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# 2-blocks with abelian hyperfocal subgroup of rank 3

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ABSTRACT

In this paper, we calculate the numbers of irreducible ordinary characters and irreducible Brauer characters in a block of a finite group $G$, whose associated fusion system over a 2-subgroup $P$ of $G$ (which is a defect group of the block) has an abelian hyperfocal subgroup of rank 3.



## 1. Introduction

Throughout this paper, $p$ is a prime and $\mathcal{O}$ is a complete discrete valuation ring with residue field $k$ and with fraction field $\mathcal{K}$; we always assume that $\mathcal{K}$ is of characteristic 0 and big enough for finite groups discussed below and that $k$ is algebraically closed of characteristic $p$.

Let $G$ be a finite group and $b$ a block (idempotent) of $G$ over $\mathcal{O}$. Let $(P, b_P)$ be a maximal $b$-Brauer pair. Denote by $Q$ the hyperfocal subgroup of $P$ with respect to $(P, b_P)$ (see [9]), which is generated by the subsets $[U, x]$, where $(U, b_U)$ runs on the

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set of $b$-Brauer pairs such that $(U, b_U) \subseteq (P, b_P)$, $x$ runs on the set of $p'$-elements of $N_G(U, b_U)$ and $[U, x]$ denotes the set of commutators $uxu^{-1}x^{-1}$ for $u \in U$.

Let $c$ be the Brauer correspondent of $b$ in $N_G(Q)$. Rouquier conjectures that the block algebras $\mathcal{O}Gb$ and $\mathcal{O}N_G(Q)c$ are basically Rickard equivalent when $Q$ is abelian (see [8]). Denote by $l(b)$ and $k(b)$ the number of irreducible Brauer and ordinary characters belonging to the block $b$ respectively. The conjecture implies that $l(b) = l(c)$ and $k(b) = k(c)$ when the hyperfocal subgroup $Q$ is abelian. The two equations have been investigated by Watanabe when $Q$ is cyclic (see [13]) and by us when $Q$ is $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$ (see [6]), where $\mathbb{Z}_{2^n}$ denotes the cyclic group of order $2^n$. In the latter case, we only get a partial result.

In this paper, we continue to investigate the two equations when $Q$ is an abelian 2-group of rank 3. We use the strategy in [6,13], but the computation here is more difficult. In order to state our main result, we need the Brauer category of the block $b$ and the fusion control. The Brauer category $\mathcal{F}_{(P,\, b_P)}(G, b)$ of the block $b$ is a category with objects all $b$-Brauer pairs contained in $(P, b_P)$ and with morphisms induced by conjugations of elements of $G$. The block $b$ is controlled by a subgroup $H$ of $G$ if any morphism in $\mathcal{F}_{(P,\, b_P)}(G, b)$ is induced by the conjugation of some element in $H$ (see [12, §49]).

**Theorem 1.1.** *Keep the notation as above and assume that $Q$ is an abelian 2-group of rank 3. Then the block $b$ is controlled by $N_G(P, b_P)$ and the inertial index of the block $b$ has only two possibilities: 7 or 21. Moreover, supposing $|Q| \leq |Z(P)|$ and denoting by $b_0$ the Brauer correspondent of $b$ in $N_G(P)$, we have*

**(i)** *$l(b) = l(b_0) = 7$ and $k(b) = k(b_0)$, if the inertial index of the block $b$ is 7;*

**(ii)** *$l(b) = l(b_0) = 5$ and $k(b) = k(b_0)$, if the inertial index of the block $b$ is 21.*

**Remark 1.2.** By Propositions 4.1 and 5.1 below, the condition $|Q| \leq |Z(P)|$ in Theorem 1.1 is automatically satisfied when $Q$ is an elementary abelian 2-group of order 8.

**Example 1.3.** There are examples where the hypotheses of Theorem 1.1 are not satisfied. Assume that $Q$ is $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$. There is an involution $t$ in the center of the automorphism group $\mathrm{Aut}(Q)$ of $Q$, of which the fixed points are the subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ of $Q$. Let $F_7$ be a Sylow 7-subgroup of $\mathrm{Aut}(Q)$. Set $L = Q \rtimes (F_7 \times \langle t \rangle)$. Take any $2'$-group $H$. Denote by $G$ the wreath product $H \wr L$. Let $b$ be the principal block of $G$ and $P$ a Sylow 2-sugbroup containing $Q$. Then $Q$ is a hyperfocal subgroup of the block $b$ and the inertial index of the block $b$ is 7. But it is easy to check that $Z(P)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Similarly, with the notation above, there is a subgroup of $\mathrm{Aut}(Q)$ of order 21. Denote it by $F_{21}$. Set $K = Q \rtimes (F_{21} \times \langle t \rangle)$. Denote by $\tilde{G}$ the wreath product $H \wr K$. Consider the principal block $\tilde{b}$ of $\tilde{G}$ and a Sylow 2-subgroup $\tilde{P}$ containing $Q$. Then $Q$ is a hyperfocal subgroup of the block $\tilde{b}$ and the inertial index of the block $\tilde{b}$ is 21. But the center $Z(\tilde{P})$ of $\tilde{P}$ is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

The notation and assumption in this section will be kept in the rest of the paper. In particular, $p$ is 2.

## 2. Preliminaries

Denote by $Q_0$ the subgroup of $Q$ which is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The following lemma is well-known.

**Lemma 2.1.** ([2, Chapter 6, 6.2]) *The automorphism group* $\mathrm{Aut}(Q_0)$ *of* $Q_0$*, denoted by* $A$*, is isomorphic to* $\mathrm{GL}(3, 2)$*, the simple group of order* 168*. Denote by* $A_2, A_3, A_7$ *a Sylow* 2*-subgroup and Sylow* 3*-subgroup and Sylow* 7*-subgroup of* $A$ *respectively. Then the following holds.*

**(1).** *If* $\theta \in \mathrm{Aut}(Q_0)$ *has order* 7*, then* $\theta$ *acts freely on* $Q_0 - \{1\}$*;*

**(2).** $|N_A(A_2)| = 8$, $|N_A(A_3)| = 6$ *and* $|N_A(A_7)| = 21$*;*

**(3).** $A_2$ *is isomorphic to the dihedral group* $\mathrm{D}_8$*, and* $N_A(A_3)$ *is isomorphic to the symmetric group* $\mathrm{S}_3$ *on* 3 *symbols;*

**(4).** $A$ *has no proper subgroup of index less than* 7*;*

**(5).** $A$ *has no element of order* 6*;*

**(6).** *Let* $U$ *be a Klein four group of* $A_2$*. Then* $N_A(U) \cong \mathrm{S}_4$*, the symmetric group on* 4 *symbols;*

**(7).** $A$ *has two isomorphism classes of maximal subgroups, one of which is* $\mathrm{S}_4$*, another one of which has order* 21*.*

As in [13], a $p$-complement of $N_G(P, b_P)/C_G(P)$ is called an inertial group of the block $b$. Inertial groups of the block $b$ are isomorphic to $N_G(P, b_P)/PC_G(P)$ and their order is the inertial index of the block $b$. Let $R$ be a normal $p$-subgroup of $G$ such that $|G : C_G(R)|$ is a $p$-power. For any subset $X$ of $G$, denote by $\bar{X}$ the image of $X$ under the canonical surjective $G \longrightarrow G/R$.

