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Classification of finite irreducible conformal modules over Lie conformal superalgebras of Block type

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A R T I C L E I N F O

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A B S T R A C T

We introduce a class of infinite Lie conformal superalgebras $\mathfrak{S}(p)$ of Block type, and classify their finite irreducible conformal modules for any nonzero parameter p. In particular, we show that such a conformal module admits a nontrivial extension of a finite conformal module over $\mathfrak{NS}$ if $p = -1$, where $\mathfrak{NS}$ is a Neveu-Schwarz conformal subalgebra of $\mathfrak{S}(p)$. As a byproduct, we also obtain the classification of finite irreducible conformal modules over a series of finite Lie conformal superalgebras $\mathfrak{s}(n)$ for $n \geq 1$.

1. Introduction

Lie conformal superalgebras, introduced by Kac [9], encode the singular part of the operator product expansion of chiral fields in conformal field theory.

Many advances have been made in the theory of finite Lie conformal superalgebras over the years [1–5,7,8,10,15,16]. Finite simple Lie conformal superalgebras were classi-

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fied in [7]. The list consists of current Lie conformal superalgebras $\mathrm{Cur}\,\mathfrak{g}$ over a simple finite-dimensional Lie superalgebra $\mathfrak{g}$, four series of Lie conformal superalgebras of Cartan type, and the exceptional Lie conformal superalgebra CK_6. Their finite irreducible conformal modules (FICMs) were classified in a series of papers [1–5] by a description of extremal vectors and degenerate modules. Remarkably, the classification of FICMs over CK_6 was also given by Martínez and Zelmanov [10] by a distinctive approach.

However, work towards the theory of infinite Lie conformal superalgebras is only at its initial level. Recently, in [13], we studied a class of infinite Lie conformal algebras $\mathfrak{B}(p)$ of Block type, where p is a nonzero complex number. In this paper, we introduce the super analogue $\mathfrak{S}(p)$ of $\mathfrak{B}(p)$ by analyzing certain module structures of $\mathfrak{B}(p)$, and then study its representation theory. Naturally, we refer to $\mathfrak{S}(p)$'s as *Lie conformal superalgebras of Block type*. As one can see later, $\mathfrak{S}(p) = \mathfrak{S}(p)_{\bar{0}} \oplus \mathfrak{S}(p)_{\bar{1}}$ with $\mathfrak{S}(p)_{\bar{0}} = \oplus_{i\in\mathbb{Z}_+}\mathbb{C}[\partial]L_i$, $\mathfrak{S}(p)_{\bar{1}} = \oplus_{i\in\mathbb{Z}_+}\mathbb{C}[\partial]G_i$ and λ-brackets

$$[L_i\,{}_\lambda\,L_j] = ((i+p)\partial + (i+j+2p)\lambda)L_{i+j}, \tag{1.1}$$

$$[L_i\,{}_\lambda\,G_j] = ((i+p)\partial + (i+j+\frac{3}{2}p)\lambda)G_{i+j}, \tag{1.2}$$

$$[G_i\,{}_\lambda\,G_j] = 2L_{i+j}. \tag{1.3}$$

Some interesting features of $\mathfrak{S}(p)$ deserve to be mentioned. Firstly, each $\mathfrak{S}(p)$ contains a Neveu-Schwarz conformal subalgebra. Set $L = \frac{1}{p}L_0$, $G = \frac{1}{\sqrt{p}}G_0$. By (1.1)–(1.3), one can check that the subalgebra

$$\mathfrak{NS} = \mathbb{C}[\partial]L \oplus \mathbb{C}[\partial]G \tag{1.4}$$

of $\mathfrak{S}(p)$ is exactly the Neveu-Schwarz conformal algebra [7]. Secondly, there are embedding relations among $\mathfrak{S}(p)$'s. For any integer $n \geq 1$, $\mathfrak{S}(p)$ can be embedded into $\mathfrak{S}(np)$ via $L_i \mapsto \frac{1}{n}L'_{ni}$, $G_i \mapsto \frac{1}{\sqrt{n}}G'_{ni}$. Thirdly, $\mathfrak{S}(-n)$ contains a series of finite Lie conformal quotient algebras (cf. (2.6))

$$\mathfrak{s}(n) = \mathfrak{S}(-n)/\mathfrak{S}(-n)_{\langle n+1\rangle}. \tag{1.5}$$

Due to their interior conformal structures, the special cases $\mathfrak{s}(1)$ and $\mathfrak{s}(2)$ will be referred to as *Heisenberg-Neveu-Schwarz conformal algebra* and *Schrödinger-Neveu-Schwarz conformal algebra*, respectively. See Subsection 2.3 for more details.

