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Structural decomposition of monomial resolutions

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ABSTRACT

We express the multigraded Betti numbers of an arbitrary monomial ideal in terms of the multigraded Betti numbers of two basic classes of ideals. This decomposition has multiple applications. In some concrete cases, we use it to construct minimal resolutions of classes of monomial ideals; in other cases, we use it to compute projective dimensions. To illustrate the effectiveness of the structural decomposition, we give a new proof of a classic theorem by Charalambous that states the following: let k be a field, and M an Artinian monomial ideal in $S = k[x_1, \ldots, x_n]$; then, for all i, $b_i(S/M) \geq \binom{n}{i}$.

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1. Introduction

The problem of finding the minimal resolution of an arbitrary monomial ideal in closed form has been deemed utopic by many a mathematician. As a consequence, people have tried to restrict the study of minimal resolutions to particular classes of ideals. Borel ideals, minimally resolved by the Eliahou–Kervaire resolution [6]; generic ideals, minimally resolved by the Scarf complex [2]; and dominant ideals, minimally resolved by the Taylor resolution [1], are examples of this restrictive approach.

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In the first half of this paper, however, we turn to the general problem, and decompose the minimal resolution of an arbitrary monomial ideal in terms of the minimal resolutions of two basic classes that we call dominant, and purely nondominant ideals. More precisely, we express the multigraded Betti numbers of an ideal as the sum of the multigraded Betti numbers of some dominant and some purely nondominant ideals. Since dominant ideals are minimally resolved by their Taylor resolutions, our decomposition reduces the study of minimal monomial resolutions to the study of minimal resolutions of purely nondominant ideals.

Unfortunately, the resolutions of purely nondominant ideals involve the same challenges that we encounter in the general context. Some of these difficulties are the existence of ghost terms (that is, the minimal resolution contains basis elements of equal multidegree in consecutive homological degrees), characteristic dependence, and the striking fact that some of the simplest purely nondominant ideals cannot be minimally resolved by any subcomplex of the Taylor resolution. Thus, in the second half of this work we focus our efforts on one particular case: monomial ideals whose structural decomposition has no purely nondominant part. As a result of this study, we obtain the multigraded Betti numbers of two families that we call 2-semidominant and almost generic ideals.

The structural decomposition is also a useful tool to compute projective dimensions. We prove, for instance, that if an ideal M satisfies certain conditions, $\mathrm{pd}(S/M) = 2$, and, under some other conditions, $\mathrm{pd}(S/M) = n$, where n is the number of variables in the polynomial ring. Another result, also related to projective dimensions, is a new proof of a classic theorem of Charalambous [3] (see also [12, Corollary 21.6]), stating: let k be a field, and M an Artinian monomial ideal in $S = k[x_1, \ldots, x_n]$; then, for all i, $\mathrm{b}_i(S/M) \geq \binom{n}{i}$. While the original proof relies on the radical of an ideal, ours is based on the structural decomposition.

The organization of the article is as follows. Section 2 is about background and notation. Section 3 is technical; it contains some isomorphism theorems that will be used to construct the structural decomposition of section 4. In sections 5 and 6 we compute Betti numbers and projective dimensions using the structural decomposition. Section 7 relates the structural decomposition to the lcm-lattice, and describes connections to some classes of ideals combinatorially defined. Section 8 is the conclusion; it includes some comments, questions, and conjectures.

2. Background and notation

Throughout this paper S represents a polynomial ring over an arbitrary field k, in a finite number variables. The letter M always denotes a monomial ideal in S. With minor modifications, the constructions that we give below can be found in [11,12].

Construction 2.1. Let M be generated by a set of monomials $\{l_1, \ldots, l_q\}$. For every subset $\{l_{i_1}, \ldots, l_{i_s}\}$ of $\{l_1, \ldots, l_q\}$, with $1 \leq i_1 < \ldots < i_s \leq q$, we create a formal symbol $[l_{i_1}, \ldots, l_{i_s}]$, called a **Taylor symbol**. The Taylor symbol associated to $\{\}$ is denoted by $[\varnothing]$.

For each $s = 0, \ldots, q$, set F_s equal to the free S-module with basis $\{[l_{i_1}, \ldots, l_{i_s}] : 1 \leq i_1 < \ldots < i_s \leq q\}$ given by the $\binom{q}{s}$ Taylor symbols corresponding to subsets of size s. That is, $F_s = \bigoplus_{i_1 < \ldots < i_s} S[l_{i_1}, \ldots, l_{i_s}]$ (note that $F_0 = S[\varnothing]$). Define

$$f_0 : F_0 \to S/M$$
$$s[\varnothing] \mapsto f_0(s[\varnothing]) = s$$

For $s = 1, \ldots, q$, let $f_s : F_s \to F_{s-1}$ be given by

$$f_s([l_{i_1}, \ldots, l_{i_s}]) = \sum_{j=1}^{s} \frac{(-1)^{j+1} \operatorname{lcm}(l_{i_1}, \ldots, l_{i_s})}{\operatorname{lcm}(l_{i_1}, \ldots, \widehat{l_{i_j}}, \ldots, l_{i_s})} [l_{i_1}, \ldots, \widehat{l_{i_j}}, \ldots, l_{i_s}]$$

and extended by linearity. The **Taylor resolution** $\mathbb{T}_{l_1, \ldots, l_q}$ of S/M is the exact sequence

$$\mathbb{T}_{l_1, \ldots, l_q} : 0 \to F_q \xrightarrow{f_q} F_{q-1} \to \cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} S/M \to 0.$$

We define the **multidegree** of a Taylor symbol $[l_{i_1}, \ldots, l_{i_s}]$, denoted $\operatorname{mdeg}[l_{i_1}, \ldots, l_{i_s}]$, as follows: $\operatorname{mdeg}[l_{i_1}, \ldots, l_{i_s}] = \operatorname{lcm}(l_{i_1}, \ldots, l_{i_s})$ (if $\operatorname{lcm}(l_{i_1}, \ldots, l_{i_s}) = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, then we have $\operatorname{mdeg}[l_{i_1}, \ldots, l_{i_s}] = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$; however, many authors define the multidegree as the n-tuple $\operatorname{mdeg}[l_{i_1}, \ldots, l_{i_s}] = (\alpha_1, \ldots, \alpha_n)$). The Taylor symbols $[l_{i_1}, \ldots, l_{i_s}]$ are called **faces**. A Taylor symbol of the form $[l_{i_1}, \ldots, \widehat{l_{i_j}}, \ldots, l_{i_s}]$ is referred to as a **facet** of the face $[l_{i_1}, \ldots, l_{i_s}]$.

Note: In our construction above, the generating set $\{l_1, \ldots, l_q\}$ is not required to be minimal. Thus, S/M has many Taylor resolutions. We reserve the notation $\mathbb{T}_M$ for the Taylor resolution of S/M, determined by the minimal generating set of M. (Although some authors define a single Taylor resolution of S/M, our construction is general, like in [5, Exercise 17.11].)

Construction 2.2. Let M be minimally generated by $\{l_1, \ldots, l_q\}$. Let A be the set of Taylor symbols of $\mathbb{T}_M$ whose multidegrees are not common to other Taylor symbols; that is, a Taylor symbol $[\sigma]$ is in A if and only if $\operatorname{mdeg}[\sigma] \neq \operatorname{mdeg}[\sigma']$, for every Taylor symbol $[\sigma'] \neq [\sigma]$. For each $s = 0, \ldots, q$, set G_s equal to the free S-module with basis $\{[l_{i_1}, \ldots, l_{i_s}] \in A : 1 \leq i_1 < \ldots < i_s \leq q\}$. For each $s = 0, \ldots, q$, let $g_s = f_s \upharpoonright_{G_s}$. It can be proven that the g_s are well defined (more precisely, that $g_s(G_s) \subseteq G_{s-1}$) and that

$$0 \to G_q \xrightarrow{g_q} G_{q-1} \to \cdots \to G_1 \xrightarrow{g_1} G_0 \xrightarrow{g_0} S/M \to 0$$

is a subcomplex of $\mathbb{T}_M$, which is called the **Scarf complex** of S/M.

Definition 2.3. Let M be a monomial ideal, and let

$$\mathbb{F} : \cdots \to F_i \xrightarrow{f_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} S/M \to 0$$

be a free resolution of S/M. We say that a basis element $[\sigma]$ of $\mathbb{F}$ has **homological degree i**, denoted $\mathrm{hdeg}[\sigma] = i$, if $[\sigma] \in F_i$. $\mathbb{F}$ is said to be a **minimal resolution** if for every i, the differential matrix (f_i) of $\mathbb{F}$ has no invertible entries.

Definition 2.4. Let M be a monomial ideal, and let

$$\mathbb{F} : \cdots \to F_i \xrightarrow{f_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} S/M \to 0$$

be a minimal free resolution of S/M.

- For every $i \geq 0$, the ith **Betti number** $\mathrm{b}_i (S/M)$ of S/M is $\mathrm{b}_i (S/M) = \mathrm{rank}(F_i)$.
- For every $i \geq 0$, and every monomial l, the **multigraded Betti number** $\mathrm{b}_{i,l} (S/M)$ of S/M, in homological degree i and multidegree l, is

$$\mathrm{b}_{i,l} (S/M) = \#\{\text{basis elements } [\sigma] \text{ of } F_i : \mathrm{mdeg}[\sigma] = l\}.$$

- The **projective dimension** $\mathrm{pd}\,(S/M)$ of S/M is

$$\mathrm{pd}\,(S/M) = \max\{i : \mathrm{b}_i (S/M) \neq 0\}.$$

Definition 2.5. Let M be minimally generated by a set of monomials G.

- A monomial $m \in G$ is called **dominant** (in G) if there is a variable x, such that for all $m' \in G \setminus \{m\}$, the exponent with which x appears in the factorization of m is larger than the exponent with which x appears in the factorization of m'. The set G is called **dominant** if each of its elements is dominant. The ideal M is called **dominant** if G is dominant.
- G is called p-**semidominant** if G contains exactly p nondominant monomials. The ideal M is p-**semidominant** if G is p-semidominant.
- We say that G is **purely nondominant** when all the elements of G are nondominant. In this case, we also say that M is **purely nondominant**.

Example 2.6. Let M_1, M_2, and M_3 be minimally generated by $G_1 = \{a^2, b^3, ab\}$, $G_2 = \{ab, bc, ac\}$, and $G_3 = \{a^2b, ab^3c, bc^2\}$, respectively. Note that a^2 and b^3 are dominant in G_1, but ab is not. Thus, both the set G_1 and the ideal M_1 are 1-semidominant. On the other hand, ab, bc, and ac are nondominant in G_2. Therefore, G_2 and M_2 are purely nondominant (as well as 3-semidominant). Finally, a^2b, ab^3c, and bc^2 are dominant in G_3. Thus, G_3 and M_3 are dominant.

3. Isomorphism theorems

The main goal of this paper is to express the multigraded Betti numbers of an ideal M as the sum of the multigraded Betti numbers of some ideals M_m, where each m is

a monomial carefully chosen. In this section we construct the M_m, and prove that the basis of the minimal resolution of S/M_m is in bijective correspondence f with a subset of the minimal resolution of S/M. We also show that there is a strong connection between the homological degrees and multidegrees of the elements in the domain and range of f. More precisely, we prove that there is an integer j such that $\operatorname{hdeg} f[\sigma] = j + \operatorname{hdeg}[\sigma]$, and $\operatorname{mdeg} f[\sigma] = m \operatorname{mdeg}[\sigma]$ (Theorem 3.13 (ii)).

The notation that we introduce below retains its meaning until the end of this section. Let M be a monomial ideal with minimal generating set $G = \{m_1, \ldots, m_q, n_1, \ldots, n_p\}$, where $m_1, \ldots, m_q$ are dominant, and $n_1, \ldots, n_p$ are nondominant. Let $1 \le d \le q$, and let $H = \{h_1, \ldots, h_c\} = \{m_{d+1}, \ldots, m_q, n_1, \ldots, n_p\}$. Then G can be expressed in the form $G = \{m_1, \ldots, m_d, h_1, \ldots, h_c\}$. Let $m = \operatorname{lcm}(m_{r_1}, \ldots, m_{r_j})$, where $1 \le r_1 < \ldots < r_j \le d$. By convention, if $j = 0$, $m = 1$. For all $s = 1, \ldots, c$, let $h'_s = \dfrac{\operatorname{lcm}(m, h_s)}{m}$. Let $M_m = (h'_1, \ldots, h'_c)$.

Example 3.1. Let $M = (a^3b^2, c^3d, ac^2, a^2c, b^2d, abc, bcd)$. Note that M is 5-semidominant, with $m_1 = a^3b^2$, $m_2 = c^3d$, $n_1 = ac^2$, $n_2 = a^2c$, $n_3 = b^2d$, $n_4 = abc$, $n_5 = bcd$. If we set $d = 2$, then $H = \{h_1, \ldots, h_5\} = \{n_1, \ldots, n_5\}$. Suppose that $m = \operatorname{lcm}(m_1) = a^3b^2$. Then $M_m = (h'_1, \ldots, h'_5)$, where $h'_1 = \dfrac{\operatorname{lcm}(a^3b^2, ac^2)}{a^3b^2} = c^2$; $h'_2 = \dfrac{\operatorname{lcm}(a^3b^2, a^2c)}{a^3b^2} = c$; $h'_3 = \dfrac{\operatorname{lcm}(a^3b^2, b^2d)}{a^3b^2} = d$; $h'_4 = \dfrac{\operatorname{lcm}(a^3b^2, abc)}{a^3b^2} = c$; $h'_5 = \dfrac{\operatorname{lcm}(a^3b^2, bcd)}{a^3b^2} = cd$. Thus, $M_{a^3b^2} = (c^2, c, d, c, cd)$. Although $\{c^2, c, d, c, cd\}$ does not generate $M_{a^3b^2}$ minimally, sometimes, nonminimal generating sets like this will serve our purpose.

Proposition 3.2. *Let $1 \le s_1 < \ldots < s_i \le c$. The Taylor symbols $[h'_{s_1}, \ldots, h'_{s_i}]$ of $\mathbb{T}_{h'_1, \ldots, h'_c}$, and $[m_{r_1}, \ldots, m_{r_j}, h_{s_1}, \ldots, h_{s_i}]$ of $\mathbb{T}_M$, satisfy*

$$\operatorname{mdeg}[m_{r_1}, \ldots, m_{r_j}, h_{s_1}, \ldots, h_{s_i}] = m \operatorname{mdeg}[h'_{s_1}, \ldots, h'_{s_i}].$$

Proof. Let $s \in \{s_1, \ldots, s_i\}$. Note that $h_s \mid \operatorname{lcm}(m, h_s) = mh'_s$, and $mh'_s \mid m \operatorname{lcm}(h'_{s_1}, \ldots, h'_{s_i})$. Thus, $h_s \mid m \operatorname{lcm}(h'_{s_1}, \ldots, h'_{s_i})$. Since s is arbitrary, $\operatorname{lcm}(m, h_{s_1}, \ldots, h_{s_i}) \mid m \operatorname{lcm}(h'_{s_1}, \ldots, h'_{s_i})$. We will show that $m \operatorname{lcm}(h'_{s_1}, \ldots, h'_{s_i}) \mid \operatorname{lcm}(m, h_{s_1}, \ldots, h_{s_i})$.

