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An Auslander–Buchweitz approximation approach to (pre)silting subcategories in triangulated categories

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We apply the Auslander–Buchweitz approximation triangles to study (pre)silting subcategories in a triangulated category $\mathcal{T}$. An Auslander–Reiten type correspondence between the class of silting subcategories of $\mathcal{T}$ and that of certain covariantly finite subcategories of $\mathcal{T}$ is established. We introduce a relation on presilting subcategories of $\mathcal{T}$, which can be used to obtain another silting subcategory from a given one. We also give a Bazzoni's characterization for two presilting subcategories satisfying such a relation. The results can be used to improve the Auslander–Reiten type correspondence and Bazzoni's characterization for small silting complexes established by Wei.

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1. Introduction

The silting theory, which can be viewed as a generalization of the tilting theory in triangulated categories, originates from the study of t-structures in the representation theory (see, for example, [13,16]). A known disadvantage of tilting mutations for tilting objects is that, some direct summands of a tilting object cannot be replaced to get a new tilting object. To overcome this disadvantage, Aihara and Iyama introduced in [1] the notions of (pre)silting subcategories (objects) in triangulated categories. By extending the class of tilting objects to the wider class of silting objects, they define a concept of silting mutations that always works (see [1, Theorem 2.31]). According to [26], the silting objects play the same role in the bounded homotopy category of projective modules as the tilting modules in module categories. Silting subcategories (objects) are closely related to t-structures and co-t-structures in triangulated categories (see, for example, [15,21]). Thus one can better understand the structure of a triangulated category from the viewpoint of the silting theory.

Originated from the concept of injective envelopes, the approximation theory has attracted increasing interest and, hence, obtained considerable development especially in the context of module categories (see, for example, [6,7,12]). Inspired by the ideas of injective envelopes and projective covers, Auslander and Buchweitz studied in [5] the maximal Cohen–Macaulay approximations for certain modules. Indeed, they established their theory in the context of abelian categories, and provided several important applications. Following their work, Mendoza Hernández, Sáenz Valadez, Santiago Vargas and Souto Salorio developed in [19,20] an analogous theory of approximations for triangulated categories. The main purpose of this manuscript is to apply this Auslander–Buchweitz approximation theory to investigate (pre)silting subcategories (objects) in triangulated categories.

Covariantly finite subcategories were firstly introduced by Auslander and Smalø [7] (note that a subcategory is called covariantly finite precisely when it is preenveloping in the sense of Enochs and Jenda [12]). In 1991, Auslander and Reiten showed in [6, Theorem 5.5] that there exists a one-to-one correspondence between certain covariantly finite subcategories of the module category and isomorphism classes of basic tilting modules. This result gives a characterization of tilting modules in terms of subcategories, and is known as the "*Auslander–Reiten correspondence*" in the literature. In 2013, Mendoza Hernández, Sáenz Valadez, Santiago Vargas and Souto Salorio established a bijective correspondence between bounded co-t-structures on $\mathcal{T}$ and silting subcategories of $\mathcal{T}$ (see [20, Corollary 5.9]). By virtue of this bijection, we obtain the following result which gives the "Auslander–Reiten correspondence" for silting subcategories of $\mathcal{T}$ (see Corollary 3.7). We refer the reader to Section 2 and Subsection 3.1 for unspecified notation and notions.

Theorem 1.1 *(Auslander–Reiten correspondence for silting subcategories). Let $\mathcal{T}$ be a triangulated category. Then the assignments*

$$\mathcal{M} \mapsto \mathcal{M}^{\perp_{i>0}} \quad \textit{and} \quad \mathcal{H} \mapsto {}^{\perp_{i>0}}\mathcal{H} \cap \mathcal{H}$$

give mutually inverse bijections between the following classes:

(1) *Silting subcategories $\mathcal{M}$ of $\mathcal{T}$.*

(2) *Subcategories $\mathcal{H}$ of $\mathcal{T}$, which is specially covariantly finite and coresolving in $\mathcal{T}$, such that $\check{\mathcal{H}} = \mathcal{T}$ and for any object $H \in \mathcal{H}$, there exists an integer i making $Hom_{\mathcal{T}}(H, \mathcal{H}[\geqslant i]) = 0$.*

This result gives a new characterization of a silting subcategory $\mathcal{M}$, and reflects more homological properties of $\mathcal{M}$. In particular, subcategories of $\mathcal{T}$ satisfying the conditions in Theorem 1.1(2) can be used to obtain bounded co-t-structures on $\mathcal{T}$ (see Proposition 3.6).

Recall that a complex is called *silting* [16] (resp., *tilting* [23]) if it is a silting (resp., tilting) object in $K^{\mathrm{b}}(\mathrm{proj}\, R)$ (here, the symbol $K^{\mathrm{b}}(\mathrm{proj}\, R)$ denotes the bounded homotopy category of finitely generated projective modules). It is easy to see that every tilting complex in $K^{\mathrm{b}}(\mathrm{proj}\, R)$ is always silting. However, the converse does not hold true in general (see [26, Section 6] for examples). As a consequence of Theorem 1.1, we get the following Auslander–Reiten type correspondence for silting complexes (see Corollary 3.13 and Subsection 3.3 for unspecified notation and notions). We thank the referee for pointing out to us an interesting work of Marks and Vitória, who proved that there exists a bijection between silting complexes up to equivalence and co-intermediate and coresolving subcategories of $K^{\mathrm{b}}(\mathrm{proj}\, R)$ (see [18, Theorem 3.6]).

Corollary 1.2 *(Auslander–Reiten correspondence for silting complexes). Let R be an arbitrary ring. Then for any integer n, the assignments*

$$T \mapsto T^{\perp_{i>0}} \quad \textit{and} \quad \mathcal{H} \mapsto N,$$

where $\mathrm{add}\, N = {}^{\perp_{i>0}}\mathcal{H} \cap \mathcal{H}$, give mutually inverse bijections between the following classes:

(1) *Equivalent classes of silting complexes T in $K^{\leqslant n}(\mathrm{proj}\, R)$.*

(2) *Subcategories $\mathcal{H}$ of $K^{\leqslant n}(\mathrm{proj}\, R)$, which is specially covariantly finite and coresolving in $K^{\mathrm{b}}(\mathrm{proj}\, R)$, such that $\check{\mathcal{H}} = K^{\mathrm{b}}(\mathrm{proj}\, R)$.*

Recently, Angeleri Hügel, Marks and Vitória [3] introduced and subsequently studied in [2,4] the notion of silting modules. These modules generalize large tilting modules in the sense of Colpi and Trlifaj [11]. They showed that there exists a bijective correspondence between equivalence classes of silting modules and equivalence classes of 2-term "silting complexes". Note that, in this case, a "silting complex" C is defined as a bounded complex of large projective modules satisfying that $\mathrm{Hom}_{K^{\mathrm{b}}(\mathrm{Proj}\, R)}(C, C^{(I)}[\geqslant 1]) = 0$ for all sets I and $\langle \mathrm{Add}\, C\rangle = K^{\mathrm{b}}(\mathrm{Proj}\, R)$ (here, the symbol $K^{\mathrm{b}}(\mathrm{Proj}\, R)$ denotes the bounded homotopy category of large projective modules, and $\langle \mathrm{Add}\, C\rangle$ denotes the smallest thick subcategory of $K^{\mathrm{b}}(\mathrm{Proj}\, R)$ containing $\mathrm{Add}\, C$). To avoid confusion, we call such a "silting complex" a *large silting complex*. In 2013, an Auslander–Reiten type correspondence for large silting complexes was established in [26, Theorem 5.3]. In terms of this correspondence, Angeleri Hügel, Marks and Vitória obtained a bijection between equivalence classes of large silting complexes and certain intermediate co-t-structures (see [3, Theorem 4.6]). It is worth mentioning, by a careful reading of [3] and [26], that if one wants to get our Corollary 1.2 by the method in the proof of [3, Theorem 4.6] or [26, Theorem 5.3], then the condition that $\mathrm{Hom}_{K^{\mathrm{b}}(\mathrm{proj}\,R)}$'s are finitely generated is necessary (see [26, Remark 3.3, Remark 5.4 and Lemma 3.11]).

In 2004, Bazzoni [8] proved a very interesting characterization for infinitely generated tilting modules of finite projective dimension. More precisely, she showed that for a non-negative integer n, a module L is n-tilting if and only if $L^{\perp_{i>0}} = \mathrm{Pres}^n(\mathrm{Add}\,L)$, where $\mathrm{Pres}^n(\mathrm{Add}\,L)$ denotes the category of all modules K such that there is an exact sequence $L_{n-1} \to \cdots \to L_0 \to K \to 0$ with each $L_i \in \mathrm{Add}\,L$ and $L^{\perp_{i>0}}$ stands for the right orthogonal category of L with respect to 'Ext'. This result is known as the "*Bazzoni's characterization*" in the literature. Motivated by her contribution, the fourth author gave corresponding characterizations for several important tilting objects in different categories, such as classical n-tilting modules [24], tilting pairs [27], tilting complexes and large silting complexes [26]. In particular, the Bazzoni's characterization for large silting complexes [26, Theorem 4.4] holds true for an arbitrary ring. However, the corresponding result for tilting complexes [26, Corollary 4.5] holds under the condition that $\mathrm{Hom}_{K^{\mathrm{b}}(\mathrm{proj}\,R)}$'s are finitely generated (see [26, Remark 3.3 and Lemma 3.11]). To remove this condition, we introduce in this manuscript a relation '$\preceq^n$' of presilting subcategories of $\mathcal{T}$.