Our main goal in this paper is to classify FICMs over $\mathfrak{S}(p)$. Obviously, any conformal module over $\mathfrak{NS} \subset \mathfrak{S}(p)$ can be trivially extended to a conformal module over $\mathfrak{S}(p)$. Our main result indicates that a FICM over $\mathfrak{S}(p)$ admits a nontrivial extension of a finite conformal module over $\mathfrak{NS}$ if $p = -1$ (see Table 1). As a byproduct, we also obtain the classification of FICMs over the finite Lie conformal superalgebra $\mathfrak{s}(n)$ (see Table 2).

In the process of our proof, we also give the classifications of all the free conformal modules of rank $(1+1)$ over $\mathfrak{S}(p)$ and $\mathfrak{s}(n)$, and characterize their simplicities.

Table 1
Nontrivial FICMs over $\mathfrak{S}(p)$.

$\mathfrak{S}(p)$	FICMs	Reference
$p \neq -1$	$T_{\Delta,\alpha}$ or $T'_{\Delta,\alpha}$	Theorem 6.1
$p = -1$	$T_{\Delta,\alpha,\beta}$ or $T'_{\Delta,\alpha,\beta}$	Theorem 6.1

Table 2
Nontrivial FICMs over $\mathfrak{s}(n)$.

$\mathfrak{s}(n)$	FICMs	Reference
$n > 1$	$T_{\Delta,\alpha}$ or $T'_{\Delta,\alpha}$	Corollary 6.5
$n = 1$	$T_{\Delta,\alpha,\beta}$ or $T'_{\Delta,\alpha,\beta}$	Corollary 6.5

To achieve our goal, we need to adapt the techniques developed in [13] to super setting. Particularly, one of key steps will be proved in a more conceptual way (see Section 3). Besides, the Schur lemma for Lie superalgebras and some analytical techniques will be employed.

The rest of this paper is organized as follows. In Section 2, we first recall some basic definitions on Lie conformal superalgebras. Then, by analyzing certain module structures of $\mathfrak{B}(p)$, we introduce its super analogue $\mathfrak{S}(p)$. At last, we construct the quotient algebra $\mathfrak{s}(n)$ of $\mathfrak{S}(-n)$.

In Section 3, we classify the irreducible modules over a subquotient algebra $\mathfrak{g}$ of the annihilation superalgebra of $\mathfrak{S}(p)$. To do this in a more conceptual way (compare with the non-super case [13]), we introduce the so-called *row*, *column* and *hook ideals* of $\mathfrak{g}$ defined by (3.1)–(3.3), respectively.

In Section 4, we prove the equivalence between the finite conformal modules over $\mathfrak{S}(p)$ and those over its quotient algebra $\mathfrak{S}(p)_{[n]}$ (cf. (2.5)) by using the classification of FICMs over the Neveu-Schwarz conformal algebra $\mathfrak{NS}$.

In Section 5, by some analytical techniques, we classify all the free conformal modules of rank $(1+1)$ over $\mathfrak{S}(p)$. The simplicities of these conformal modules will be also given.

Finally, in Section 6, we complete the classification of FICMs over $\mathfrak{S}(p)$ by showing that they must be free of rank $(1+1)$ by using results in Sections 3–5. As an application of our main result, we also obtain the classification of FICMs over $\mathfrak{s}(n)$ by the feature (1.5) of $\mathfrak{S}(-n)$.

2. Preliminaries

2.1. Basic definitions

We first recall some basic definitions and notations for the sake of completeness, see [6,7,9] for more details.

