I^s$, then I/I^s is a weak complete intersection ideal in R.*

Other useful tools in computing Poincaré series are the notions of a Golod homomorphism, which was introduced by Levin [14], and of a Golod ring, introduced by Golod [6]. Golod homomorphisms are ring homomorphisms over which the Poincaré series of the residue field has the fastest growth allowed by 1.0.1. It is known that if R is I is a complete intersection ideal in R then the natural surjection $R \to R/I^s$ is a Golod homomorphism, for all $s \geq 2$ and if in addition R is regular then R/I^s is a Golod ring; see [11, Folgerung 2]. The following statement, which we prove in Section 4, provides families of Golod rings and Golod homomorphisms; see Theorem 4.7 and Corollary 4.8.

Theorem C. *Let R be a commutative local ring and let I be a complete intersection ideal in R. For all integers $s \geq 2$, and $t \geq 2$ the following statements hold.*

(i) The homomorphism

$$R \times_{R/I} R \to R/I^s \times_{R/I} R/I^t$$

induced by the natural surjections, is a Golod map.

(ii) If R is regular then $R/I^s \times_{R/I} R/I^s$ is a Golod ring.

The relevance of Theorem C also comes from the fact that the fiber products of Golod rings over the residue field are Golod; see [12, 4.1].

We finish the introduction by recalling some notations and definitions. The rank of the vector space $\operatorname{Tor}_i^R(M, k)$ is called the ith Betti number of M and is denoted by $\beta_i(M)$. We will denote the ith syzygy of M by $\Omega_i^R(M)$.

Given a sequence $\mathbf{x} = x_1, \ldots, x_n$ of elements in R, we denote by $\mathrm{K}(\mathbf{x}; R)$ the Koszul complex of $\mathbf{x}$ with coefficients in R. If I is an ideal of R, the Koszul complex of I, with coefficients in R, is computed using a minimal set of generators of I and is denoted by $\mathrm{K}(I; R)$.

We denote by $\hat{R}$ the $\mathfrak{m}$-adic completion of R and by $\hat{M}$ the $\hat{R}$-module $M \otimes_R \hat{R}$. Note that, since $\hat{R}$ is a faithfully flat R-module, we have $P_M^R = P_{\hat{M}}^{\hat{R}}$.

The embedding dimension of a local ring is the minimal number of generator of the maximal ideal. If R is a complete ring, then by Cohen's Structure Theorem, there is a surjective map $\varphi : Q \to R$ such that Q is a regular local ring; this map is called a Cohen presentation of R. A Cohen presentation is minimal if R and Q have the same embedding dimension.

2. Weak Complete intersection ideals

In this section, we introduce the notion of a weak complete intersection ideal and provide a family of examples.

Definition 2.1. Let R be a commutative local ring and let I be an ideal of R. We say that I is a *weak complete intersection ideal*, if $\operatorname{Tor}_i^R(R/I, R/I)$ is a free R/I-module for all i.

The next lemma shows that weak complete intersection ideals are exactly the ideals with the property that every differential in their minimal free resolutions can be represented by a matrix whose entries are in I.

Lemma 2.2. *Let $(R, \mathfrak{m})$ be a commutative local ring, let M be a finite R-module and let F be the minimal free resolution of M. If I is an ideal of R, the following conditions are equivalent.*

(i) Every differential ∂_i of F satisfies $\partial_i(F_i) \subseteq IF_{i-1}$.
(ii) The module $\operatorname{Tor}_i^R(M, R/I)$ is a free R/I-module for every i.

Proof. Clearly *(i)* implies *(ii)*. Now, suppose that the module $\operatorname{Tor}_i^R(M, R/I)$ is a free R/I-module for all i. We show, by induction, that $\operatorname{Im}\partial_i^F \subseteq IF_{i-1}$. Since $\operatorname{Tor}_0^R(M, R/I) \cong M/IM$ is a free, one has $F_0 \otimes R/I \cong M/IM$ and $\partial_1 \otimes R/I = 0$, which imply that $\operatorname{Im}\partial_1^F \subseteq IF_0$.

Now, assume that $\operatorname{Im}\partial_n^F \subseteq IF_{n-1}$. Since $\operatorname{Tor}_n^R(M, R/I)$ is a free R/I- module and $\partial_n^F \otimes R/I = 0$, the map $\partial_{n+1}^F \otimes R/I$ must split. However, the image of $\partial_{n+1}^F \otimes R/I$ is in $\mathfrak{m}F_n \otimes R/I$ and hence it must be the zero map. Therefore, $\operatorname{Im}\partial_{n+1}^F \subseteq IF_n$. □

Recall that an ideal I of a commutative local ring R is a complete intersection ideal if it is generated by a regular sequence.

Example 2.3. (i) The maximal ideal is weak complete intersection.

(ii) If I is complete intersection then R/I is resolved by the Koszul complex on a minimal set of generators of I, hence I is weak complete intersection.

(iii) Let $R = k[x]/(x^5)$, where k is a field. If I is the ideal generated by x^2 then the minimal free resolution of R/I is of the form

$$\cdots \xrightarrow{x^2} R \xrightarrow{x^3} R \xrightarrow{x^2} R \to 0.$$

Thus I is weak complete intersection.

(iv) Ulrich ideals are weak complete intersection. An ideal in a Cohen-Macaulay local ring R is called an Ulrich ideal if I/I^2 is a free R/I-module and $I^2 = JI$ where J is a parameter ideal in I that is a reduction of I; see [7, 7.1].

Remark 2.4. In [18, Corollary 1], Vasconcelos shows that I is a complete intersection ideal if and only if I/I^2 is free and the projective dimension of I is finite. Since I/I^2 is isomorphic to $\operatorname{Tor}_1^R(R/I, R/I)$, the ideal I is complete intersection if and only if I is weak complete intersection and the projective dimension of I is finite.

The next theorem gives a family of weak complete intersection ideals.

Theorem 2.5. *Let S be a commutative local ring, let I be an ideal of S that is generated by a regular sequence of length c. Let $s \geq 2$ be an integer and let $R = S/I^s$. Then the image of I in R is weak complete intersection. Moreover, the Poincaré series of this ideal is given by*

$$\begin{aligned} P^R_{R/IR} &= (1+z)^c(1 - z^2 P^S_{I^s})^{-1} \\ &= (1+z)^c \left(1 - \sum_{i=1}^{c} \binom{c+s-1}{s+i-1}\binom{s+i-2}{s-1} z^{i+1}\right)^{-1}. \end{aligned}$$

We devote the rest of this section to proving this theorem. In fact, we prove a more general statement; see Theorem 2.9. The main ingredient in our proof is the notion of a trivial Massey operation on a differential graded algebra.

Definition 2.6. A *differential graded algebra over R* is a complex A of R-modules equipped with a morphism of complexes, called the product of the differential graded algebra

$$A \otimes_R A \to A, \;\; a \otimes b \to ab$$

that satisfies the following properties.

i) The equality $A_i = 0$ holds for all $i < 0$.
ii) The product is associative.
iii) There is an element $1 \in A_0$ such that $1a = a1 = a$ for all $a \in A$.
iv) If $a, b \in A$ then $ab = (-1)^{|a||b|}ba$ and $a^2 = 0$ if the degree of a is odd.
v) The differential of A satisfies the *Leibniz rule*; that is

$$\partial(ab) = \partial(a)b + (-1)^{|a|}a\partial(b), \text{ for all } a, b \in A.$$

Here $|a|$ denotes the degree of a. Note that if A is a differential graded algebra, then $H(A)$ is a graded $H_0(A)$-algebra.