**Lemma 2.2.** *Keep the notation and the assumption in the paragraph above. Let* $\bar{b}$ *be the block of* $\bar{G}$ *determined by the block* $b$*. Then* $\bar{Q}$ *is a hyperfocal subgroup of the block* $\bar{b}$ *and an inertial group of the block* $\bar{b}$ *is isomorphic to that of the block* $b$*. Moreover, the block* $b$ *is controlled by the normalizer of its maximal Brauer pair if and only if the block* $\bar{b}$ *is controlled by the normalizer of its maximal Brauer pair.*

**Proof.** It is well known that $\bar{P}$ is a defect group of the block $\bar{b}$. For any subgroup $T$ of $P$ containing $R$, by the proof of [13, Lemma 8] the $b$-Brauer pair $(T, b_T)$ determines a unique $\bar{b}$-Brauer pair $(\bar{T}, \bar{b}_{\bar{T}})$. Let $\hat{C}_G(T)$ be the preimage of $C_{\bar{G}}(\bar{T})$ in $G$ and $\hat{N}_G(T, b_T)$ the preimage of $N_{\bar{G}}(\bar{T}, \bar{b}_{\bar{T}})$ in $G$. Put $\bar{Q}'$ be the hyperfocal subgroup of $\bar{P}$

with respect to $(\bar{P}, \bar{b}_{\bar{P}})$. Then by [9, 1.7], $\bar{Q}' = \langle [\bar{T}, O^p(N_{\bar{G}}(\bar{T}, \bar{b}_{\bar{T}})] \mid R \le T \le P\rangle$ and $Q = \langle [T, O^p(N_G(T, b_T))] \mid R \le T \le P\rangle$ since $R$ is normal in $G$. It is clear that $O^p(N_G(T, b_T)) \le O^p(\hat{N}_G(T, b_T))$. We have $\bar{Q} \le \bar{Q}'$. At the same time $\bar{Q}' \le \bar{Q}$ by the proof of [13, Lemma 8]. So we have $\bar{Q}' = \bar{Q}$.

It is clear that $C_G(T) \le \hat{C}_G(T)$ and $|\hat{C}_G(T) : C_G(T)|$ is a $p$-power and $\hat{N}_G(T, b_T) = \hat{C}_G(T)N_G(T, b_T)$ (see [13, Lemma 8]). So an inertial group of the block $\bar{b}$ is isomorphic to $N_G(P, b_P)/(N_G(P, b_P) \cap P\hat{C}_G(P))$. Since $PC_G(P)/C_G(P)$ is a Sylow $p$-subgroup of $N_G(P, b_P)/C_G(P)$ and $\hat{C}_G(P)/C_G(P)$ is a $p$-group, we have $N_G(P, b_P) \cap P\hat{C}_G(P) = PC_G(P)$. So the two blocks $b$ and $\bar{b}$ have isomorphic inertial groups.

Suppose that the block $b$ is controlled by $N_G(P, b_P)$. Assume that $(\bar{S}, \bar{b}_{\bar{S}})$ is an essential $\bar{b}$-Brauer pair which is contained in $(\bar{P}, \bar{b}_{\bar{P}})$. Then the $\bar{b}$-Brauer pair $(\bar{S}, \bar{b}_{\bar{S}})$ is uniquely determined by a $b$-Brauer pair $(S, b_S)$ such that $R \le S \le P$. Since the block $b$ is controlled by $N_G(P, b_P)$, we have

$$\hat{N}_G(S, b_S) = \hat{C}_G(S)N_G(S, b_S) = \hat{C}_G(S)\big(N_G(P, b_P) \cap N_G(S)\big).$$

This implies that $N_{\bar{P}}(\bar{S})C_{\bar{G}}(\bar{S})/C_{\bar{G}}(\bar{S})$ is a normal $p$-subgroup of $N_{\bar{G}}(\bar{S}, \bar{b}_{\bar{S}})/C_{\bar{G}}(\bar{S})$. Then we have $N_{\bar{P}}(\bar{S})C_{\bar{G}}(\bar{S}) = \bar{S}C_{\bar{G}}(\bar{S})$. Since $C_{\bar{P}}(\bar{S}) \le \bar{S}$, we have $N_{\bar{P}}(\bar{S}) = \bar{S}$ which is impossible. So the block $\bar{b}$ is controlled by $N_{\bar{G}}(\bar{P}, \bar{b}_{\bar{P}})$. Conversely, assume that $(D, b_D)$ is an essential $b$-Brauer pair. So $R \le D$ since $R$ is normal in $G$. Since the block $\bar{b}$ is controlled by $N_{\bar{G}}(\bar{P}, \bar{b}_{\bar{P}})$, $N_{\bar{G}}(\bar{D}, \bar{b}_{\bar{D}}) = C_{\bar{G}}(\bar{D})\big(N_{\bar{G}}(\bar{P}, \bar{b}_{\bar{P}}) \cap N_{\bar{G}}(\bar{D}, \bar{b}_{\bar{D}})\big)$. So $\hat{N}_G(D, b_D) = \hat{C}_G(D)\big(\hat{N}_G(P, b_P) \cap \hat{N}_G(D, b_D)\big)$. This means that $N_P(D)\big(\hat{C}_G(D) \cap N_G(D, b_D)\big)/C_G(D)$ is a normal $p$-subgroup of $N_G(D, b_D)/C_G(D)$ since $\hat{C}_G(D)/C_G(D)$ is a $p$-group. So $N_P(D)\big(\hat{C}_G(D) \cap N_G(D, b_D)\big) = DC_G(D)$. Hence $N_P(D) = D$ since $C_P(D) \le D$. That is impossible. We are done. □

**Lemma 2.3.** *The block $b_{Q_0}$ of $C_G(Q_0)$ is nilpotent.*

**Proof.** By the proof of [6, Lemma 2.2], we may assume that $G$ is equal to $N_G(Q, b_Q)$. Suppose that $C_G(Q_0)/C_G(Q)$ has a nontrivial $2'$-subgroup $F$. The group $F$ can be viewed as a subgroup of $\mathrm{Aut}(Q)$. By [5, Chapter 5, Theorem 2.3], $Q = [Q, F] \times C_Q(F)$. But $Q_0$ is contained in $C_Q(F)$. That is impossible. So $C_G(Q_0)/C_G(Q)$ has to be a 2-group. Since the block $b_{Q_0}$ of $C_G(Q_0)$ covers the block $b_Q$ of $C_G(Q)$ which is nilpotent by [9, Proposition 4.2], the block $b_{Q_0}$ is nilpotent. □

For any subgroup $H$ of $G$ and block $d$ of $H$, denote by $d^G$ the induced block of $d$ if it exists (see [1, §14]).