Let $h'_{s_1} = x_1^{\alpha_{11}} \ldots x_n^{\alpha_{1n}}, \ldots, h'_{s_i} = x_1^{\alpha_{i1}} \ldots x_n^{\alpha_{in}}$, and let $\gamma_1 = \max(\alpha_{11}, \ldots, \alpha_{i1}), \ldots, \gamma_n = \max(\alpha_{1n}, \ldots, \alpha_{in})$. Then $\operatorname{lcm}(h'_{s_1}, \ldots, h'_{s_i}) = x_1^{\gamma_1} \ldots x_n^{\gamma_n}$. Notice that $mx_1^{\gamma_1}$ divides one of $mh'_{s_1} = \operatorname{lcm}(m, h_{s_1}), \ldots, mh'_{s_i} = \operatorname{lcm}(m, h_{s_i})$, and therefore, $mx_1^{\gamma_1} \mid \operatorname{lcm}(m, h_{s_1}, \ldots, h_{s_i})$. Similarly, $mx_2^{\gamma_2}, \ldots, mx_n^{\gamma_n} \mid \operatorname{lcm}(m, h_{s_1}, \ldots, h_{s_i})$. Thus, $x_1^{\gamma_1}, \ldots, x_n^{\gamma_n} \mid \dfrac{\operatorname{lcm}(m, h_{s_1}, \ldots, h_{s_i})}{m}$. It follows that $\operatorname{lcm}(h'_{s_1}, \ldots, h'_{s_i}) = x_1^{\gamma_1} \ldots x_n^{\gamma_n} \mid \dfrac{\operatorname{lcm}(m, h_{s_1}, \ldots, h_{s_i})}{m}$, which is equivalent to saying that $m \operatorname{lcm}(h'_{s_1}, \ldots, h'_{s_i}) \mid \operatorname{lcm}(m, h_{s_1}, \ldots, h_{s_i})$. Finally,

$$m \operatorname{mdeg}[h'_{s_1}, \ldots, h'_{s_i}] = m \operatorname{lcm}(h'_{s_1}, \ldots, h'_{s_i})$$
$$= \operatorname{lcm}(m, h_{s_1}, \ldots, h_{s_i})$$

$$= \operatorname{lcm}(\operatorname{lcm}(m_{r_1}, \ldots, m_{r_j}), h_{s_1}, \ldots, h_{s_i})$$
$$= \operatorname{lcm}(m_{r_1}, \ldots, m_{r_j}, h_{s_1}, \ldots, h_{s_i})$$
$$= \operatorname{mdeg}[m_{r_1}, \ldots, m_{r_j}, h_{s_1}, \ldots, h_{s_i}]. \quad \square$$

Example 3.3. Let M, m, and H be as in Example 3.1. The Taylor symbols $[h'_2, h'_3] = [c, d]$ of $\mathbb{T}_{c^2, c, d, c, cd}$, and $[m_1, h_2, h_3] = [a^3 b^2, a^2 c, b^2 d]$ of $\mathbb{T}_M$, have multidegrees cd and $a^3 b^2 cd$, respectively. Therefore, $\operatorname{mdeg}[a^3 b^2, c, d] = a^3 b^2 \operatorname{mdeg}[c, d]$, which is consistent with Proposition 3.2.

Note: We will say that a monomial l **occurs** in a resolution $\mathbb{F}$ if there is a basis element of $\mathbb{F}$ with multidegree l. If a is an entry of a differential matrix of a resolution $\mathbb{F}$ and $[\sigma]$ is an element of the basis of $\mathbb{F}$, by abusing the language we will often say that a **is an entry of** $\mathbb{F}$ and $[\sigma]$ **is an element of** $\mathbb{F}$. Moreover, sometimes we will use the notation $[\sigma] \in \mathbb{F}$.

Theorem 3.4. *Let m' be a multidegree that occurs in $\mathbb{T}_{h'_1, \ldots, h'_c}$.*

(i) There are no basis elements of $\mathbb{T}_M$, with multidegree mm' and homological degree less than j.

(ii) For every i, there is a bijective correspondence between the basis elements of $\mathbb{T}_{h'_1, \ldots, h'_c}$, with multidegree m' and homological degree i, and the basis elements of $\mathbb{T}_M$, with multidegree mm' and homological degree $i + j$.

Proof. (i) Let $[h'_{s_1}, \ldots, h'_{s_i}]$ be a basis element of $\mathbb{T}_{h'_1, \ldots, h'_c}$, with multidegree m'. By Proposition 3.2,

$$mm' = m \operatorname{mdeg}[h'_{s_1}, \ldots, h'_{s_i}] = \operatorname{mdeg}[m_{r_1}, \ldots, m_{r_j}, h_{s_1}, \ldots, h_{s_i}].$$

It follows that every basis element $[\sigma]$ of $\mathbb{T}_M$, with multidegree mm', must contain the same dominant monomials $m_{r_1}, \ldots, m_{r_j}$ [1, Lemma 4.3]. This means that $\operatorname{hdeg}[\sigma] \geq j$.

(ii) Let $A_{i,m'} = \{ [\sigma] \in \mathbb{T}_{h'_1, \ldots, h'_c} : \operatorname{hdeg}[\sigma] = i; \operatorname{mdeg}[\sigma] = m' \}$.

Let $B_{i,m'} = \{ [\sigma] \in \mathbb{T}_M : \operatorname{hdeg}[\sigma] = i + j; \operatorname{mdeg}[\sigma] = mm' \}$.

Let $f_{i,m'} : A_{i,m'} \to B_{i,m'}$ be defined by $f_{i,m'}[h'_{s_1}, \ldots, h'_{s_i}] = [m_{r_1}, \ldots, m_{r_j}, h_{s_1}, \ldots, h_{s_i}]$.

Notice that $f_{i,m'}$ is well defined:

if $[\sigma] \in A_{i,m'}$, then $\operatorname{hdeg} f_{i,m'}[\sigma] = i + j$ and by Proposition 3.2, $mm' = m \operatorname{mdeg}[\sigma] = \operatorname{mdeg} f_{i,m'}[\sigma]$.

Besides that, $f_{i,m'}$ is one to one:

$$f_{i,m'}[h'_{s_1}, \ldots, h'_{s_i}] = f_{i,m'}[h'_{t_1}, \ldots, h'_{t_i}]$$
$$\Rightarrow [m_{r_1}, \ldots, m_{r_j}, h_{s_1}, \ldots, h_{s_i}] = [m_{r_1}, \ldots, m_{r_j}, h_{t_1}, \ldots, h_{t_i}]$$

$$\Rightarrow h_{s_1} = h_{t_1}, \cdots, h_{s_i} = h_{t_i}$$
$$\Rightarrow [h'_{s_1}, \ldots, h'_{s_i}] = [h'_{t_1}, \ldots, h'_{t_i}].$$

Finally, $f_{i,m'}$ is onto:

Suppose that $[\tau]$ is in $B_{i,m'}$. Let $[h'_{t_1}, \ldots, h'_{t_k}]$ be an element in $\mathbb{T}_{h'_1, \ldots, h'_c}$, with multi-degree m'. By Proposition 3.2,

$$mm' = m \operatorname{mdeg}[h'_{t_1}, \ldots, h'_{t_k}] = \operatorname{mdeg}[m_{r_1}, \ldots, m_{r_j}, h_{t_1}, \ldots, h_{t_k}].$$

Since $[\tau]$ and $[m_{r_1}, \ldots, m_{r_j}, h_{t_1}, \ldots, h_{t_k}]$ are basis elements of equal multidegree, they must contain the same dominant monomials [1, Lemma 4.3] and, given that $[\tau]$ has homological degree $i + j$, $[\tau]$ must be of the form $[\tau] = [m_{r_1}, \ldots, m_{r_j}, h_{s_1}, \ldots, h_{s_i}]$. By Proposition 3.2, $[h'_{s_1}, \ldots, h'_{s_i}] \in A_{i,m'}$, and $f_{i,m'}[h'_{s_1}, \ldots, h'_{s_i}] = [\tau]$. $\square$

Note: Since consecutive cancellations take place between Taylor symbols of equal multidegree, and by [1, Lemma 4.3] two Taylor symbols of equal multidegree contain the same dominant generators, it is reasonable to associate the Taylor symbols whose multidegrees are divisible by the same dominant generators in a common class. For example, if $m = \operatorname{lcm}(m_{r_1}, \ldots, m_{r_j})$, we can define the class $B_m = \{[m_{r_1}, \ldots, m_{r_j}, h_{s_1}, \ldots, h_{s_i}] : 1 \leq s_1 < \ldots < s_i \leq c\}$. If a consecutive cancellation eliminates two Taylor symbols $[\sigma_1], [\sigma_2] \in \mathbb{T}_M$, and $[\sigma_1] \in B_m$, then $[\sigma_2] \in B_m$. That is, consecutive cancellations take place between Taylor symbols of the same class B_m. The class B_m motivated the construction of M_m; in fact, the basis of the Taylor resolution $\mathbb{T}_{h'_1, \ldots, h'_c}$ is analogous to B_m as Theorem 3.4 shows (recall the bijective correspondence $[h'_{s_1}, \ldots, h'_{s_i}] \rightarrow [m_{r_1}, \ldots, m_{r_j}, h_{s_1}, \ldots, h_{s_i}]$).

Example 3.5. Let M, m, and H be as in Example 3.1. Let $m' = cd$. Since the only basis element of $\mathbb{T}_M$ in homological degree 0 is $[\varnothing]$, and $\operatorname{mdeg}[\varnothing] = 1$, there are no basis elements of $\mathbb{T}_M$ in homological degree 0 and multidegree $mm' = a^3 b^2 cd$. This illustrates Theorem 3.4(i). On the other hand, the basis elements of $\mathbb{T}_{c^2, c, d, c, cd}$ with multidegree $m' = cd$ are

$[h'_5]$, in homological degree 1;
$[h'_2, h'_5], [h'_3, h'_5], [h'_4, h'_5], [h'_2, h'_3], [h'_3, h'_4]$ in homological degree 2;
$[h'_2, h'_3, h'_5], [h'_2, h'_4, h'_5], [h'_3, h'_4, h'_5], [h'_2, h'_3, h'_4]$ in homological degree 3; and
$[h'_2, h'_3, h'_4, h'_5]$ in homological degree 4.

Similarly, the basis elements of $\mathbb{T}_M$ with multidegree $mm' = a^3 b^2 cd$ are

$[m_1, h_5]$ in homological degree 2;
$[m_1, h_2, h_5], [m_1, h_3, h_5], [m_1, h_4, h_5], [m_1, h_2, h_3], [m_1, h_3, h_4]$, in homological degree 3;
$[m_1, h_2, h_3, h_5], [m_1, h_2, h_4, h_5], [m_1, h_3, h_4, h_5], [m_1, h_2, h_3, h_4]$ in homological degree 4; and
$[m_1, h_2, h_3, h_4, h_5]$ in homological degree 5, which illustrates the bijective correspondence of Theorem 3.4(ii).

Notation: for every multidegree m' that occurs in $\mathbb{T}_{h'_1,\ldots,h'_c}$ and every $i = 0,\ldots,c$, let $f_{i,m'} : A_{i,m'} \to B_{i,m'}$ be the bijection constructed in Theorem 3.4. Let $A_i = \bigcup_{m'} A_{i,m'}$ and $B_i = \bigcup_{m'} B_{i,m'}$. Let us define $f_i : A_i \to B_i$ by $f_i[\sigma] = f_{i,m'}[\sigma]$, if $\mathrm{mdeg}[\sigma] = m'$. Let $A = \bigcup_i A_i$; $B = \bigcup_i B_i$. Let us define $f : A \to B$ by $f[\sigma] = f_i[\sigma]$, if $\mathrm{hdeg}[\sigma] = i$. Note that A is the basis of $\mathbb{T}_{h'_1,\ldots,h'_c}$, and f is a bijection that sends an element with multidegree m' and homological degree i to an element with multidegree mm' and homological degree $i + j$.

Recall that a differential matrix (f_k) associated to a differential map $f_k : F_k \to F_{k-1}$ is determined by the basis elements of F_k and F_{k-1}. That is, if $\{[\sigma_1],\ldots,[\sigma_u]\}$ and $\{[\tau_1],\ldots,[\tau_v]\}$ are bases of F_k and F_{k-1}, respectively, and $f_k[\sigma_j] = \sum_{i=1}^{v} a_{\tau_i\sigma_j}[\tau_i]$, then the entry of (f_k) that appears in the jth column and the ith row is $a_{\tau_i\sigma_j}$. Therefore, we can say that $a_{\tau_i\sigma_j}$ is determined by $[\sigma_j]$ and $[\tau_i]$. In the next lemma we make reference to this identification.

Lemma 3.6. *If $a_{\pi\theta}$ is an entry of $\mathbb{T}_{h'_1,\ldots,h'_c}$, determined by elements $[\theta], [\pi] \in A$, then $f[\theta], f[\pi]$ determine an entry $b_{\pi\theta}$ of $\mathbb{T}_M$ such that $b_{\pi\theta} = (-1)^j a_{\pi\theta}$.*

Proof. Since $[\theta], [\pi]$ appear in consecutive homological degrees, so do $f[\theta], f[\pi]$. Thus, $f[\theta], f[\pi]$ determine an entry $b_{\pi\theta}$ of $\mathbb{T}_M$.

If $[\pi]$ is a facet of $[\theta]$, then $f[\pi]$ is also a facet of $f[\theta]$ and, these elements are of the form:

$$[\theta] = [h'_{s_1},\ldots,h'_{s_i}]; \quad [\pi] = [h'_{s_1},\ldots,\widehat{h'_{s_t}},\ldots,h'_{s_i}];$$

$$f[\theta] = [m_{r_1},\ldots,m_{r_j},h_{s_1},\ldots,h_{s_i}]; \quad f[\pi] = [m_{r_1},\ldots,m_{r_j},h_{s_1},\ldots,\widehat{h_{s_t}},\ldots,h_{s_i}]$$

Thus $b_{\pi\theta} = (-1)^{j+t+1}\dfrac{\mathrm{mdeg}\, f[\theta]}{\mathrm{mdeg}\, f[\pi]} = (-1)^j(-1)^{t+1}\dfrac{m\,\mathrm{mdeg}[\theta]}{m\,\mathrm{mdeg}[\pi]} = (-1)^j a_{\pi\theta}$.

On the other hand, if $[\pi]$ is not a facet of $[\theta]$, $f[\pi]$ cannot be a facet of $f[\theta]$, either. Thus $a_{\pi\theta} = 0 = b_{\pi\theta}$.   □

Notation: let $\mathbb{F}_0 = \mathbb{T}_{h'_1,\ldots,h'_c}$. If there is an invertible entry $a^{(0)}_{\pi_0\theta_0}$ of $\mathbb{F}_0$, determined by elements $[\theta_0], [\pi_0] \in \mathbb{F}_0$, let $\mathbb{F}_1$ be the resolution of S/M_m such that

$$\mathbb{F}_0 = \mathbb{F}_1 \oplus (0 \to S[\theta_0] \to S[\pi_0] \to 0).$$

Let us assume that $\mathbb{F}_{k-1}$ has been defined for $k \geq 2$. If there is an invertible entry $a^{(k-1)}_{\pi_{k-1}\theta_{k-1}}$ of $\mathbb{F}_{k-1}$, determined by elements $[\theta_{k-1}], [\pi_{k-1}]$ of $\mathbb{F}_{k-1}$, let $\mathbb{F}_k$ be the resolution of S/M_m such that

$$\mathbb{F}_{k-1} = \mathbb{F}_k \oplus (0 \to S[\theta_{k-1}] \to S[\pi_{k-1}] \to 0).$$

Theorem 3.7. *Suppose that $\mathbb{F}_0, \ldots, \mathbb{F}_u$ are resolutions of S/M_m, defined as above. Then*

(i) It is possible to define resolutions $\mathbb{G}_0, \ldots, \mathbb{G}_u$ of S/M, as follows:

$$\mathbb{G}_0 = \mathbb{T}_M; \quad \mathbb{G}_{k-1} = \mathbb{G}_k \oplus \left(0 \to Sf[\theta_{k-1}] \to Sf[\pi_{k-1}] \to 0\right).$$

(ii) If $a_{\tau\sigma}^{(u)}$ is an entry of $\mathbb{F}_u$, determined by elements $[\sigma], [\tau]$ of $\mathbb{F}_u$, then $f[\sigma], f[\tau]$ are in the basis of $\mathbb{G}_u$ and determine an entry $b_{\tau\sigma}^{(u)}$ of $\mathbb{G}_u$, such that $b_{\tau\sigma}^{(u)} = (-1)^j a_{\tau\sigma}^{(u)}$.

Proof. The proof is by induction on u. If $u = 0$, (i) and (ii) are the content of Lemma 3.6.

Let us assume that parts (i) and (ii) hold for $u - 1$. We will prove parts (i) and (ii) for u.

(i) We need to show that $\mathbb{G}_u$ can be defined by the rule

$$\mathbb{G}_{u-1} = \mathbb{G}_u \oplus \left(0 \to Sf[\theta_{u-1}] \to Sf[\pi_{u-1}] \to 0\right).$$

In other words, we must show that $f[\theta_{u-1}], f[\pi_{u-1}]$ are in the basis of $\mathbb{G}_{u-1}$, and the entry $b_{\pi_{u-1}\theta_{u-1}}^{(u-1)}$ of $\mathbb{G}_{u-1}$, determined by them, is invertible. But this follows from induction hypothesis and the fact that $a_{\pi_{u-1}\theta_{u-1}}^{(u-1)}$ is invertible.