An example of a differential graded algebra, that we use in this paper, is the Koszul complex $\mathrm{K}(I; R)$ of an ideal I in R. In this case, $H(\mathrm{K}(I; R))$ forms a graded R/I-algebra.

Definition 2.7. Let R be a commutative local ring and let A be a differential graded algebra over R such that $\mathrm{H}_0(A) \cong R/I$ for some ideal I of R and that $\mathrm{H}_i(A)$ is a free R/I-module for every $i \geq 1$. We say that A admits a *trivial Massey operation*, if for some R/I-basis $\mathcal{B} = \{h_\lambda\}_{\lambda \in \Lambda}$ of $\mathrm{H}_{\geq 1}(A)$ there exists a function

$$\mu \colon \coprod_{n=1}^{\infty} \mathcal{B}^n \to A$$

such that

$$\mu(h_\lambda) = z_\lambda \in Z(A) \text{ where } \mathrm{cls}(z_\lambda) = h_\lambda,$$

$$\partial^A \mu(h_{\lambda_1}, \cdots, h_{\lambda_p}) = \sum_{j=1}^{p-1} \overline{\mu(h_{\lambda_1}, \cdots, h_{\lambda_j})}\mu(h_{\lambda_{j+1}}, \cdots, h_{\lambda_p}),$$

where $Z(A)$ denotes the set of cycles of A, and $\overline{a} = (-1)^{|a|+1}a$.

If I is an ideal of a commutative local ring R with the property that I/I^2 is a free R/I-module, the existence of a trivial Massey operation guarantees that I is weak complete intersection. This is the content of Theorem 2.9, whose proof requires the following lemma.

Lemma 2.8. *Let $(R, \mathfrak{m})$ be a commutative local ring and I be an ideal of R. Let $K = \mathrm{K}(I, R)$ with differentials ∂ and let i be a positive integer. The following statements hold.*

(i) *If $y \in K_i$ satisfies $\partial_i(y) \in \mathfrak{m}IK_{i-1}$, then $y \in \mathfrak{m}K_i$.*

(ii) *Suppose that I/I^2 is free as an R/I-module and $y \in K_i$. If $\partial_i(y) \in I^2K_{i-1}$ then $y \in IK_i$.*

Proof. (i) Let $x_1, \cdots, x_c$ be a minimal set of generators for I and let $y \in K_i$ then

$$y = \sum_{\substack{J' \subset \{1,\ldots,c\}, \\ |J'|=i}} b_{J'} e_{J'},$$

where $b_{J'} \in R$ and the elements $e_{J'}$ form a basis of K_i. Then

$$\partial_i(y) = \sum_{\substack{J \subseteq \{1,\ldots,c\}, \\ |J|=i-1}} a_J e_J,$$

where $a_J \in R$ and the elements e_J form a basis for K_{i-1}. In particular, $a_J = \sum_{i=1, i \notin J}^{c} \pm b_{J_i} x_i$, where the $J_i = J \cup \{i\}$. Since $\partial_i(y) \in \mathfrak{m}IK_{i-1}$, and K_{i-1} is a free module, $a_J \in \mathfrak{m}I$ for all $J \subseteq \{1, \ldots, c\}$ such that $|J| = i - 1$. This implies that $\sum_{i=1, i \notin J}^{c} \pm b_{J_i} x_i \in \mathfrak{m}I$ and $b_{J_i} \in \mathfrak{m}$, since the elements x_i are minimal generators for the ideal I. Since this holds for every J we obtain that $b_{J'} \in \mathfrak{m}$ for all $J' \subset \{1, \ldots, c\}$ and $|J'| = i$.

(ii) The argument is similar to the proof of part (i), as $\sum_{i=1, i \notin J}^{c} \pm b_{J_i} x_i \in I^2$ implies that $b_{J_i} \in I$, since I/I^2 is a free R/I module. $\square$

In the proof of the next theorem, we use an analogue of a construction of free resolutions first introduced by Eagon. Eagon's resolution, which can be found in [2], [8], and [17], uses the Koszul complex of the maximal ideal, whose homology modules are clearly vector spaces. Our proof is modeled after the proofs that can be found in the references mentioned above, but new difficulties arise since we are not working with vector spaces.

Theorem 2.9. *Let R be a commutative local ring, let I be an ideal of R such that I/I^2 is a free R/I-module and let $K = \mathrm{K}(I, R)$. If $\mathrm{H}_i(K)$ is a free R/I-module for every i and if K admits a trivial Massey operation μ, then I is weak complete intersection. Moreover, one has*

$$P^R_{R/I} = \frac{(1+z)^c}{1 - \epsilon_1 z^2 - \epsilon_2 z^3 - \cdots - \epsilon_c z^{c+1}},$$

where c is the minimal number of generators of I and ϵ_i is the rank of the free R/I-module $\mathrm{H}_i(K)$.

Proof. First recall that $\mathrm{H}_0(K) \cong R/I$ and that $\mathrm{H}_i(K)$ is a free R/I-module for every $i \geq 1$. Let $\{h_\lambda\}_{\lambda \in \Lambda}$ be a basis for $\mathrm{H}_{\geq 1}(K)$. For every $i \geq 2$, let V_i be a free R-module with basis $\mathcal{B}_i = \{v_\lambda \mid |h_\lambda| = i - 1\}$. Note that the rank of V_i is the same as the rank of $\mathrm{H}_{i-1}(K)$.

We construct the free resolution F of R/I recursively. Set $F_0 = K_0$, $F_1 = K_1$ and $\partial_1^F = \partial_1^K$. For $n \geq 2$, let

$$\begin{aligned} F_n &= K_n \oplus (F_{n-2} \otimes_R V_2) \oplus (F_{n-3} \otimes_R V_3) \oplus \cdots \oplus (F_1 \otimes_R V_{n-1}) \oplus (F_0 \otimes_R V_n) \\ &= \bigoplus_{i+i_1+\cdots+i_p=n} K_i \otimes_R V_{i_1} \otimes_R \cdots \otimes_R V_{i_p} \end{aligned}$$

and define the differential as

$$\begin{aligned} \partial_n^F(a \otimes v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_p}) &= \partial_i^K(a) \otimes v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_p} \\ &\quad + (-1)^i \sum_{j=1}^{p} a\mu(h_{\lambda_1}, \cdots, h_{\lambda_j}) \otimes v_{\lambda_{j+1}} \otimes \cdots \otimes v_{\lambda_p}, \end{aligned}$$

where $v_{\lambda_j} \in \mathcal{B}_{i_j}$ for each $j = 1, \ldots, p$ and $a \in K_i$. First, we show that (F, ∂^F) is a complex and the composition of differential is the zero homomorphism:

$$\begin{aligned} &\partial_{n-1}^F(\partial_n^F(a \otimes v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_p})) = \partial^K(\partial^K(a)) \otimes v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_p} \\ &+ (-1)^{i-1} \sum_{j=1}^{p} \partial^K(a)\mu(h_{\lambda_1}, \cdots, h_{\lambda_j}) \otimes v_{\lambda_{j+1}} \otimes \cdots \otimes v_{\lambda_p} \\ &+ (-1)^{i} \sum_{j=1}^{p} \partial^K(a)\mu(h_{\lambda_1}, \cdots, h_{\lambda_j}) \otimes v_{\lambda_{j+1}} \otimes \cdots \otimes v_{\lambda_p} \\ &+ \sum_{j=1}^{p} a\partial^K(\mu(h_{\lambda_1}, \cdots, h_{\lambda_j})) \otimes v_{\lambda_{j+1}} \otimes \cdots \otimes v_{\lambda_p} \\ &- \sum_{j=1}^{p} \sum_{l=j+1}^{p} a\overline{\mu(h_{\lambda_1}, \cdots, h_{\lambda_j})}\mu(h_{\lambda_{j+1}}, \cdots, h_{\lambda_l}) \otimes v_{\lambda_{l+1}} \otimes \cdots \otimes v_{\lambda_p} \\ &= 0. \end{aligned}$$