**Lemma 2.4.** *Assume that the block $b$ is controlled by $N_G(P, b_P)$ and $Q_0$ is in the center of $P$. Let $R$ be a subgroup of $P$. For any subgroup $K$ of $N_G(R, b_R)$ containing $C_G(R)$, set $d = (b_R)^K$. Then there exists an inertial group $E/C_G(P)$ of the block $b$ such that an inertial group of the block $d$ is isomorphic to $(K \cap E)/C_G(P)$.*

**Proof.** Since the block $b$ is controlled by $N_G(P, b_P)$, $P \cap K$ is a defect group of the block $d$. Set $\hat{R} = P \cap K$. Denote by $\mathcal{F}$ the Brauer category of the block $b$ with respect to the maximal $b$-Brauer pair $(P, b_P)$. Let $N_{\mathcal{F}}^K(R)$ be the $K$-normalizer of $R$ in $\mathcal{F}$ (see [10, 2.14]). By [10, Corollary 3.6], we may choose a suitable $d_{\hat{R}}$, so that $N_{\mathcal{F}}^K(R)$ is the Brauer category $\mathcal{F}_{(\hat{R},\, d_{\hat{R}})}(K, d)$. By the definition of $N_{\mathcal{F}}^K(R)$, $N_K(\hat{R}, d_{\hat{R}}) = C_K(\hat{R})\big(K \cap N_G(P, b_P)\big)$. Set $X = K \cap N_G(P, b_P)$. Since $N_G(P, b_P)/C_G(P)$ has a normal Sylow $p$-subgroup $PC_G(P)/C_G(P)$ and $C_G(P) \leq K$, $X/C_G(P) = \Big(\big(X/C_G(P)\big) \cap \big(PC_G(P)/C_G(P)\big)\Big) \cdot \Big(\big(X/C_G(P)\big) \cap \big(E/C_G(P)\big)\Big)$ for some inertial group $E/C_G(P)$ of the block $b$. So $X = (X \cap P) \cdot (X \cap E)$. Hence $N_K(\hat{R}, b_{\hat{R}})/\hat{R}C_G(\hat{R})$ is isomorphic to $(K \cap E)/\big((K \cap E) \cap \hat{R}C_K(\hat{R})\big)$. Since $E/C_G(P)$ is a $p'$-group, $E \cap \hat{R}C_K(\hat{R}) = E \cap C_K(\hat{R})$. Since $Q_0$ is in the center of $P$, $Q_0$ belongs to $\hat{R}$. By Lemma 2.3, we have $E \cap C_G(Q_0) = E \cap N_G(P, b_P) \cap C_G(Q_0) = E \cap PC_G(P) = C_G(P)$. So $E \cap C_K(\hat{R}) \leq E \cap C_G(Q_0) = C_G(P)$. Therefore we have $K \cap E \cap \hat{R}C_K(\hat{R}) = C_G(P)$. Then we can get $N_K(\hat{R}, b_{\hat{R}})/\hat{R}C_K(\hat{R})$ is isomorphic to $(K \cap E)/C_G(P)$. $\square$

Let $R$ be a lower defect group (see [4, Chapter V]) of $b$ associated with the identity element of $G$, and denote by $m(b, R)$ its multiplicity.

**Lemma 2.5.** ([11, Lemma 2.4]) *If $Q < P$ and $|Q| \leq |Z(P)|$, then $m(b, 1) = 0$.*

**Proof.** Suppose that there is a simple $\mathcal{O}Gb$-module $M$ with a vertex $R$ which is contained in $Q$. By [12, Theorem (41.6)], there exists a self-centralizing $b$-Brauer pair $(R, h)$. So there is an element $x$ of $G$ such that $(R, h)^x \subseteq (P, b_P)$. Then $|Z(P)| \leq |C_P(R^x)| \leq |R^x| = |R| \leq |Q| \leq |Z(P)|$. Hence $R^x = Z(P)$ which implies that $R = P$. This is a contradiction. By the proof of [13, Theorem 4], any Cartan integer of the block $b$ is divisible by $p$. So we have $m(b, 1) = 0$. $\square$

Let $(Q, b_Q)$ be the $b$-Brauer pair contained in $(P, b_P)$. For simplicity, we will call $|N_G(Q, b_Q)/C_G(Q)|_{p'}$ the hyperfocal inertial index of the block $b$ and denote it by $f$.

**Lemma 2.6.** *Keep the notation and assumption above. If the block $b$ is controlled by $N_G(P, b_P)$, the inertial index of the block $b$ is equal to $f$.*

**Proof.** Since the block $b_Q$ is nilpotent (see [9, Proposition 4.2]), we have $N_G(P, b_P) \cap C_G(Q) = C_P(Q)C_G(P)$. At the same time, the block $b$ is controlled by $N_G(P, b_P)$. Hence $N_G(Q, b_Q)/C_G(Q) = N_G(P, b_P)C_G(Q)/C_G(Q) \cong N_G(P, b_P)/C_P(Q)C_G(P)$. Then the inertial index of the block $b$ is equal to $f$ since $PC_G(P)/C_G(P)$ is the Sylow $p$-subgroup of $N_G(P, b_P)/C_G(P)$. $\square$

By assumption, $Q$ is $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^m} \times \mathbb{Z}_{2^s}$, where $n, m, s$ are positive integers. If $n, m, s$ are different, $\mathrm{Aut}(Q)$ is a 2-group, and by [13, Theorem 2] the block $b$ is nilpotent. That contradicts with $|Q|$ bigger than 1. So $Q$ has to be $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^m}$. In this case, $f$ has three possibilities: 3, or 7, or 21. In sequel, we treat the three cases separately.

## 3. The case $f = 3$

In this section, we show that the case $f = 3$ does not happen.

**Proposition 3.1.** *The hyperfocal inertial index of the block $b$ is not* 3.

**Proof.** In order to prove the proposition, by [13, Theorem 2], we may assume that $G = N_G(Q, b_Q)$ and $b = b_Q$. In particular, $C_G(Q_0)$ is normal in $G$. Suppose $f = 3$.

Firstly, we assume that the block $b$ is controlled by $N_G(P, b_P)$. Let $E$ be a subgroup of $N_G(P, b_P)$ such that $E/C_G(P)$ is an inertial group of the block $b$. By Alperin's fusion theorem, the hyperfocal subgroup $Q$ is equal to $[P, E]$. By [5, Chapter 5, Theorem 3.6], $[Q, E] = [P, E, E] = [P, E] = Q$. Since the block $b_Q$ of $C_G(Q)$ is nilpotent, we have $E \cap C_G(Q) = C_G(P)$. Since $f = 3$, $EC_G(Q)/C_G(Q)$ has order 3. Hence $C_Q(E)$ is not trivial. But this contradicts $Q = [Q, E]$.

Secondly, we assume that the block $b$ is not controlled by $N_G(P, b_P)$. Let $(S, b_S)$ be an essential $b$-Brauer pair contained in $(P, b_P)$. Since $Q_0$ is normal in $G$, we have $Q_0 \leq S$. Set $N = N_G(S, b_S)$ and $P_0 = C_P(Q_0)$. Since the block $b_{Q_0}$ is $G$-stable, $P_0$ is a defect group of the block $b_{Q_0}$. By Lemma 2.3, we have $N \cap C_G(Q_0) = N_{P_0}(S)C_G(S)$. Since $(S, b_S)$ is essential, $P_0$ has to be contained in $S$.