(ii) Notice that the basis of $\mathbb{F}_u$ is obtained from the basis of $\mathbb{F}_{u-1}$, by eliminating $[\theta_{u-1}], [\pi_{u-1}]$. This means that $[\sigma], [\tau]$ are in the basis of $\mathbb{F}_{u-1}$, and the pairs $([\sigma], [\tau])$, $([\theta_{u-1}], [\pi_{u-1}])$ are disjoint. Then by induction hypothesis, $f[\sigma], f[\tau]$ are in the basis of $\mathbb{G}_{u-1}$, and because f is a bijection, $(f[\sigma], f[\tau]), (f[\theta_{u-1}], f[\pi_{u-1}])$ are disjoint pairs. Since the basis of $\mathbb{G}_u$ is obtained from the basis of $\mathbb{G}_{u-1}$, by eliminating $f[\theta_{u-1}], f[\pi_{u-1}]$, we must have that $f[\sigma], f[\tau]$ are in the basis of $\mathbb{G}_u$. Finally, we need to prove that $b_{\tau\sigma}^{(u)} = (-1)^j a_{\tau\sigma}^{(u)}$. By [1, Lemma 3.2(iv)], if $\operatorname{hdeg}[\sigma] \neq \operatorname{hdeg}[\theta_{u-1}]$, then $a_{\tau\sigma}^{(u)} = a_{\tau\sigma}^{(u-1)}$. In this case, we must also have that $\operatorname{hdeg} f[\sigma] \neq \operatorname{hdeg} f[\theta_{u-1}]$, which implies that $b_{\tau\sigma}^{(u)} = b_{\tau\sigma}^{(u-1)}$, by the same lemma. Then, by induction hypothesis, $b_{\tau\sigma}^{(u)} = b_{\tau\sigma}^{(u-1)} = (-1)^j a_{\tau\sigma}^{(u-1)} = (-1)^j a_{\tau\sigma}^{(u)}$. On the other hand, if $\operatorname{hdeg}[\sigma] = \operatorname{hdeg}[\theta_{u-1}]$, then $\operatorname{hdeg} f[\sigma] = \operatorname{hdeg} f[\theta_{u-1}]$. Combining the induction hypothesis with [1, Lemma 3.2(iii)], we obtain

$$b_{\tau\sigma}^{(u)} = b_{\tau\sigma}^{(u-1)} - \frac{b_{\tau\theta_{u-1}}^{(u-1)} b_{\pi_{u-1}\sigma}^{(u-1)}}{b_{\pi_{u-1}\theta_{u-1}}^{(u-1)}} = (-1)^j \left(a_{\tau\sigma}^{(u-1)} - \frac{a_{\tau\theta_{u-1}}^{(u-1)} a_{\pi_{u-1}\sigma}^{(u-1)}}{a_{\pi_{u-1}\theta_{u-1}}^{(u-1)}} \right) = (-1)^j a_{\tau\sigma}^{(u)} \quad \square$$

Since the process of making standard cancellations must eventually terminate, there is an integer $u \geq 0$, such that $\mathbb{F}_0, \ldots, \mathbb{F}_u$ are defined as above and $\mathbb{F}_u$ is a minimal resolution of S/M_m. For the rest of this section u is such an integer and $\mathbb{F}_u$ is such a minimal resolution. Moreover, the resolutions $\mathbb{F}_0, \ldots, \mathbb{F}_u$ and $\mathbb{G}_0, \ldots, \mathbb{G}_u$ are also fixed for the rest of this section.

Notation: Let $A' = A \setminus \{[\theta_0], [\pi_0], \ldots, [\theta_{u-1}], [\pi_{u-1}]\}$ and $B' = B \setminus \{f[\theta_0], f[\pi_0], \ldots, f[\theta_{u-1}], f[\pi_{u-1}]\}$. Notice that A' is the basis of the minimal resolution $\mathbb{F}_u$.

Corollary 3.8. *If $b_{\pi\theta}^{(u)}$ is an entry of $\mathbb{G}_u$, determined by elements $f[\theta], f[\pi] \in B'$, then $b_{\pi\theta}^{(u)}$ is noninvertible.*

Proof. Since $f[\theta], f[\pi] \in B'$, $[\theta], [\pi] \in A'$ and thus, the entry $a_{\pi\theta}^{(u)}$ of $\mathbb{F}_u$ is noninvertible. Now, by Theorem 3.7(ii), $b_{\pi\theta}^{(u)}$ is noninvertible. $\square$

Theorem 3.9. *Let m' be a multidegree that occurs in $\mathbb{T}_{M_m}$.*

(i) *There are no basis elements of $\mathbb{G}_u$, with multidegree mm' and homological degree less than j.*

(ii) *For every $i = 0, \ldots, c$, there is a bijective correspondence between the basis elements of $\mathbb{F}_u$, with multidegree m' and homological degree i, and the basis elements of $\mathbb{G}_u$, with multidegree mm' and homological degree $i + j$.*

Proof. (i) Since the basis of $\mathbb{G}_u$ is contained in that of $\mathbb{T}_M$, the statement follows from Theorem 3.4(i).

(ii) The set of basis elements of $\mathbb{F}_u$, with multidegree m' and homological degree i is $A'_{i,m'} = A_{i,m'} \setminus \{[\theta_0], [\pi_0], \ldots, [\theta_{u-1}], [\pi_{u-1}]\}$. Similarly, the set of basis elements of $\mathbb{G}_u$, with multidegree mm' and homological degree $i + j$ is $B'_{i,m'} = B_{i,m'} \setminus \{f[\theta_0], f[\pi_0], \ldots, f[\theta_{u-1}], f[\pi_{u-1}]\}$. Notice that $[\theta_k] \in A_{i,m'}$ if and only if $f[\theta_k] \in B_{i,m'}$. Likewise, $[\pi_k] \in A_{i,m'}$ if and only if $f[\pi_k] \in B_{i,m'}$. Therefore, if we restrict $f_{i,m'} : A_{i,m'} \to B_{i,m'}$ to $A'_{i,m'}$, we get a bijection between $A'_{i,m'}$ and $B'_{i,m'}$. $\square$

Notation: If $b_{\gamma_0\delta_0}^{(u)}$ is an invertible entry of $\mathbb{G}_u$, determined by basis elements $[\delta_0], [\gamma_0]$ of $\mathbb{G}_u$, let $\mathbb{G}_{u+1}$ be the resolution of S/M such that

$$\mathbb{G}_u = \mathbb{G}_{u+1} \oplus (0 \to S[\delta_0] \to S[\gamma_0] \to 0).$$

Assume that $\mathbb{G}_{u+(k-1)}$ has been defined. If $b_{\gamma_{k-1}\delta_{k-1}}^{(u+k-1)}$ is an invertible entry of $\mathbb{G}_{u+(k-1)}$, determined by basis elements $[\delta_{k-1}], [\gamma_{k-1}]$ of $\mathbb{G}_{u+(k-1)}$, let $\mathbb{G}_{u+k}$ be the resolution of S/M such that

$$\mathbb{G}_{u+(k-1)} = \mathbb{G}_{u+k} \oplus (0 \to S[\delta_{k-1}] \to S[\gamma_{k-1}] \to 0).$$

Theorem 3.10. *Suppose that $\mathbb{G}_u$, $\mathbb{G}_{u+1}$ are defined as above. If $b_{\pi\theta}^{(u)}$ is an entry of $\mathbb{G}_u$, determined by elements $f[\theta], f[\pi] \in B'$, then $f[\theta], f[\pi]$ are in the basis of $\mathbb{G}_{u+1}$. Moreover, the entry $b_{\pi\theta}^{(u+1)}$ of $\mathbb{G}_{u+1}$, determined by $f[\theta]$ and $f[\pi]$ is noninvertible.*

Proof. Since $\mathbb{G}_u = \mathbb{G}_{u+1} \oplus (0 \to S[\delta_0] \to S[\gamma_0] \to 0)$, we have that $b_{\gamma_0\delta_0}^{(u)}$ is invertible, $[\delta_0], [\gamma_0]$ are in $\mathbb{G}_u$, in consecutive homological degrees, and $\text{mdeg}[\delta_0] = \text{mdeg}[\gamma_0]$. Suppose that $[\delta_0], [\gamma_0] \in B'$. Then there are elements $[\sigma], [\tau] \in A'$, in consecutive homological degrees, such that $f[\sigma] = [\delta_0]$ and $f[\tau] = [\gamma_0]$. By Theorem 3.9(ii), $[\sigma], [\tau]$ are in $\mathbb{F}_u$ and

determine an entry $a_{\tau\sigma}^{(u)}$. Now, it follows from Theorem 3.7(ii), that $b_{\gamma_0\delta_0}^{(u)} = (-1)^j a_{\tau\sigma}^{(u)}$. This means that $a_{\tau\sigma}^{(u)}$ is invertible and $\mathbb{F}_u$ is not minimal, a contradiction. On the other hand, if only one of $[\delta_0], [\gamma_0]$ is in B', then $\mathrm{mdeg}[\delta_0] \neq \mathrm{mdeg}[\gamma_0]$; another contradiction. We conclude that neither $[\delta_0]$ nor $[\gamma_0]$ is in B' and therefore, the pairs $(f[\theta], f[\pi])$; $([\delta_0], [\gamma_0])$ are disjoint. This proves that $f[\theta], f[\pi]$ are in the basis of $\mathbb{G}_{u+1}$.

Let us finally prove that $b_{\pi\theta}^{(u+1)}$ is noninvertible. If $\mathrm{hdeg}\, f[\theta] \neq \mathrm{hdeg}[\delta_0]$, then $b_{\pi\theta}^{(u+1)} = b_{\pi\theta}^{(u)}$ by [1, Lemma 3.2(iv)], and by Corollary 3.8, $b_{\pi\theta}^{(u+1)}$ is noninvertible. If $\mathrm{hdeg}\, f[\theta] = \mathrm{hdeg}[\delta_0]$, by [1, Lemma 3.2(iii)], we have $b_{\pi\theta}^{(u+1)} = b_{\pi\theta}^{(u)} - \dfrac{b_{\pi\delta_0}^{(u)} b_{\gamma_0\theta}^{(u)}}{b_{\gamma_0\delta_0}^{(u)}}$. Since $b_{\pi\theta}^{(u)} = (-1)^j a_{\pi\theta}^{(u)}$, $b_{\pi\theta}^{(u)}$ is noninvertible. Since $\mathrm{mdeg}\, f[\pi] \neq \mathrm{mdeg}[\delta_0]$, the entry $b_{\pi\delta_0}^{(u)}$ of $\mathbb{G}_u$, determined by $[\delta_0], f[\pi]$, is noninvertible. Hence, the product $b_{\pi\delta_0}^{(u)} b_{\gamma_0\theta}^{(u)}$ must be noninvertible. This means that the quotient $\dfrac{b_{\pi\delta_0}^{(u)} b_{\gamma_0\theta}^{(u)}}{b_{\gamma_0\delta_0}^{(u)}}$ is noninvertible. Finally, $b_{\pi\theta}^{(u+1)}$ is noninvertible, for the difference of two noninvertible monomials is noninvertible.   □

Theorem 3.11. *Suppose that $\mathbb{G}_u, \ldots, \mathbb{G}_{u+v}$ are defined as above. If $b_{\pi\theta}^{(u)}$ is an entry of $\mathbb{G}_u$, determined by elements $f[\theta], f[\pi] \in B'$, then $f[\theta], f[\pi]$ are in the basis of $\mathbb{G}_{u+v}$, and the entry $b_{\pi\theta}^{(u+v)}$ of $\mathbb{G}_{u+v}$, determined by $f[\theta], f[\pi]$ is noninvertible.*

Proof. The proof is by induction on v. If $v = 1$, the statement is the content of Theorem 3.10. Let us assume that the statement holds for $v - 1$. Since $\mathbb{G}_{u+(v-1)} = \mathbb{G}_{u+v} \oplus (0 \to S[\delta_{v-1}] \to S[\gamma_{v-1}] \to 0)$, it follows that the entry $b_{\gamma_{v-1}\delta_{v-1}}^{(u+v-1)}$ of $\mathbb{G}_{u+(v-1)}$, determined by $[\delta_{v-1}], [\gamma_{v-1}]$, is invertible. If we had that $[\delta_{v-1}], [\gamma_{v-1}] \in B'$, then, by induction hypothesis, $b_{\gamma_{v-1}\delta_{v-1}}^{(u+v-1)}$ would be noninvertible, a contradiction. On the other hand, if exactly one of $[\delta_{v-1}], [\gamma_{v-1}]$ were in B', their multidegrees would be different, another contradiction. Hence, neither $[\delta_{v-1}]$ nor $[\gamma_{v-1}]$ is in B'. This means that the pairs $(f[\theta], f[\pi])$, $([\delta_{v-1}], [\gamma_{v-1}])$ are disjoint. Thus, $f[\theta], f[\pi]$ are in the basis of $\mathbb{G}_{u+v}$.

Let us now prove that $b_{\pi\theta}^{(u+v)}$ is noninvertible. If $\mathrm{hdeg}\, f[\theta] \neq \mathrm{hdeg}[\delta_{v-1}]$, then $b_{\pi\theta}^{(u+v)} = b_{\pi\theta}^{(u+v-1)}$ by [1, Lemma 3.2 (iv)], and the result follows from induction hypothesis. Now, if $\mathrm{hdeg}\, f[\theta] = \mathrm{hdeg}[\delta_{v-1}]$,

$$b_{\pi\theta}^{(u+v)} = b_{\pi\theta}^{(u+v-1)} - \frac{b_{\pi\delta_{v-1}}^{(u+v-1)} b_{\gamma_{v-1}\theta}^{(u+v-1)}}{b_{\gamma_{v-1}\delta_{v-1}}^{(u+v-1)}}$$

by [1, Lemma 3.2(iii)]. Notice that $b_{\pi\theta}^{(u+v-1)}$ is noninvertible, by induction hypothesis. Since $\mathrm{mdeg}\, f[\pi] \neq \mathrm{mdeg}[\delta_{v-1}]$, it follows that the entry $b_{\pi\delta_{v-1}}^{(u+v-1)}$ of $\mathbb{G}_{u+(v-1)}$, determined by $[\delta_{v-1}], f[\pi]$, is noninvertible. This implies that the product $b_{\pi\delta_{v-1}}^{(u+v-1)} b_{\gamma_{v-1}\theta}^{(u+v-1)}$ is noninvertible. Moreover, since $b_{\gamma_{v-1}\delta_{v-1}}^{(u+v-1)}$ is invertible, the quotient $\dfrac{b_{\pi\delta_{v-1}}^{(u+v-1)} b_{\gamma_{v-1}\theta}^{(u+v-1)}}{b_{\gamma_{v-1}\delta_{v-1}}^{(u+v-1)}}$

is noninvertible. Finally, $b_{\pi\theta}^{(u+v)}$ is noninvertible, for the difference of two noninvertible monomials is noninvertible. $\quad\square$

Since the process of making standard cancellations must eventually terminate, there is an integer $v \geq 0$, such that $\mathbb{G}_u, \ldots, \mathbb{G}_{u+v}$ are defined as above, and $\mathbb{G}_{u+v}$ is a minimal resolution of S/M.

For the rest of this section, v is such an integer and $\mathbb{G}_{u+v}$ is such a minimal resolution. Moreover, the resolutions $\mathbb{G}_u, \ldots, \mathbb{G}_{u+v}$ are fixed for the rest of this section.

Theorem 3.12. *Let m' be a multidegree that occurs in $\mathbb{T}_{M_m}$. For each i, there is a bijective correspondence between the basis elements of $\mathbb{G}_u$, with multidegree mm' and homological degree $i + j$, and the basis elements of $\mathbb{G}_{u+v}$, with multidegree mm' and homological degree $i + j$.*

Proof. Since the basis of $\mathbb{G}_{u+v}$ is contained in that of $\mathbb{G}_u$, every basis element of $\mathbb{G}_{u+v}$, with multidegree mm' and homological degree $i + j$ is in $\mathbb{G}_u$. Conversely, every basis element of $\mathbb{G}_u$, with multidegree mm' and homological degree $i + j$ is in $\mathbb{G}_{u+v}$, by Theorem 3.11. $\quad\square$

Theorem 3.13. *Let $\mathbb{F}$ be a minimal resolution of S/M_m, and let $\mathbb{G}$ be a minimal free resolution of S/M. Let m' be a multidegree that occurs in $\mathbb{T}_{M_m}$. Then*

(i) *There are no basis elements of $\mathbb{G}$, with multidegree mm' and homological degree less than j.*
(ii) *For each i, there is a bijective correspondence between the basis elements of $\mathbb{F}$, with multidegree m' and homological degree i, and the basis elements of $\mathbb{G}$, with multidegree mm' and homological degree $i + j$.*

Proof. (i) Since the basis of $\mathbb{G}$ is contained in that of $\mathbb{T}_M$, this part follows from Theorem 3.4(i).