Let $\mathcal{M}$ and $\mathcal{N}$ be two presilting subcategories of $\mathcal{T}$. We write '$\mathcal{M} \preceq^n \mathcal{N}$' provided that every object in $\mathcal{M}$ has a finite resolution of length n by objects in $\mathcal{N}$ and every object in $\mathcal{N}$ has a finite coresolution by objects in $\mathcal{M}$ (see Definition 4.1). Note that if T is a silting complex in $K^{\mathrm{b}}(\mathrm{proj}\,R)$ with $\inf\{s \in \mathbb{Z} \,|\, T_s \neq 0\} = l$ and $\sup\{s \in \mathbb{Z} \,|\, T_s \neq 0\} = k$, then one has $\mathrm{add}\,R \preceq^{k-l} \mathrm{add}\,(T[k])$ (see Corollary 4.8). Let E and F be two finitely generated modules over an artin algebra R. If the pair (E, F) forms an n-tilting pair, then $\mathrm{add}\,E \preceq^n \mathrm{add}\,F$ in $D^{\mathrm{b}}(\mathrm{mod}\,R)$ whenever we consider both E and F as stalk complexes (see Example 4.2(2) for more details). Thus, such a relation is a common generalization of silting complexes and tilting pairs. One can obtain a silting subcategory from another one by means of such a relation (see Proposition 4.4). In particular, we obtain the following result, which gives a Bazzoni's characterization for two presilting subcategories satisfying the relation '$\preceq^n$' (see Theorem 4.7). We refer the reader to Section 2 and Section 4 for unspecified notation and notions.

Theorem 1.3. *Let $\mathcal{M}, \mathcal{N}$ be two presilting subcategories of $\mathcal{T}$ and n a non-negative integer. Suppose that $\mathcal{M}$ is silting. Then $\mathcal{M} \preceq^n \mathcal{N}$ if and only if $\mathrm{Pres}^n_{\mathcal{M}^{\perp_{i>0}}}(\mathcal{N}) = {}_{\mathcal{N}}\mathscr{X}$ and $\mathcal{N} \subseteq \mathcal{M}^{\perp_{i>0}}$.*

As an application of Theorem 1.3, we obtain the next result (see Corollary 4.9), which improves [26, Corollary 4.5] by removing the condition that $\mathrm{Hom}_{K^{\mathrm{b}}(\mathrm{proj}\,R)}$'s are finitely generated.

Corollary 1.4 *(Bazzoni's characterization for tilting complexes). Let R be an arbitrary ring and T a pretilting complex in $K^{\mathrm{b}}(\mathrm{proj}\, R)$. Suppose that $\sup T = n$ and $\inf T = l$, where n and l are two integers. Then the following statements are equivalent:*

(1) *T is a tilting complex.*

(2) *$add\, R \preceq^{n-l} add\,(T[n])$.*

(3) *$Pres^{n-l}_{R^{\perp_{i>0}}}(add\,(T[n])) = {}_{add\,(T[n])}\mathscr{X}$.*

We conclude the section by summarizing the contents of this article. Section 2 contains necessary notions and results for the later sections. In Section 3, we establish Auslander–Reiten type correspondences for silting subcategories and objects, and prove Theorem 1.1 and Corollary 1.2. The relation '$\preceq$' on presilting subcategories is introduced and investigated in Section 4, where we also prove Theorem 1.3 and Corollary 1.4.

2. Preliminaries

In this section, we fix some notation. We recall the Auslander–Buchweitz approximation triangles in a triangulated category. We also recall the definitions of (pre)silting subcategories (objects) and (bounded) co-t-structures, and give some necessary facts about these notions.

2.1. Some notation

Throughout this article, by the term *"subcategory"* we always mean a full additive subcategory of an additive category closed under isomorphisms and direct summands.

Throughout this article, let R be an associative ring with identity. Denote by $\mathrm{mod}\, R$ the category of all finitely generated right R-modules, by $\mathrm{Proj}\, R$ the category of all projective right R-modules, and by $\mathrm{proj}\, R$ the category of all finitely generated projective right R-modules.

Throughout this article, let $\mathcal{T}$ be a triangulated category. We will denote by [1] the shift functor of any triangulated category. Suppose that $\mathcal{C}$ is a subcategory of $\mathcal{T}$. Denote by $\langle \mathcal{C} \rangle$ the *smallest thick subcategory* of $\mathcal{T}$ containing $\mathcal{C}$. For any integer n, set

$$\mathcal{C}^{\perp_{i>n}} = \{N \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(M, N[>n]) = 0 \text{ for all } M \in \mathcal{C}\},$$
$${}^{\perp_{i>n}}\mathcal{C} = \{N \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(N, M[>n]) = 0 \text{ for all } M \in \mathcal{C}\}.$$

Following the notions in [26], the subcategory $\mathcal{C}$ is called *extension-closed* if for any triangle

$$U \to V \to W \to U[1]$$

in $\mathcal{T}$ with $U, W \in \mathcal{C}$, it holds that $V \in \mathcal{C}$. It is *resolving* (resp., *coresolving*) if it is further closed under the functor $[-1]$ (resp., [1]). Note that $\mathcal{C}$ is resolving (resp., coresolving) if and only if, for any triangle

$$U \to V \to W \to U[1] \qquad (\text{resp., } W \to V \to U \to W[1])$$

in $\mathcal{T}$ with $W \in \mathcal{C}$, it holds that $U \in \mathcal{C} \Leftrightarrow V \in \mathcal{C}$. It is easy to see that $\mathcal{C}^{\perp_{i>0}}$ (resp., ${}^{\perp_{i>0}}\mathcal{C}$) is coresolving (resp., resolving).

For two subcategories $\mathcal{S}$ and $\mathcal{S}'$ of $\mathcal{T}$, define

$$\begin{aligned}\mathcal{S} * \mathcal{S}' = \mathcal{S} *_{\mathcal{T}} \mathcal{S}' = \{X \in \mathcal{T} \mid & \text{ there is a triangle} \\ & S \to X \to S' \to S[1] \text{ in } \mathcal{T} \text{ with } S \in \mathcal{S} \text{ and } S' \in \mathcal{S}'\}.\end{aligned}$$

In case $\mathcal{S} = \{S\}$, we use the notation $S * \mathcal{Y}$ instead of $\{S\} * \mathcal{Y}$. There are some analogues such as $\mathcal{S} * S'$ and $S * S'$.

Let $\mathcal{A}$ be an abelian category. A complex X is often displayed as a sequence

$$\cdots \longrightarrow X_{n-1} \xrightarrow{\delta^X_{n-1}} X_n \xrightarrow{\delta^X_n} X_{n+1} \longrightarrow \cdots$$

of objects in $\mathcal{A}$ with $\delta^X_n \delta^X_{n-1} = 0$ for all $n \in \mathbb{Z}$. Set $\inf X = \inf\{s \in \mathbb{Z} \mid X_s \neq 0\}$ and $\sup X = \sup\{s \in \mathbb{Z} \mid X_s \neq 0\}$. We say that two complexes X and Y are *equivalent*, and denoted by $X \simeq Y$ [10, A.1.11, p. 164], if they can be linked by a sequence of quasi-isomorphisms with arrows in alternating directions.

Let $\mathcal{E}$ be a subcategory of $\mathcal{A}$. Denote by $D^{\mathrm{b}}(\mathcal{A})$ the bounded derived category of $\mathcal{A}$ and by $K^{\mathrm{b}}(\mathcal{E})$ the bounded homotopy category with each complex constructed by objects in $\mathcal{E}$.

2.2. Auslander–Buchweitz approximation triangles

We describe in this subsection the Auslander–Buchweitz approximation triangles established by Mendoza Hernández, Sáenz Valadez, Santiago Vargas and Souto Salorio in [19].

Let $\mathcal{W}$ and $\mathcal{X}$ be subcategories of $\mathcal{T}$. For a non-negative integer n, denote by $(\widehat{\mathcal{X}})_n$ (resp., $(\check{\mathcal{X}})_n$) the subclass of $\mathcal{T}$ consisting of all objects T satisfying that there exists a series of triangles

$$T_{i+1} \to X_i \to T_i \to T_{i+1}[1] \qquad (\text{resp., } T_i \to X_i \to T_{i+1} \to T_i[1])$$

in $\mathcal{T}$ with $0 \leqslant i \leqslant n$ such that $T_0 = T$, $T_{n+1} = 0$ and each $X_i \in \mathcal{X}$. We use the symbol $\widehat{\mathcal{X}}$ (resp., $\check{\mathcal{X}}$) to denote the subclass of $\mathcal{T}$ consisting of all objects K satisfying that there is a non-negative integer m such that $K \in (\widehat{\mathcal{X}})_m$ (resp., $K \in (\check{\mathcal{X}})_m$). Note that $0 \in \mathcal{X}$ by assumption. It is easy to see that $\widehat{\mathcal{X}}$ (resp., $\check{\mathcal{X}}$) is closed under the functor [1] (resp., [−1]).

Recall that $\mathcal{W}$ is called a *weak-cogenerator* in $\mathcal{X}$ [19, Definition 5.1] if $\mathcal{W} \subseteq \mathcal{X}$ and $\mathcal{X} \subseteq \mathcal{X}[-1] * \mathcal{W}$, that is, for any object $X \in \mathcal{X}$, there exists a triangle

$$X \to W \to X' \to X[1]$$

in $\mathcal{T}$ with $X' \in \mathcal{X}$ and $W \in \mathcal{W}$. Dually, one has the notion of a *weak-generator*. The subcategory $\mathcal{W}$ is said to be $\mathcal{X}$-*injective* (resp., $\mathcal{X}$-*projective*) if $\mathrm{Hom}_{\mathcal{T}}(X, W[\geqslant 1]) = 0$ (resp., $\mathrm{Hom}_{\mathcal{T}}(W, X[\geqslant 1]) = 0$) for any object $W \in \mathcal{W}$ and object $X \in \mathcal{X}$. We say that $\mathcal{W}$ is a *weak-generator-cogenerator* in $\mathcal{X}$ if it is both an $\mathcal{X}$-projective weak-generator and an $\mathcal{X}$-injective weak-cogenerator in $\mathcal{X}$.

The following two results will be used frequently in the sequel.

Theorem 2.1. *([19, Theorem 5.4]) Let $\mathcal{W} \subseteq \mathcal{X}$ be subcategories of $\mathcal{T}$. Suppose that $\mathcal{X}$ is closed under extensions and $\mathcal{W}$ is a weak-cogenerator in $\mathcal{X}$. Then for any object $M \in \widehat{\mathcal{X}}$, there exist triangles*

$$K_M \to X_M \to M \to K_M[1] \quad \textit{and} \quad M \to K^M \to X^M \to M[1]$$

in $\mathcal{T}$ with $X_M, X^M \in \mathcal{X}$ and $K_M, K^M \in \widehat{\mathcal{W}}$.