Suppose $P_0 < S$. The inclusion $N \subset G$ induces an injective homomorphism

$$N/N_{P_0}(S)C_G(S) \longrightarrow G/C_G(Q_0).$$

Since $N/P_0C_G(S)$ is not a 2-group and $|G/C_G(Q)|_{2'}$ equals 3, $SC_G(Q_0)/C_G(Q_0)$ is a nontrivial proper subgroup of $PC_G(Q_0)/C_G(Q_0)$ normalized by a Sylow 3-subgroup. By Lemma 2.1, $G/C_G(Q_0)$ is isomorphic to $\mathrm{S}_4$, $S/P_0$ is isomorphic to the Klein four group, $P/P_0$ is isomorphic to $\mathrm{D}_8$, and the index of $S$ in $P$ is 2. So $S$ is normal in $P$ and the block $b_S$ is $P$-stable. Therefore $PC_G(S)/P_0C_G(S)$ is a Sylow 2-subgroup of $N/P_0C_G(S)$. Then we can get $N/P_0C_G(S) \cong G/C_G(Q_0) \cong \mathrm{S}_4$. So there exists a subgroup $K$ of $N$ such that $K$ contains $C_G(S)$, $K/C_G(S)$ has order 3 and $K \cap S = Z(S)$. Take an element $x \in K$ such that $xC_G(S)$ has order 3 in $N/C_G(S)$. Then we have $[s, x] \in Q$ for any $s \in S$ and $[sC_G(Q_0), xC_G(Q_0)] = 1$ in $G/C_G(Q_0)$. But $xC_G(Q_0)$ has order 3 in $G/C_G(Q_0)$. This is impossible by Lemma 2.1.

Hence $S = P_0$. Then $(S, b_S)$ is the unique essential $b$-Brauer pair contained in $(P, b_P)$. Since the block $b_{Q_0}$ is nilpotent by Lemma 2.3, $N_G(P, b_P) \cap C_G(Q_0) = P_0C_G(P)$. The inclusion $N_G(P, b_P) \subset G$ induces an injective homomorphism $N_G(P, b_P)/P_0C_G(P) \longrightarrow G/C_G(Q_0)$. Suppose that the inertial index of the block $b$ is not 1. Since $|G/C_G(Q_0)|_{2'} = 3$, the homomorphism is an isomorphism and $PC_G(Q_0)/C_G(Q_0)$ is normal in $G/C_G(Q_0)$. Since the preimage of $PC_G(Q_0)/C_G(Q_0)$ in $N/SC_G(S)$ is $PC_G(S)/SC_G(S)$, $PC_G(S)/$

$SC_G(S)$ is normal in $N/SC_G(S)$. This is impossible. So $N_G(P, b_P) = PC_G(P)$. Then the block $b$ is controlled by $N_G(S)$ by Alperin's fusion theorem. Consequently, we have $N/SC_G(S) \cong G/C_G(Q_0)$ since $N \cap C_G(Q_0) = N_{P_0}(S)C_G(S) = SC_G(S)$. There is a subgroup $F$ of $N$ such that $F$ contains $C_G(S)$, $F/C_G(S)$ is a Sylow 3-subgroup of $N/C_G(S)$ and $F \cap S = Z(S)$. By [9, 1.7], the hyperfocal subgroup $Q$ is equal to $[S, F]$, which is equal to $[Q, F]$ by [5, Chapter 5, Theorem 3.6]. But this is impossible since $C_Q(F)$ is nontrivial. We are done. □

## 4. The case $f = 7$

In this section, we always assume that $f$ is equal to 7. Then the hyperfocal subgroup $Q$ has to be $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$ for some positive integer $n$.

**Proposition 4.1.** *The block $b$ is controlled by $N_G(P, b_P)$, $Q_0 \leq Z(P)$ and the inertial index of the block $b$ is* 7.

**Proof.** In order to prove the proposition, by [13, Theorem 2] we may assume that $G = N_G(Q, b_Q)$ and $b = b_Q$. Then $C_G(Q_0)$ is normal in $G$. Since $f = 7$, $G/C_G(Q_0)$ is a $\{2, 7\}$-subgroup of $\mathrm{Aut}(Q_0)$. By Lemma 2.1, the order of $G/C_G(Q_0)$ is either 7 or 14. If $G/C_G(Q_0)$ has order 14, the Sylow 7-subgroup of $G/C_G(Q_0)$ has to be normal. That contradicts Lemma 2.1 (2). So $G/C_G(Q_0)$ is a 7-group. Since the block $b_{Q_0}$ of $C_G(Q_0)$ is nilpotent by Lemma 2.3, the block $b$ is inertial by [16, Theorem]. In particular, the block $b$ is controlled by $N_G(P, b_P)$. Again, since the block $b_{Q_0}$ of $C_G(Q_0)$ is nilpotent, $N_G(P, b_P) \cap C_G(Q_0) = P_0C_G(P)$, where $P_0$ is $C_P(Q_0)$. So the inclusion $N_G(P, b_P) \subset G$ induces an injective homomorphism $N_G(P, b_P)/P_0C_G(P) \to G/C_G(Q_0)$, which has to be an isomorphism. Consequently, $P_0 = P$ and the inertial index of the block $b$ is 7 by Lemma 2.6. □

**Lemma 4.2.** *For any subgroup $R$ of $P$, let $K$ be a subgroup of $N_G(R, b_R)$ containing $C_G(R)$. Set $d = (b_R)^K$. Then the hyperfocal subgroup $\hat{Q}$ of the block $d$ is either trivial or $\mathbb{Z}_{2^m} \times \mathbb{Z}_{2^m} \times \mathbb{Z}_{2^m}$, where $m$ is a positive integer not more than $n$. Furthermore the block $d$ has the hyperfocal inertial index either* 1 *or* 7.

**Proof.** By [6, Lemmas 2.5 and 2.6], the block $d$ has a defect group $\hat{R} = P \cap K$ and a hyperfocal subgroup $\hat{Q}$ contained in $Q$, and it is controlled by $N_K(\hat{R})$. By Lemma 2.4, there exists an inertial group $E/C_G(P)$ of the block $b$ such that $(E \cap K)/C_G(P)$ is isomorphic to an inertial group of the block $d$. By Proposition 4.1, the inertial index of the block $b$ is 7. So the inertial index of the block $d$ is 1 or 7. Hence $\hat{Q}$ is either trivial or $\mathbb{Z}_{2^m} \times \mathbb{Z}_{2^m} \times \mathbb{Z}_{2^m}$ for a suitable positive integer $n$, and the hyperfocal inertial index of the block $d$ is 1 or 7 by Lemma 2.6. □

**Lemma 4.3.** *Assume that $P = Q$. For a subgroup $R$ of $P$, we have*

$$m(b,\, R) = \begin{cases} 1 & \textit{if } R \textit{ is conjugate to } P; \\ 6 & \textit{if } R \textit{ is equal to } 1; \\ 0 & \textit{otherwise.} \end{cases}$$

**Proof.** Since $Q \neq 1$, $l(b) = 7$ by [3, Corollary 1.2] and [14, Corollary]. In order to prove the lemma, by [6, Equation (3.2)] it suffices to prove $m\big((b_R)^{N_G(R,\, b_R)},\, R\big) = 0$ for any nontrivial proper subgroup $R$ of $P$. Let $R$ be a nontrivial proper subgroup of $P$. Suppose that $m\big((b_R)^{N_G(R,\, b_R)},\, R\big) \geq 1$. Set $d = (b_R)^{N_G(R,\, b_R)}$ and $g = (b_R)^{RC_G(R)}$. Clearly the block $g$ is nilpotent, $l(g) = 1$ and $m(g,\, R) = 0$. By [7, Theorem 5.12], $m(d,\, R) \leq m(g,\, R)$. This is a contradiction. □

Let $E$ be a subgroup of $N_G(P, b_P)$ such that $E/C_G(P)$ is an inertial group of the block $b$. Then we have $P = [P, E] \rtimes C_P(E)$. By Proposition 4.1, $E/C_G(P)$ has order 7. Since the block $b$ is controlled by $N_G(P,\, b_P)$ and $f = 7$, $EC_G(Q)/C_G(Q)$ has to be of order 7. Then $Q = [P, E] = [Q, E]$ since $EC_G(Q)/C_G(Q)$ acts freely on the set of nonidentity elements of $Q$.