(ii) This part follows immediately from Theorem 3.9(ii) and Theorem 3.12. $\quad\square$

Example 3.14. Consider Example 3.1, again. Recall that $M = (m_1, m_2, h_1, h_2, h_3, h_4, h_5) = (a^3b^2, c^3d, ac^2, a^2c, b^2d, abc, bcd)$, and $m = \text{lcm}(m_1) = a^3b^2$. Since $M_m = (c^2, c, d, c, cd) = (c, d)$, the minimal resolution of S/M_m is of the form

$$\mathbb{F} : 0 \to S[c, d] \to \begin{array}{c} S[c] \\ \oplus \\ S[d] \end{array} \to S[\varnothing] \to S/M_m \to 0.$$

Thus, $b_{0,1}(S/M_m) = b_{1,c}(S/M_m) = b_{1,d}(S/M_m) = b_{2,cd}(S/M_m) = 1$. By Theorem 3.13(ii) (with $m = a^3b^2$, and $j = 1$), $b_{1,a^3b^2}(S/M) = b_{2,a^3b^2c}(S/M) = b_{2,a^3b^2d}(S/M) = b_{3,a^3b^2cd}(S/M) = 1$. By Theorem 3.13(i), $b_{0,a^3b^2}(S/M) =$

$b_{0,a^3b^2c}(S/M) = b_{0,a^3b^2d}(S/M) = b_{0,a^3b^2cd}(S/M) = 0$. (In the next section we will give the entire list of multigraded Betti numbers of S/M.)

4. Structural decomposition theorems

The notation below retains its meaning until the end of this section.

Let M be an ideal with minimal generating set $G = \{m_1, \ldots, m_q, n_1, \ldots, n_p\}$, where $m_1, \ldots, m_q$ are dominant, $n_1, \ldots, n_p$ are nondominant, $q \geq 1$, and $p \geq 0$. Let $1 \leq d \leq q$, and let $H = \{m_{d+1}, \ldots, m_q, n_1, \ldots, n_p\}$. Then G can be expressed in the form $G = \{m_1, \ldots, m_d, h_1, \ldots, h_c\}$, where $H = \{h_1, \ldots, h_c\}$.

- If $c > 0$, let $C = \{(j, m) \in \mathbb{Z}^+ \times S : \text{ there are integers } 1 \leq r_1 < \cdots < r_j \leq d, \text{ such that } m = \mathrm{lcm}(m_{r_1}, \ldots, m_{r_j})\} \bigcup \{(0, 1)\}$. For each $(j, m) \in C$, let $M_m = (h'_1, \ldots, h'_c)$, where $h'_i = \dfrac{\mathrm{lcm}(m, h_i)}{m}$.
- If $c = 0$, let $C = \{(0, 1)\}$ and let $M_1 = M$.

Theorem 4.1. *For each integer k and each monomial l,*

$$b_{k,l}(S/M) = \sum_{(j,m) \in C} b_{k-j, l/m}(S/M_m).$$

Proof. If $c = 0$, the theorem is trivial. Let us consider the case $c > 0$.

If $b_{k,l}(S/M) = 0$, then $\sum_{(j,m) \in C} b_{k-j, l/m}(S/M_m) = 0$, by Theorem 3.13(ii). Suppose now that $b_{k,l}(S/M) \neq 0$. Then there is an element $[\tau]$ in the basis of a minimal resolution of S/M, such that $\mathrm{hdeg}[\tau] = k$ and $\mathrm{mdeg}[\tau] = l$. Let $m_{r_1}, \ldots, m_{r_j}$ be the dominant monomials that are contained in $[\tau]$, and such that $\{m_{r_1}, \ldots, m_{r_j}\}$ is a subset of $\{m_1, \ldots, m_d\}$. Since all basis elements of $\mathbb{T}_M$ with equal multidegree must contain the same dominant monomials [1, Lemma 4.3], every basis element of $\mathbb{T}_M$ in homological degree k and multidegree l must be of the form $[m_{r_1}, \ldots, m_{r_j}, h_{s_1}, \ldots, h_{s_{k-j}}]$. Let $m = \mathrm{lcm}(m_{r_1}, \ldots, m_{r_j})$. Then $(j, m) \in C$, and $b_{k,l}(S/M) = b_{k-j, l/m}(S/M_m)$, by Theorem 3.13(ii).

We will complete the proof by showing that $b_{k-j', l/m}(S/M_{m'}) = 0$, for all $(j', m') \in C \setminus \{(j, m)\}$. Let $(j', m') \in C$. Then there are dominant monomials $m_{u_1}, \ldots, m_{u_j}$, such that $m' = \mathrm{lcm}(m_{u_1}, \ldots, m_{u_{j'}})$. Suppose that $b_{k-j, l/m}(S/M_{m'}) \neq 0$. Then $\mathbb{T}_{M_{m'}}$ has a basis element $[h'_{t_1}, \ldots, h'_{t_{k-j'}}]$ with multidegree l/m. By Proposition 3.2, $l = m\, \mathrm{mdeg}[h'_{t_1}, \ldots, h'_{t_{k-j'}}] = \mathrm{mdeg}[m_{u_1}, \ldots, m_{u_{j'}}, h_{t_1}, \ldots, h_{t_{k-j'}}]$. Since the basis elements of $\mathbb{T}_M$ in homological degree k and multidegree l are of the form $[m_{r_1}, \ldots, m_{r_j}, h_{s_1}, \ldots, h_{s_{k-j}}]$, we must have that $\{m_{u_1}, \ldots, m_{u_{j'}}\} = \{m_{r_1}, \ldots, m_{r_j}\}$. In particular, $j' = j$, and $m' = \mathrm{lcm}(m_{u_1}, \ldots, m_{u_{j'}}) = \mathrm{lcm}(m_{r_1}, \ldots, m_{r_j}) = m$. Thus $(j', m') = (j, m)$. $\square$

Definition 4.2. Recall that $G = \{m_1, \ldots, m_q, n_1, \ldots, n_p\} = \{m_1, \ldots, m_d, h_1, \ldots, h_c\}$ is the minimal generating set of M. If $d = q$, the equation

$$\mathrm{b}_{k,l}(S/M) = \sum_{(j,m)\in C} \mathrm{b}_{k-j,l/m}(S/M_m),$$

given by Theorem 4.1, will be called the **first structural decomposition** of M.

Note that when $d = q$, we have that $c = p$, and $\{h_1, \ldots, h_c\} = \{n_1, \ldots, n_p\}$.

Example 4.3. Consider Example 3.1, again. Recall that $M = (m_1, m_2, n_1, n_2, n_3, n_4, n_5) = (a^3 b^2, c^3 d, ac^2, a^2 c, b^2 d, abc, bcd)$, where $\{h_1, \ldots, h_5\} = \{n_1, \ldots, n_5\}$, and hence, $d = q$. By definition,

$$C = \{(2, \mathrm{lcm}(a^3 b^2, c^3 d)); (1, \mathrm{lcm}(a^3 b^2)); (1, \mathrm{lcm}(c^3 d)); (0, 1)\}$$
$$= \{(2, a^3 b^2 c^3 d); (1, a^3 b^2); (1, c^3 d); (0, 1)\}.$$

Now, each ordered pair (j, m) in C, determines a monomial ideal M_m. Namely, $(2, a^3 b^2 c^3 d)$ defines $M_{a^3 b^2 c^3 d} = (h_1', h_2', h_3', h_4', h_5')$, where $h_1' = \dfrac{\mathrm{lcm}(a^3 b^2 c^3 d, ac^2)}{a^3 b^2 c^3 d} = 1$. Therefore, $M_{a^3 b^2 c^3 d} = (1) = S$.

Likewise, $(1, a^3 b^2)$ defines $M_{a^3 b^2} = (c, d)$ (recall Example 3.14).

Also, $(1, c^3 d)$ defines $M_{c^3 d} = (h_1', h_2', h_3', h_4', h_5')$, where $h_1' = \dfrac{\mathrm{lcm}(c^3 d, ac^2)}{c^3 d} = a$; $h_2' = \dfrac{\mathrm{lcm}(c^3 d, a^2 c)}{c^3 d} = a^2$; $h_3' = \dfrac{\mathrm{lcm}(c^3 d, b^2 d)}{c^3 d} = b^2$; $h_4' = \dfrac{\mathrm{lcm}(c^3 d, abc)}{c^3 d} = ab$; $h_5' = \dfrac{\mathrm{lcm}(c^3 d, bcd)}{c^3 d} = b$. Thus, $M_{c^3 d} = (a, a^2, b^2, ab, b) = (a, b)$.

Finally, $(0, 1)$ defines $M_1 = (ac^2, a^2 c, b^2 d, abc, bcd)$. Therefore, the first structural decomposition of M is

$$\mathrm{b}_{k,l}(S/M) = \mathrm{b}_{k-1,l/a^3 b^2}\left(\frac{S}{(c,d)}\right) + \mathrm{b}_{k-1,l/c^3 d}\left(\frac{S}{(a,b)}\right) + \mathrm{b}_{k,l}\left(\frac{S}{(ac^2, a^2 c, b^2 d, abc, bcd)}\right).$$

Theorem 4.4. *There is a family $\mathscr{D}$ of dominant ideals and a family $\mathscr{N}$ of purely non-dominant ideals, such that*

$$\mathrm{b}_{k,l}(S/M) = \sum_{D\in\mathscr{D}} \mathrm{b}_{k-j_D,l/m_D}(S/D) + \sum_{N\in\mathscr{N}} \mathrm{b}_{k-j_N,l/m_N}(S/N),$$

where j_D, j_N are integers that depend on D and N, respectively, and m_D, m_N are monomials that depend on D and N, respectively.

Proof. Let

$$\mathrm{b}_{k,l}(S/M) = \sum_{(j,m)} \mathrm{b}_{k-j,l/m}(S/M_m)$$

be the first structural decomposition of M. If some $M_m = (h'_1, \ldots, h'_p)$ (recall that $c = p$) is neither dominant nor purely nondominant, then its minimal generating set is of the form $\{u_1, \ldots, u_{q_1}, v_1, \ldots, v_{p_1}\}$, where $u_1, \ldots, u_{q_1}$ are dominant, $v_1, \ldots, v_{p_1}$ are nondominant, $q_1 \geq 1$, $p_1 \geq 1$, and $q_1 + p_1 \leq p$. In particular, $p_1 \leq p-1$. Let $b_{k,l}(S/M_m) = \sum_{(j',m')} b_{k-j',l/m'}(S/M_{m,m'})$ be the first structural decomposition of M_m. Combining the last two identities, we obtain

$$b_{k,l}(S/M) = \sum_{(j,m)} b_{k-j,l/m}(S/M_m) = \sum_{(j,m)} \sum_{(j',m')} b_{k-j-j',l/mm'}(S/M_{m,m'}).$$

If some $M_{m,m'}$ is neither dominant nor purely nondominant, then $M_{m,m'} = (v'_1, \ldots, v'_{p_1})$ (where $v'_i = \dfrac{\mathrm{lcm}(m', v_i)}{m'}$), and the number p_2 of nondominant generators in its minimal generating set is less than p_1 (because $M_{m,m'}$ is minimally generated by at most p_1 monomials). In particular, $p_2 \leq p_1 - 1 \leq p - 2$. Suppose that, after applying Theorem 4.1 r times, we obtain a decomposition $b_{k,l}(S/M) = \sum_{(j_1,l_1)} \cdots \sum_{(j_r,l_r)} b_{k-j_1-\ldots-j_r,l/l_1\ldots l_r}(S/M_{l_1,\ldots,l_r})$, such that if some $M_{l_1,\ldots,l_r}$ is neither dominant nor purely nondominant, then $M_{l_1,\ldots,l_r} = (w'_1, \ldots, w'_{p_{r-1}})$, with $p_{r-1} \leq p - (r-1)$.

If some $M_{l_1,\ldots,l_r}$ is neither dominant nor purely nondominant, then the number p_r of nondominant generators in its minimal generating set is less than p_{r-1}. In particular, $p_r \leq p_{r-1} - 1 \leq p - r$. Therefore, after applying Theorem 4.1 p times, we obtain a decomposition

$$b_{k,l}(S/M) = \sum_{(j_1,l_1)} \cdots \sum_{(j_p,l_p)} b_{k-j_1-\ldots-j_p,l/l_1\ldots l_p}(S/M_{l_1,\ldots,l_p}).$$

If we assume that there is an ideal $M_{l_1,\ldots,l_p}$ which is neither dominant nor purely nondominant, then $M_{l_1,\ldots,l_p} = (z'_1, \ldots, z'_{p_{p-1}})$, with $p_{p-1} \leq p - (p-1) = 1$.

But this scenario is not possible, for the minimal generating set of such an ideal must contain at least one dominant generator and at least one nondominant generator.

We conclude that each $M_{l_1,\ldots,l_p}$ is either dominant or purely nondominant. $\square$

Definition 4.5. The equation

$$b_{k,l}(S/M) = \sum_{D \in \mathscr{D}} b_{k-j_D,l/m_D}(S/D) + \sum_{N \in \mathscr{N}} b_{k-j_N,l/m_N}(S/N)$$

constructed in the proof of Theorem 4.4 will be called **second structural decomposition** of M. The sum $\sum_{D \in \mathscr{D}} b_{k-j_D,l/m_D}(S/D)$ will be called **dominant part** of the second structural decomposition, and the sum $\sum_{N \in \mathscr{N}} b_{k-j_N,l/m_N}(S/N)$ will be called **purely nondominant part** of the second structural decomposition.

Note: Although Theorem 4.4 states the existence of a decomposition of the form

$$\mathrm{b}_{k,l}(S/M) = \sum_{D \in \mathscr{D}} \mathrm{b}_{k-j_D, l/m_D}(S/D) + \sum_{N \in \mathscr{N}} \mathrm{b}_{k-j_N, l/m_N}(S/N),$$

the proof of Theorem 4.4 is constructive. In fact, we show that

$$\mathrm{b}_{k,l}(S/M) = \sum_{(j_1, l_1)} \cdots \sum_{(j_p, l_p)} \mathrm{b}_{k-j_1-\ldots-j_p, l/l_1 \ldots l_p}(S/M_{l_1, \ldots, l_p}),$$

where the ideals $M_{l_1, \ldots, l_p}$ are either dominant or purely nondominant, and they determine the dominant and purely nondominant part of the second structural decomposition.

Recall that if D is a dominant ideal, its minimal resolution is given by $\mathbb{T}_D$ [1, Theorem 4.4]. Therefore, when the second structural decomposition of M has no purely nondominant part, we can immediately compute the multigraded Betti numbers $\mathrm{b}_{k,l}(S/M)$. Such is the case in the next example.

Example 4.6. In Example 3.1 we introduced the ideal $M = (a^3b^2, c^3d, ac^2, a^2c, b^2d, abc, bcd)$ and, in Example 4.3 we gave its first structural decomposition. We would like to read off the Betti numbers of S/M from the Betti numbers of the three ideals on the right side of that decomposition. The first two of these ideals, namely $M_{a^3b^2} = (c, d)$ and $M_{c^3d} = (a, b)$, are dominant. Hence, their minimal resolutions are $\mathbb{T}_{M_{a^3b^2}}$ and $\mathbb{T}_{M_{c^3d}}$, respectively. However, the third ideal, $M_1 = (ac^2, a^2c, b^2d, abc, bcd)$, is not dominant. In order to obtain the multigraded Betti numbers of S/M_1 we compute the first structural decomposition of M_1 (we leave the details to the reader):

$$\mathrm{b}_{k,l}\left(\frac{S}{M_1}\right) = \mathrm{b}_{k-2, l/a^2c^2}\left(\frac{S}{(b)}\right) + \mathrm{b}_{k-1, l/ac^2}\left(\frac{S}{(b)}\right) + \mathrm{b}_{k-1, l/a^2c}\left(\frac{S}{(b)}\right)$$
$$+ \mathrm{b}_{k-1, l/b^2d}\left(\frac{S}{(c)}\right) + \mathrm{b}_{k,l}\left(\frac{S}{(abc, bcd)}\right).$$

Now, if we combine this equation with the first structural decomposition of M, given in Example 4.3, we obtain

$$\mathrm{b}_{k,l}(S/M) = \mathrm{b}_{k-1, l/a^3b^2}\left(\frac{S}{(c, d)}\right) + \mathrm{b}_{k-1, l/c^3d}\left(\frac{S}{(a, b)}\right) + \mathrm{b}_{k-2, l/a^2c^2}\left(\frac{S}{(b)}\right)$$
$$+ \mathrm{b}_{k-1, l/ac^2}\left(\frac{S}{(b)}\right) + \mathrm{b}_{k-1, l/a^2c}\left(\frac{S}{(b)}\right) + \mathrm{b}_{k-1, l/b^2d}\left(\frac{S}{(c)}\right)$$
$$+ \mathrm{b}_{k,l}\left(\frac{S}{(abc, bcd)}\right).$$

Note that this is the second structural decomposition of M, for each ideal on the right side of this decomposition is dominant. In order to compute $\mathrm{b}_{k,l}(S/M)$, it would

be unwise to choose integers k and monomials l at random. We might take many guesses and still not find any nonzero multigraded Betti numbers. The right way to compute $b_{k,l}(S/M)$ is by first computing the minimal resolutions of the dominant ideals on the right side of the decomposition, which we do next.