Dually, one has

Theorem 2.2. *Let $\mathcal{V} \subseteq \mathcal{Y}$ be subcategories of $\mathcal{T}$. Suppose that $\mathcal{Y}$ is closed under extensions and $\mathcal{V}$ is a weak-generator in $\mathcal{Y}$. Then for any object $N \in \check{\mathcal{Y}}$, there exist triangles*

$$N \to Y^N \to L^N \to N[1] \quad \textit{and} \quad Y_N \to L_N \to N \to Y_N[1]$$

in $\mathcal{T}$ with $Y^N, Y_N \in \mathcal{Y}$ and $L^N, L_N \in \check{\mathcal{V}}$.

2.3. (Pre)silting and thick subcategories

In this subsection, we mainly recall the definitions of (pre)silting subcategories (objects) and give some necessary facts on subcategories arising from a presilting subcategory.

Definition 2.3. ([1, Definition 2.1]) Let $\mathcal{M}$ be a subcategory of $\mathcal{T}$ and M an object in $\mathcal{T}$.

(1) $\mathcal{M}$ is called *presilting* (resp., *pretilting*) if

$$\mathrm{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{M}[\geqslant 1]) = 0 \qquad (\text{resp.}, \mathrm{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{M}[\neq 0]) = 0).$$

(2) $\mathcal{M}$ is called *silting* (resp., *tilting*) if it is presilting (resp., pretilting) and $\mathcal{T} = \langle \mathcal{M} \rangle$.

(3) M is called *silting* (resp., *tilting*) if the subcategory $\mathrm{add}\, M$ is silting (resp., tilting).

The following result is easy to obtain.

Lemma 2.4. *Let $\mathcal{M}$ be a presilting subcategory of $\mathcal{T}$. Then $Hom_{\mathcal{T}}(\breve{\mathcal{M}}, \mathcal{M}^{\perp_{i>0}}[\geqslant 1]) = 0$.*

Triangulated categories with silting subcategories have the following property.

Lemma 2.5. *([14, Lemma 2.4]) Let $\mathcal{T}$ be a triangulated category with a silting subcategory $\mathcal{M}$.*

(1) *For all objects $X, Y \in \mathcal{T}$, there exists an integer i such that $Hom_{\mathcal{T}}(X, Y[\geqslant i]) = 0$.*

(2) *For any object $X \in \mathcal{T}$, there exist integers j and k such that*

$$Hom_{\mathcal{T}}(\mathcal{M}, X[\geqslant j]) = 0 \quad and \quad Hom_{\mathcal{T}}(X, \mathcal{M}[\geqslant k]) = 0.$$

By virtue of [1, Proposition 2.17] and [15, Proposition 3.3], we see that for any silting subcategory $\mathcal{M}$ of $\mathcal{T}$,

$$\mathcal{M}^{\perp_{i>0}} = \bigcup_{l \geqslant 0} \mathcal{M} * \mathcal{M}[1] * \cdots * \mathcal{M}[l].$$

This leads to the following lemma.

Lemma 2.6. *Let $\mathcal{M}$ be a silting subcategory of $\mathcal{T}$. Then $\mathcal{M}$ is a weak-generator in $\mathcal{M}^{\perp_{i>0}}$.*

Proof. Let N be an object in $\mathcal{M}^{\perp_{i>0}}$. According to [1, Proposition 2.17] and [15, Proposition 3.3], we know that there exists a non-negative integer l such that $N \in \mathcal{M} * \mathcal{M}[1] * \cdots * \mathcal{M}[l]$. This induces a triangle

$$N' \to M \to N \to N'[1]$$

in $\mathcal{T}$ with $M \in \mathcal{M}$ and $N' \in \mathcal{M} * \mathcal{M}[1] * \cdots * \mathcal{M}[l-1]$. Clearly, $N' \in \mathcal{M}^{\perp_{i>0}}$ as well. Therefore, the result follows. □

Let $\mathcal{M}$ be a presilting subcategory of $\mathcal{T}$. We use the symbol ${}_{\mathcal{M}}\mathscr{X}$ (resp., $\mathscr{X}_{\mathcal{M}}$) to denote the subcategory of $\mathcal{M}^{\perp_{i>0}}$ (resp., ${}^{\perp_{i>0}}\mathcal{M}$) consisting of all objects N such that there exist triangles

$$N_{i+1} \to M_i \to N_i \to N_{i+1}[1] \quad \text{(resp., } N_i \to M_i \to N_{i+1} \to N_i[1])$$

in $\mathcal{T}$ such that $N_0 = N$, $N_i \in \mathcal{M}^{\perp_{i>0}}$ (resp., $N_i \in {}^{\perp_{i>0}}\mathcal{M}$) and $M_i \in \mathcal{M}$ for all $i \geqslant 0$. It is easy to see that $\widehat{\mathcal{M}} \subseteq {}_{\mathcal{M}}\mathscr{X} \subseteq \mathcal{M}^{\perp_{i>0}}$ and $\widecheck{\mathcal{M}} \subseteq \mathscr{X}_{\mathcal{M}} \subseteq {}^{\perp_{i>0}}\mathcal{M}$.

Lemma 2.7. *([25, Lemma 2.2]) Let $\mathcal{M}$ be a presilting subcategory of $\mathcal{T}$. Then the following statements hold:*

(1) *$\widecheck{\mathcal{M}}$ and $\mathscr{X}_{\mathcal{M}}$ are resolving.*

(2) *$\widehat{\mathcal{M}}$ and ${}_{\mathcal{M}}\mathscr{X}$ are coresolving.*

Next, we consider some thick subcategories of $\mathcal{T}$. Let $\mathcal{H}$ be a subcategory of $\mathcal{T}$. Define

$$(\mathcal{H})_+ := \{N \in \mathcal{T} \mid N \cong L[i] \text{for some object } L \in \mathcal{H} \text{ and some integer } i \geqslant 0\}.$$
$$(\mathcal{H})_- := \{N \in \mathcal{T} \mid N \cong L[i] \text{ for some object } L \in \mathcal{H} \text{ and some integer } i \leqslant 0\}.$$

Lemma 2.8. *[25, Lemma 2.1] Let $\mathcal{H}$ be a subcategory of $\mathcal{T}$.*
(1) *If $\mathcal{H}$ is resolving, then $\langle\mathcal{H}\rangle = (\mathcal{H})_+ = \widehat{\mathcal{H}}$.*
(2) *If $\mathcal{H}$ is coresolving, then $\langle\mathcal{H}\rangle = (\mathcal{H})_- = \check{\mathcal{H}}$.*

Corollary 2.9. *Let $\mathcal{C}$ be a subcategory of $\mathcal{T}$ closed under extensions. If $\mathcal{C}$ admits a weak-generator-cogenerator, then $\langle\mathcal{C}\rangle = (\check{\mathcal{C}})_+ = (\widehat{\mathcal{C}})_-$.*

Proof. Note that $\mathcal{C}$ admits a weak-generator-cogenerator by assumption. It is easy to see that both $\check{\mathcal{C}}$ and $\widehat{\mathcal{C}}$ are closed under extensions and direct summands. Hence, $\check{\mathcal{C}}$ (resp., $\widehat{\mathcal{C}}$) is resolving (resp., coresolving), and so $(\check{\mathcal{C}})_+ = \langle\check{\mathcal{C}}\rangle$ and $(\widehat{\mathcal{C}})_+ = \langle\widehat{\mathcal{C}}\rangle$ by Lemma 2.8. However, it is clear that $\langle\check{\mathcal{C}}\rangle = \langle\mathcal{C}\rangle$ (resp., $\langle\widehat{\mathcal{C}}\rangle = \langle\mathcal{C}\rangle$). Therefore, the result follows. □

Suppose that $\mathcal{M}$ is a presilting subcategory of $\mathcal{T}$. According to [20, Lemma 5.3(2)], we know that $\mathcal{M}$ is closed under extensions. Moreover, it is obvious that $\mathcal{M}$ admits itself as a weak-generator-cogenerator. Thus, by Corollary 2.9, we obtain

Corollary 2.10. *Let $\mathcal{M}$ be a presilting subcategory of $\mathcal{T}$. Then $\langle\mathcal{M}\rangle = (\check{\mathcal{M}})_+ = (\widehat{\mathcal{M}})_-$.*

Finally, we recall the definition of a (bounded) co-t-structure.

Definition 2.11. [9,22] A *co-t-structure* on $\mathcal{T}$ is a pair $(\mathcal{A}, \mathcal{B})$ of subcategories of $\mathcal{T}$ such that
(1) $\mathcal{A}[-1] \subseteq \mathcal{A}$ and $\mathcal{B}[1] \subseteq \mathcal{B}$,
(2) $\mathrm{Hom}_{\mathcal{T}}(\mathcal{A}[-1], \mathcal{B}) = 0$, and
(3) $\mathcal{T} = \mathcal{A}[-1] * \mathcal{B}$.

In this case, the *co-heart* is defined as the intersection $\mathcal{A} \cap \mathcal{B}$, which is a presilting subcategory of $\mathcal{T}$.

A co-t-structure $(\mathcal{A}, \mathcal{B})$ is said to be *bounded* if

$$\bigcup_{n\in\mathbb{Z}} \mathcal{A}[n] = \mathcal{T} = \bigcup_{n\in\mathbb{Z}} \mathcal{B}[n].$$

Fact 2.12. *Let $\mathcal{D}$ be a presilting subcategory of $\mathcal{T}$. According to [20, Theorem 5.5], we know that the pair $({}_{\mathcal{D}}\mathcal{U}, \mathcal{U}_{\mathcal{D}})$ forms a bounded co-t-structure on $\langle\mathcal{D}\rangle$. Here, the symbol ${}_{\mathcal{D}}\mathcal{U}$ (resp., $\mathcal{U}_{\mathcal{D}}$) stands for the smallest extension closed subcategory of $\mathcal{T}$ containing $\mathcal{D}[\leqslant 0]$ (resp., $\mathcal{D}[\geqslant 1]$).*

3. Auslander–Reiten type correspondences

In this section, we apply the bijective correspondence between bounded co-t-structures on $\mathcal{T}$ and silting subcategories of $\mathcal{T}$ established by Mendoza Hernández, Sáenz Valadez, Santiago Vargas and Souto Salorio in [20] to obtain Auslander–Reiten type correspondences for silting subcategories and objects of $\mathcal{T}$ (Corollary 3.7 and Corollary 3.8). As an application, the Auslander–Reiten type correspondence for silting complexes in [26] is improved by removing the condition that $\mathrm{Hom}_{K^{\mathrm{b}}(\mathrm{proj}\ R)}$'s are finitely generated (Corollary 3.13).