**Lemma 4.4.** *Set $R = C_P(E)$, $N = N_G(R,\, b_R)$ and $d = (b_R)^N$. Then $m(d,\, R) = 6$.*

**Proof.** Set $\hat{R} = N_P(R)$. Then $\hat{R}$ is a defect group of the block $d$ and $N = C_G(R)\hat{R}$ by Proposition 4.1. By Lemma 2.4, the inertial index of the block $d$ is 7. So by [6, Lemma 3.1], $m(d,\, R) = m(\bar{d},\, 1)$, where $\bar{d}$ is the block of $N/R$ determined by the block $d$. The block $\bar{d}$ has a defect group $\hat{R}/R$, which is isomorphic to $N_Q(R)$. By Lemma 2.2, the inertial index of the block $\bar{d}$ is 7. Then by Lemma 2.1, $\hat{R}/R$ is a hyperfocal subgroup of the block $\hat{d}$. By Lemma 4.3, we are done. □

**Lemma 4.5.** *Assume that $|Q| \leq |Z(P)|$. We have $l(b) = l(b_0) = 7$.*

**Proof.** By [13, Theorem 2] and Proposition 4.1, the blocks $b$ and $b_0$ have equivalent Brauer categories. So in order to prove the lemma, it suffices to prove that $l(b) = 7$. By [3, Corollary 1.2] and [14, Corollary], we have $l(b) = 7$ when $Q = P$. So we assume that $Q$ is a proper subgroup of $P$. We prove $l(b) = 7$ by induction on $|G|$.

Let $R$ be a proper subgroup of $P$ such that $m\big((b_R)^{N_G(R,\, b_R)},\, R\big) \neq 0$. We set $N = N_G(R,\, b_R)$ and $d = (b_R)^{N_G(R,\, b_R)}$. By Proposition 4.1, $\hat{R} = N_P(R)$ is a defect group of the block $d$. Since $m(d,\, R) \neq 0$, the block $d$ is not nilpotent. By Lemma 4.2, the hyperfocal subgroup $\hat{Q}$ of the block $d$ is $\mathbb{Z}_{2^m} \times \mathbb{Z}_{2^m} \times \mathbb{Z}_{2^m}$ for a suitable positive integer and the hyperfocal inertial index of it is 7. Then by Proposition 4.1, the inertial index of the block $d$ is 7. By Lemma 2.4, there is a suitable inertial group $E/C_G(P)$ of the block $b$ such that $(E \cap N)/C_G(P)$ is isomorphic to an inertial group of the block $d$. Since the two blocks $b$ and $d$ have the same inertial index, we have $E \subseteq N$ and $E/C_G(P)$ is isomorphic to an inertial group of the block $d$.

Suppose that $N < G$. By induction, we have $l(d) = 7$. Then by Lemma 4.4 and [6, Equation (3.2)], $R = C_{\hat{R}}(E) = N_{C_P(E)}(R)$. Hence $R = C_P(E)$. Again, by Lemma 4.4 and [6, Equation (3.2)], we have $l(b) = 7$.

Assume that $G = N$ and $b = d$. Set $C = C_G(R)$ and $\tilde{R} = C_P(R)$. Then $\tilde{R}$ is a defect group of the block $b_R$ by Proposition 4.1. By [6, Lemma 2.6] and Lemma 2.4, the block $b_R$ is controlled by $N_C(\tilde{R})$ and the inertial group of the block $b_R$ is isomorphic to $C_E(R)/C_G(P)$ for some inertial group $E/C_G(P)$ of the block $b$.

If $C_E(R)$ is equal to $C_G(P)$, the block $b_R$ is nilpotent and so is the block $(b_R)^{PC_G(R)}$. Since the block $b$ is controlled by $N_G(P, b_P)$, we have $G = PC_G(R) \cdot E$. So $G/PC_G(R)$ is a $2'$-group. Then by [16, Theorem], the block $b$ is inertial. In particular, we have $l(b) = 7$.

If $C_E(R)$ is not equal to $C_G(P)$, then $C_E(R) = E$ and $G = PC_G(R)$. Set $\bar{G} = G/R$. For any subgroup $X$ of $G$, denote by $\bar{X}$ the image of $X$ in $\bar{G}$. Denote by $\bar{b}$ the unique block of $\bar{G}$ determined by the block $b$. By Lemma 2.2, the block $\bar{b}$ has a hyperfocal subgroup which is contained in $\bar{Q}$ and inertial index 7. Since the block $b$ is controlled by $N_G(P, b_P)$, by Lemma 2.2, the block $\bar{b}$ is also controlled by the normalizer of its maximal Brauer pair. Hence the hyperfocal inertial index of the block $\bar{b}$ is 7. Since $|Q| \le |Z(P)|$ and $m(d, R) \neq 0$, by Lemma 2.5 $R \neq 1$ and thus $|\bar{G}| < |G|$. Then by induction, we have $l(b) = l(\bar{b}) = 7$. The proof is done. □

**Lemma 4.6.** *Assume that $|Q| \le |Z(P)|$. We have $k(b) = k(b_0)$.*

**Proof.** For any $u \in P$, set $b_u = b_{\langle u \rangle}$. Denote by $e_u$ the inertial index of the block $b_u$. $N_G(P, b_P)$, $C_P(u)$ is a defect group of the block $b_u$. Set $P_u = C_P(u)$. By [6, Lemma 2.6], the block $b_u$ is controlled by $N_{C_G(u)}(P_u)$. So by Lemma 2.4, $e_u$ is either 1 or 7. If $e_u$ is 1, the block $b_u$ is nilpotent. So $l(b_u) = 1$. If $e_u$ is 7, we have $l(b_u) = 7$ by Lemma 4.5. In conclusion, $l(b_u) = e_u$. On the other hand, denote by $b_u^\circ$ the block of $C_{N_G(P)}(u)$ satisfying that $(u, b_u^\circ)$ belongs to the maximal $b_0$-Brauer pair $(P, b_P)$. Since the block $b$ is controlled by $N_G(P, b_P)$, the blocks $b$ and $b_0$ have the same Brauer categories. This means that for any $u, v \in P$, $(u, b_u)$ and $(v, b_v)$ are $G$-conjugate if and only if $(u, b_u^\circ)$ and $(v, b_v^\circ)$ are $N_G(P)$-conjugate. By [10, Corollary 3.6], the blocks $b_u$ and $b_u^\circ$ have the same Brauer categories. In particular they have the same inertial index $e_u$. By the structure theorem of the blocks with normal defect group, the equation $l(b_u^\circ) = e_u$ holds. Hence we have $l(b_u) = l(b_u^\circ)$ for any $u \in P$. Therefore $k(b) = k(b_0)$. □

## 5. The case $f = 21$

In this section, we always assume that $f$ is equal to 21. In this case, the hyperfocal subgroup $Q$ is $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$ for some positive integer $n$. We borrow the notation in Section 4.