- The multigraded Betti numbers of $S/(c, d)$ are

$$b_{0,1}(S/(c,d)) = b_{1,c}(S/(c,d)) = b_{1,d}(S/(c,d)) = b_{2,cd}(S/(c,d)) = 1.$$

Therefore, $b_{k-1,l/a^3b^2}\left(\dfrac{S}{(c,d)}\right) = 1$ when $(k-1, l/a^3b^2)$ equals one of $(0,1)$, $(1,c)$, $(1,d)$, $(2,cd)$; that is, when (k,l) equals one of $(1, a^3b^2)$, $(2, a^3b^2c)$, $(2, a^3b^2d)$, $(3, a^3b^2cd)$.

- The multigraded Betti numbers of $S/(a, b)$ are

$$b_{0,1}(S/(a,b)) = b_{1,a}(S/(a,b)) = b_{1,b}(S/(a,b)) = b_{2,ab}(S/(a,b)) = 1.$$

Therefore, $b_{k-1,l/c^3d}\left(\dfrac{S}{(a,b)}\right) = 1$ when $(k-1, l/c^3d)$ equals one of $(0,1)$, $(1,a)$, $(1,b)$, $(2,ab)$; that is, when (k,l) equals one of $(1, c^3d)$, $(2, ac^3d)$, $(2, bc^3d)$, $(3, abc^3d)$.

- The multigraded Betti numbers of $S/(b)$ are $b_{0,1}(S/(b)) = b_{1,b}(S/(b)) = 1$.
Therefore, $b_{k-2,l/a^2c^2}\left(\dfrac{S}{(b)}\right) = 1$, or $b_{k-1,l/ac^2}\left(\dfrac{S}{(b)}\right) = 1$, or $b_{k-1,l/a^2c}\left(\dfrac{S}{(b)}\right) = 1$, when $(k-2, l/a^2c^2)$ equals one of $(0,1)$, $(1,b)$, or when $(k-1, l/ac^2)$ equals one of $(0,1)$, $(1,b)$, or when $(k-1, l/a^2c)$ equals one of $(0,1)$, $(1,b)$; that is, when (k,l) equals one of $(2, a^2c^2)$, $(3, a^2bc^2)$, $(1, ac^2)$, $(2, abc^2)$, $(1, a^2c)$, $(2, a^2bc)$.

- The multigraded Betti numbers of $S/(c)$ are $b_{0,1}(S/(c)) = b_{1,c}(S/(c)) = 1$.
Therefore, $b_{k-1,l/b^2d}\left(\dfrac{S}{(c)}\right) = 1$ when $(k-1, l/b^2d)$ equals one of $(0,1)$, $(1,c)$; that is, when (k,l) equals one of $(1, b^2d)$, $(2, b^2cd)$.

- The multigraded Betti numbers of $S/(abc, bcd)$ are

$$b_{0,1}(S/(abc,bcd)) = b_{1,abc}(S/(abc,bcd)) = b_{1,bcd}(S/(abc,bcd))$$
$$= b_{2,abcd}(S/(abc,bcd)) = 1.$$

Therefore, $b_{k,l}\left(\dfrac{S}{(abc,bcd)}\right) = 1$ when (k,l) equals one of $(0,1)$, $(1, abc)$, $(1, bcd)$, $(2, abcd)$.

Thus, the nonzero multigraded Betti numbers of S/M are

$$b_{3,a^3b^2cd} = b_{3,abc^3d} = b_{3,a^2bc^2} = b_{2,a^3b^2c} = b_{2,a^3b^2d} = b_{2,ac^3d} = b_{2,bc^3d} = b_{2,a^2c^2}$$

$$= b_{2,abc^2} = b_{2,a^2bc} = b_{2,b^2cd} = b_{2,abcd} = b_{1,a^3b^2} = b_{1,c^3d} = b_{1,ac^2}$$

$$= b_{1,a^2c} = b_{1,b^2d} = b_{1,abc} = b_{1,bcd} = b_{0,1} = 1.$$

Definition 4.7. Recall that $G = \{m_1, \ldots, m_d, h_1, \ldots, h_c\}$ is the minimal generating set of M. If $d = 1$, the equation

$$\mathrm{b}_{k,l}(S/M) = \sum_{(j,m) \in C} \mathrm{b}_{k-j,l/m}(S/M_m),$$

given by Theorem 4.1, will be called the **third structural decomposition** of M.

Note that when $d = 1$, the right hand side of the equation above has only two terms. The third structural decomposition will be instrumental in the proof of Charalambous theorem, in Section 6.

5. Decompositions without purely nondominant part

When the second structural decomposition of M has no purely nondominant part, the numbers $\mathrm{b}_{k,l}(S/M)$ can be easily computed, as illustrated in Example 4.6. In this section, however, our aim is to compute Betti numbers of classes of ideals rather than single ideals. More specifically, we will introduce two families of ideals whose decompositions have no purely nondominant part, and will give their multigraded Betti numbers explicitly.

Definition 5.1. Let L be the set of all monomials l such that the number of basis elements of $\mathbb{T}_M$, with multidegree l is odd.

(i) We say that M has **characteristic Betti numbers**, if for each monomial l

$$\sum_k \mathrm{b}_{k,l}(S/M) = \begin{cases} 1 \text{ if } l \in L, \\ 0 \text{ otherwise.} \end{cases}$$

(ii) For each $l \in L$, let

$$f(l) = \min\{\mathrm{hdeg}[\sigma] : [\sigma] \in \mathbb{T}_M \text{ and } \mathrm{mdeg}[\sigma] = l\}.$$

We say that M has **characteristic Betti numbers in minimal homological degrees**, if

$$\mathrm{b}_{k,l}(S/M) = \begin{cases} 1 \text{ if } l \in L \text{ and } k = f(l), \\ 0 \text{ otherwise.} \end{cases}$$

Lemma 5.2. *Let*

$$\mathrm{b}_{k,l}(S/M) = \sum_{(j,m)} \mathrm{b}_{k-j,l/m}(S/M_m)$$

be the first structural decomposition of M. Then, the second structural decomposition of M has no purely nondominant part if and only if the second structural decomposition of each M_m has no purely nondominant part.

Proof. Let

$$b_{k,l}(S/M_m) = \sum_{D \in \mathscr{D}} b_{k-j_D, l/m_D}(S/D) + \sum_{N \in \mathscr{N}} b_{k-j_N, l/m_N}(S/N),$$

be the second structural decomposition of M_m. Then

$$b_{k,l}(S/M) = \sum_{(j,m)} b_{k-j, l/m}(S/M_m) =$$

$$\sum_{(j,m)} \left(\sum_{D \in \mathscr{D}_m} b_{k-j-j_D, l/mm_D}(S/D) + \sum_{N \in \mathscr{N}_m} b_{k-j-j_N, l/mm_N}(S/N) \right)$$

is the second structural decomposition of M. □

Theorem 5.3. *If the second structural decomposition of M has no purely nondominant part, then M has characteristic Betti numbers.*

Proof. The proof is by induction on the cardinality of the minimal generating set G of M.

If $\#G = 1$ or $\#G = 2$, then M is dominant and, by [1, Corollary 4.5], M is Scarf. Now, Scarf ideals have characteristic Betti numbers.

Suppose now that the theorem holds for ideals with minimal generating sets of cardinality $\leq q - 1$.

Let us assume that $\#G = q$. By hypothesis, the second structural decomposition of M has no purely nondominant part, which implies that M itself is not purely nondominant. Therefore, G must be of the form $G = \{m_1, \ldots, m_s, n_1, \ldots, n_t\}$, where $m_1, \ldots, m_s$ are dominant, $n_1, \ldots, n_t$ are nondominant, $s > 0$, and $s + t = q$. In particular, $t \leq q - 1$. Now, the first structural decomposition of M is

$$b_{k,l}(S/M) = \sum_{(j,m) \in C} b_{k-j, l/m}(S/M_m),$$

where each M_m is minimally generated by at most $q - 1$ monomials. Then,

$$\sum_k b_{k,l}(S/M) = \sum_k \sum_{(j,m)} b_{k-j, l/m}(S/M_m) = \sum_{(j,m)} \sum_k b_{k-j, l/m}(S/M_m).$$

Suppose that, for some monomial l, $\sum_k b_{k,l}(S/M) \neq 0$. Then, there must be a pair $(j', m') \in C$ such that $\sum_k b_{k-j', l/m'}(S/M_{m'}) \neq 0$. By Lemma 5.2, the second structural

decomposition of $M_{m'}$ has no purely nondominant part and, by induction hypothesis, $M_{m'}$ has characteristic Betti numbers. Hence, $\sum_k \mathrm{b}_{k-j',l/m'}(S/M_{m'}) = 1$.

Suppose, by means of contradiction, that there is a pair $(j,m) \in C \setminus \{(j',m')\}$ such that $\sum_k \mathrm{b}_{k-j,l/m}(S/M_m) \neq 0$. Then, there exist basis elements $[\sigma] \in \mathbb{T}_{M_{m'}}$ and $[\tau] \in \mathbb{T}_{M_m}$, such that $\mathrm{mdeg}[\sigma] = l/m'$, and $\mathrm{mdeg}[\tau] = l/m$. Recall that $[\sigma]$, $[\tau]$, m', and m are of the form $[\sigma] = [n'_{a_1}, \ldots, n'_{a_c}]$; $[\tau] = [n'_{b_1}, \ldots, n'_{b_d}]$; $m' = \mathrm{lcm}(m_{u_1}, \ldots, m_{u_{j'}})$; $m = \mathrm{lcm}(m_{v_1}, \ldots, m_{v_j})$. By Proposition 3.2 we have that

$$l = m'l/m' = m'\,\mathrm{mdeg}[\sigma] = m'\,\mathrm{mdeg}[n'_{a_1}, \ldots, n'_{a_c}] = \mathrm{mdeg}[m_{u_1}, \ldots, m_{u_{j'}}, n_{a_1}, \ldots, n_{a_c}].$$

Similarly,

$$l = ml/m = m\,\mathrm{mdeg}[\tau] = m\,\mathrm{mdeg}[n'_{b_1}, \ldots, n'_{b_d}] = \mathrm{mdeg}[m_{v_1}, \ldots, m_{v_j}, n_{b_1}, \ldots, n_{b_d}].$$

Hence, $\mathrm{mdeg}[m_{u_1}, \ldots, m_{u_{j'}}, n_{a_1}, \ldots, n_{a_c}]$ and $\mathrm{mdeg}[m_{v_1}, \ldots, m_{v_j}, n_{b_1}, \ldots, n_{b_d}]$ are two basis elements of $\mathbb{T}_M$, with the same multidegree l. By [1, Lemma 4.3], these basis elements must contain the same dominant generators. However, since $(j',m') \neq (j,m)$, we must have that $\{m_{u_1}, \ldots, m_{u'_j}\} \neq \{m_{v_1}, \ldots, m_{v_j}\}$, a contradiction. Therefore, $\sum_k \mathrm{b}_{k-j,l/m}(S/M_m) = 0$, for all $(j,m) \in C \setminus \{(j',m')\}$. Thus,

$$\sum_k \mathrm{b}_{k,l}(S/M) = \sum_{(j,m)} \sum_k \mathrm{b}_{k-j,l/m}(S/M_m) = \sum_k \mathrm{b}_{k-j',l/m'}(S/M_{m'}) = 1.$$

We have proven that $\sum_k \mathrm{b}_{k,l}(S/M) \leq 1$, for each monomial l.

Since a minimal resolution $\mathbb{F}$ of S/M can be obtained from $\mathbb{T}_M$ by making series of consecutive cancellations, and given that each consecutive cancellation involves a pair of basis elements of equal multidegree, the number of basis elements of $\mathbb{T}_M$ with a given multidegree l is even if and only if the number of basis elements of $\mathbb{F}$ with multidegree l is even. But the number of basis elements of $\mathbb{F}$ with multidegree l is $\sum_k \mathrm{b}_{k,l}(S/M) \leq 1$, which proves the theorem. $\square$

Note that Theorem 5.3 does not depict the entire family of ideals with characteristic Betti numbers. In fact, $M = (a^2bc, b^2c^2, a^2b^2, abc^2)$ is purely nondominant and has characteristic Betti numbers.

In Example 4.6, we computed the second structural decomposition of the ideal $M = (a^3b^2, c^3d, ac^2, a^2c, b^2d, abc, bcd)$, and noticed that it has no purely nondominant part. Right after, we found the numbers $\mathrm{b}_{k,l}(S/M)$ and proved that, with the language of this section, M has characteristic Betti numbers. This is consistent with Theorem 5.3.

Lemma 5.4. *Let* $\mathrm{b}_{k,l}(S/M) = \sum_{(j,m)} \mathrm{b}_{k-j,l/m}(S/M_m)$ *be the first structural decomposition of M. Suppose that M has characteristic Betti numbers, and each M_m has characteristic*

Betti numbers in minimal homological degrees. Then M has characteristic Betti numbers in minimal homological degrees.

Proof. Let l be a monomial that is the common multidegree of an odd number of basis elements of $\mathbb{T}_M$. Let r be such that $\mathrm{b}_{r,l}(S/M) = 1$. Then, there is a pair $(j, m) \in C$ such that $\mathrm{b}_{r-j,l/m}(S/M_m) = 1$. It follows that there is a basis element $[n'_{a_1}, \ldots, n'_{a_{r-j}}]$ of $\mathbb{T}_{M_m}$ with multidegree l/m. Recall that m is of the form $m = \mathrm{lcm}(m_{b_1}, \ldots, m_{b_j})$, with $m_{b_1}, \ldots, m_{b_j} \in \{m_1, \ldots, m_s\}$. By Proposition 3.2, $l = m\dfrac{l}{m} = m\,\mathrm{mdeg}[n'_{a_1}, \ldots, n'_{a_{r-j}}] = \mathrm{mdeg}[m_{b_1}, \ldots, m_{b_j}, n_{a_1}, \ldots, n_{a_{r-j}}]$. Suppose that $[\sigma]$ is a basis element of $\mathbb{T}_M$, such that $\mathrm{mdeg}[\sigma] = l$. We will show that $r \leq \mathrm{hdeg}[\sigma]$. By [1, Lemma 4.3], $[\sigma]$ must be of the form $[\sigma] = [m_{b_1}, \ldots, m_{b_j}, n_{c_1}, \ldots, n_{c_d}]$. By Proposition 3.2, $l = \mathrm{mdeg}[\sigma] = m\,\mathrm{mdeg}[n'_{c_1} \ldots, n'_{c_d}]$. Thus, the basis element $[n'_{c_1}, \ldots, n'_{c_d}]$ of $\mathbb{T}_{M_m}$ has multidegree l/m. Since M_m has Betti numbers in minimal homological degrees, $\mathrm{hdeg}[n'_{a_1}, \ldots, n'_{a_{r-j}}] \leq \mathrm{hdeg}[n'_{c_1}, \ldots, n'_{c_d}]$. It follows that $r = \mathrm{hdeg}[m_{b_1}, \ldots, m_{b_j}, n_{a_1}, \ldots, n_{a_{r-j}}] \leq \mathrm{hdeg}[m_{b_1}, \ldots, m_{b_j}, n_{c_1}, \ldots, n_{c_d}] = \mathrm{hdeg}[\sigma]$, which proves that M itself has characteristic Betti numbers in minimal homological degrees. $\square$

Recall that a monomial ideal is called generic if no variable appears with the same nonzero exponent in the factorization of two minimal generators [2]. By slightly relaxing the defining condition of generic ideal, we next introduce the concept of almost generic ideal.

Definition 5.5. Suppose that the polynomial ring S has n variables $x_1, \ldots, x_n$. We will say that M is **almost generic**, if there is an index i such that no variable among $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ appears with the same nonzero exponent in the factorization of two minimal generators of M.

Example 5.6. $M = (a^2b^2cd^2, a^3b^3c, cd^4)$ is almost generic because no variable among a, b, d appears with the same nonzero exponent in the factorization of two minimal generators of M.