To prove further that the complex F is a free resolution of R/I, we first note that K is a subcomplex of F, and $F/K \cong F \otimes V$, where V is the complex

$$\cdots \xrightarrow{0} V_n \xrightarrow{0} V_{n-1} \xrightarrow{0} \cdots \xrightarrow{0} V_2 \to 0 \to 0.$$

The short exact sequence of complexes

$$0 \to K \to F \to F \otimes V \to 0$$

yields a long exact sequence on homology

$$\begin{aligned} \cdots \to \bigoplus_i (\mathrm{H}_{n+1-i}(F) \otimes_R V_i) &\xrightarrow{\eth_{n+1}} \mathrm{H}_n(K) \to \mathrm{H}_n(F) \\ \to \bigoplus_i (\mathrm{H}_{n-i}(F) \otimes_R V_i) &\xrightarrow{\eth_n} \mathrm{H}_{n-1}(K) \to \cdots . \end{aligned}$$

First note that the equalities $\mathrm{H}_0(F) = R/I$ and $\mathrm{H}_1(F \otimes_R V) = 0$ holds. Also, one can easily see $\eth_n(1 \otimes v_\lambda) = h_\lambda$, thus $\eth_n$ is a surjective homomorphism, which is split since the module $\mathrm{H}_n(K)$ is free.

Now we prove, by induction, that $\mathrm{H}_i(F) = 0$ for all $i \geq 1$. Since $\mathrm{H}_0(K) \otimes V_2 \cong \mathrm{H}_1(K)$ and $\eth_2$ is surjective, the exact sequence

$$0 \to \mathrm{H}_2(F) \to \mathrm{H}_0(K) \otimes V_2 \xrightarrow{\eth_2} \mathrm{H}_1(K) \to \mathrm{H}_1(F) \to 0$$

yields the equalities $\mathrm{H}_1(F) = 0 = \mathrm{H}_2(F)$.

Suppose that $n \geq 3$ and $\mathrm{H}_i(F) = 0$ for all $1 \leq i < n$. The short exact sequence

$$0 \to \mathrm{H}_n(F) \to \mathrm{H}_0(K) \otimes V_n \xrightarrow{\eth_n} \mathrm{H}_{n-1}(K) \to 0$$

and the isomorphism $\mathrm{H}_0(K) \otimes V_n \cong \mathrm{H}_{n-1}(K)$ imply the equality $\mathrm{H}_n(F) = 0$. Therefore, F is a free resolution of R/I.

To show the inclusion $\partial(F) \subseteq IF$, it suffices to prove that $\mu(h_{\lambda_1}, \cdots, h_{\lambda_p})$ is in IK for all $p \geq 1$. We prove this by induction on p. If $p = 1$, then $\mu(h_{\lambda_1}) \in Z(K) \subseteq IK$, by Lemma 2.8. Suppose $p > 1$ and $\mu(h_{\lambda_1}, \cdots, h_{\lambda_j}) \in IK$ for all $1 \leq j \leq p-1$ then by the definition $\partial^K(\mu(h_{\lambda_1}, \cdots, h_{\lambda_p})) \in I^2K$. Therefore, by Lemma 2.8, $\mu(h_{\lambda_1}, \cdots, h_{\lambda_p})$ is in IK.

Finally, note that, by the construction of the resolution, one has

$$P^R_{R/I} = (1+z)^c + \epsilon_1 z^2 P^R_{R/I} + \epsilon_2 z^3 P^R_{R/I} + \cdots + \epsilon_c z^{c+1} P^R_{R/I}$$

which implies the last assertion of the theorem. $\square$

Remark 2.10. Note that if $I/I^2 \cong \mathrm{Tor}^R_1(R/I, R/I)$ is a free R/I-module, then I is not necessarily weak complete intersection. Indeed, if $R = \mathbb{Q}[x,y]/(x^4, x^3y^3, y^4)$ and $I = (x^2, y^2)$, one can see that the minimal free resolution of R/I has the form of

$$\cdots \to R^8 \xrightarrow{d_3} R^4 \xrightarrow{d_2} R^2 \xrightarrow{d_1} R \to 0$$

with differentials $d_1 = \begin{bmatrix} x^2 & y^2 \end{bmatrix}$ and

$$d_2 = \begin{bmatrix} 0 & -y^2 & x^2 & xy^3 \\ y^2 & x^2 & 0 & 0 \end{bmatrix}, \; d_3 = \begin{bmatrix} y^2 & -x^2 & 0 & 0 & 0 & 0 & x^3y & 0 \\ 0 & y^2 & x^2 & 0 & 0 & x^2y & 0 & x^3y \\ 0 & 0 & y^2 & x^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y & x & 0 & 0 \end{bmatrix}$$

Therefore, $\mathrm{Tor}^R_1(R/I, R/I)$ is free but $\mathrm{Tor}^R_2(R/I, R/I)$ is not, by Lemma 2.2, .

We use the next lemma to show existence of trivial Massey operations on the Koszul complex of certain ideals.

Lemma 2.11. *Let R be a commutative local ring, let I be an ideal of R that is generated by a regular sequence and let M be an R-module such that $IM = 0$. Let (F, ∂) be a free resolution of M. For every positive integers i and every integer s such that $s \geq 2$, one has*

$$I^sF_{i-1} \cap \operatorname{Im} \partial_i \subseteq \partial_i(I^{s-1}F_i).$$

Proof. First note that if J is ideal in R then for every integer i, there is an isomorphism

$$\mathrm{Tor}^R_i(M, R/J) \cong \frac{\operatorname{Im} \partial^F_i \cap JF_{i-1}}{J \operatorname{Im} \partial_i}. \tag{2.11.1}$$

Let I be minimally generated by a regular sequence $x_1, \ldots, x_c$, let I_j denote the ideal generated by $x_j, \ldots, x_c$ for all $j = 1, \ldots, c$ and let $s \geq 2$ be an integer. We first show that for every j, the inclusion

$$I^s_jF_{i-1} \cap \operatorname{Im} \partial_i \subseteq I^{s-1}_j \operatorname{Im} \partial_i + (I^s_{j+1}F_{i-1} \cap \operatorname{Im} \partial_i) \tag{2.11.2}$$

holds. If $y \in I^s_jF_{i-1} \cap \operatorname{Im} \partial_i$, then by 2.11.1, x_jy is in $I^s_j \operatorname{Im} \partial_i$. So, we can write

$$x_jy = x_ju + v, \tag{2.11.3}$$

where $u \in I_j^{s-1} \operatorname{Im} \partial_i$ and $v \in I_{j+1}^s \operatorname{Im} \partial_i$. So, one has

$$x_j(y-u) \in I_{j+1}^s F_{i-1} \cap (x_j)F_{i-1} = (I_{j+1}^s \cap (x_j))F_{i-1} = I_{j+1}^s (x_j) F_{i-1},$$

which implies that $x_j(y-u) = x_j w$ for some $w \in I_{j+1}^s F_{i-1}$. Since x_j is a non-zerodivisor, one must have

$$\begin{aligned} y = u + w &\in (I_j^{s-1} \operatorname{Im} \partial_i + I_{j+1}^s F_{i-1}) \cap \operatorname{Im} \partial_i \\ &\subseteq I_j^{s-1} \operatorname{Im} \partial_i + (I_{j+1}^s F_{i-1} \cap \operatorname{Im} \partial_i), \end{aligned}$$

which proves the inclusion 2.11.2. Now, using 2.11.2, c times, we get the following sequence of inclusions

$$\begin{aligned} I^s F_{i-1} \cap \operatorname{Im} \partial_i &\subseteq I^{s-1} \operatorname{Im} \partial_i + (I_2^s F_{i-1} \cap \operatorname{Im} \partial_i) \\ &\subseteq I^{s-1} \operatorname{Im} \partial_i + I_2^{s-1} \operatorname{Im} \partial_i + (I_3^s F_{i-1} \cap \operatorname{Im} \partial_i) \\ &\vdots \\ &\subseteq I^{s-1} \operatorname{Im} \partial_i + I_2^{s-1} \operatorname{Im} \partial_i + \cdots + I_c^s \operatorname{Im} \partial_i. \end{aligned}$$