Definition 2.1. A *Lie conformal superalgebra* $R = R_{\bar{0}} \oplus R_{\bar{1}}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map $R \otimes R \to \mathbb{C}[\lambda] \otimes R$, $a \otimes b \to [a\,{}_\lambda\, b]$ called λ-bracket, and satisfying the following axioms (a, b, $c \in R$):

$$\begin{aligned} &\text{(conformal sesquilinearity)} \quad [\partial a\,{}_\lambda\, b] = -\lambda[a\,{}_\lambda\, b], \qquad [a\,{}_\lambda\, \partial b] = (\partial+\lambda)[a\,{}_\lambda\, b],\\ &\text{(skew-symmetry)} \quad [a\,{}_\lambda\, b] = -(-1)^{|a||b|}[b\,{}_{-\lambda-\partial}\, a],\\ &\text{(Jacobi identity)} \quad [a\,{}_\lambda\,[b\,{}_\mu\, c]] = [[a\,{}_\lambda\, b]\,{}_{\lambda+\mu}\, c] + (-1)^{|a||b|}[b\,{}_\mu\,[a\,{}_\lambda\, c]]. \end{aligned}$$

Here and further, we use the notation $|a| \in \mathbb{Z}/2\mathbb{Z}$ to denote the parity of a, and we always assume that a is homogeneous if $|a|$ appears in an expression. Let R be a Lie conformal superalgebra. We call R *finite* if it is finitely generated over $\mathbb{C}[\partial]$; $\mathbb{Z}$*-graded* if $R = \oplus_{i\in\mathbb{Z}} R_i$, where R_i is a $\mathbb{C}[\partial]$-submodule and $[R_i\,{}_\lambda\, R_j] \subset R_{i+j}[\lambda]$ for $i, j \in \mathbb{Z}$.

Definition 2.2. A *conformal module* $M = M_{\bar 0} \oplus M_{\bar 1}$ over a Lie conformal superalgebra R is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map $R \otimes M \to \mathbb{C}[\lambda] \otimes M$, $a \otimes v \to a\,{}_\lambda\, b$ called λ-action, such that (a, $b \in R$, $v \in M$)

$$\begin{aligned} &(\partial a)\,{}_\lambda\, v = -\lambda a\,{}_\lambda\, v, \qquad a\,{}_\lambda\,(\partial v) = (\partial+\lambda) a\,{}_\lambda\, v,\\ &[a\,{}_\lambda\, b]\,{}_{\lambda+\mu}\, v = a\,{}_\lambda\,(b\,{}_\mu\, v) - (-1)^{|a||b|} b\,{}_\mu\,(a\,{}_\lambda\, v). \end{aligned}$$

Let $M = M_{\bar 0} \oplus M_{\bar 1}$ be a conformal R-module. We call M *finite* if it is finitely generated over $\mathbb{C}[\partial]$. As $\mathbb{C}[\partial]$-modules, if $M_{\bar 0}$ has rank m and $M_{\bar 1}$ has rank n, we say that M has *rank* $(m+n)$. In case R is $\mathbb{Z}$-graded, we call M $\mathbb{Z}$*-graded* if $M = \oplus_{i\in\mathbb{Z}} M_i$, where M_i is a $\mathbb{C}[\partial]$-submodule and $R_i\,{}_\lambda\, M_j \subset M_{i+j}[\lambda]$ for $i, j \in \mathbb{Z}$. Furthermore, if each M_i is freely generated by one element over $\mathbb{C}[\partial]$, we call M a $\mathbb{Z}$*-graded free intermediate series module.*

Obviously, for any fixed $\alpha \in \mathbb{C}$, the $\mathbb{C}[\partial]$-module $\mathbb{C}c_\alpha$ with $\partial c_\alpha = \alpha c_\alpha$, $a\,{}_\lambda c_\alpha = 0$ for $a \in R$, is a conformal R-module, which will be referred to as the *even* (respectively, *odd*) *one-dimensional trivial module* if $|c_\alpha| = 0$ (respectively, $|c_\alpha| = 1$).

Definition 2.3. The *annihilation superalgebra* $\mathcal{A}(R)$ of a Lie conformal superalgebra R is a Lie superalgebra with $\mathbb{C}$-basis $\{a_n \,|\, a \in R,\, n \in \mathbb{Z}_+\}$ and relations

$$[a_m, b_n] = \sum_{k\in\mathbb{Z}_+} \binom{m}{k} (a_{(k)} b)_{m+n-k}, \qquad (\partial a)_n = -n a_{n-1}, \tag{2.1}$$

where $a_{(k)}b$ is called the k-product, given by the following inversion formula:

$$[a\,{}_\lambda\, b] = \sum_{k\in\mathbb{Z}_+} \lambda^{(k)} a_{(k)} b \quad \text{with} \quad \lambda^{(k)} = \frac{\lambda^k}{k!}.$$

Here, the reason why (2.1) gives a Lie superalgebra can be found in the book by Kac [9, pages 41 and 42]. The parity $|a_n|$ of $a_n \in \mathcal{A}(R)$ is the same as $|a|$ for any homogeneous element $a \in R$ and $n \in \mathbb{Z}_+$. Note that $\mathcal{A}(R)$ admits a derivation T given by $T(a_n) = -n a_{n-1}$ for any $a_n \in \mathcal{A}(R)$.