Lemma 5.7. *Let $\mathrm{b}_{k,l}(S/M) = \displaystyle\sum_{(j,m)} \mathrm{b}_{k-j,l/m}(S/M_m)$ be the first structural decomposition of M. Suppose that M is almost generic. Then, each M_m is almost generic.*

Proof. Let G be the minimal generating set of M. By definition, there is an index i such that no variable among $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ appears with the same nonzero exponent in the factorization of two generators in G. Let $G = \{m_1, \ldots, m_s, n_1, \ldots, n_t\}$, where $m_1, \ldots, m_s$ are dominant, and $n_1, \ldots, n_t$ are nondominant. Then, each M_m in the first structural decomposition of M is of the form $M_m = (n'_1, \ldots, n'_t)$. Suppose, by means of contradiction, that some M_m is not almost generic. Then, there is a variable $x \neq x_i$ that appears with the same nonzero exponent α in the factorization of two generators

$n'_a, n'_b \in \{n'_1, \ldots, n'_t\}$. Recall that $n'_a = \dfrac{\text{lcm}(m, n_a)}{m}$, $n'_b = \dfrac{\text{lcm}(m, n_b)}{m}$. Let u, v, w be the exponents with which x appears in the factorizations of m, n_a, n_b, respectively. Note that x appears with exponents $u + \alpha$ and $\max(u, v)$ in the factorizations of mn'_a and $\text{lcm}(m, n_a)$, respectively. Since $mn'_a = \text{lcm}(m, n_a)$, we must have that $u + \alpha = \max(u, v)$. It follows that $v = u + \alpha$. Likewise, x appears with exponents $u + \alpha$ and $\max(u, w)$ in the factorizations of mn'_b and $\text{lcm}(m, n_b)$, respectively. Since $mn'_b = \text{lcm}(m, n_b)$, we must have that $u + \alpha = \max(u, w)$. It follows that $w = u + \alpha$. Combining these identities, we deduce that $v = w$, which implies that x appears with the same nonzero exponent in the factorizations of n_a and n_b, which contradicts the fact that M is almost generic. $\square$

Lemma 5.8. *If M is almost generic, its second structural decomposition has no purely nondominant part.*

Proof. Let G be the minimal generating set of M. The proof is by induction on the cardinality of G.

If $\#G = 1$ or $\#G = 2$, then M is dominant and the theorem holds.

Suppose that the theorem holds whenever $\#G \leq q - 1$. Let us now assume that $\#G = q$. Since M is almost generic, there is an index i such that no variable among $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ appears with the same nonzero exponent in the factorization of two generators in G. Let $j \neq i$, and let k be the greatest exponent with which x_j appears in the factorization of a generator in G. Then there is a unique element in G, divisible by x_j^k, and such an element must be dominant (in x_j). Hence, G can be represented in the form $G = \{m_1, \ldots, m_s, n_1, \ldots, n_t\}$, where $m_1, \ldots, m_s$ are dominant, $n_1, \ldots, n_t$ are nondominant, $s + t = q$, and $s \geq 1$. In particular $t \leq q - 1$. By Lemma 5.7, each M_m in the first structural decomposition of M is almost generic, and since $M_m = (n'_1, \ldots, n'_t)$, M_m is minimally generated by at most $q - 1$ elements. Then, by induction hypothesis, the second structural decomposition of M_m has no purely nondominant part. Finally, the theorem follows from Lemma 5.2. $\square$

Corollary 5.9. *If M is almost generic, it has characteristic Betti numbers.*

Proof. Immediate from Theorem 5.3 and Lemma 5.8. $\square$

Theorem 5.10. *If M is almost generic, it has characteristic Betti numbers in minimal homological degrees.*

Proof. By induction on the cardinality of the minimal generating set G of M.

If $\#G = 1$ or $\#G = 2$, M is dominant, and the Theorem holds. Let us assume that the theorem holds whenever $\#G \leq q - 1$.

Suppose now that $\#G = q$. By Lemma 5.8, M is not purely nondominant. Then, G can be represented in the form $G = \{m_1, \ldots, m_s, n_1, \ldots, n_t\}$, where $m_1, \ldots, m_s$ are

dominant, $n_1, \ldots, n_t$ are nondominant, $s + t = q$, and $s \geq 1$. In particular, $t \leq q - 1$. Let $\mathrm{b}_{k,l}(S/M) = \sum\limits_{(j,m)} \mathrm{b}_{k-j,l/m}(S/M_m)$ be the first structural decomposition of M. Since $M_m = (n'_1, \ldots, n'_t)$, M_m is minimally generated by at most $q - 1$ monomials. By Lemma 5.7, M_m is almost generic. By induction hypothesis, M_m has characteristic Betti numbers in minimal homological degrees. Now, the result follows from Lemma 5.4. $\quad\square$

Theorem 5.11. *If M is 2-semidominant, it has characteristic Betti numbers in minimal homological degrees.*

Proof. Let $G = \{m_1, \ldots, m_q, n_1, n_2\}$ be the minimal generating set of M, where n_1, n_2 are nondominant. Then, the first structural decomposition of M is

$$\mathrm{b}_{k,l}(S/M) = \sum_{(j,m)} \mathrm{b}_{k-j,l/m}(S/M_m),$$

where $M_m = (n'_1, n'_2)$. If $n'_1 = 1$ or $n'_2 = 1$, then $M_m = S$, and $\mathrm{b}_{k-j,l/m}(S/M_m) = 0$. Otherwise, (n'_1, n'_2) is minimally generated by either one or two monomials, which implies that M_m is dominant. Note that the first and second structural decompositions agree, and have no purely nondominant part. By Theorem 5.3, M has characteristic Betti numbers. Moreover, since each M_m is dominant, it has characteristic in minimal homological degrees. Now, the result follows from Lemma 5.4. $\quad\square$

6. Structural decomposition and projective dimension

If the minimal generating set G of M has at least two monomials, and the ring S has n variables, there are two natural bounds for the projective dimension $\mathrm{pd}\,(S/M)$, namely, $2 \leq \mathrm{pd}\,(S/M) \leq n$. In this section we discuss some cases where the lower and upper bounds are achieved.

Hilbert–Burch theorem [5, Theorem 20.15] describes the structure of the minimal resolutions of ideals M, when $\mathrm{pd}(S/M) = 2$. The next theorem gives sufficient conditions for the lower bound $\mathrm{pd}(S/M) = 2$ to be achieved.

Theorem 6.1. *Let M be either 2- or 3-semidominant. Suppose that there exists a minimal generator of M that divides the lcm of every pair of minimal generators of M. Then $\mathrm{pd}(S/M) = 2$.*

Proof. Let $G = \{m_1, \ldots, m_s, n_1, \ldots, n_t\}$ be the minimal generating set of M, where $m_1, \ldots, m_s$ are dominant, and $n_1, \ldots, n_t$ are nondominant. By hypothesis, there is an element n in G that divides the lcm of every pair of elements in G. However, since each m_i is dominant, $m_i \nmid \mathrm{lcm}(n_1, n_2)$. It follows that n must be one of the $n_1, \ldots, n_t$; say $n = n_1$. Let $\mathrm{b}_{k,l}(S/M) = \sum\limits_{(j,m)\in C} \mathrm{b}_{k-j,l/m}(S/M_m)$ be the first structural decomposition

of M. Let $(j, m) \in C$. We will prove that $\mathrm{b}_{k-j,l/m}(S/M_m) = 0$ for all $k \geq 3$, and all monomials l.

First, let us assume that $j \geq 2$. Then there are monomials $m_{i_1}, \ldots, m_{i_j} \in G$ such that $m = \mathrm{lcm}(m_{i_1}, \ldots, m_{i_j})$, and $M_m = (n'_1, \ldots, n'_t)$, where $n'_i = \dfrac{\mathrm{lcm}(m, n_i)}{m}$. In particular, $n_1 \mid \mathrm{lcm}(m_{i_1}, m_{i_2})$, and hence, $n_1 \mid m$. This implies that $\mathrm{lcm}(m, n_1) = m$, and thus $n'_1 = \dfrac{\mathrm{lcm}(m, n_1)}{m} = 1$. Therefore, $M_m = S$, and $\mathrm{b}_{k-j,l/m}(S/M_m) = 0$ for all $k \geq 3$ and all monomials l.

Now, let us assume that $j = 1$. Then $m = m_{i_1}$, and by hypothesis, $n_1 \mid \mathrm{lcm}(m_{i_1}, n_k)$, for all $k = 2, \ldots, t$. Then $\mathrm{lcm}(m_{i_1}, n_1) \mid (m_{i_1}, n_k)$, for all $k = 2, \ldots, t$, and therefore, $n'_1 \mid n'_k$, for all $k = 2, \ldots, t$. This means that $M_m = (n'_1, \ldots, n'_t) = (n'_1)$, and thus, $\mathrm{pd}(S/M_m) \leq 1$. It follows that $\mathrm{b}_{k-j,l/m}(S/M_m) = \mathrm{b}_{k-1,l/m}(S/M_m) = 0$, for all $k \geq 3$, and all monomials l. Finally, suppose that $j = 0$. Then $m = 1$, and $M_m = (n'_1, \ldots, n'_t) = (n_1, \ldots, n_t)$. Since M is either 2-semidominant or 3-semidominant, $t = 2$ or $t = 3$. If $t = 2$, then $M_m = (n_1, n_2)$, and thus $\mathrm{b}_{k-j,l/m}(S/M_m) = \mathrm{b}_{k,l}(S/M_m) = 0$, for all $k \geq 3$, and all monomials l. On the other hand, if $t = 3$, since $n_1 \mid \mathrm{lcm}(n_2, n_3)$, $M_m = (n_1, n_2, n_3)$ is not dominant, and we must have $\mathrm{pd}(S/M_m) \leq 2$. It follows that $\mathrm{b}_{k-j,l}(S/M_m) = \mathrm{b}_{k,l}(S/M_m) = 0$, for all $k \geq 3$, and all monomials l. Therefore, for all $k \geq 3$, and all monomials l the first structural decomposition of M gives $\mathrm{b}_{k,l}(S/M) = \sum\limits_{(j,m) \in C} \mathrm{b}_{k-j,l/m}(S/M_m) = 0$, which means that $\mathrm{pd}(S/M) \leq 2$. However, since $\#G \geq 2$, we must have $\mathrm{pd}(S/M) = 2$. $\square$

Example 6.2. Let

$$m_1 = a^3 c^2 d^2 e^2 f^2 g^2 \qquad m_2 = a^2 b^3 d^2 e^2 f^2 g^2 \qquad m_3 = a^2 b^2 c^3 e^2 f^2 g^2$$

$$m_4 = a^2 b^2 c^2 d^3 f^2 g^2 \qquad m_5 = a^2 b^2 c^2 d e^3 g^2 \qquad m_6 = b^2 c^2 d^2 e^2 g^3$$

$$n_1 = abcdefg \qquad n_2 = a^2 b^2 c^2 d^2 e^2 f \qquad n_3 = a^2 b^2 c^2 d^2 e f^2.$$

Let $M = (m_1, \ldots, m_6)$; $M_2 = (m_1, \ldots, m_6, n_1, n_2)$; $M_3 = (m_1, \ldots, m_6, n_1, n_2, n_3)$. It is clear that M is dominant; M_2 is 2-semidominant, and M_3 is 3-semidominant. Note that n_1 divides the lcm of every pair of monomials in $\{m_1, \ldots, m_6, n_1, n_2, n_3\}$. By Theorem 6.1, $\mathrm{pd}(S/M) = 6$; $\mathrm{pd}(S/M_2) = \mathrm{pd}(S/M_3) = 2$. (We see how adding a few monomials to the minimal generating set can change the projective dimension dramatically.)

The fact that Artinian monomial ideals have maximum projective dimension (in the sense of Hilbert Syzygy theorem) was proven by Charalambous [3] (see also [12, Corollary 21.6]), using the radical of an ideal as main tool. Here we give an alternative proof of this fact that relies entirely on the first structural decomposition.

Theorem 6.3. *If M is Artinian in $S = k[x_1, \ldots, x_n]$, then $\mathrm{pd}(S/M) = n$.*

Proof. By induction on n. If $n = 1$, the result is trivial. Suppose that $\operatorname{pd}(S/M) = n - 1$, for Artinian ideals M in $n - 1$ variables.

Let M be Artinian in $S = k[x_1, \ldots, x_n]$. Then, for each $i = 1, \ldots, n$, the minimal generating set G of M contains a monomial $x_i^{\alpha_i}$, $\alpha_i \geq 1$. Notice that each $x_i^{\alpha_i}$ is dominant. Let $\mathrm{b}_{k,l}(S/M) = \sum_{(j,m) \in C} \mathrm{b}_{k-j,l/m}(S/M_m)$ be the third structural decomposition of M, where $m_1 = x_1^{\alpha_1}$, and $H = G \setminus \{x_1^{\alpha_1}\}$. Since $x_1^{\alpha_1}$ is dominant, $(1, x_1^{\alpha_1}) \in C$. Thus, $\mathrm{b}_{k-1,l/x_1^{\alpha_1}}(S/M_{x_1^{\alpha_1}})$ is one of the terms on the right side of the third structural decomposition.

Let $G = \{x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}, l_1, \ldots, l_q\}$. By construction,

$$
M_{x_1^{\alpha_1}} = \left(\frac{\operatorname{lcm}(x_1^{\alpha_1}, x_2^{\alpha_2})}{x_1^{\alpha_1}}, \ldots, \frac{\operatorname{lcm}(x_1^{\alpha_1}, x_n^{\alpha_n})}{x_1^{\alpha_1}}, \frac{\operatorname{lcm}(x_1^{\alpha_1}, l_1)}{x_1^{\alpha_1}}, \ldots, \frac{\operatorname{lcm}(x_1^{\alpha_1}, l_q)}{x_1^{\alpha_1}} \right)
$$
$$
= (x_2^{\alpha_2}, \ldots, x_n^{\alpha_n}, l_1', \ldots, l_q').
$$

Since $x_1^{\alpha_1}$ is dominant, x_1 does not appear in the factorization of l_i'. Thus, $M_{x_1^{\alpha_1}}$ is an Artinian ideal in $n - 1$ variables. By induction hypothesis, $\operatorname{pd}(S/M_{x_1^{\alpha_1}}) = n - 1$. Therefore, there is a monomial l, such that $\mathrm{b}_{n-1,l}(S/M_{x_1^{\alpha_1}}) \neq 0$. Let $l' = l x_1^{\alpha_1}$. Then, $\mathrm{b}_{n-1,l'/x_1^{\alpha_1}}(S/M_{x_1^{\alpha_1}}) \neq 0$. Finally,

$$
\mathrm{b}_{n,l'}(S/M) = \sum_{(j,m) \in C \setminus \{(1,x_1^{\alpha_1})\}} \mathrm{b}_{k-j,l'/m}(S/M_m) + \mathrm{b}_{n-1,l'/x_1^{\alpha_1}}(S/M_{x_1^{\alpha_1}}) \neq 0,
$$

which implies that $\operatorname{pd}(S/M) = n$. $\quad\square$

Theorem 6.4. *Let M be Artinian in $S = k[x_1, \ldots, x_n]$. Let $\mathbb{F}_M$ be a minimal resolution of S/M, obtained from $\mathbb{T}_M$ by means of consecutive cancellations. Then there is a basis element $[\sigma]$ of $\mathbb{F}_M$, such that $\operatorname{hdeg}[\sigma] = n$, and $\operatorname{mdeg}[\sigma]$ is divisible by each variable $x_1, \ldots, x_n$.*

Proof. By Theorem 6.3, there is a basis element $[\sigma]$ in the basis of $\mathbb{F}_M$, such that $\operatorname{hdeg}[\sigma] = n$.

By means of contradiction, suppose that the set $\{x_{j_1}, \ldots, x_{j_i}\}$ of all variables dividing $\operatorname{mdeg}[\sigma]$ is a proper subset of $\{x_1, \ldots, x_n\}$; that is $i \leq n - 1$.

Let $m = \operatorname{mdeg}[\sigma]$; let G be the minimal generating set of M, and let M_m be the ideal generated by $\{l \in G : l \mid m\}$. By [8, Theorem 2.1], there is a subcomplex $(\mathbb{F}_M)_{\leq m}$ of $\mathbb{F}_M$, such that $[\sigma]$ is a basis element of $(\mathbb{F}_M)_{\leq m}$, and $(\mathbb{F}_M)_{\leq m}$ is a minimal resolution of S/M_m. Therefore, $\operatorname{pd}(S/M_m) \geq \operatorname{hdeg}[\sigma] = n$. However, since M_m is a monomial ideal in $k[x_{j_1}, \ldots, x_{j_i}]$, it follows from Hilbert Syzygy theorem that $\operatorname{pd}(S/M_m) \leq i \leq n - 1$, a contradiction.

We conclude that $\operatorname{mdeg}[\sigma]$ is divisible by $x_1, \ldots, x_n$. $\quad\square$

Now we are ready to prove Charalambous theorem.