Therefore, $I^s F_{i-1} \cap \operatorname{Im} \partial_i \subseteq I^{s-1} \operatorname{Im} \partial_i = \partial_i(I^{s-1} F_i)$. □

Remark 2.12. Let I is an ideal in R generated by be a regular sequence and let M be an R-module such that $IM = 0$. Lemma 2.11 implies that the map

$$\operatorname{Tor}_i^R(M, I^s) \to \operatorname{Tor}_i^R(M, I^{s-1})$$

induced by the inclusion map $I^s \to I^{s-1}$ is zero for all i and $s \geq 2$. Thus the map

$$\operatorname{Tor}_i^R(M, I^s) \to \operatorname{Tor}_i^R(M, I)$$

induced by the inclusion map $I^s \to I$ is also zero. Therefore, for every $s \geq 2$ the short exact sequence

$$0 \to I^s \to I \to I/I^s \to 0$$

gives a short exact sequence

$$0 \to \operatorname{Tor}_i^R(M, I) \to \operatorname{Tor}_i^R(M, I/I^s) \to \operatorname{Tor}_{i-1}^R(M, I^s) \to 0$$

for all i.

2.13. Let I be an ideal of R that is generated by a regular sequence of length c and let s be a positive integer. In [3], Buchsbaum and Eisenbud show that $\operatorname{Tor}_i^R(R/I^s, R/I)$ is a free R/I-module; see [3, 3.2]. They also show that the Betti numbers of R/I^s are given by

$$\beta_i(R/I^s) = \binom{c+s-1}{s+i-1}\binom{s+i-2}{s-1}, \text{ for } i \geq 1$$

see [3, 2.5(c)].

Now, we give the proof of Theorem 2.5.

Proof of Theorem 2.5. Let $\bar{I}$ denote the image of I in R and note that, by [16, Theorem 16.2], the $R/\bar{I}$-module $\bar{I}^n/\bar{I}^{n+1}$ is free for every $n \geq 0$, in particular, $\bar{I}/\bar{I}^2$ is free. Now, let $K = \mathrm{K}(\bar{I}; R)$. Since $K \cong \mathrm{K}(I; S) \otimes_S R$, for every integer i, the R-modules $\mathrm{H}_i(K)$ and $\operatorname{Tor}_i^S(S/I, R)$ are isomorphic. Hence, by 2.13, $\mathrm{H}_i(K)$ is a free $R/\bar{I}$-module. Also, applying Lemma 2.11 to K, one can choose a set of cycles

$\{z_\lambda\}_{\lambda\in\Lambda}$ with $z_\lambda \in \bar{I}^{s-1}K$ such that the set $\mathcal{B} = \{h_\lambda = \operatorname{cls}(z_\lambda)\}_{\lambda\in\Lambda}$ forms a basis for $\mathrm{H}_{\geq 1}(K)$. So, the function $\mu : \bigsqcup_{n=1}^{\infty} \mathcal{B}^n \to K$ defined by

$$\mu(h_\lambda) = z_\lambda \text{ and } \mu(h_{\lambda_1}, \cdots, h_{\lambda_p}) = 0 \text{ for } p \geq 2$$

trivially satisfies the equalities in Definition 2.7. Thus K admits a trivial Massey operation and therefore, by Theorem 2.9, $\bar{I}$ is weak complete intersection.

Finally, the formulas for the Poincaré series of $R/\bar{I}$ can be obtained from Theorem 2.9 and 2.13. □

We finish this section, in view of 2.13, by raising the following question.

Question 2.14. Let I be a weak complete intersection ideal of R given in Theorem 2.5 and let $s \geq 2$ be a positive integer. Are the R/I-modules $\operatorname{Tor}_i^R(R/I^s, R/I)$ free, for all i?

3. Fiber products

In this section, we study fiber products over a weak complete intersection ideal. In particular, we are interested in Poincaré series of modules over such fiber products and the connections to the Poincaré series over the individual rings. Many of the results in this section are analogues of statements that are known to hold for fiber products over the maximal ideal.

Definition 3.1. Let R, S, and T be commutative rings. The *fiber product* of the diagram of ring homomorphisms

(3.1.1)
$$\begin{tikzcd} R \arrow[dr, "\varepsilon_R"] & \\ & T \\ S \arrow[ur, "\varepsilon_S"'] & \end{tikzcd}$$

is the subring of $R \times S$, defined as

(3.1.2) $$R \times_T S = \{(x, y) \in R \times S \mid \varepsilon_R(x) = \varepsilon_S(y)\}.$$

In this paper we assume that the rings R, S, and T are local rings with the same residue field k and ε_R and ε_S are surjective local homomorphisms. In this case, if $\mathfrak{r}$, $\mathfrak{s}$ and $\mathfrak{t}$ are maximal ideals of R, S and T, respectively, then $R \times_T S$ is local, with maximal ideal

$$\mathfrak{r} \times_{\mathfrak{t}} \mathfrak{s} = \{(x, y) \in \mathfrak{r} \times \mathfrak{s} \mid \varepsilon_R(x) = \varepsilon_S(y)\}$$

see [1, 1.2].

Remark 3.2. Let $R \times_T S \to R \times S$ and $R \times S \to R$ be the canonical maps. The composition of these maps gives a ring homomorphism $\pi_R : R \times_T S \to R$. Thus every R-module has a $R\times_T S$-module structure. Also, there is a short exact sequence of $R \times_T S$-modules

(3.2.1) $$0 \to \operatorname{Ker} \varepsilon_S \to R \times_T S \xrightarrow{\pi_R} R \to 0$$

Similarly, every S-module has a $R\times_T S$-module structure and there is a corresponding short exact sequence for S.

Lemma 3.3. *Let $P = R \times_T S$ be the fiber product of the diagram $R \xrightarrow{\varepsilon_R} T \xleftarrow{\varepsilon_S} S$ and let $I = \operatorname{Ker} \varepsilon_R$ be a weak complete intersection ideal in R. If n is a nonnegative integer and $\{x_1, \cdots, x_{\beta_n}\}$ is a minimal set of generators of $\Omega_n^R(I)$ then*

$$\begin{aligned}\Omega_1^P(\Omega_n^R(I)) &= \{\sum_{i=1}^{\beta_n} e_i(r_i,0) | \sum_{i=1}^{\beta_n} r_i x_i = 0, r_i \in I\} + \{\sum_{i=1}^{\beta_n} e_i(0,s_i) \mid s_i \in \operatorname{Ker} \varepsilon_S\} \\ &\cong \Omega_{n+1}^R(I) \oplus (\oplus_{i=1}^{\beta_n} \operatorname{Ker} \varepsilon_S)\end{aligned}$$

where $\{e_1, \cdots, e_{\beta_n}\}$ is the standard basis for the free module P^{β_n}.