We begin with the following subsection, which shows that certain specially covariantly finite and coresolving subcategories of $\mathcal{T}$ can be used to construct silting subcategories (Corollary 3.4).

3.1. Specially covariantly finite and coresolving subcategories

Let $\mathcal{H}$ be a subcategory of $\mathcal{T}$. Recall that $\mathcal{H}$ is said to be *covariantly finite* in $\mathcal{T}$ [7] provided that for any object $T \in \mathcal{T}$, there is a morphism $f : T \to H$ with $H \in \mathcal{H}$ such that $\mathrm{Hom}_{\mathcal{T}}(f, H')$ is surjective for any object $H' \in \mathcal{H}$ (see also [6]). The subcategory $\mathcal{H}$ is called *specially covariantly finite* in $\mathcal{T}$ [26] provided that for any object $T \in \mathcal{T}$, there is a triangle

$$T \to H \to K \to T[1]$$

in $\mathcal{T}$ such that $H \in \mathcal{H}$ and $\mathrm{Hom}_{\mathcal{T}}(K, \mathcal{H}[1]) = 0$. Clearly, if $\mathcal{H}$ is closed under [1], then $\mathrm{Hom}_{\mathcal{T}}(K, \mathcal{H}[\geqslant 1]) = 0$ in this case.

Proposition 3.1. *Let $\mathcal{M}$ be a silting subcategory of $\mathcal{T}$. Then $\mathcal{M}^{\perp_{i>0}}$ is specially covariantly finite and coresolving in $\mathcal{T}$ such that $\widetilde{\mathcal{M}^{\perp_{i>0}}} = \mathcal{T}$.*

Proof. Firstly, we show that $\widetilde{\mathcal{M}^{\perp_{i>0}}} = \mathcal{T}$. To this end, let X be an object of $\mathcal{T}$. Then in view of Lemma 2.5(2), we see that there exists an integer i such that $\mathrm{Hom}_{\mathcal{T}}(\mathcal{M}, X[\geqslant i]) = 0$. If $i \leqslant 1$ then $X \in \mathcal{M}^{\perp_{i>0}}$. Hence, $X \in \widetilde{\mathcal{M}^{\perp_{i>0}}}$, as desired. Assume now that $i > 1$. Then $X[1-i] \in \mathcal{M}^{\perp_{i>0}}$, which implies that $X \in \widetilde{\mathcal{M}^{\perp_{i>0}}}$ by noting that $0 \in \mathcal{M}^{\perp_{i>0}}$.

Next, we show that $\mathcal{M}^{\perp_{i>0}}$ is specially covariantly finite in $\mathcal{T} = \widetilde{\mathcal{M}^{\perp_{i>0}}}$. Note that $\mathcal{M}$ is a weak-generator in $\mathcal{M}^{\perp_{i>0}}$ by Lemma 2.6. Then Theorem 2.2 tells us that there exists a triangle

$$X \to Y \to Z \to X[1]$$

in $\mathcal{T}$ with $Y \in \mathcal{M}^{\perp_{i>0}}$ and $Z \in \widecheck{\mathcal{M}}$. To complete the proof, we need $Z \in {}^{\perp_{i>0}}(\mathcal{M}^{\perp_{i>0}})$, which is guaranteed by Lemma 2.4.

Finally, it is easy to see that $\mathcal{M}^{\perp_{i>0}}$ is coresolving. $\square$

One can conclude from the following result that any specially covariantly finite and coresolving subcategory of $\mathcal{T}$ admits a projective weak-generator. It will be used in the proofs of Corollary 3.4 and Lemma 3.12.

Lemma 3.2. *Let $\mathcal{H}$ be a subcategory of $\mathcal{T}$. Suppose that $\mathcal{H}$ is specially covariantly finite and coresolving in $\mathcal{T}$. Set $\mathcal{M} = {}^{\perp_{i>0}}\mathcal{H} \cap \mathcal{H}$. Then*

(1) *$\mathcal{M}$ is presilting.*

(2) *For each object $H \in \mathcal{H}$, there exists a triangle*

$$H' \to M \to H \to H'[1]$$

in $\mathcal{T}$ with $M \in \mathcal{M}$ and $H' \in \mathcal{H}$.

In particular, $\mathcal{M}$ is an $\mathcal{H}$-projective weak-generator in $\mathcal{H}$.

Proof. (1) It is obvious.

(2) Since $\mathcal{H}$ is specially covariantly finite and coresolving in $\mathcal{T}$ by assumption, there exists a triangle

$$H[-1] \to H' \to M \to H$$

in $\mathcal{T}$ such that $H' \in \mathcal{H}$ and $M \in {}^{\perp_{i>0}}\mathcal{H}$. This induces the triangle $H' \to M \to H \to H'[1]$ in $\mathcal{T}$. Note that $\mathcal{H}$ is closed under extensions. It follows that M is also in $\mathcal{H}$. Thus, $M \in {}^{\perp_{i>0}}\mathcal{H} \cap \mathcal{H}$. This completes the proof of (2).

The last statement follows from (2) and the trivial fact $\mathcal{H} \in \mathcal{M}^{\perp_{i>0}}$. $\square$

The following result, which is the dual version of [19, Proposition 5.9(b)], will play a key role in the proof of Corollary 3.4.

Lemma 3.3. *Let $\mathcal{M} \subseteq \mathcal{H}$ be subcategories of $\mathcal{T}$. Suppose that $\mathcal{H}$ is coresolving and $\mathcal{M}$ is an $\mathcal{H}$-projective weak-generator in $\mathcal{H}$. Then for any object $K \in \check{\mathcal{H}}$, $K \in \check{\mathcal{M}}$ if and only if $K \in {}^{\perp_{i>0}}\mathcal{H} \cap \check{\mathcal{H}}$.*

We present a method to construct silting subcategories in the next corollary.

Corollary 3.4. *Let $\mathcal{H}$ be a subcategory of $\mathcal{T}$. Suppose that $\mathcal{H}$ is specially covariantly finite and coresolving in $\mathcal{T}$ such that $\check{\mathcal{H}} = \mathcal{T}$ and for any object $H \in \mathcal{H}$ there exists an integer i making $Hom_{\mathcal{T}}(H, \mathcal{H}[\geqslant i]) = 0$. Then ${}^{\perp_{i>0}}\mathcal{H} \cap \mathcal{H}$ is a silting subcategory of $\mathcal{T}$.*

Proof. Let $\mathcal{M} = {}^{\perp_{i>0}}\mathcal{H} \cap \mathcal{H}$ for convenience. It is clear that $\mathcal{M}$ is presilting. Hence, it remains to show $\langle \mathcal{M} \rangle = \mathcal{T}$.

To this end, let X be an object of $\mathcal{T} = \check{\mathcal{H}}$. Since $\mathcal{H}$ is closed under extensions and admits a weak-generator $\mathcal{M}$ by Lemma 3.2, it follows from Theorem 2.2 that there exists a triangle

$$X \to H \to K \to X[1]$$

in $\mathcal{T}$ with $H \in \mathcal{H}$ and $K \in \check{\mathcal{M}} \subseteq \langle \mathcal{M} \rangle$. To complete the proof, we only need to show that $H \in \langle \mathcal{M} \rangle$.

Indeed, note that there exists an integer i such that $\mathrm{Hom}_{\mathcal{T}}(H, \mathcal{H}[\geqslant i]) = 0$ by assumption. If $i \leqslant 1$, then $H \in {}^{\perp_{i>0}}\mathcal{H}$. This yields that $H \in \mathcal{H} \cap {}^{\perp_{i>0}}\mathcal{H} = \mathcal{M} \subseteq \langle\mathcal{M}\rangle$, as desired. Assume now that $i > 1$. Then $\mathrm{Hom}_{\mathcal{T}}(H[-i+1], \mathcal{H}[\geqslant 1]) \cong \mathrm{Hom}_{\mathcal{T}}(H, \mathcal{H}[\geqslant i]) = 0$. This implies that $H[-i+1] \in {}^{\perp_{i>0}}\mathcal{H}$. Meanwhile, we see that $H[-i+1] \in \check{\mathcal{H}}$ as well. Hence, in view of Lemma 3.3, we have $H[-i+1] \in \breve{\mathcal{M}} \subseteq \langle\mathcal{M}\rangle$. This yields also $H \in \langle\mathcal{M}\rangle$, as needed. □

3.2. Auslander–Reiten type correspondence for silting subcategories

Mendoza Hernández, Sáenz Valadez, Santiago Vargas and Souto Salorio established in [20] the following bijective correspondence between bounded co-t-structures on $\mathcal{T}$ and silting subcategories of $\mathcal{T}$. By virtue of this bijection, we give in this subsection a generalized version of the Auslander–Reiten type correspondence for silting subcategories of $\mathcal{T}$.

Theorem 3.5. *[20, Corollary 5.9] The assignments*

$$\mathcal{M} \mapsto (\breve{\mathcal{M}}, \mathcal{M}^{\perp_{i>0}}) \quad \textit{and} \quad (\mathcal{A}, \mathcal{B}) \mapsto \mathcal{A} \cap \mathcal{B}$$

give mutually inverse bijections between the following classes:

(1) Silting subcategories $\mathcal{M}$ of $\mathcal{T}$.

(2) Bounded co-t-structures $(\mathcal{A}, \mathcal{B})$ on $\mathcal{T}$.

Now, we give a new description for bounded co-t-structures on $\mathcal{T}$, which will play a key role in obtaining Corollary 3.7.

Proposition 3.6. *The assignments*

$$(\mathcal{A}, \mathcal{B}) \mapsto \mathcal{B} \quad \textit{and} \quad \mathcal{H} \mapsto (\breve{\mathcal{M}}, \mathcal{H}),$$

where $\mathcal{M} = {}^{\perp_{i>0}}\mathcal{H} \cap \mathcal{H}$, give mutually inverse bijections between the following classes:

(1) Bounded co-t-structures $(\mathcal{A}, \mathcal{B})$ on $\mathcal{T}$.