Theorem 6.5. *Let M be Artinian in $S = k[x_1, \ldots, x_n]$. Then, for all $i = 0, \ldots, n$, $\mathrm{b}_i(S/M) \geq \binom{n}{i}$.*

Proof. Let $0 \leq i \leq n$. Let $\mathbb{X}_i$ be the class of all subsets of $\{x_1, \ldots, x_n\}$, of cardinality i. Then $\#\mathbb{X}_i = \binom{n}{i}$. Let $\{x_{j_1}, \ldots, x_{j_i}\} \in \mathbb{X}_i$. Since M is Artinian, its minimal generating set G contains generators of the form $x_{j_1}^{\alpha_{j_1}}, \ldots, x_{j_i}^{\alpha_{j_i}}$. Let $m = \mathrm{lcm}(x_{j_1}^{\alpha_{j_1}}, \ldots, x_{j_i}^{\alpha_{j_i}})$. Let G_m be the set of all minimal generators of G dividing m, and let M_m be the ideal generated by G_m. Then M_m is Artinian in $k[x_{j_1}, \ldots, x_{j_i}]$, and by Theorem 6.4, there is a basis element $[\sigma_{j_1, \ldots, j_i}]$ of a minimal resolution $(\mathbb{F}_M)_{\leq m}$ of S/M_m, such that $\mathrm{hdeg}[\sigma_{j_1, \ldots, j_i}] = i$, and $\mathrm{mdeg}[\sigma_{j_1, \ldots, j_i}]$ is divisible by $x_{j_1}, \ldots, x_{j_i}$. Since M_m is an ideal in $k[x_{j_1}, \ldots, x_{j_i}]$, $\mathrm{mdeg}[\sigma_{j_1, \ldots, j_i}]$ is not divisible by any variable of $\{x_1, \ldots, x_n\} \setminus \{x_{j_1}, \ldots, x_{j_i}\}$. By [8, Theorem 2.1], $(\mathbb{F}_M)_{\leq m}$ can be regarded as a subcomplex of a minimal resolution $\mathbb{F}_M$ of S/M. Therefore, $[\sigma_{j_1, \ldots, j_i}]$ is a basis element of $\mathbb{F}_M$. Notice that if $\{x_{j_1}, \ldots, x_{j_i}\}$ and $\{x_{k_1}, \ldots, x_{k_i}\}$ are different elements of $\mathbb{X}_i$, the basis elements $[\sigma_{j_1, \ldots, j_i}]$ and $[\sigma_{k_1, \ldots, k_i}]$, determined by these sets must be different too. In fact, the sets of variables dividing $\mathrm{mdeg}[\sigma_{j_1, \ldots, j_i}]$ and $\mathrm{mdeg}[\sigma_{k_1, \ldots, k_i}]$ are $\{x_{j_1}, \ldots, x_{j_i}\}$ and $\{x_{k_1}, \ldots, x_{k_i}\}$, respectively. It follows that $\mathrm{b}_i(S/M) \geq \#\mathbb{X}_i = \binom{n}{i}$. $\square$

Theorem 6.6. *Let M be an ideal in $S = k[x_1, \ldots, x_n]$, with minimal generating set G. Suppose that for all $i = 1, \ldots, n$ there are integers $\alpha_i \geq 0$, $\beta_i \geq 1$, such that the set $G' = \{x_1^{\alpha_1} x_1^{\beta_1}, x_1^{\alpha_2} x_2^{\beta_2}, \ldots, x_1^{\alpha_n} x_n^{\beta_n}\}$ is contained in G. Then $\mathrm{pd}(S/M) = n$.*

Proof. Let $G = G' \cup \{l_1, \ldots, l_q\}$. Let $\mathrm{b}_{k,l}(S/M) = \sum_{(j,m) \in C} \mathrm{b}_{k-j,l/m}(S/M_m)$ be the first structural decomposition of M. Let $\gamma = \alpha_1 + \beta_1$. Then $x_1^{\gamma} = x_1^{\alpha_1} x_1^{\beta_1}$ is a minimal generator in G. Notice that the exponent γ with which x_1 appears in the factorization of x_1^{γ}, must be larger than the exponent with which x_1 appears in the factorization of any other minimal generator l in G; otherwise, l would be a multiple of x_1^{γ}. Hence, x_1^{γ} is dominant in G; which implies that $(1, x_1^{\gamma}) \in C$. Thus, $\mathrm{b}_{k-1, l/x_1^{\gamma}}(S/M_{x_1^{\gamma}})$ is one of the terms on the right side of the first structural decomposition. Now,

$$M_{x_1^{\gamma}} = \left(\frac{\mathrm{lcm}(x_1^{\gamma}, x_1^{\alpha_2} x_2^{\beta_2})}{x_1^{\gamma}}, \ldots, \frac{\mathrm{lcm}(x_1^{\gamma}, x_1^{\alpha_n} x_n^{\beta_n})}{x_1^{\gamma}}, \frac{\mathrm{lcm}(x_1^{\gamma}, l_1)}{x_1^{\gamma}}, \ldots, \frac{\mathrm{lcm}(x_1^{\gamma}, l_q)}{x_1^{\gamma}} \right)$$

$$= (x_2^{\beta_2}, \ldots, x_n^{\beta_n}, l_1', \ldots, l_q'),$$

where $l_i' = \dfrac{\mathrm{lcm}(x_1^{\gamma}, l_i)}{x_1^{\gamma}}$.

Since x_1^{γ} is dominant, x_1 does not appear in the factorization of l_i'. Thus, $M_{x_1^{\gamma}}$ is an Artinian monomial ideal in $k[x_2, \ldots, x_n]$. It follows that $\mathrm{pd}(S/M_{x_1^{\gamma}}) = n - 1$. Therefore, there is a monomial l, such that $\mathrm{b}_{n-1,l}(S/M_{x_1^{\gamma}}) \neq 0$. Let $l' = l x_1^{\gamma}$. Then, $\mathrm{b}_{n-1, l'/x_1^{\gamma}}(S/M_{x_1^{\gamma}}) \neq 0$. Finally,

$$\mathrm{b}_{n,l'}(S/M) = \sum_{(j,m)\in C\setminus\{(1,x_1^\gamma)\}} \mathrm{b}_{n-j,l'/m}(S/M_m) + \mathrm{b}_{n-1,l'/x_1^\gamma}(S/M_{x_1^\gamma}) \neq 0,$$

which implies that $\mathrm{pd}(S/M) = n$. $\square$

Example 6.7. Let $M = (x_1^3, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2x_4, x_3x_5)$. Then, the subset $G' = \{x_1^3, x_1x_2, x_1x_3, x_1x_4, x_1x_5\}$ of the minimal generating set of M, satisfies the hypotheses of Theorem 6.6. Hence, $\mathrm{pd}(S/M) = 5$.

The hypothesis of Theorem 6.6 is more common than it may seem. For instance, Artinian ideals satisfy this condition. The hypothesis of Theorem 6.6 is also satisfied if M is the smallest Borel ideal containing a monomial $x_1^{\alpha_1}\ldots x_n^{\alpha_n}$, with $\alpha_n \geq 1$ (such an M is sometimes called principal Borel. A comprehensive study of principal Borel ideals can be found in [13]). Below we prove our last assertion.

A monomial m' is said to be in the **big shadow** of a monomial m if $m' = \dfrac{x_i m}{x_j}$, for some x_j dividing m and some $i < j$.

M is a **Borel ideal** if and only if any of the two equivalent conditions hold:

(i) for every $m \in M$, M contains all monomials in the big shadow of m.
(ii) M contains all monomials in the big shadows of its minimal generators.

(See [12, Pages 109 and 110].)

Lemma 6.8. *Let* $m = x_1^{\alpha_1}\ldots x_n^{\alpha_n}$, *with* $\alpha_n \geq 1$. *Let* M *be the smallest monomial ideal containing* m. *Let* $G = \{x_1^{\beta_1}\ldots x_n^{\beta_n} : \sum_{i=1}^n \beta_i = \sum_{i=1}^n \alpha_i,$ *and* $\sum_{i=k}^n \beta_i \leq \sum_{i=k}^n \alpha_i,$ *for all* $k = 2,\ldots,n\}$.

Then for each $m' \in G$, *either* $m' \in M$ *or there exists* $m'' \in G$ *such that* m' *is in the big shadow of* m''.

Proof. Let $m' = x_1^{\beta_1}\ldots x_n^{\beta_n} \in G$. Suppose that for all k, $\beta_k \geq \alpha_k$. If for one particular value of k we had that $\beta_k > \alpha_k$, then $\sum_{i=k}^n \beta_i > \sum_{i=k}^n \alpha_i$, which contradicts the definition of G. It follows that $\beta_1 = \alpha_1,\ldots,\beta_n = \alpha_n$, and thus, $m' = m \in M$. Now suppose there is an index k such that $\beta_k < \alpha_k$, and let $r = \max\{k : \beta_k < \alpha_k\}$. Then for all $k = r+1,\ldots,n$, $\beta_k \geq \alpha_k$ and, since $\sum_{i=r+1}^n \beta_i \leq \sum_{i=r+1}^n \alpha_i$, we must have that $\beta_k = \alpha_k$ for all $k = r+1,\ldots,n$. Since $\sum_{i=1}^n \beta_i = \sum_{i=1}^n \alpha_i$, there must be an index $1 \leq s < r$, such that $\beta_s > \alpha_s$; in particular, $\beta_s \geq 1$. Let $m'' = x_1^{\gamma_1}\ldots x_n^{\gamma_n}$, where $\gamma_s = \beta_s - 1, \gamma_r = \beta_r + 1$, and $\gamma_k = \beta_k$, if $k \neq s, r$. Then $\sum_{i=1}^n \gamma_i = \sum_{i=1}^n \beta_i = \sum_{i=1}^n \alpha_i$; also, for all $k = 2,\ldots,s$, and all $k = r+1,\ldots,n$, $\sum_{i=k}^n \gamma_i = \sum_{i=k}^n \beta_i \leq \sum_{i=k}^n \alpha_i$; and for all $k = s+1,\ldots,r$, $\sum_{i=k}^n \gamma_i < 1 + \sum_{i=k}^n \gamma_i =$

$\sum\limits_{i=k}^{n} \beta_i \leq \sum\limits_{i=k}^{n} \alpha_i$. Therefore, $m'' \in G$. Moreover, $m' = \dfrac{x_s m''}{x_r}$, which implies that m' is in the big shadow of m''. $\square$

Theorem 6.9. *Let $m = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, with $\alpha_n \geq 1$. Let M be the smallest monomial ideal containing m. Then the minimal generating set of M is $G = \{x_1^{\beta_1} \ldots x_n^{\beta_n} : \sum\limits_{i=1}^{n} \beta_i = \sum\limits_{i=1}^{n} \alpha_i,$ and $\sum\limits_{i=k}^{n} \beta_i \leq \sum\limits_{i=k}^{n} \alpha_i$, for all $k = 2, \ldots, n\}$.*

Proof. First, we will prove that $G \subseteq M$. Let $m_1 \in G$. Suppose, by means of contradiction, that $m_1 \notin M$. By Lemma 6.8, there exists $m_2 \in G$, such that m_1 is in the big shadow of m_2. If m_2 were an element of M, m_1 would also be in M, since M is Borel. It follows that $m_1, m_2 \in S \setminus M$. Suppose that $m_1, \ldots, m_k$ are monomials of $S \setminus M$ such that, for all $i = 1, \ldots, k-1$, m_i is in the big shadow of m_{i+1}. By Lemma 6.8, there is a monomial $m_{k+1} \in G$ such that m_k is in the big shadow of m_{k+1}. If m_{k+1} were an element of M, then m_k would also be in M, since M is Borel. Hence, $m_{k+1} \in S \setminus M$. Thus, by recurrence, we can construct an infinite sequence $m_1, m_2, \ldots, m_k, \ldots$ of elements of G, such that each m_i is in the big shadow of m_{i+1}, and each m_i is in $S \setminus M$. But this is not possible as G is a finite set; a contradiction. We conclude that $m_1 \in M$, and then $G \subseteq M$. Moreover, $< G > \subseteq M$.

Let us now show that $< G > = M$. Let $m'' = x_1^{\beta_1} \ldots x_n^{\beta_n}$ be an element of G. Suppose that a monomial m' is in the big shadow of m''. That is there is a variable x_r dividing m'', and $s < r$, such that $m' = \dfrac{x_s m''}{x_r}$. Therefore, m' is of the form $m' = x_1^{\gamma_1} \ldots x_n^{\gamma_n}$, where $\gamma_s = \beta_s + 1$, $\gamma_r = \beta_r - 1$, and $\gamma_k = \beta_k$, if $k \neq s, r$. It easy to check (and analogous to what we did in the proof of Lemma 6.8) that $m' \in G$. Therefore, the ideal $< G >$ contains all monomials in the big shadow of its generators. It follows that G is Borel. Moreover, since $m \in G$, and $< G > \subseteq M$, $< G > = M$. $\square$

Corollary 6.10. *Let $m = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, with $\alpha_n \geq 1$. Let $M = < G >$ be the smallest Borel ideal containing m. Let $\alpha = \alpha_1 + \ldots + \alpha_n$. Then the minimal generating set G contains the subset $G' = \{x_1^{\alpha-1} x_i : i = 1, \ldots, n\}$.*

Proof. Let $m' = x_1^{\beta_1} \ldots x_n^{\beta_n}$, where $\beta_1 = \alpha - 1$, $\beta_r = 1$, and $\beta_i = 0$ for all $i \neq 1, r$. Then $\sum\limits_{i=1}^{n} \beta_i = \alpha = \sum\limits_{i=1}^{n} \alpha_i$. On the other hand, if $2 \leq k \leq n$, then $\sum\limits_{i=k}^{n} \beta_i \leq 1 \leq \sum\limits_{i=k}^{n} \alpha_i$. Therefore, $m' = x_1^{\alpha-1} x_r \in G$. $\square$

7. Structural decomposition in context

In this section we set the structural decomposition in context by explaining how it relates to some of the important concepts in the field. Specifically, we describe connections

to the lcm-lattice [9], and to two combinatorially defined classes of ideals: rigid ideals [4], and homologically monotonic ideals [7].

For the sake of clarity we reintroduce the notation adopted in Sections 3 and 4. Let M be a monomial ideal with minimal generating set $G = \{m_1, \ldots, m_q, n_1, \ldots, n_p\}$, where $q \geq 1$, $p \geq 0$, $m_1, \ldots, m_q$ are dominant, and $n_1, \ldots, n_p$ are nondominant. Let $1 \leq d \leq q$, and let $H = \{m_{d+1}, \ldots, m_q, n_1, \ldots, n_p\}$. Then G can be expressed in the form $G = \{m_1, \ldots, m_d, h_1, \ldots, h_c\}$, where $H = \{h_1, \ldots, h_c\}$. Let $C = \{(j, m) \in \mathbb{Z}^+ \times S$: there are integers $1 \leq r_1 < \ldots < r_j \leq d$, such that $m = \operatorname{lcm}(m_{r_1}, \ldots, m_{r_j})\} \cup \{(0, 1)\}$. For each $(j, m) \in C$, let $M_m = (h_1', \ldots, h_c')$, where $h_i' = \dfrac{\operatorname{lcm}(m, h_i)}{m}$.

Theorem 7.1. *For every $(j, m) \in C$, let Λ and Γ_m be the* lcm-*lattices of M and M_m, respectively, and let $\Lambda_m = \{w \in \Lambda$: for each $i = 1, \ldots, d$, $m_i \mid w$ if and only if $m_i \mid m\}$. Then,*

(i) Each Λ_m is a sublattice of Λ.

(ii) The application

$$\rho : \Gamma_m \to \Lambda_m$$
$$w \to mw$$

is an injection that preserves lcm *and* gcd*; that is, $\rho(\operatorname{lcm}(w, w')) = \operatorname{lcm}(\rho(w), \rho(w'))$, and $\rho(\gcd(w, w')) = \gcd(\rho(w), \rho(w'))$.*

(iii) Λ is the disjoint union of all the Λ_m; that is, $\Lambda = \bigcup\limits_{(j,m) \in C} \Lambda_m$, and $\Lambda_m \cap \Lambda_{m'} = \varnothing$ if $(j, m) \neq (i, m')$.

Proof. (i) If $(j, m) = (0, 1)$, then $\Lambda_m = \Lambda_1 = \{\operatorname{lcm}(A) : A \subseteq \{h_1, \ldots, h_c\}\}$; that is, Λ_m is the lcm-lattice of $(h_1, \ldots, h_c)$, which is a sub-lattice of Λ.

Suppose now that $(j, m) \neq (0, 1)$. Then there are integers $1 \leq r_1 < \ldots < r_j \leq d$, such that $m = \operatorname{lcm}(m_{r_1}, \ldots, m_{r_j})$. Let $1 \leq i \leq d$. Since m_i is dominant (in G), $m_i \mid m$ if and only if $m_i \in \{m_{r_1}, \ldots, m_{r_j}\}$. Let $w, w' \in \Lambda_m \subseteq \Lambda$. Since Λ is the lcm-lattice of M, w, w' have a unique join and a unique meet in Λ; namely, $\operatorname{lcm}(w, w')$ and $\gcd(w, w')$, respectively. We will show that $\operatorname{lcm}(w, w'), \gcd(w, w') \in \Lambda_m$. Let $m_k \in \{m_1, \ldots, m_d\} \setminus \{m_{r_1}, \ldots, m_{r_j}\}$. Since m_k is dominant in G, there is a variable x, such that the exponent α with which x appears in the factorization of m_k is larger than the exponent with which x appears in the factorization of each monomial of $G \setminus \{m_k\}$. Hence, the exponents with which x appears in the factorizations of w and w' are strictly less than α. It follows that neither $\operatorname{lcm}(w, w')$ nor $\gcd(w, w')$ is divisible by m_k. On the other hand, since both w and w' are divisible by each of $m_{r_1}, \ldots, m_{r_j}$, it follows that both $\operatorname{lcm}(w, w')$ and $\gcd(w, w')$ are divisible by each of $m_{r_1}, \ldots, m_{r_j}$. Hence, $\operatorname{lcm}(w, w'), \gcd(w, w') \in \Lambda_m$.