Proof. Define the homomorphism $\phi : P^{\beta_n} \to \Omega_n^R(I)$ by $\phi(e_i(r,s)) = rx_i$. Since ε_R is surjective, it is straightforward to verify that ϕ is surjective and

$$\begin{aligned}\Omega_1^P(\Omega_n^R(I)) = \operatorname{Ker}(\phi) &= \{\Sigma e_i(r_i,s_i) \mid r_i \in R, s_i \in S, \varepsilon_R(r_i) = \varepsilon_S(s_i), \text{ and } \Sigma r_i x_i = 0\} \\ &= \{\Sigma e_i(r_i,s_i) \mid r_i \in I, s_i \in S, \varepsilon_S(s_i) = 0, \text{ and } \Sigma r_i x_i = 0\} \\ &= \{\Sigma e_i(r_i,0) | \Sigma r_i x_i = 0, r_i \in I\} + \{\Sigma e_i(0,s_i) \mid s_i \in \operatorname{Ker} \varepsilon_S\}\end{aligned}$$

where the third equality holds because $R/I \otimes_R \Omega_n^P(R/I)$ is a free, as I is a weak complete intersection ideal. □

Lemma 3.4. *Let $R \times_T S$ be the fiber product of the diagram $R \xrightarrow{\varepsilon_R} T \xleftarrow{\varepsilon_S} S$. If $I = \operatorname{Ker} \varepsilon_R$ is a weak complete intersection ideal in R then the following equalities hold*

(i) $P_I^{R\times_T S} = P_I^R P_R^{R\times_T S}$, *and*
(ii) $P_S^{R\times_T S} = 1 + P_T^R P_R^{R\times_T S} - P_R^{R\times_T S}$.

Proof. Let $P_I^R = \sum \beta_i z^i$, let $P = R \times_T S$ and let $J = \operatorname{Ker} \varepsilon_S$. Thus by Lemma 3.3 one has

$$\Omega_1^P(\Omega_n^R(I)) \cong \Omega_{n+1}^R(I) \oplus (\oplus_{i=1}^{\beta_n} J)$$

for every nonnegative n. Thus

$$P_I^P = \beta_0 + z(P^P_{\Omega_1^R(I)} + \beta_0 P_J^P) = \beta_0(1 + zP_J^P) + zP^P_{\Omega_1^R(I)} = \beta_0 P_R^P + zP^P_{\Omega_1^R(I)},$$

where the last equality follows from 3.2.1. Repeating this process one gets

$$P^P_{\Omega_n^R(I)} = \beta_n P_R^P + zP^P_{\Omega_{n+1}^R(I)},$$

for all n and therefore

$$P_I^P = \sum_{n=0}^{\infty} z^n \beta_n P_R^P = P_I^R P_R^P.$$

which completes the proof of part (i). Now by 3.2.1 and part (i), one has

$$P_S^P = 1 + zP_I^P = 1 + zP_I^R P_R^P = 1 + (P_T^R - 1)P_R^P = 1 + P_T^R P_R^P - P_R^P$$

□

As a corollary we get the following statement.

Corollary 3.5. *Let $R \times_T S$ be the fiber product of the diagram $R \xrightarrow{\varepsilon_R} T \xleftarrow{\varepsilon_S} S$. If $I = \operatorname{Ker} \varepsilon_R$ and $J = \operatorname{Ker} \varepsilon_S$ are weak complete intersections in R and S, respectively, then the equality*

$$P_R^{R\times_T S} = \frac{P_T^S}{P_T^R + P_T^S - P_T^R P_T^S}$$

holds.

Proof. Let P denote the fiber product $R \times_T S$. By Lemma 3.4(ii), there are equalities $P_R^P = 1 + P_T^S P_S^P - P_S^P$ and $P_S^P = 1 + P_T^R P_R^P - P_R^P$. Now, substituting P_S^P from the second equality into the first one and then solving for P_R^P yields the desired equality. □

Using the proof of Lemma 3.4 we obtain new weak complete intersection ideals in the fiber product.

Proposition 3.6. *Let $R \times_T S$ be the fiber product of the diagram $R \xrightarrow{\varepsilon_R} T \xleftarrow{\varepsilon_S} S$. Let $I = \ker \varepsilon_R$ and $J = \ker \varepsilon_S$ be weak complete intersection ideals of R and S, respectively. The ideal $I \times_T J$ is weak complete intersection in $R \times_T S$.*

Proof. Let $P = R \times_T S$ and observe that $I \times_T J = I \times J \cong (I \times \{0\}) \oplus (\{0\} \times J)$ as ideals of P. Since I and J are weak complete intersection ideals, by Lemma 3.3, one has

$$\Omega_1^P(\Omega_n^R(I)) \cong \Omega_{n+1}^R(I) \oplus (\oplus_{i=1}^{\beta_n} J) \text{ and } \Omega_1^P(\Omega_n^S(J)) \cong \Omega_{n+1}^S(J) \oplus (\oplus_{i=1}^{\gamma_n} I)$$

for every integer $n \geq 0$, where β_n and γ_n are the minimal number of generators of $\Omega_n^R(I)$ and $\Omega_n^S(J)$, respectively. Again by Lemmas 3.3 every differential in the minimal free resolution of I as a P-module can be represented by a matrix whose entries are in $I \times J$. Hence the $P/(I \times_T J)$-module $\operatorname{Tor}_i^P(I, P/(I \times_T J))$ is free for all i. Similarly one can show that $\operatorname{Tor}_i^P(J, P/(I \times_T J))$ is also a free $P/(I \times_T J)$-module for all i. Therefore, $I \times_T J$ is a weak complete intersection ideal in P. □

Example 3.7. Let k be a field. Consider

$$R = k[a,b]/(a^2,b^2)^2, \; S = k[a,b]/(a^2,b^2)^3, \text{ and } T = k[a,b]/(a^2,b^2)$$

and let $\varepsilon_R \colon R \to T$ and $\varepsilon_S \colon S \to T$ be the natural surjections. By Theorem 2.5, the ideals $I = \ker \varepsilon_R$ and $J = \ker \varepsilon_S$ are weak complete intersection in R and S, respectively.

Set $P = R \times_T S$. It is straightforward to show that P is generated by (a,a), (b,b), $(b^2,0)$, and $(a^2,0)$. Now, consider the ring $Q = k[x,y,z,w]/L$ where

$$L = (x^6, x^4y^2, x^2y^4, y^6, z^2, zw, w^2, x^2z - y^2w, x^2w, y^2z, x^2z)$$

Let $\phi \colon Q \to P$ be the map defined by

$$x \mapsto (a,a), \; y \mapsto (b,b), \; z \mapsto (b^2,0), \text{ and } w \mapsto (a^2,0)$$

Clearly, ϕ is a surjective ring homomorphism. It is an easy computation to see that the length of Q is 32 and by [1, 1.1], the length of P is given by

$$\operatorname{length}(P) = \operatorname{length}(R) + \operatorname{length}(S) - \operatorname{length}(T) = 32$$

Therefore, P is isomorphic to Q.

Now, the ideal $I \times_T J$ is generated by $(a^2,0)$, $(b^2,0)$, $(0,a^2)$ and $(0,b^2)$ and under ϕ, corresponds to the ideal (x^2,y^2,z,w) in S. Therefore, (x^2,y^2,z,w) is a weak complete intersection ideal in Q.

We recall some definitions that we use in the rest of this section. Suppose that Q and R be local rings with the same residue field k and $\varphi \colon Q \to R$ is a local homomorphism.