**Proposition 5.1.** *The block $b$ is controlled by $N_G(P, b_P)$, $Q_0 \le Z(P)$ and the inertial index of the block $b$ is* 21.

**Proof.** In order to prove the proposition, by [13, Theorem 2], we may assume that $G = N_G(Q, b_Q)$ and $b = b_Q$. Then $Q_0$ is normal in $G$, the block $b_{Q_0}$ is $G$-stable, and $P_0 = C_P(Q_0)$ is a defect group of the block $b_{Q_0}$. Since $f = 21$, $|G/C_G(Q_0)|$ is divided by 21. By Lemma 2.1, we have $|G/C_G(Q_0)|$ is 21 or 168.

Suppose that $G/C_G(Q_0)$ is of order 168. Then $G/C_G(Q_0)$ is the simple group of order 168. Set $\bar{G} = G/C_G(Q_0)$. For any subset $X$ of $G$, denote by $\bar{X}$ the image of $X$ in $\bar{G}$. It is clear that $\bar{P}$ is a Sylow 2-subgroup of $\bar{G}$, which is isomorphic to $P/P_0$. Let $R$ be a normal subgroup of $P$ such that $R$ contains $P_0$ and $R/P_0$ is the Klein four group. By Lemma 2.1, $N_{\bar{G}}(\bar{R})$ is $\mathrm{S}_4$. Take an element $\bar{x}$ of $\bar{G}$ such that $\langle \bar{x} \rangle$ is a Sylow 3-subgroup of $N_{\bar{G}}(\bar{R})$. Then we have $x$ belongs to $N_G\big(RC_G(Q_0)\big)$. Set $e = (b_{Q_0})^{RC_G(Q_0)}$. Since $R$ contains $P_0$, $(R, b_R)$ is a maximal $e$-Brauer pair. Since the block $b_{Q_0}$ is $G$-stable, by the Frattini argument, there exist an element $y$ of $N_G(R, b_R)$ and an element $z$ of $C_G(Q_0)$ such that $x = yz$. So $\bar{y} = \bar{x}$ is an element of $\bar{G}$ of order 3. Since $N_G(R, b_R) \cap C_G(Q_0) = P_0C_G(R)$ by Lemma 2.3, the inclusion $N_G(R, b_R) \subset G$ induces an injective group homomorphism $N_G(R, b_R)/P_0C_G(R) \to \bar{G}$. Therefore there is an integer $t$ such that $y^tC_G(R)$ is of order 3 in $N_G(R, b_R)/C_G(R)$. By the definition of the hyperfocal subgroup, $[r, y^t]$ belongs to $Q$ for any $r \in R$. Therefore $[\bar{r}, \bar{y}^t] = 1$. But $\bar{y}^t$ has order 3 in $\bar{G}$. By Lemma 2.1, the equality $[\bar{r}, \bar{y}^t] = 1$ is impossible.

So the order of $\bar{G}$ is 21. Then the block $b$ is inertial by [16, Theorem]. In particular the block $b$ is controlled by $N_G(P, b_P)$. Since $\bar{P}$ is a Sylow 2-subgroup of $\bar{G}$, this forces $P = P_0$, namely, $Q_0$ is in the center of $P$. By Lemma 2.6, the inertial index of the block $b$ is 21. $\square$

Let $E/C_G(P)$ be an inertial group of the block $b$. We have $P = [P, E] \rtimes C_P(E)$. Set $H = N_G(Q, b_Q)$. The inclusion $E \subset H$ induces homomorphisms $E/C_G(P) \to H/C_G(Q)$ and $E/C_G(P) \to H/C_H(Q_0)$ such that the following diagram commutes

$$\begin{tikzcd} E/C_G(P) \arrow[d] \arrow[dr] & \\ H/C_G(Q) \arrow[r] & H/C_H(Q_0) \end{tikzcd}$$

where the bottom homomorphism is the canonical homomorphism. By the last paragraph, the homomorphism $E/C_G(P) \to H/C_H(Q_0)$ is an isomorphism. So the homomorphism $E/C_G(P) \to H/C_G(Q)$ must be injective. Viewing $E/C_G(P)$ as a subgroup of the automorphism group $\mathrm{Aut}(Q)$ of $Q$, we have $Q = [P, E] = [Q, E]$ since $C_Q(E) = 1$. Let $E_3$ and $E_7$ be subgroups of $E$ such that $E_3/C_G(P)$ and $E_7/C_G(P)$ are Sylow 3- and Sylow 7-subgroups of $E/C_G(P)$ respectively.

**Lemma 5.2.** *Keep the notation as above. Then $C_P(E_7) = C_P(E)$.*

**Proof.** Obviously $C_P(E) \le C_P(E_7)$. Since $P = [P, E] \rtimes C_P(E) = Q \rtimes C_P(E)$, we have $C_P(E_7) = C_Q(E_7) \rtimes C_P(E)$. Since $E_7/C_G(P)$ is a subgroup of $\mathrm{Aut}(Q)$ of order 7, by Lemma 2.1 we have $C_Q(E_7) = 1$. The proof is done. $\square$

**Lemma 5.3.** *Assume that $P = Q$. For a subgroup $R$ of $P$, we have*

$$m(b,\, R) = \begin{cases} 1 & \textit{if } R \textit{ is conjugate to } P; \\ 2 & \textit{if } R \textit{ is equal to } 1; \\ 2 & \textit{if } R \textit{ is conjugate to } C_P(E_3); \\ 0 & \textit{otherwise.} \end{cases}$$

**Proof.** By [3, Corollary 1.2] and [14, Corollary], $l(b) = 5$ because the block $b$ is not nilpotent. Set $R = C_P(E_3)$ and $N = N_G(R, b_R)$. Then $R$ is a cyclic 2-group and $N_{E_7}(R)$ must be $C_G(P)$ since $E_7/C_G(P)$ acts freely on $Q_0 - \{1\}$. So we have $N = C_G(R)$. The block $b_R$ has defect group $P$ and its inertial index is index 3. Denote by $\bar{b}_R$ the block of $N/R$ determined by the block $b_R$. Then $m(b_R,\, R) = m(\bar{b}_R,\, 1)$ by [6, Lemma 3.1]. The block $\bar{b}_R$ has a defect group $P/R$ isomorphic to $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$ and inertial index 3. Hence $m(b_R,\, R) = m(\bar{b}_R,\, 1) = 2$ by [6, Lemma 3.2]. Suppose that there is another block $b'_R$ of $C_G(R)$ associated with the block $b$. We have $N' = N_G(R,\, b_{R'}) = C_G(R)$ and the inertial group of the block $b'_R$ is 1 or cyclic of order 3. If the inertial index of the block $b'_R$ is 1, then the block $b'_R$ is nilpotent and thus $m(b'_R,\, R) = 0$. If the inertial index of the block $b'_R$ is 3, we have $m(b'_R,\, R) = 2$ as above. In conclusion, we have $m(b,\, R)$ is a nonzero even positive integer by [13, Equation (7)].