(ii) Let w, w' be elements of Γ_m. Then there are sequences $1 \leq s_1 < \ldots < s_u \leq c$, and $1 \leq t_1 < \ldots < t_v \leq c$, such that $w = \operatorname{lcm}(h_{s_1}', \ldots, h_{s_u}')$, and $w' = \operatorname{lcm}(h_{t_1}', \ldots, h_{t_v}')$.

By Proposition 3.2, $w = \operatorname{lcm}(h'_{s_1}, \ldots, h'_{s_u}) = \dfrac{1}{m} \operatorname{lcm}(m_{r_1}, \ldots, m_{r_j}, h_{s_1}, \ldots, h_{s_u})$, and $w' = \operatorname{lcm}(h'_{t_1}, \ldots, h'_{t_v}) = \dfrac{1}{m} \operatorname{lcm}(m_{r_1}, \ldots, m_{r_j}, h_{t_1}, \ldots, h_{t_v})$. Therefore,

$$\rho(\operatorname{lcm}(w, w')) =$$
$$= m \operatorname{lcm}(w, w')$$
$$= m \operatorname{lcm}\left(\frac{1}{m} \operatorname{lcm}(m_{r_1}, \ldots, m_{r_j}, h_{s_1}, \ldots, h_{s_u}), \frac{1}{m} \operatorname{lcm}(m_{r_1}, \ldots, m_{r_j}, h_{t_1}, \ldots, h_{t_v})\right)$$
$$= \operatorname{lcm}\left(\operatorname{lcm}(m_{r_1}, \ldots, m_{r_j}, h_{s_1}, \ldots, h_{s_u}), \operatorname{lcm}(m_{r_1}, \ldots, m_{r_j}, h_{t_1}, \ldots, h_{t_v})\right)$$
$$= \operatorname{lcm}(mw, mw')$$
$$= \operatorname{lcm}(\rho(w), \rho(w')).$$

Also, $\rho(gcd(w, w')) = m \gcd(w, w') = m \dfrac{ww'}{\operatorname{lcm}(w, w')} = \dfrac{mwmw'}{m \operatorname{lcm}(w, w')} = \dfrac{\rho(w)\rho(w')}{\rho(\operatorname{lcm}(w, w'))} = \dfrac{\rho(w)\rho(w')}{\operatorname{lcm}(\rho(w), \rho(w'))} = \gcd(\rho(w), \rho(w'))$.

In addition, if $w \neq w'$, then $\rho(w) = mw \neq mw' = \rho(w')$.

(iii) Let $(j, m) \neq (i, m')$. Then there are sequences $1 \leq r_1 < \ldots < r_j \leq d$ and $1 \leq s_1 < \ldots < s_i \leq d$, such that $m = \operatorname{lcm}(m_{r_1}, \ldots, m_{r_j})$ and $m' = \operatorname{lcm}(m_{s_1}, \ldots, m_{s_i})$. Suppose that $m = m'$. Then each m_{s_k} divides $\operatorname{lcm}(m_{r_1}, \ldots, m_{r_j})$ and, since m_{s_k} is dominant, we must have that $m_{s_k} \in \{m_{r_1}, \ldots, m_{r_j}\}$. Likewise, we can conclude that $m_{r_k} \in \{m_{s_1}, \ldots, m_{s_i}\}$. Hence, $\{m_{r_1}, \ldots, m_{r_j}\} = \{m_{s_1}, \ldots, m_{s_i}\}$, and $i = j$. In other words, if $m = m'$, then $(j, m) = (i, m')$. Since we are under the assumption that $(j, m) \neq (i, m')$, it follows that $m \neq m'$. Then, m and m' must be of the form $m = \operatorname{lcm}(m_{r_1}, \ldots, m_{r_j})$; $m' = \operatorname{lcm}(m_{s_1}, \ldots, m_{s_i})$, where one of $\{m_{r_1}, \ldots, m_{r_j}\}$, $\{m_{s_1}, \ldots, m_{s_i}\}$ is not contained in the other. Without loss of generality, we may assume that there is $m_r \in \{m_{r_1}, \ldots, m_{r_j}\}$, such that $m_r \notin \{m_{s_1}, \ldots, m_{s_i}\}$. Let $w \in \Lambda_m$. By definition of Λ_m, $m_r \mid w$. By definition of $\Lambda_{m'}$, no element of $\Lambda_{m'}$ is divisible by m_r. It follows that $w \in \Lambda_m \setminus \Lambda_{m'}$. We have shown that the sublattices Λ_m of Λ are disjoint. Finally, if $w \in \Lambda$, then w is of the form $w = \operatorname{lcm}(m_{a_1}, \ldots, m_{a_u}, h_{b_1}, \ldots, h_{b_v})$. Let $m_a = \operatorname{lcm}(m_{a_1}, \ldots, m_{a_u})$. Then $(u, m_a) \in C$, and $w \in \Lambda_{m_a}$. Thus, $\Lambda = \bigcup_{(j,m) \in C} \Lambda_m$. $\square$

According to Theorem 4.1, the Betti numbers of S/M can be computed in terms of the Betti numbers of the ideals M_m. Since the lcm-lattices Γ_m of the ideals M_m can be embedded in the sublattices Λ_m of Λ (that is $\Gamma_m \cong \rho(\Gamma_m) \subseteq \Lambda_m$), and given that the Λ_m are disjoint, it follows from Theorem 7.1 that in order to compute the Betti numbers of S/M we can use the collection of disjoint sublattices $\rho(\Gamma_m)$ of Λ.

In the particular case when the structural decomposition gives the Betti numbers of S/M in terms of the Betti numbers of dominant ideals, the lcm-lattice can be used as a practical tool to simplify computations. We illustrate this point in the next example.
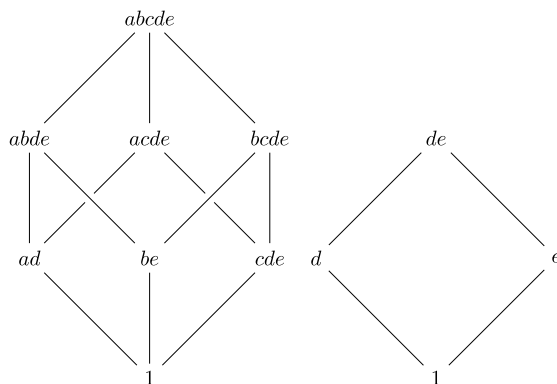
Fig. 1. The lcm-lattices Γ_1 and Γ_{a^2bc}.

Example 7.2. Let $M = (a^2bc, ad, be, cde)$. Let us compute the first structural decomposition of M. Note that a^2bc is the only dominant generator of M. Thus, $C = \{(0,1), (1, a^2bc)\}$. The pair $(0,1)$ determines the ideal $M_1 = (ad, be, cde)$. The pair $(1, a^2bc)$ determines the ideal $M_{a^2bc} = (d, e, de) = (d, e)$. Hence, the first structural decomposition of M is

$$\mathrm{b}_{k,l}(S/M) = \mathrm{b}_{k-0, \frac{l}{1}}(S/M_1) + \mathrm{b}_{k-1, \frac{l}{a^2bc}}(S/M_{a^2bc})$$

Since the ideals M_1 and M_{a^2bc} are dominant, the computations are easy. However, there is an even easier and faster way to compute the $\mathrm{b}_{k,l}(S/M)$. Since M_1 and M_{a^2bc} are dominant, their lcm-lattices Γ_1, Γ_{a^2bc} (described in Fig. 1), are Boolean. Thus, the labels of Γ_1 and Γ_{a^2bc} are the multidegrees of the basis elements of M_1 and M_{a^2bc}, respectively.

For instance, the label d of Γ_{a^2bc} is an atom. Thus, there must be an element $[\tau]$ in the basis of the minimal resolution of S/M_{a^2bc} such that $\mathrm{hdeg}[\tau] = 1$, and $\mathrm{mdeg}[\tau] = d$. According to the first structural decomposition of M, there is an element $[\sigma]$ in the basis of the minimal resolution of S/M such that $\mathrm{hdeg}[\sigma] = 1+1$ and $\mathrm{mdeg}[\sigma] = a^2bcd$. (Notice that the components of $(1, a^2bcd)$ indicate a shift in homological degree and multidegree, respectively.)

Once we understand this reasoning, we can compute the $\mathrm{b}_{k,l}(S/M)$ very quickly, as follows. The labels $1, d, e, de$ of Γ_{a^2bc} determine the following multigraded Betti numbers:

$$\mathrm{b}_{1,a^2bc}(S/M) = \mathrm{b}_{2,a^2bcd}(S/M) = \mathrm{b}_{2,a^2bce}(S/M) = \mathrm{b}_{3,a^2bcde}(S/M) = 1.$$

The labels $1, ad, be, cde, abde, bcde, abcde$ of Γ_1 determine the following multigraded Betti numbers:

$$\mathrm{b}_{0,1}(S/M) = \mathrm{b}_{1,ad}(S/M) = \mathrm{b}_{1,be}(S/M) = \mathrm{b}_{1,cde}(S/M) = \mathrm{b}_{2,abde}(S/M)$$
$$= \mathrm{b}_{2,bcde}(S/M) = \mathrm{b}_{2,acde}(S/M) = \mathrm{b}_{3,abcde}(S/M) = 1.$$

We leave the details to the reader.

We now turn our attention to monomial ideals with characteristic Betti numbers, a class that originated in our study of the structural decomposition.

Proposition 7.3. *M has characteristic Betti numbers if and only if for each multidegree m, $\sum_i \mathrm{b}_{i,m}(S/M)$ is either 0 or 1.*

Proof. Recall that the minimal resolution of S/M can be obtained from $\mathbb{T}_M$ by doing consecutive cancellations. Since each consecutive cancellation eliminates a pair of basis elements of equal multidegree, the number of basis elements of a given multidegree m in the new resolution will either be the same as the number of basis elements of multidegree m in the preceding resolution, or it will decrease by 2. In either case, we observe that the parity of the number of basis elements of multidegree m does not change by doing consecutive cancellations. In other words, if the number of basis elements of $\mathbb{T}_M$ of multidegree m is even (respectively, odd), then the number of basis elements of multidegree m of the minimal resolution of S/M is also even (respectively, odd).

Suppose that $\sum_i \mathrm{b}_{i,m}(S/M)$ is either 0 or 1 for each multidegree m. Then, in light of the previous observation, when the number of basis elements of multidegree m in $\mathbb{T}_M$ is even (respectively, odd), $\sum_i \mathrm{b}_{i,m}(S/M) = 0$ (respectively, $\sum_i \mathrm{b}_{i,m}(S/M) = 1$). That is, M has characteristic Betti numbers. The converse is obvious. □

The following class of ideals was studied in [4].

Definition 7.4. A monomial ideal M is called **rigid** if it satisfies the following two properties:

(1) $\mathrm{b}_{i,m}(S/M)$ is either 0 or 1 for all i and all multidegrees m.
(2) If $m \neq m'$, and $\mathrm{b}_{i,m}(S/M) = 1 = \mathrm{b}_{i,m'}(S/M)$, then $m \nmid m'$ and $m' \nmid m$.

Proposition 7.5. *If M is rigid then M has characteristic Betti numbers.*

Proof. Let M be rigid. Suppose that there is a monomial m, and integers $i < j$, such that $\mathrm{b}_{i,m}(S/M) = 1 = \mathrm{b}_{j,m}(S/M)$. Choose a basis element of the minimal resolution of S/M with homological degree j and multidegree m, and apply the differential map. The result is an S-linear combination of basis elements in homological degree $j - 1$, each with multidegree strictly dividing m. Pick one such basis element and repeat. After $(j - i)$ steps we will have found a basis element in homological degree i with multidegree m' strictly dividing m. Thus, $m' \neq m$, $m' \mid m$, and $\mathrm{b}_{i,m'}(S/M) = 1 = \mathrm{b}_{i,m}(S/M)$; a contradiction. We have proven that two basis elements of the minimal resolution of S/M in different homological degrees must have different multidegrees. Combining this with the fact that each $\mathrm{b}_{i,m}(S/M)$ is either 0 or 1, yields the following: for each multidegree m, $\sum_i \mathrm{b}_{i,m}(S/M)$ is either 0 or 1. The result now follows from Proposition 7.3. □

We are interested in one last family of ideals which extends the class of rigid ideals. The following concept was first introduced (with more generality) in [7].

Definition 7.6. We say that M is homologically monotonic (HM) if it satisfies the following: for every i and every pair $m \neq m'$ such that $m \mid m'$, either $b_{i,m}(S/M) = 0$ or $b_{i,m'}(S/M) = 0$.

The classes of Scarf, rigid, and HM ideals are related as follows: Scarf $\subsetneq$ Rigid $\subsetneq$ HM [7, Section 4]. Proposition 7.5, combined with the fact that M_1 in Example 7.7 has characteristic Betti numbers but is not rigid, tells us that Scarf $\subsetneq$ Rigid $\subsetneq$ Characteristic Betti Numbers.

The next example shows that the classes HM and Characteristic Betti numbers are incomparable.

Example 7.7. Let $M_1 = (a^2b^2, a^2c, bc)$. Then M_1 has characteristic Betti numbers, but M_1 is not HM because a^2bc strictly divides a^2b^2c, and $b_{2,a^2bc} = 1 = b_{2,a^2b^2c}$.

On the other hand, $M_2 = (ab, ac, bc)$ is HM but does not have characteristic Betti numbers, for $b_{2,abc} = 2$.

8. Conclusion

We close this article with some remarks, questions, and conjectures.

The structural decomposition is one of the very few techniques that allow us to compute Betti numbers by hand, not for arbitrary monomial ideals, but for a wide class of them. As a matter of fact, the ideal M in Example 4.6, is minimally generated by 7 monomials and even so we were able to compute the numbers $b_{k,l}(S/M)$. (Starting with $\mathbb{T}_M$, we could also calculate the $b_{k,l}(S/M)$ by means of consecutive cancellations. But since the basis of $\mathbb{T}_M$ contains $\sum \binom{7}{i} = 128$ elements, and the basis of the minimal resolution of S/M contains 20 elements, we should make $\dfrac{128 - 20}{2} = 54$ consecutive cancellations, which obviously requires the use of software.)

On a different note, the structural decomposition of an ideal M generates a finite family $\{M_m\}$ of ideals that usually has these two properties: a) the minimal generating set of each M_m has smaller cardinality than the minimal generating set of M; b) if M is an ideal in $S = k[x_1, \ldots, x_n]$, then M_m is an ideal in a polynomial ring with less than n variables. As a consequence, the structural decomposition works well when one wants to prove facts by induction. Theorems 5.3 and 5.10, for instance, are proven by induction on the cardinality of the minimal generating set. On the other hand, Theorem 6.3 is proven by induction on the number of variables.

We close this article with some conjectures.

Conjecture 8.1. *Suppose that for some ideal M, there are indices $i < j$ such that no variable among $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n$ appears with the same nonzero*

exponent in the factorization of two minimal generators. Then M has characteristic Betti numbers.

The definition of purely nondominant ideal involves many overlaps. By this we mean that it is common to find that one variable appears with the same nonzero exponent in the factorization of two minimal generators. On the other hand, the hypothesis of Conjecture 8.1 prevents overlaps (unless the variables are x_i or x_j). Thus, following our intuition, we suggest that the second structural decomposition of an ideal M satisfying the hypothesis of Conjecture 8.1 will have no purely nondominant part. If this is true, then Conjecture 8.1 will follow from Theorem 5.3.

Conjecture 8.2. *Let M be minimally generated by $\{m_1, \ldots, m_q, n_1, \ldots, n_p\}$, where the m_i are dominant and the n_i are nondominant. If the ideal $M' = (n_1, \ldots, n_p)$ is almost generic, then M has characteristic Betti numbers in minimal homological degrees.*

Conjecture 8.3. *If M is minimally generated by more than one monomial, and there is a minimal generator that divides the* lcm *of every pair of generators, then* $\mathrm{pd}(S/M) = 2$.

Conjectures 8.2 and 8.3 were inspired by Theorems 5.3 and 6.1, respectively, and were verified for many examples run in Macaulay2 [10].

Acknowledgments

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