Serre shows that for every finite R-module M, one has a coefficient-wise inequality of formal power series

$$P_M^R \preccurlyeq \frac{P_M^Q}{1+z-zP_R^Q} \tag{3.7.1}$$

see also [2, 3.3.2]. If equality holds for $M = k$ then φ is called a *Golod homomorphism*. A finite R-module M is called a *Golod R-module* if equality holds for $\hat{M}$ when φ is a minimal Cohen presentation of $\hat{R}$. The ring R is called a *Golod ring* if k is a Golod R-module.

Lescot, in [13, 2.7], shows that for every finite R-module M, there is an inequality

$$P_k^Q P_M^R \preccurlyeq P_k^R P_M^Q$$

If equality holds, we say that M is *inert by φ*. We say that M is an *inert R-module* if $\hat{M}$ is inert by a minimal Cohen presentation of $\hat{R}$; see Introduction for the definition of a minimal Cohen presentation.

In general, if φ is surjective then for every finite R-module M, there is an inequality

$$P_M^Q \preccurlyeq P_M^R P_R^Q$$

If equality holds then φ is said to be *large*. See [15, 1.1].

We say that the homomorphism $\alpha\colon R \to Q$ is an *algebra retract* if there is a ring homomorphisms $\beta\colon Q \to R$ such that $\beta \circ \alpha = \mathrm{id}_R$. In this case, we say that R is an algebra retract of Q.

In [11, Folgerung 2] , Herzog and Steurich give a family of Golod homomorphisms. In particular, they show the following

3.8. If R is a local ring and if I is a complete intersection ideal in R, then the natural surjection $R \to R/I^s$ is a Golod homomorphism for all $s \geq 2$. In particular, if R is regular then R/I^s is a Golod ring.

The next result, by Herzog, shows that there is always a large map from a ring to an algebra retract of it; see [10, Theorem 1].

3.9. Let $\alpha : R \to Q$ be an algebra retract and let $\beta : Q \to R$ be a homomorphism such that $\beta \circ \alpha = \mathrm{id}_R$. Then β is large.

The next statement is proved by Lescot; see [13, 6.2].

3.10. For any local ring $(R, \mathfrak{m})$, R is Golod if and only if the maximal ideal $\mathfrak{m}$ is an inert R-module.

Our next goal is to study the fiber products in the following setup.

3.11. Consider the diagram

$$\begin{tikzcd} & R \arrow[dr, "\varepsilon_R"] & \\ & & T \\ S \arrow[r, "\pi"] & R \arrow[ur, "\varepsilon_R"'] & \end{tikzcd} \tag{3.11.1}$$

where π is a surjective homomorphism and set $\varepsilon_S = \varepsilon_R \pi$. It is straightforward to see that the map $\alpha\colon S \to R \times_T S$ defined by $\alpha(s) = (\pi(s), s)$ is an algebra retract and $\pi_S \alpha = \mathrm{id}_S$. Therefore, by 3.9, the homomorphism π_S is large.

Using 3.11, we show that in this case Corollary 3.5 can be extended to every S-module when only $\operatorname{Ker}\varepsilon_R$ is a weak complete intersection.

Theorem 3.12. *If $R \times_T S$ is the fiber product of a diagram of the form 3.11.1 and if $I = \operatorname{Ker}\varepsilon_R$ is a weak complete intersection ideal in R, then for every finite S-module M, the equality*

$$P_M^{R\times_T S} = \frac{P_M^S P_T^R}{P_T^R + P_T^S - P_T^R P_T^S}$$

holds.

Proof. Set $P = R \times_T S$ and $J = \operatorname{Ker}\varepsilon_S$. If we show that

$$P_R^P = 1 + P_T^S P_S^P - P_S^P \text{ and } P_S^P = 1 + P_T^R P_R^P - P_R^P, \tag{3.12.1}$$

then substituting P_S^P from the second equality into the first one and then solving for P_R^P yields

$$P_S^P = \frac{P_T^R}{P_T^R + P_T^S - P_T^R P_T^S}$$

which, using 3.11, yields $P_M^P = P_M^S P_S^P$.

The second equality in 3.12.1 holds by Lemma 3.4 and the first equality holds since

$$P_R^P = 1 + zP_J^P = 1 + zP_J^S P_S^P = 1 + (P_T^S - 1)P_S^P,$$

where the first equality holds by 3.2.1, the second one follows from 3.11 and the third one holds since J is the first syzygy of T over S. □

We give a family of inert modules that we use in this section to construct Golod homomorphisms and Golod rings.

Theorem 3.13. *Let R be a commutative local ring and let I be an ideal in R that is generated by a regular sequence of length c. Let $s \geq 2$ be an integer and let $\pi : R \to R/I^s$ be the natural surjection. The following statements hold.*

(i) The R/I^s-modules R/I and I/I^s are inert by π; that is

$$P_k^R P_{R/I}^{R/I^s} = P_k^{R/I^s} P_{R/I}^R \text{ and } P_k^R P_{I/I^s}^{R/I^s} = P_k^{R/I^s} P_{I/I^s}^R.$$

(ii) The R/I-module $\operatorname{Tor}_i^R(R/I, I/I^s)$ is free.

Proof. (i) Let $I = (x_1, \ldots, x_c)$, where $x_1, \ldots, x_c$ is a regular sequence. Let $K = \mathrm{K}(x_1, \ldots, x_c; R)$ and for $i \geq 1$, let ϵ_i denote the rank of the free R/I^s-module $\mathrm{H}_i(K \otimes_R R/I^s) \cong \operatorname{Tor}_i^R(R/I, R/I^s)$.

By [3, 3.2], every differential in the minimal free resolution of R/I^s, as an R-module, can be represented by a matrix whose entries are in I, thus $\epsilon_i = \dim_k \operatorname{Tor}_i^R(k, R/I^s)$ for all $i \geq 1$ and hence

$$P_{R/I^s}^R = 1 + \epsilon_1 z + \epsilon_2 z^2 + \cdots + \epsilon_c z^c \tag{3.13.1}$$

Since $s \geq 2$, the map π is Golod by 3.8, thus one has

$$P_k^{R/I^s} = \frac{P_k^R}{1 + z - zP_{R/I^s}^R} = \frac{P_k^R}{1 - \epsilon_1 z^2 - \epsilon_2 z^3 - \cdots - \epsilon_c z^{c+1}}.$$

So, using Theorem 2.5, we have

$$P_{R/I}^{R/I^s} = \frac{(1+z)^c}{1-\epsilon_1 z^2-\epsilon_2 z^3-\cdots-\epsilon_c z^{c+1}} = (1+z)^c \frac{P_k^{R/I^s}}{P_k^R} \tag{3.13.2}$$

Therefore R/I is inert by π.

Now, by Remark 2.12, one has a short sequence

$$0 \to \operatorname{Tor}_i^R(k, I) \to \operatorname{Tor}_i^R(k, I/I^s) \to \operatorname{Tor}_{i-1}^R(k, I^s) \to 0,$$

for every i. Therefore, $P_{I/I^s}^R = P_I^R + zP_{I^s}^R$ and by 3.13.1 one has

$$P_{I/I^s}^R = \frac{(1+z)^c-(1-\epsilon_1 z^2-\epsilon_2 z^3-\cdots-\epsilon_c z^{c+1})}{z}, \tag{3.13.3}$$

and therefore

$$\begin{aligned} P_k^{R/I^s} P_{I/I^s}^R &= P_{R/I}^{R/I^s} P_k^R (1+z)^{-c} \frac{(1+z)^c-(1-\epsilon_1 z^2-\epsilon_2 z^3-\cdots-\epsilon_c z^{c+1})}{z} \\ &= P_k^R (P_{R/I}^{R/I^s}-1)\frac{1}{z} \\ &= P_k^R P_{I/I^s}^{R/I^s}, \end{aligned}$$

where the first equality follows from 3.13.2 and 3.13.3 and the second one follows from 3.13.2. Therefore, I/I^s is inert by π as desired.