Let $T$ be a nontrivial proper subgroup of $P$ which is not $C_P(E_3)$, up to conjugation. By Lemma 2.4 the inertial block of the block $b_T$ of $C_G(T)$ is isomorphic to $C_E(T)/C_G(P)$. By Lemma 2.1 (1), $|C_E(T)/C_G(P)|$ is either 1 or 3. If $|C_E(T)/C_G(P)|$ is 1, the block $b_T$ is nilpotent and thus $m(b_T,\, T) = 0$. Then by [7, Theorem 5.12], we have $m\big((b_T)^{N_G(T,\, b_T)},\, T\big) = 0$. If $|C_E(T)/C_G(P)|$ is 3, we have $T \le C_P(E_3)$, up to conjugation. Then it is easily concluded that $T$ is a cyclic 2-group, that $N_{E_7}(T) = C_G(P)$ (see Lemma 2.1), that $E_3/C_G(P)$ is isomorphic to an inertial group of the block $(b_T)^{N_G(T,\, b_T)}$, and that the hyperfocal subgroup of the block $(b_T)^{N_G(T,\, b_T)}$ with respect to the maximal $(b_T)^{N_G(T,\, b_T)}$-Brauer pair $(P,\, b_P)$ is isomorphic to $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$. By the proof of [6, Lemma 4.1], $m\big((b_T)^{N_G(T,\, b_T)},\, T\big) = 0$ if $T < C_P(E_3)$, up to conjugation.

By [3, Corollary 1.2] and [14, Corollary], the block $b$ and $b_0$ are derived equivalent. It is well known that the derived equivalence preserves the elementary divisors of the Cartan matrices (see [15, Proposition 6.8.9]). It is straightforward to calculate that 1 appears as an elementary divisor of the Cartan matrix of the block $b_0$ by the structure theorem of the blocks with normal defect group. So $m(b,\, 1) \ne 0$.

By [13, Equations (7), (9) and (10)], we have $m\big(b,\, C_P(E_3)\big) = l(b) - m(b,\, P) - m(b,\, 1) \le 3$. Since $m\big(b,\, C_P(E_3)\big)$ is a nonzero even positive integer, it has to be 2. Then $m(b,\, 1) = 2$. We are done. $\square$

**Lemma 5.4.** *Set $R = C_P(E)$. Then $m\big((b_R)^{N_G(R,\, b_R)}, R\big) = 2$.*

**Proof.** Set $N = N_G(R, b_R)$ and $d = (b_R)^{N_G(R, b_R)}$. Then $\hat{R} = N_P(R)$ is a defect group of the block $d$ and $N = C_G(R)\big(N_G(R) \cap N_G(P, b_P)\big) = \hat{R}C_G(R)$ by Proposition 5.1. By [6, Lemma 3.1], $m(d, R) = m(\bar{d}, 1)$, where $\bar{d}$ is the unique block of $\bar{N} = N/R$ determined by the block $d$. The block $\bar{d}$ has a defect group $\bar{\hat{R}} = \hat{R}/R$ which is isomorphic to a subgroup of $Q$. By Lemma 2.4, an inertial group of the block $d$ is $E/C_G(P)$. So the block $d$ has inertial index 21. Then by Lemma 2.2, the block $\bar{d}$ still has inertial index 21. This implies that $\bar{\hat{R}}$ is isomorphic to $\mathbb{Z}_{2^m} \times \mathbb{Z}_{2^m} \times \mathbb{Z}_{2^m}$ for some positive integer $m$. So $\bar{\hat{R}}$ is a hyperfocal subgroup of the block $\bar{d}$. Then by Lemma 5.3, we have $m(d, R) = m(\bar{d}, 1) = 2$. $\square$

**Lemma 5.5.** *Set $R = C_P(E_3)$. Then $m\big((b_R)^{N_G(R, b_R)}, R\big) = 2$.*

**Proof.** Set $N = N_G(R, b_R)$ and $d = (b_R)^{N_G(R, b_R)}$. By Lemma 2.4, there exists an inertial group $E'/C_G(P)$ such that $(E' \cap N)/C_G(P)$ is isomorphic to an inertial group of the block $d$. Without loss of generality, we may assume that $E' = E$. Since $P = Q \rtimes C_P(E)$, $R = C_Q(E_3) \rtimes C_P(E)$. By Lemma 2.1, $E_7/C_G(P)$ acts freely on $Q_0 - \{1\}$. Since $C_Q(E_3) \neq 1$, $N_E(R)$ has to be $E_3$. So the inertial index of the block $d$ is 3.

By Proposition 5.1 $\hat{R} = N_P(R)$ is a defect group of the block $f$. By [6, Lemmas 2.5 and 2.6], we may assume that there is a hyperfocal subgroup of the block $d$ contained in $Q$ and that the block $d$ is controlled by $N_N(\hat{R})$. By Proposition 3.1, the hyperfocal subgroup of the block $d$ has to be isomorphic to $\mathbb{Z}_{2^s} \times \mathbb{Z}_{2^s}$ for some positive integer $s$. Since $C_{\hat{R}}(E_3) = N_{C_P(E_3)}(R) = R$, we have $m(d, R) = 2$ by [6, Lemma 3.3]. $\square$

**Lemma 5.6.** *Assume that $|Q| \leq |Z(P)|$. We have $l(b) = l(b_0) = 5$.*

**Proof.** By [13, Theorem 2] and Proposition 4.1, the blocks $b$ and $b_0$ have equivalent Brauer categories. So in order to prove the lemma, it suffices to prove that $l(b) = 5$. By [3, Corollary 1.2] and [14, Corollary], we have $l(b) = 5$ when $Q = P$. So we assume that $Q$ is a proper subgroup of $P$. We prove $l(b) = 5$ by induction on $|G|$.

Let $R$ be a proper subgroup of $P$ such that $m\big((b_R)^{N_G(R, b_R)}, R\big) \neq 0$. Set $N = N_G(R, b_R)$ and $d = (b_R)^{N_G(R, b_R)}$. By Proposition 5.1 $\hat{R} = N_P(R)$ is a defect group of the block $d$. By Lemma 2.4, $(E \cap N)/C_G(P)$ is an inertial group of the block $d$ for some suitable inertial group $E/C_G(P)$ of the block $b$. We are going to prove that $R \leq C_P(E_3)$. Assume that $R \not\leq C_P(E_3)$.

Suppose that $E \cap N = C_G(P)$. The inertial index of the block $d$ is 1. By [6, Lemma 2.6], the block $d$ is controlled by $N_N(\hat{R})$. So the block $d$ is nilpotent. This is a contradiction.

Suppose that $(E \cap N)/C_G(P)$ has order 3. Then by Lemma 2.6, the hyperfocal inertial index of the block $d$ is 3 since the block $d$ is controlled by $N_N(\hat{R})$. By [6, Lemma 2.5] and Proposition 3.1, the hyperfocal subgroup of the block $d$ is isomorphic to $\mathbb{Z}_{2^m} \times \mathbb{Z}_{2^m}$ for some positive integer $m$. By the proof of [6, Lemma 4.1], we have $R = C_P(E_3)$. This is a contradiction.

Suppose that $(E\cap N)/C_G(P)$ has order 7. Then by Lemma 2.6, the hyperfocal inertial index of the block $d$ is 7. By [6, Equation (3.2)], Lemmas 4.4 and 4.5, we have $R = C_{\hat{R}}(E_7) = C_P(E_7)$. Then by Lemma 5.2, we have $R = C_P(E)$. This is a contradiction.