(2) Subcategories $\mathcal{H}$ of $\mathcal{T}$, which is specially covariantly finite and coresolving in $\mathcal{T}$, such that $\check{\mathcal{H}} = \mathcal{T}$ and for any object $H \in \mathcal{H}$, there exists an integer i making $\mathrm{Hom}_{\mathcal{T}}(H, \mathcal{H}[\geqslant i]) = 0$.

Proof. Let $(\mathcal{A}, \mathcal{B})$ be a bounded co-t-structure on $\mathcal{T}$. Then for any object $X \in \mathcal{T}$, there exists a triangle $X \to B \to A \to X[1]$ in $\mathcal{T}$ with $B \in \mathcal{B}$ and $A \in \mathcal{A}$. This implies that $\mathcal{B}$ is specially covariantly finite in $\mathcal{T}$. Note that $\mathcal{B}$ is obviously coresolving in $\mathcal{T}$. In view of Lemma 2.8(2), we see that $\check{\mathcal{B}} = \bigcup_{n \leqslant 0} \mathcal{B}[n] = \mathcal{T}$. Thus, to prove that $\mathcal{B}$ satisfies the requirements in (2), it remains to show that for any object $B \in \mathcal{B}$, there exists an integer i such that $\mathrm{Hom}_{\mathcal{T}}(B, \mathcal{B}[\geqslant i]) = 0$. Indeed, since $B \in \bigcup_{n \geqslant 0} \mathcal{A}[n]$ as well, it follows that there exist an object $A \in \mathcal{A}$ and a nonnegative integer n such that $B \cong A[n]$. This yields that $\mathrm{Hom}_{\mathcal{T}}(B, \mathcal{B}[n+1]) \cong \mathrm{Hom}_{\mathcal{T}}(A[n], \mathcal{B}[n+1]) = 0$. Clearly, the integer $n+1$ is the appropriate candidate for the required i.

Assume now that $\mathcal{H}$ is a subcategory of $\mathcal{T}$ satisfying the conditions in (2). Since $\check{\mathcal{H}} = \mathcal{T}$ and $\mathcal{M}$ is a weak-generator in $\mathcal{H}$ (see Lemma 3.2), it follows from Theorem 2.2 that for any object $X \in \mathcal{T}$ there exists a triangle

$$K \to X \to H \to K[1]$$

in $\mathcal{T}$ with $K \in \breve{\mathcal{M}}[-1]$ and $H \in \mathcal{H}$. This implies that $\mathcal{T} = \breve{\mathcal{M}}[-1] * \mathcal{H}$. Moreover, it is clear that $\mathcal{H} \subseteq \mathcal{M}^{\perp_{i>0}}$. Hence, we have $\mathrm{Hom}_{\mathcal{T}}(\breve{\mathcal{M}}, \mathcal{H}[1]) = 0$ by Lemma 2.4. Thus, $(\breve{\mathcal{M}}, \mathcal{H})$ forms a co-t-structure on $\mathcal{T}$. On the other hand, note that $\mathcal{H}$ is coresolving by assumption. We see that $\mathcal{T} = \check{\mathcal{H}} = \bigcup_{n \leqslant 0} \mathcal{H}[n]$ by Lemma 2.8(2) again. According to Corollary 3.4, we know that $\mathcal{M}$ is a silting subcategory of $\mathcal{T}$. Hence, $\mathcal{T} = \langle \mathcal{M} \rangle$, and so $\mathcal{T} = \bigcup_{n \geqslant 0} \breve{\mathcal{M}}[n]$ by Corollary 2.10. Thus, the co-t-structure $(\breve{\mathcal{M}}, \mathcal{H})$ is bounded. This completes the proof. $\square$

As a consequence of Theorem 3.5 and Proposition 3.6, we obtain the following promised Auslander–Reiten type correspondence for silting subcategories of $\mathcal{T}$.

Corollary 3.7. *The assignments*

$$\mathcal{M} \mapsto \mathcal{M}^{\perp_{i>0}} \quad \textit{and} \quad \mathcal{H} \mapsto {}^{\perp_{i>0}}\mathcal{H} \cap \mathcal{H}$$

give mutually inverse bijections between the following classes:

(1) *Silting subcategories $\mathcal{M}$ of $\mathcal{T}$.*

(2) *Subcategories $\mathcal{H}$ of $\mathcal{T}$, which is specially covariantly finite and coresolving in $\mathcal{T}$, such that $\check{\mathcal{H}} = \mathcal{T}$ and for any object $H \in \mathcal{H}$, there exists an integer i making $\mathrm{Hom}_{\mathcal{T}}(H, \mathcal{H}[\geqslant i]) = 0$.*

3.3. Auslander–Reiten type correspondences for silting objects

We say that two objects M and M' in $\mathcal{T}$ are *equivalent* provided that $\mathrm{add}M = \mathrm{add}M'$. The following result is a consequence of Corollary 3.7, which gives an Auslander–Reiten type correspondence for silting objects in $\mathcal{T}$.

Corollary 3.8. *The assignments*

$$M \mapsto M^{\perp_{i>0}} \quad \textit{and} \quad \mathcal{H} \mapsto N$$

where $\mathrm{add}\,N = {}^{\perp_{i>0}}\mathcal{H} \cap \mathcal{H}$, give mutually inverse bijections between the following classes:

(1) *Equivalent classes of silting objects M in $\mathcal{T}$.*

(2) *Subcategories $\mathcal{H}$ of $\mathcal{T}$, which is specially covariantly finite and coresolving in $\mathcal{T}$, such that $\check{\mathcal{H}} = \mathcal{T}$, ${}^{\perp_{i>0}}\mathcal{H} \cap \mathcal{H}$ is additively generated by an object N, and for any object $H \in \mathcal{H}$, there exists an integer i making $\mathrm{Hom}_{\mathcal{T}}(H, \mathcal{H}[\geqslant i]) = 0$.*

Recall that a complex $T \in K^{\mathrm{b}}(\operatorname{proj} R)$ is called *silting* [26] (resp., *tilting* [23]) if T is a silting (resp., tilting) object in $K^{\mathrm{b}}(\operatorname{proj} R)$. It is easy to see that every tilting complex in $K^{\mathrm{b}}(\operatorname{proj} R)$ is always silting. However, the converse does not hold true in general (see [26, Section 6]).

Suppose that T is a presilting complex in $K^{\mathrm{b}}(\operatorname{proj} R)$. According to [26, Corollary 2.6(1) and Remark 2.9], we know that $\widecheck{\operatorname{add} T} = {}^{\perp_{i>0}}T \cap \langle \operatorname{add} T\rangle$. This gives a characterization for a silting complex in $K^{\mathrm{b}}(\operatorname{proj} R)$.

Proposition 3.9. *Let T be a presilting complex in $K^{\mathrm{b}}(\operatorname{proj} R)$. Suppose that $\sup T = n$ for some integer n. Then T is a silting complex if and only if $R[-n] \in \widecheck{\operatorname{add} T}$.*

Proof. ($\Leftarrow$) Note that T is a presilting complex in $K^{\mathrm{b}}(\operatorname{proj} R)$ by assumption. We need only to show that $K^{\mathrm{b}}(\operatorname{proj} R) \subseteq \langle \operatorname{add} T\rangle$. Indeed, since $R[-n] \in \widecheck{\operatorname{add} T}$ by assumption, we see that $R \in \langle \operatorname{add} T\rangle$. Hence, $\operatorname{add} R \subseteq \langle \operatorname{add} T\rangle$. This yields that $K^{\mathrm{b}}(\operatorname{proj} R) = \langle \operatorname{add} R\rangle \subseteq \langle \operatorname{add} T\rangle$, as desired.

($\Rightarrow$) Since T is silting, we see that $R \in K^{\mathrm{b}}(\operatorname{proj} R) = \langle \operatorname{add} T\rangle$. This implies that $R[-n] \in \langle \operatorname{add} T\rangle$. On the other hand, it is obvious that $R[-n] \in {}^{\perp_{i>0}}(\operatorname{add} T)$ because $\sup T = n$ by assumption. Thus, by [26, Corollary 2.6(1) and Remark 2.9], we conclude that $R[-n] \in \widecheck{\operatorname{add} T}$. $\square$

As an immediate consequence of Proposition 3.9, we obtain the following characterization for a tilting complex in $K^{\mathrm{b}}(\operatorname{proj} R)$.

Corollary 3.10. *Let T be a pretilting complex in $K^{\mathrm{b}}(\operatorname{proj} R)$. Suppose that $\sup T = n$ for some integer n. Then T is a tilting complex if and only if $R[-n] \in \widecheck{\operatorname{add} T}$.*

For an integer n, denote by $K^{\leqslant n}(\operatorname{proj} R)$ the subcategory of $K^{\mathrm{b}}(\operatorname{proj} R)$ consisting of all complexes T such that $T_i = 0$ for all $i > n$.

Lemma 3.11. *Let T be a silting complex in $K^{\mathrm{b}}(\operatorname{proj} R)$. Suppose that $T \in K^{\leqslant n}(\operatorname{proj} R)$ for some integer n. Then we have $T^{\perp_{i>0}} \subseteq K^{\leqslant n}(\operatorname{proj} R)$.*

Proof. In view of Proposition 3.9, we see that $R[-n] \in \widecheck{\operatorname{add} T}$. This implies that

$$T^{\perp_{i>0}} \subseteq (R[-n])^{\perp_{i>0}} = R^{\perp_{i>n}}.$$

On the other hand, it is easy to check that $R^{\perp_{i>n}} = K^{\leqslant n}(\operatorname{proj} R)$. Thus, we obtain $T^{\perp_{i>0}} \subseteq K^{\leqslant n}(\operatorname{proj} R)$, as desired. $\square$

Let T be a presilting complex in $K^{\mathrm{b}}(\operatorname{proj} R)$. By [26, Corollary 2.6 (2) and Remark 2.9], we have $\widehat{\operatorname{add} T} = T^{\perp_{i>0}} \cap \langle \operatorname{add} T\rangle$.