(ii) Another application of Remark 2.12, gives a short sequence

$$0 \to \operatorname{Tor}_i^R(R/I, I) \to \operatorname{Tor}_i^R(R/I, I/I^s) \to \operatorname{Tor}_{i-1}^R(R/I, I^s) \to 0.$$

for every i. We know that the R/I-module $\operatorname{Tor}_i^R(R/I, I)$ is free for all i because I is complete intersection and $\operatorname{Tor}_i^R(R/I, I^s)$ is free for all i by 2.13. Therefore, the R/I-module $\operatorname{Tor}_i^R(R/I, I/I^s)$ is free for all i. □

We give a formula for the Poincaré series of a fiber product over a weak complete intersection ideal from Theorem 2.5. We use this formula in the next section.

Proposition 3.14. *Let Q be a commutative local ring and let I be an ideal generated by a regular sequence of length c. Let $R = Q/I^s$, $S = Q/I^t$, and $T = Q/I$, where $s \geq 2$ and $t \geq 2$ are integers. If $R \times_T S$ is the fiber product of the diagram $R \xrightarrow{\varepsilon_S} T \xleftarrow{\varepsilon_S} S$, where ε_R and ε_S are natural surjections, then*

$$\begin{aligned} (P_k^{R\times_T S})^{-1} &= (P_k^R)^{-1} + (P_k^S)^{-1} - P_T^Q (P_k^Q)^{-1} \\ &= (P_k^R)^{-1} + (P_k^S)^{-1} - (1+z)^c (P_k^Q)^{-1}. \end{aligned}$$

Proof. First note that the same argument used to show equality 3.13.2 in the proof of Theorem 3.13 gives

$$P_T^R = P_k^R (1+z)^c (P_k^Q)^{-1} \text{ and } P_T^S = P_k^S (1+z)^c (P_k^Q)^{-1}. \tag{3.14.1}$$

Now, the proposition follows from the equalities

$$\begin{aligned} (P_k^{R\times_T S})^{-1} &= \frac{P_T^R + P_T^S - P_T^R P_T^S}{P_k^R P_T^S} \\ &= \frac{P_k^R + P_k^S - P_k^R P_k^S (1+z)^c (P_k^Q)^{-1}}{P_k^R P_k^S} \\ &= (P_k^R)^{-1} + (P_k^S)^{-1} - (1+z)^c (P_k^Q)^{-1}, \end{aligned}$$

where the first equality holds by 2.5 and 3.12, and the second one follows from the equalities 3.14.1. □

Remark 3.15. The formula in Proposition 3.14 can be written more explicitly using the equality

$$P_k^R = P_k^Q \left(1 - \sum_{i=1}^{c} \binom{c+s-1}{s+i-1} \binom{s+i-2}{s-1} z^{i+1} \right)^{-1}$$

see 3.8.

4. Golod rings

In this section we investigate conditions under which a fiber product is a Golod ring or a homomorphism of fiber products is Golod.

4.1. Let $\alpha : R' \to R$ and $\beta : S' \to S$ be algebra retracts and let $\alpha' : R \to R'$ and $\beta' : S \to S'$ be ring homomorphisms such that $\alpha'\alpha = \mathrm{id}_{R'}$ and $\beta'\beta = \mathrm{id}_{S'}$. Consider the diagram

$$\begin{tikzcd} R \arrow[r, "\alpha'"] & R' \arrow[dr, "\varepsilon_{R'}"] & \\ & & T \\ S \arrow[r, "\beta'"] & S' \arrow[ur, "\varepsilon_{S'}"'] & \end{tikzcd}$$

where $\varepsilon_{R'}$ and $\varepsilon_{S'}$ are surjective local homomorphisms. If $R' \times_T S'$ and $R \times_T S$ are fiber products obtained from this diagram then the induced homomorphism $\alpha \times_T \beta : R' \times_T S' \to R \times_T S$ is an algebra retract. Indeed, it is straightforward to show that $(\alpha' \times_T \beta')(\alpha \times_T \beta) = \mathrm{id}_{R' \times_T S'}$.

In [9, 5.5], Gupta proves that

4.2. For every commutative local ring R, the ring $R \times_T R$ is Golod if and only if $\ker \varepsilon_R$ is a Golod R-module.

In particular, if R is a regular local ring then $R \times_T R$ is a Golod ring. Gupta then asks the following question: If R and S are regular rings, is $R \times_T S$ a Golod ring? By [4, 1.11] an algebra retract of a regular ring is regular, so the following observation gives a partial answer to this question.

Proposition 4.3. *If R and S are both algebra retracts of the same regular local ring, then the fiber product $R \times_T S$ is Golod.*

Proof. Suppose that R and S are algebra retracts of a regular local ring Q. By Remark 4.1, $R \times_T S$ is an algebra retract of $Q \times_T Q$ so, by 3.9 there is a large map from $Q \times_T Q$ to $R \times_T S$. By 4.2, $Q \times_T Q$ is a Golod ring and therefore, [9, 5.8] implies that $R \times_T S$ is also Golod. □

We prove that a similar formula, to that in Proposition 3.14, holds for a fiber products of algebra retracts of a ring.

Proposition 4.4. *Suppose that T is an algebra retract of both R and S and $\alpha\colon R \to T$ and $\beta\colon S \to T$ are the corresponding surjective homomorphisms. Let I be an ideal of T and $R \times_{T/I} S$ be the fiber product of the diagram $R \xrightarrow{\varepsilon_R} T/I \xleftarrow{\varepsilon_S} S$ where ε_R*

and ε_S are the compositions of α and β with the natural surjection, respectively. Then T/I is inert by $\pi_R\colon R\times_{T/I} S \to R$ and by $\pi_S\colon R\times_{T/I} S\to S$, and the equality

$$(P_k^{R\times_{T/I}S})^{-1} = (P_k^R)^{-1} + (P_k^S)^{-1} - P^T_{T/I}(P^T_k)^{-1}$$

holds.

Proof. Set $Q = T\times_{T/I} T$ and $P = R\times_{T/I} S$. By 3.11, T is an algebra retract of Q with the corresponding surjective map $\phi : Q \to T$ define by $\phi((a, b)) = a$. By 4.1, Q is an algebra retract of P with the corresponding surjective map $\psi : P \to Q$ defined as $\psi((r, s)) = (\alpha(r), \beta(s))$. Therefore, T is an algebra retract of P with the corresponding surjective map $\phi\psi$. Note that $\phi\psi = \alpha\pi_R$. By 3.9 the map $\alpha\pi_R$ is large so, [13, 3.3] implies that T/I is inert by $\alpha\pi_R$. Thus it follows from [13, 3.6(1)] that T/I is inert by π_R. Similarly, one can show that T/I is inert by π_S. Now [13, 7.1] yields the equality

$$(P^P_k)^{-1} = (P^R_k)^{-1} + (P^S_k)^{-1} - P^S_{T/I}(P^S_k)^{-1}.$$

and the desired equality follows since

$$P^S_{T/I}(P^S_k)^{-1} = P^T_{T/I}P^S_T(P^T_kP^S_T)^{-1} = P^T_{T/I}(P^T_k)^{-1}\,.$$

□

Remark 4.5. Let I be an ideal of R and let M be a finite R-module. Set $P = R\times_T R$. By 3.11 and 3.9, the map π_R is large and hence, by [13, 3.3], R/I is inert by π_R. Thus one has $P^P_M P^R_k = P^R_M P^P_k$. On the other hand, it follows from 4.4 that

$$P^P_k = \frac{P^R_k}{2 - P^R_{R/I}}.$$

Therefore, there is an equality

$$P^P_M = \frac{P^R_M}{2 - P^R_{R/I}}.$$

Proposition 4.6. *Consider the following diagram of local homomorphisms*

$$\begin{tikzcd} S \arrow[r, "\varphi"] & R \arrow[dr, "\varepsilon_R"] & \\ & & T \\ S \arrow[r, "\varphi"] & R \arrow[ur, "\varepsilon_R"'] & \end{tikzcd}$$

where φ is a surjective map. Let $S\times_T S$ and $R\times_T R$ be fiber products obtained from this diagram and let $\varphi\times_T\varphi$ be the map induced by φ . Suppose that T is inert by φ. The map $\varphi\times_T\varphi$ is Golod if and only if φ is Golod.