Suppose that $(E\cap N)/C_G(P)$ has order 21. Then we have $E\cap N = E$. We divide the proof of the case into two cases. Suppose that $N < G$. By induction, we have $l(d) = 5$. By Lemmas 5.4 and 5.5 $R = C_{\hat{R}}(E_3)$ or $R = C_{\hat{R}}(E)$. In both cases, we have $R \le C_P(E_3)$. This is a contradiction. Suppose that $G = N$ and $b = d$. Set $C = C_G(R)$ and $\tilde{R} = C_P(R)$. Then the block $b_R$ is controlled by $N_C(\tilde{R})$ by [6, Lemma 2.6]. The inertial group of the block $b_R$ is isomorphic to $C_E(R)/C_G(P)$. Since $R \nleq C_P(E_3)$, the order of $C_E(R)/C_G(P)$ is either 1 or 7. If $C_E(R)$ is equal to $C_G(P)$, the block $b_R$ is nilpotent and so is the block $(b_R)^{PC_G(R)}$. Since the block $b$ is controlled by $N_G(P, b_P)$, we have $G = PC_G(R)\cdot E$. Then by [16, Theorem], the block $b$ is inertial. In particular $l(b) = 5$. Then by Lemmas 5.4 and 5.5, we have $R \le C_P(E_3)$. This is a contradiction.

Up to now, we proved that $R \le C_P(E_3)$. Next we prove that $R$ is either $C_P(E_3)$ or $C_P(E)$.

Suppose that $R < C_P(E_3)$. Assume that $N < G$. The inertial index of the block $d$ is either 3 or 21. If the inertial index of the block $d$ is equal to 3, the hyperfocal subgroup of the block $d$ is $\mathbb{Z}_{2^t}\times\mathbb{Z}_{2^t}$ for some positive integer $t$ by [6, Lemma 2.5] and Proposition 3.1. By the proof of [6, Lemma 4.1], $R$ has to be $C_P(E_3)$. This is a contradiction. So the inertial index of the block $d$ is 21. Then by induction, we have $l(d) = 5$. By Lemmas 5.4 and 5.5 and [6, Equation (3.2)], we have $R = C_{N_P(R)}(E) = N_{C_P(E)}(R) = C_P(E)$.

Assume that $N = G$ and $b = d$. Suppose that $R \le C_P(E_7)$. Then we have $G = PC_G(R)$ since the block $b$ is controlled by $N_G(P, b_P)$ by Proposition 5.1 and Lemma 5.2. By Lemma 2.5, we have $R \ne 1$. Let $\bar{b}$ be the block of $G/R$ determined by the block $b$. By the proof of [13, Lemma 8], there is a hyperfocal subgroup of the block $\bar{b}$ contained in $\bar{Q}$. Then by Lemma 2.2 and induction, we have $l(\bar{b}) = 5$ and hence $l(b) = 5$. By Lemmas 5.4 and 5.5 and [6, Equation (3.2)], we have $R = C_P(E)$.

Suppose that $R \nleq C_P(E_7)$. Set $C = RC_G(R)$ and $g = (b_R)^C$. By Proposition 5.1, $RC_P(R)$ is a defect group of the block $g$. By Lemma 2.4, the block $g$ has an inertial group isomorphic to $\big(E\cap RC_G(R)\big)/C_G(P)$ which has order 3. Then by [6, Lemmas 2.5 and 2.6] and Proposition 3.1, the hyperfocal subgroup of the block $g$ is $\mathbb{Z}_{2^s}\times\mathbb{Z}_{2^s}$ for some positive integer $s$ and the block $g$ is controlled by the normalizer of its defect group. So by [6, Theorem 1.1], we have $l(g) = 3$. On the other hand, by [7, Theorem 5.12] we have $m(g, R) \ge m(d, R) \ge 1$. By [6, Lemma 3.2] and the equality $l(g) = 3$, we deduce that $R = C_P(E_3)$. This is a contradiction.

Summarizing the above, the subgroup $R$ such that $m\big((b_R)^{N_G(R, b_R)}, b_R\big) \ne 0$ is either $C_P(E_3)$ or $C_P(E)$ for some suitable inertial group $E/C_G(P)$ of the block $b$. By Lemmas 5.4 and 5.5 and [6, Equation (3.2)], $l(b) = 5$. We are done. $\square$

**Lemma 5.7.** *Assume that* $|Q| \le |Z(P)|$. *We have* $k(b) = k(b_0)$.

**Proof.** We will borrow the notation in the proof of Lemma 4.6. Similar to the proof of Lemma 4.6, the block $b_u$ has the following properties: $P_u$ is a defect group; there is a hyperfocal subgroup contained in $Q$; the block is controlled by $N_{C_G(u)}(P_u)$; the inertial index $e_u$ is 1, 3, 7 or 21. If $e_u$ is 1, the block $b_u$ is nilpotent. So $l(b_u) = 1$. If $e_u$ is 3, by Lemma 2.6 the hyperfocal inertial index of the block $b_u$ is 3. By Proposition 3.1, the hyperfocal subgroup of the block $b_u$ is isomorphic to $\mathbb{Z}_{2^s} \times \mathbb{Z}_{2^s}$ for some positive integer $s$. By [6, Theorem 1.1], $l(e_u) = 3$. If $e_u$ is 7, the hyperfocal subgroup of the block $b_u$ is isomorphic to $\mathbb{Z}_{2^t} \times \mathbb{Z}_{2^t} \times \mathbb{Z}_{2^t}$ for some positive integer $t$ and the hyperfocal inertial index is 7 by Lemma 2.6. By Lemma 4.5, $l(b_u) = 7$. If $e_u$ is 21, the hyperfocal subgroup of the block $b_u$ is isomorphic to $\mathbb{Z}_{2^l} \times \mathbb{Z}_{2^l} \times \mathbb{Z}_{2^l}$ for some positive integer $l$ and the hyperfocal inertial index is 21 by Lemma 2.6. By Lemma 5.6, we have $l(b_u) = 5$. On the other hand, by the structure theorem of the blocks with normal defect group, $l(b_u^\circ) = 1$ if its inertial index is 1; $l(b_u^\circ) = 3$ if its inertial index is 3; $l(b_u^\circ) = 7$ if its inertial index is 7; $l(b_u^\circ) = 5$ if its inertial index is 21. By [10, Corollary 3.6], the blocks $b_u$ and $b_u^\circ$ have the same Brauer categories from which we can deduce that both have the same inertial indices. Hence we have $l(b_u) = l(b_u^\circ)$ for any $u \in P$. Since the block $b$ is controlled by $N_G(P, b_P)$, the blocks $b$ and $b_0$ have the same Brauer categories. This means that for any $u, v \in P$, $(u, b_u)$ and $(v, b_v)$ are $G$-conjugate if and only if $(u, b_u^\circ)$ and $(v, b_v^\circ)$ are $N_G(P)$-conjugate. Therefore $k(b) = k(b_0)$. □

Then we can prove Theorem 1.1.

**Proof of Theorem 1.1.** We denote by $e$ the inertial index of the block $b$ with respect to the maximal $b$-Brauer pair $(P, b_P)$. By Lemma 2.6 and Propositions 4.1 and 5.1, we have $e = f$. Hence Theorem 1.1 will follow from Proposition 3.1 and Lemmas 4.5, 4.6, 5.6, 5.7. □

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