Lemma 3.12. *Let $\mathcal{H}$ be a subcategory of $K^{\leqslant n}(\operatorname{proj} R)$, where n is an integer. Suppose that $\mathcal{H}$ is specially covariantly finite and coresolving in $K^{\rm b}(\operatorname{proj} R)$ such that $\check{\mathcal{H}} = K^{\rm b}(\operatorname{proj} R)$. Then ${}^{\perp_{i>0}}\mathcal{H} \cap \mathcal{H} = \operatorname{add} N$ for some silting complex N in $K^{\rm b}(\operatorname{proj} R)$.*

Proof. Let $\mathcal{M} = {}^{\perp_{i>0}}\mathcal{H} \cap \mathcal{H}$. Since $\mathcal{H}$ is specially covariantly finite and coresolving in $K^{\rm b}(\operatorname{proj} R)$ by assumption, it follows from Lemma 3.2 that $\mathcal{M}$ is a projective weak-generator in $\mathcal{H}$. Note that $R[-n] \in K^{\rm b}(\operatorname{proj} R) = \check{\mathcal{H}}$. It follows from Theorem 2.2 that there exists a family of triangles

$$\{\, X_i \to N_i \to X_{i+1} \to X_i[1] \,\}_{i=0}^{r}$$

in $K^{\rm b}(\operatorname{proj} R)$, where r is the coresolution dimension of $R[-n]$ with respect to $\mathcal{H}$, such that $X_0 = R[-n]$, $X_1 \in \check{\mathcal{M}}$, $X_{r+1} = 0$, $N_0 \in \mathcal{H}$ and $N_i \in \mathcal{M}$ for all $1 \leqslant i \leqslant r$. Since $\mathcal{H} \subseteq K^{\leqslant n}(\operatorname{proj} R)$ by assumption, we see that $R[-n] \in {}^{\perp_{i>0}}\mathcal{H}$. Note that X_1 is also in ${}^{\perp_{i>0}}\mathcal{H}$ (see Lemma 2.4). It follows that $N_0 \in \mathcal{M}$.

Take $N = \oplus_{i=0}^{r} N_i$. Then N is a presilting complex in $K^{\rm b}(\operatorname{proj} R)$. Moreover, according to the above argument, we see that $R[-n] \in \widecheck{\operatorname{add} N} \subseteq \langle \operatorname{add} N \rangle$. Hence, $R \in \langle \operatorname{add} N \rangle$. This implies that $K^{\rm b}(\operatorname{proj} R) = \langle \operatorname{add} N \rangle$. Thus, N is a silting complex in $K^{\rm b}(\operatorname{proj} R)$.

Next, we show that $\mathcal{M} \subseteq \operatorname{add} N$, which yields our desired result $\mathcal{M} = \operatorname{add} N$. To this end, let L be a complex in $\mathcal{M}$. Then $N \oplus L$ is also a silting complex in $K^{\rm b}(\operatorname{proj} R)$. This implies that $\langle \operatorname{add} N \rangle = \langle \operatorname{add}(N \oplus L) \rangle$. It follows from [26, Corollary 2.6 and Remark 2.9] that

$$L \in N^{\perp_{i>0}} \cap \langle \operatorname{add}(N \oplus L) \rangle = N^{\perp_{i>0}} \cap \langle \operatorname{add} N \rangle = \widehat{\operatorname{add} N}.$$

Now, it is easy to check that $L \in \operatorname{add} N$ by noting that L is also in ${}^{\perp_{i>0}}N$. Hence, $\mathcal{M} \subseteq \operatorname{add} N$, as desired. □

In 2013, an Auslander–Reiten type correspondence for silting complexes in $K^{\rm b}(\operatorname{proj} R)$ was established by the fourth author of this paper under the condition that $\operatorname{Hom}_{K^{\rm b}(\operatorname{proj} R)}$'s are finitely generated (see [26, Remark 3.3, Theorem 5.3 and Remark 5.4]). Now, we can remove this additional condition.

Corollary 3.13. *For any integer n, the assignments*

$$T \mapsto T^{\perp_{i>0}} \quad \textit{and} \quad \mathcal{H} \mapsto N,$$

where $\operatorname{add} N = {}^{\perp_{i>0}}\mathcal{H} \cap \mathcal{H}$, give mutually inverse bijections between the following classes:

(1) *Equivalent classes of silting complexes T in $K^{\leqslant n}(\operatorname{proj} R)$.*

(2) *Subcategories $\mathcal{H}$ of $K^{\leqslant n}(\operatorname{proj} R)$, which is specially covariantly finite and coresolving in $K^{\rm b}(\operatorname{proj} R)$, such that $\check{\mathcal{H}} = K^{\rm b}(\operatorname{proj} R)$.*

Proof. Based on Corollary 3.8, Lemma 3.11 and Lemma 3.12, we only need to show that for any complex $H \in \mathcal{H}$, there exists an integer i such that $\mathrm{Hom}_{K^{\mathrm{b}}(\mathrm{proj}\,R)}(H, \mathcal{H}[\geqslant i]) = 0$. To this end, suppose that $\inf H = l$ for some integer l. Note that $\mathcal{H}$ is contained in $K^{\leqslant n}(\mathrm{proj}\,R)$ by assumption. It is easy to check that the integer '$n - l + 1$' is the appropriate candidate for the required i. This completes the proof. $\square$

4. A relation on presilting subcategories

In this section, we introduce a relation on presilting subcategories of $\mathcal{T}$ (Definition 4.1). One can obtain another silting subcategory from a given one by means of such a relation (Proposition 4.4). We give a Bazzoni's characterization for two presilting subcategories satisfying this relation (Theorem 4.7). As an application, we obtain a Bazzoni's characterization for tilting complexes so that a corresponding result of the fourth author can be improved by removing the condition that $\mathrm{Hom}_{K^{\mathrm{b}}(\mathrm{proj}\,R)}$'s are finitely generated (Corollary 4.9).

Definition 4.1. Let $\mathcal{M}$ and $\mathcal{N}$ be two presilting subcategories of $\mathcal{T}$. We write "$\mathcal{M} \preceq \mathcal{N}$" provided that they satisfy the conditions

(1) $\mathcal{M} \subseteq \check{\mathcal{N}}$ and

(2) $\mathcal{N} \subseteq \widehat{\mathcal{M}}$.

If in addition there exists a non-negative integer n such that $\mathcal{M} \subseteq (\check{\mathcal{N}})_n$ (that is, for each object $M \in \mathcal{M}$, $M \in (\check{\mathcal{N}})_n$), then we write "$\mathcal{M} \preceq^n \mathcal{N}$".

Example 4.2. (1) Let T be a silting complex in $K^{\mathrm{b}}(\mathrm{proj}\,R)$. Suppose that $\sup T = n$ for some integer n. Then it is easy to see that $T[n] \in \widehat{\mathrm{add}\,R}$. Therefore, $\mathrm{add}\,(T[n]) \subseteq \widehat{\mathrm{add}\,R}$. On the other hand, according to Proposition 3.9, we know that $R[-n] \in \widecheck{\mathrm{add}\,T}$, which implies that $R \in \widecheck{\mathrm{add}\,(T[n])}$. This yields that $\mathrm{add}\,R \subseteq \widecheck{\mathrm{add}\,(T[n])}$. Thus, we have $\mathrm{add}\,R \preceq \mathrm{add}\,(T[n])$.

(2) Let R be an artin algebra and E, F two finitely generated right R-modules. Recall that the pair (E, F) is called *n-tilting* [17], where n is a non-negative integer, provided that both E and F are selforthogonal such that $F \in (\widehat{\mathrm{add}\,E})_n$ and $E \in (\widecheck{\mathrm{add}\,F})_n$ (here, the symbol $(\widehat{\mathrm{add}\,E})_n$ (resp., $(\widecheck{\mathrm{add}\,F})_n$) denotes the subcategory of R-mod consisting of all modules N such that there exists an exact sequence

$$0 \to E_n \to E_{n-1} \to \cdots \to E_0 \to N \to 0 \quad (\text{resp.}, 0 \to N \to F_0 \to \cdots \to F_{n-1} \to F_n \to 0)$$

with each $E_i \in \mathrm{add}\,E$ (resp., $F_i \in \mathrm{add}\,F$)). We refer the reader to [27, Example 3.2] for more examples of tilting pairs.

Let (E, F) be an n-tilting pair in $\mathrm{mod}\,R$. Consider both E and F as stalk complexes. We see that $\mathrm{add}\,E \preceq^n \mathrm{add}\,F$ in $D^{\mathrm{b}}(\mathrm{mod}\,R)$.

Proposition 4.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be two presilting subcategories of $\mathcal{T}$ such that $\mathcal{M} \preceq \mathcal{N}$. Then for a non-negative integer n, $\mathcal{N} \subseteq (\widehat{\mathcal{M}})_n$ if and only if $\mathcal{M} \subseteq (\check{\mathcal{N}})_n$.*

Proof. We only show the only-if-part. The if-part can be proved by a dual argument.

Let M be an object in $\mathcal{M}$. We want to prove that $M \in (\check{\mathcal{N}})_n$. Indeed, since $\mathcal{M} \preceq \mathcal{N}$ by assumption, we know that $M \in \check{\mathcal{N}}$. Hence, there exists a non-negative r such that $M \in (\check{\mathcal{N}})_r$. If $r \leqslant n$ then we are done. Suppose that $r > n$, and consider a family of triangles

$$\{\, M_i \to N_i \to M_{i+1} \to M_i[1] \,\}_{i=0}^{r}$$

in $\mathcal{T}$ with each $N_i \in \mathcal{N}$, $M_0 = M$ and $M_{r+1} = 0$. Apply the functor $\mathrm{Hom}_{\mathcal{T}}(M_r, -)$ to these triangles. Note that $M_r \cong N_r \in \mathcal{N}$ and $\mathcal{N}$ is presilting. It follows that

$$\begin{aligned} \mathrm{Hom}_{\mathcal{T}}(M_r, M_{r-1}[1]) &\cong \mathrm{Hom}_{\mathcal{T}}(M_r, M_{r-2}[2]) \\ &\cong \cdots \\ &\cong \mathrm{Hom}_{\mathcal{T}}(M_r, M_0[r]) = \mathrm{Hom}_{\mathcal{T}}(M_r, M[r]). \end{aligned}$$

Since $\mathcal{N} \subseteq (\widehat{\mathcal{M}})_n$ by assumption, we deduce that $\mathcal{N} \subseteq {}^{\perp_{i>n}}(\mathcal{M}^{\perp_{i>0}})$. This yields that $\mathrm{Hom}_{\mathcal{T}}(M_r, M[r]) = 0$ as $M_r \in \mathcal{N}$ and $M \in \mathcal{M}^{\perp_{i>0}}$. Hence, the triangle

$$M_{r-1} \to N_{r-1} \to M_r \to M_{r-1}[1]$$

splits. This shows that $M \in (\check{\mathcal{N}})_{r-1}$.