Proof. Set $J = \operatorname{Ker}(\varphi)$ and note that $\operatorname{Ker}(\varphi\times_T\varphi) \cong J\oplus J$ as $S\times_T S$-modules. Then one has

$$\begin{aligned} P^{S\times_T S}_{R\times_T R} &= 1 + z(2P^{S\times_T S}_J) \\ &= 1 + 2z\left(\frac{P^S_J}{2 - P^S_T}\right) \qquad (4.6.1) \\ &= 1 + 2\frac{P^S_R - 1}{2 - P^S_T}, \end{aligned}$$

where the second equality holds by 4.5. Since T is inert by φ, another application of 4.5 gives the equalities

$$\frac{1}{P_k^{R\times_T R}} = \frac{2-P_T^R}{P_k^R} = \frac{2P_k^S - P_k^S P_T^R}{P_k^S P_k^R} = \frac{2P_k^S - P_k^R P_T^S}{P_k^S P_k^R}, \tag{4.6.2}$$

$$\frac{1+z-zP_{R\times_T R}^{S\times_T S}}{P_k^{S\times_T S}} = \frac{2-P_T^S-2zP_R^S+2z}{P_k^S}. \tag{4.6.3}$$

Now, $\varphi \times_T \varphi$ is Golod if and only if the leftmost expressions in 4.6.2 and 4.6.3 are equal if and only if the rightmost expressions are equal if and only if φ is Golod. $\square$

Specialization of T in the proposition above gives a more general result about the induced homomorphism of fiber products.

Theorem 4.7. *Let Q be a commutative local ring, let I be a complete intersection ideal in Q. Set $R = Q/I^s$, $S = Q/I^t$, and $T = Q/I$ where $s \geq 2$ and $t \geq 2$ are integers. Consider the fiber products obtained from the diagram*

$$\begin{tikzcd} Q \arrow[r, "\pi_1"] & R \arrow[dr, "\varepsilon_R"] & \\ & & T \\ Q \arrow[r, "\pi_2"] & S \arrow[ur, "\varepsilon_R"'] & \end{tikzcd}$$

where all maps are natural surjections. Then the induced homomorphism

$$\pi_1 \times_T \pi_2 \colon Q \times_T Q \to R \times_T S$$

is a Golod map.

Proof. Since s and t are at least 2, by 3.8, the homomorphisms π_1 and π_2 are Golod. Also, by 3.13, T is inert by π_1 and π_2. It is straightforward to show that $\mathrm{Ker}(\pi_1 \times_T \pi_2) \cong I^s \oplus I^t$, as $Q \times_T Q$-modules. A computation similar to the proof in 4.6 gives the equality

$$P_{R\times_T S}^{Q\times_T Q} = 1 + \frac{P_R^Q + P_S^Q - 2}{2 - P_T^Q}.$$

This equality, combined with 4.5, gives

$$\frac{1+z-zP_{R\times_T S}^{Q\times_T Q}}{P_k^{Q\times_T Q}} = \frac{2-P_T^Q - z(P_R^Q + P_S^Q - 2)}{P_k^Q}$$

On the other hand, from 3.13, we obtain

$$\begin{aligned}(P_k^{R\times_T S})^{-1} &= (P_k^R)^{-1} + (P_k^S)^{-1} - P_T^Q (P_k^Q)^{-1} \\ &= \frac{1+z-zP_R^Q}{P_k^Q} + \frac{1+z-zP_S^Q}{P_k^R} - P_T^Q (P_k^Q)^{-1} \\ &= \frac{2+2z-z(P_R^Q+P_S^Q)-P_T^Q}{P_k^Q}.\end{aligned}$$

So $\pi_1 \times_T \pi_2$ is Golod by comparing the two equalities above. $\square$

Recall that Lescot [12] shows that one can construct Golod rings by taking the fiber product of two Golod rings over the residue field. We construct new Golod rings using fiber products over weak complete intersection ideals from Theorem 2.5.

Corollary 4.8. *Let Q be a regular local ring, let I be a complete intersection ideal in Q. Let $R = Q/I^s$, where $s \geq 2$ is an integer, and $T = Q/I$. Let $R \times_T R$ be the fiber product of the diagram $R \xrightarrow{\varepsilon_R} T \xleftarrow{\varepsilon_R} R$ where ε_R is the natural surjection. Then the ring $R \times_T R$ is Golod.*

Proof. By 4.2, it is enough to show I/I^s is a Golod R-module. By 3.13, I/I^s is inert by $Q \to R$ and by 3.8 the natural surjection $Q \to R$ is a Golod map. Thus there are equalities

$$P^R_{I/I^s} = \frac{P^R_k P^Q_{I/I^s}}{P^Q_k} = \frac{P^Q_k P^Q_{I/I^s}}{P^Q_k(1+z-zP^Q_R)} = \frac{P^Q_{I/I^s}}{1+z-zP^Q_R}.$$

Therefore, I/I^s is a Golod R-module. □

Remark 4.9. In [9, 5.6], Gupta shows that 4.2 can be extended to the fiber product $R \times_T \cdots \times_T R$ and hence 4.8 can also be extended to the fiber product of finitely many copies of R over T.

Example 4.10. Let k be a field. Consider

$$R = k[a,b]/(a^2,b^2)^s \text{ and } T = k[a,b]/(a^2,b^2).$$

Let $\varepsilon_R \colon R \to T$ be the natural surjection. If $s = 2$, a similar argument as in Example 3.7 shows that $R \times_T R$ is isomorphic to $S = k[x,y,z,w]/J$ where

$$J = (x^4, x^2y^2, y^4, z^2, zw, w^2, x^2z - y^2w, x^2w, y^2z, x^2z).$$

Therefore, by Theorem 4.8, S is a Golod ring.

In fact, `Macaulay2` computations suggest that for $s \geq 2$, the ring $R \times_T R$ is isomorphic to $S = k[x,y,z,w]/J$ where J is the ideal generated by

$$\{\{x^{2s-2i}y^{2i}\}_{i=0}^s, \{z^{s-i}w^i\}_{i=0}^s, x^2z - y^2w, x^2w - w^2, y^2z - z^2, x^2z - zw\}.$$

When k is of characteristic zero, [9, 5.5 and 3.6] imply that S is Golod but it follows from Corollary 4.8 that S is Golod in positive characteristic too.

We end this section by raising the following question.

Question 4.11. Let R be a regular local ring, let I be a complete intersection ideal in R and let s and t be integers greater than or equal to 2. Is the ring $R/I^s \times_{R/I} R/I^t$ Golod?

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