Repeating the above process, we can finally obtain that $M \in (\check{\mathcal{N}})_n$. Thus, $\mathcal{M} \subseteq (\check{\mathcal{N}})_n$. This completes the proof. $\square$

Motivated by Example 4.2(1), we obtain the following result. It gives us a way to obtain a silting subcategory from another one.

Proposition 4.4. *Let $\mathcal{M}$ and $\mathcal{N}$ be two presilting subcategories of $\mathcal{T}$ such that $\mathcal{M} \preceq \mathcal{N}$. Then $\mathcal{M}$ is silting if and only if $\mathcal{N}$ is silting.*

Proof. We only deal with the if-part. The only-if-part is dual.

To prove that $\mathcal{M}$ is a silting subcategory, we need only to show that $\mathcal{T} \subseteq \langle \mathcal{M} \rangle$. Indeed, since $\mathcal{N} \subseteq \widehat{\mathcal{M}}$, it follows that $\mathcal{N} \subseteq \langle \mathcal{M} \rangle$. This yields $\langle \mathcal{N} \rangle \subseteq \langle \mathcal{M} \rangle$. Note that $\mathcal{N}$ is silting by assumption. We have $\mathcal{T} = \langle \mathcal{N} \rangle$. Thus, $\mathcal{T} \subseteq \langle \mathcal{M} \rangle$, as desired. $\square$

Next, we give a Bazzoni's characterization for two presilting subcategories $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{T}$ satisfying $\mathcal{M} \preceq^n \mathcal{N}$. To archive the goal, we introduce the following subcategory of $\mathcal{T}$.

Let n be a non-negative integer. We use the symbol $\mathrm{Pres}^n_{\mathcal{M}^{\perp_{i>0}}}(\mathcal{N})$ to denote the subcategory of $\mathcal{T}$ consisting of all objects T such that there exists a family of triangles

$$\{\, T_{i+1} \to N_i \to T_i \to T_{i+1}[1] \,\}_{i=0}^{n-1}$$

in $\mathcal{T}$ with each $N_i \in \mathcal{N}$, $T_n \in \mathcal{M}^{\perp_{i>0}}$ and $T_0 = T$.

By noting that $0 \in \mathcal{N}$ and $\mathcal{M}^{\perp_{i>0}}$ is coresolving, we have the next result.

Lemma 4.5. *Let $\mathcal{M}, \mathcal{N}$ be two subcategories of $\mathcal{T}$ and n a non-negative integer. Then*
(1) *If $\mathcal{M}$ is presilting, then $(\mathcal{M}^{\perp_{i>0}})[n] \subseteq Pres^n_{\mathcal{M}^{\perp_{i>0}}}(\mathcal{N})$.*
(2) *If $\mathcal{N} \subseteq \mathcal{M}^{\perp_{i>0}}$, then $Pres^n_{\mathcal{M}^{\perp_{i>0}}}(\mathcal{N}) \subseteq \mathcal{M}^{\perp_{i>0}}$.*

The proof of the following lemma is almost the same as in [26, Proposition 2.3 (1)].

Lemma 4.6. *Let $\mathcal{M}$ be a presilting subcategory of $\mathcal{T}$ and n a non-negative integer. Then we have*

$$(\widehat{\mathcal{M}})_n = {}_{\mathcal{M}}\mathscr{X} \cap {}^{\perp_{i>n}}(\mathcal{M}^{\perp_{i>0}}).$$

We present now the promised Bazzoni's characterization for presilting subcategories.

Theorem 4.7. *Let $\mathcal{M}, \mathcal{N}$ be two presilting subcategories of $\mathcal{T}$ and n a non-negative integer. Suppose that $\mathcal{M}$ is silting. Then $\mathcal{M} \preceq^n \mathcal{N}$ if and only if $Pres^n_{M^{\perp_{i>0}}}(\mathcal{N}) = {}_{\mathcal{N}}\mathscr{X}$ and $\mathcal{N} \subseteq \mathcal{M}^{\perp_{i>0}}$.*

Proof. ($\Leftarrow$) Firstly, let N be an object in $\mathcal{N}$. We wish to show that $N \in (\widehat{\mathcal{M}})_n$, which will imply that $\mathcal{N} \subseteq (\widehat{\mathcal{M}})_n$. To this end, in view of Lemma 4.6, it suffices to prove that $N \in {}_{\mathcal{M}}\mathscr{X} \cap {}^{\perp_{i>n}}(\mathcal{M}^{\perp_{i>0}})$. Indeed, on the one hand, since $N \in \mathcal{M}^{\perp_{i>0}}$ and $\mathcal{M}$ is a silting subcategory of $\mathcal{T}$ by assumption, it follows from Lemma 2.6 that $N \in {}_{\mathcal{M}}\mathscr{X}$. On the other hand, in view of Lemma 4.5, we know that $(\mathcal{M}^{\perp_{i>0}})[n] \subseteq \mathrm{Pres}^n_{\mathcal{M}^{\perp_{i>0}}}(\mathcal{N})$. Therefore, $(\mathcal{M}^{\perp_{i>0}})[n] \subseteq {}_{\mathcal{N}}\mathscr{X} \subseteq \mathcal{N}^{\perp_{i>0}}$, and so $\mathcal{M}^{\perp_{i>0}} \subseteq (\mathcal{N}^{\perp_{i>0}})[-n] = \mathcal{N}^{\perp_{i>n}}$. It follows that ${}^{\perp_{i>n}}(\mathcal{N}^{\perp_{i>n}}) \subseteq {}^{\perp_{i>n}}(\mathcal{M}^{\perp_{i>0}})$. It is clear that $N \in {}^{\perp_{i>n}}(\mathcal{N}^{\perp_{i>n}})$ as $\mathcal{N}$ is presilting. Thus, $N \in {}^{\perp_{i>n}}(\mathcal{M}^{\perp_{i>0}})$, as desired.

Next, we show that $\mathcal{M} \subseteq (\check{\mathcal{N}})_n$. To this end, let M be an object in $\mathcal{M}$. Since $\mathcal{M}$ is presilting by assumption, it follows from Lemma 4.5 that $M[n] \in \mathrm{Pres}^n_{\mathcal{M}^{\perp_{i>0}}}(\mathcal{N})$. Hence, $M[n] \in {}_{\mathcal{N}}\mathscr{X}$. This yields that $M \in (\widecheck{{}_{\mathcal{N}}\mathscr{X}})_n$. Note that ${}_{\mathcal{N}}\mathscr{X}$ admits a weak-generator $\mathcal{N}$. It follows from Theorem 2.2 that there exists a triangle

$$M[-1] \to X \to K \to M \qquad (*)$$

in $\mathcal{T}$ with $X \in {}_{\mathcal{N}}\mathscr{X} = \mathrm{Pres}^n_{\mathcal{M}^{\perp_{i>0}}}(\mathcal{N})$ and $K \in (\check{\mathcal{N}})_n$. In view of Lemma 4.5, we deduce that $X \in \mathcal{M}^{\perp_{i>0}}$. This implies that $(*)$ splits, and hence, M is a direct summand of K. Consequently, we have $M \in (\check{\mathcal{N}})_n$, as desired.

($\Rightarrow$) Since $\mathcal{M}$ is presilting and $\mathcal{N} \subseteq \widehat{\mathcal{M}}$ by assumption, we conclude that $\mathcal{N} \subseteq \mathcal{M}^{\perp_{i>0}}$. Note that $\mathcal{M} \subseteq \check{\mathcal{N}}$. It follows that $\mathcal{N}^{\perp_{i>0}} \subseteq \mathcal{M}^{\perp_{i>0}}$. Hence, ${}_{\mathcal{N}}\mathscr{X} \subseteq \mathrm{Pres}^n_{\mathcal{M}^{\perp_{i>0}}}(\mathcal{N})$.

For the other containment, let T be an object in $\mathrm{Pres}^n_{\mathcal{M}^{\perp_{i>0}}}(\mathcal{N})$ and N an object in $\mathcal{N}$. Then there exists a family of triangles

$$\{\, T_{i+1} \to N_i \to T_i \to T_{i+1}[1] \,\}_{i=0}^{n-1}$$

in $\mathcal{T}$ with each $N_i \in \mathcal{N}$, $T_n \in \mathcal{M}^{\perp_{i>0}}$ and $T_0 = T$. Applying the functor $\operatorname{Hom}_{\mathcal{T}}(N, -)$ to these triangles, we obtain

$$\begin{aligned}\operatorname{Hom}_{\mathcal{T}}(N, T[i]) &\cong \operatorname{Hom}_{\mathcal{T}}(N, N_1[i+1]) \\ &\cong \operatorname{Hom}_{\mathcal{T}}(N, N_2[i+2]) \\ &\cong \cdots \\ &\cong \operatorname{Hom}_{\mathcal{T}}(N, N_n[i+n])\end{aligned}$$

for $i \geqslant 1$. Note that $N \in (\widehat{\mathcal{M}})_n$ and $T_n \in \mathcal{M}^{\perp_{i>0}}$. It is easy to check that $\operatorname{Hom}_{\mathcal{T}}(N, N_n[i+n]) = 0$, which yields that $T \in N^{\perp_{i>0}}$. Hence, $T \in \mathcal{N}^{\perp_{i>0}}$. This shows that $\operatorname{Pres}^n_{\mathcal{M}^{\perp_{i>0}}}(\mathcal{N}) \subseteq \mathcal{N}^{\perp_{i>0}} = {}_{\mathcal{N}}\mathscr{X}$ because $\mathcal{N}$ is also a silting subcategory of $\mathcal{T}$ (see Proposition 4.4 and Lemma 2.6). This completes the proof. $\square$

In view of Proposition 3.9 and Theorem 4.7, we obtain the following Bazzoni's characterization for any silting complex in $K^{\mathrm{b}}(\operatorname{proj} R)$.

Corollary 4.8. *Let T be a presilting complex in $K^{\mathrm{b}}(\operatorname{proj} R)$. Suppose that* $\sup T = n$ *and* $\inf T = l$, *where n and l are two integers. Then the following statements are equivalent:*

(1) *T is a silting complex.*
(2) *$\operatorname{add} R \preceq^{n-l} \operatorname{add}(T[n])$.*
(3) *$\operatorname{Pres}^{n-l}_{R^{\perp_{i>0}}}(\operatorname{add}(T[n])) = {}_{\operatorname{add}(T[n])}\mathscr{X}$.*

Proof. Since $\sup T = n$ by assumption, we see that $T[n] \in R^{\perp_{i>0}}$. This implies that $\operatorname{add}(T[n]) \subseteq (\operatorname{add} R)^{\perp_{i>0}}$. Thus, the equivalence of (2) and (3) follows from Theorem 4.7 by noting that $\operatorname{add} R$ is obviously a silting subcategory of $K^{\mathrm{b}}(\operatorname{proj} R)$.

(1) $\Rightarrow$ (2) According to Proposition 3.9, we know that $R[-n] \in \widecheck{\operatorname{add} T}$. This implies that $R \in \widecheck{\operatorname{add}(T[n])}$. Hence, $\operatorname{add} R \subseteq \widecheck{\operatorname{add}(T[n])}$. On the other hand, note that $\sup T = n$ and $\inf T = l$ by assumption. It is easy to check that $T[n] \in (\widehat{\operatorname{add} R})_{n-l}$, which implies that $\operatorname{add}(T[n]) \subseteq (\widehat{\operatorname{add} R})_{n-l}$. Thus, $\operatorname{add} R \preceq \operatorname{add}(T[n])$. In particular, by Proposition 4.3, we conclude that $\operatorname{add} R \subseteq (\widecheck{\operatorname{add}(T[n])})_{n-l}$. This yields that $\operatorname{add} R \preceq^{n-l} \operatorname{add}(T[n])$, as desired.

(2) $\Rightarrow$ (1) To prove that T is a silting complex, by Proposition 3.9, it is enough to show that $R[-n] \in \widecheck{\operatorname{add} T}$. Indeed, it follows from the fact that $R \in \widecheck{\operatorname{add}(T[n])}$ since we have $\operatorname{add} R \preceq \operatorname{add}(T[n])$ by assumption. $\square$

As an immediate consequence of Corollary 4.8, we obtain the following Bazzoni's characterization for a tilting complex. It improves the corresponding result [26, Corollary 4.5] by removing the condition that $\operatorname{Hom}_{K^{\mathrm{b}}(\operatorname{proj} R)}$'s are finitely generated (see [26, Remark 3.3]).

Corollary 4.9. *Let T be a pretilting complex in $K^{\mathrm{b}}(\operatorname{proj} R)$. Suppose that* $\sup T = n$ *and* $\inf T = l$, *where n and l are two integers. Then the following statements are equivalent:*

(1) T *is a tilting complex.*
(2) $add\, R \preceq^{n-l} add\,(T[n])$.
(3) $Pres^{n-l}_{R^{\perp_{i>0}}}(add\,(T[n])) = {}_{add\,(T[n])}\mathscr{X}$.

Lemma 4.10. *Let* $(\mathcal{A}, \mathcal{B})$ *be a co-t-structure on* $\mathcal{T}$ *with co-heart* $\mathcal{S}$ *and* n *a non-negative integer. Then there exists an equality* $(\widehat{\mathcal{S}})_n = \mathcal{A}[n] \cap \mathcal{B}$.

Proof. We proceed by induction on n. Note that the equality $(\widehat{\mathcal{S}})_i = \mathcal{S} * \mathcal{S}[1] * \cdots * \mathcal{S}[i]$ holds for all $i \geqslant 0$.

If $n = 1$, then the result follows from [14, Lemma 2.1]. Suppose now that the conclusion holds for $n-1$, that is, there exist equalities $(\widehat{\mathcal{S}})_{n-1} = \mathcal{S}*\mathcal{S}[1]*\cdots*\mathcal{S}[n-1] = \mathcal{A}[n-1]\cap\mathcal{B}$. We show next that the conclusion also holds for n.

It is easy to see that the containment $(\widehat{\mathcal{S}})_n \subseteq \mathcal{A}[n] \cap \mathcal{B}$ holds true since $\mathcal{S}[i] \subseteq \mathcal{A}[n] \cap \mathcal{B}$ for all $0 \leqslant i \leqslant n$ by the condition (1) of Definition 2.11. To prove the other containment, let K be an object in $\mathcal{A}[n] \cap \mathcal{B}$. Then $K[-1] \in \mathcal{A}[n-1] \cap \mathcal{B}[-1]$. By the condition (3) of Definition 2.11, there exists a triangle

$$A[-1] \to K[-1] \to B \to A$$

in $\mathcal{T}$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Since both $B[-1]$ and $K[-1]$ belong to $\mathcal{B}[-1]$, we deduce that $A[-1] \in \mathcal{B}[-1]$. This implies that $A[-1] \in \mathcal{A}[-1] \cap \mathcal{B}[-1]$. Hence, $A \in \mathcal{A} \cap \mathcal{B} = \mathcal{S}$. On the other hand, since both $K[-1]$ and A belong to $\mathcal{A}[n-1]$, we see that $B \in \mathcal{A}[n-1]$ as well. Hence, $B \in \mathcal{A}[n-1] \cap \mathcal{B} = \mathcal{S} * \mathcal{S}[1] * \cdots * \mathcal{S}[n-1]$ by the induction assumption. This yields that $B[1] \in \mathcal{S}[1] * \cdots * \mathcal{S}[n]$. Thus, we have $K \in \mathcal{S} * \mathcal{S}[1] * \cdots * \mathcal{S}[n] = (\widehat{\mathcal{S}})_n$, which implies that $\mathcal{A}[n] \cap \mathcal{B} \subseteq (\widehat{\mathcal{S}})_n$. This completes the proof. □

We conclude the article by the following result, which gives some equivalent conditions for two silting subcategories to satisfy the relation "$\preceq^n$".

Theorem 4.11. *Let* $\mathcal{M}, \mathcal{N}$ *be two silting subcategories of* $\mathcal{T}$ *and* n *a non-negative integer. Suppose that* $(\mathcal{A}, \mathcal{B})$ *(resp.,* $(\mathcal{A}', \mathcal{B}')$*) is the bounded co-t-structure corresponding to* $\mathcal{M}$ *(resp.,* $\mathcal{N}$*). Then the following conditions are equivalent:*
(1) $\mathcal{M} \preceq^n \mathcal{N}$.
(2) $\mathcal{N} \subseteq (\widehat{\mathcal{M}})_n$.
(3) $\mathcal{M} \subseteq (\check{\mathcal{N}})_n$.
(4) $\mathcal{A} \subseteq \mathcal{A}' \subseteq \mathcal{A}[n]$.
(5) $\mathcal{B}[n] \subseteq \mathcal{B}' \subseteq \mathcal{B}$.

Proof. It suffices to prove that the consitions (2), (3) and (4) are equivalent. We show in the following (2) ⇔ (4), the equivalence of (3) and (4) can be proved by a dual argument.

(2) ⇒ (4) Note that $\mathcal{A}'$ (resp., $\mathcal{A}$) is the smallest extension closed subcategory of $\mathcal{T}$ containing $\mathcal{N}[\leqslant 0]$ (resp., $\mathcal{M}[\leqslant 0]$) and $\mathcal{B}'$ (resp., $\mathcal{B}$) is the smallest extension closed subcategory of $\mathcal{T}$ containing $\mathcal{N}[\geqslant 1]$ (resp., $\mathcal{M}[\geqslant 1]$) (see Fact 2.12).

Since $\mathcal{N} \subseteq (\widehat{\mathcal{M}})_n = \mathcal{M} * \mathcal{M}[1] * \cdots * \mathcal{M}[n]$ by assumption, we see that $\mathcal{A}'$ is contained in the smallest extension closed subcategory of $\mathcal{T}$ containing $\mathcal{M}[\leqslant n]$. Hence, we have $\mathcal{A}' \subseteq \mathcal{A}[n]$. On the other hand, by a similar argument, one can obtain that $\mathcal{B}' \subseteq \mathcal{B}$. This implies that $\mathcal{A} \subseteq \mathcal{A}'$. Thus, we have $\mathcal{A} \subseteq \mathcal{A}' \subseteq \mathcal{A}[n]$, as desired.

(4) $\Rightarrow$ (2) Since $\mathcal{A} \subseteq \mathcal{A}' \subseteq \mathcal{A}[n]$ by assumption, we see that $\mathcal{B}[n] \subseteq \mathcal{B}' \subseteq \mathcal{B}$. Hence, $\mathcal{N} = \mathcal{A}' \cap \mathcal{B}' \subseteq \mathcal{A}[n] \cap \mathcal{B} = (\widehat{\mathcal{M}})_n$, where the last equality holds by Lemma 4.10. □

Let $(\mathcal{A}, \mathcal{B})$ be a bounded co-t-structure on $\mathcal{T}$ with co-heart $\mathcal{S}$. Following [14], a bounded co-t-structure $(\mathcal{A}', \mathcal{B}')$ is said to be *intermediate* (with respect to $(\mathcal{A}, \mathcal{B})$) if $\mathcal{A} \subseteq \mathcal{A}' \subseteq \mathcal{A}[1]$. A subcategory $\mathcal{S}'$ of $\mathcal{T}$ is called a *two-term* silting subcategory of $\mathcal{T}$ (with respect to $\mathcal{S}$) if it is a silting subcategory of $\mathcal{T}$ satisfying $\mathcal{S}' \subseteq \mathcal{S} * \mathcal{S}[1]$.

Recently, Iyama, Jørgensen and Yang showed that the assignment $(\mathcal{A}', \mathcal{B}') \mapsto \mathcal{S}'$ gives a bijective correspondence between the intermediate co-t-structures on $\mathcal{T}$ and the two-term silting subcategories of $\mathcal{T}$ (with respect to the given co-t-structure), where $\mathcal{S}'$ is the co-heart of $(\mathcal{A}', \mathcal{B}')$ (see [14, Theorem 2.3]).

Note that the condition $\mathcal{N} \subseteq (\widehat{\mathcal{M}})_n$ in the above theorem is equivalent to

$$\mathcal{N} \subseteq \mathcal{M} * \mathcal{M}[1] * \cdots * \mathcal{M}[n].$$

We see that Theorem 4.11 extends [14, Theorem 2.3] to the $(n+1)$-term case.

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