The *extended annihilation superalgebra* $\mathcal{A}(R)^e$ of a Lie conformal superalgebra R is defined by $\mathcal{A}(R)^e = \mathbb{C}T \ltimes \mathcal{A}(R)$ with $[T, a_n] = -na_{n-1}$. The representation theory of R is controlled by the representation theory of $\mathcal{A}(R)^e$ in the following sense:

Proposition 2.4. *A conformal module M over a Lie conformal superalgebra R is the same as a module over the Lie superalgebra $\mathcal{A}(R)^e$ satisfying $a_n v = 0$ for $a \in R$, $v \in M$, $n \gg 0$.*

2.2. Construction of $\mathfrak{S}(p)$

The Lie conformal algebra $\mathfrak{B}(p)$ defined by (1.1) was introduced in [13]. It is called a Lie conformal algebra of Block type due to its relation with some Lie algebras of Block type [11,12,14]. Let us now construct its nontrivial super analogue.

For any a, $b \in \mathbb{C}$, one can easily check that the following $\mathbb{C}[\partial]$-module $M(a, b)$ is a $\mathbb{Z}$-graded free intermediate series module over $\mathfrak{B}(p)$:

$$M(a, b) = \oplus_{i\in\mathbb{Z}}\mathbb{C}[\partial]m_i \quad \text{with } \lambda\text{-action} \quad L_i\,{}_\lambda\, m_j = ((i + j + a)\lambda + (i + p)(\partial + b))m_{i+j}.$$

Motivated by this, we consider a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}[\partial]$-module

$$R(a, b, \{\phi_{i,j}\}) = R_{\bar{0}} \oplus R_{\bar{1}} \quad \text{with} \quad R_{\bar{0}} = \mathfrak{B}(p),\ R_{\bar{1}} = \oplus_{i\in\mathbb{Z}_+}\mathbb{C}[\partial]G_i,$$

and satisfying

$$[L_i\,{}_\lambda\, G_j] = ((i + j + a)\lambda + (i + p)(\partial + b))G_{i+j}, \quad [G_i\,{}_\lambda\, G_j] = \phi_{i,j}(\partial, \lambda)L_{i+j},$$

where $\phi_{i,j}(\partial, \lambda) \in \mathbb{C}[\partial, \lambda]$ with $\phi_{i,j}(\partial, \lambda) \neq 0$ for some $i, j \in \mathbb{Z}_+$.

Lemma 2.5. *The $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}[\partial]$-module $R(a, b, \{\phi_{i,j}\})$ becomes a Lie conformal superalgebra if and only if $a = \frac{3}{2}p$, $b = 0$ and $\phi_{i,j}(\partial, \lambda) = c$ is a constant for all $i, j \in \mathbb{Z}_+$. Up to isomorphism, we may assume that $c = 2$. Then the resulting algebra is exactly $\mathfrak{S}(p)$.*

Proof. The sufficiency can be checked by a direct computation.

Next, we consider the necessity. Assume that $R(a, b, \{\phi_{i,j}\})$ is a Lie conformal superalgebra. For any $i, j \in \mathbb{Z}_+$, using the Jacobi identity for triple (L_0, G_i, G_j), we have

$$\begin{aligned}(p\partial + (i + j + 2p)\lambda)\phi_{i,j}(\partial + \lambda, \mu) = {}&((i + a - p)\lambda + p(b - \mu))\phi_{i,j}(\partial, \lambda + \mu)\\ &+((j + a)\lambda + p(\partial + \mu + b))\phi_{i,j}(\partial, \mu). \qquad (2.2)\end{aligned}$$

Taking $\lambda = 0$ in (2.2), we obtain $2pb\phi_{i,j}(\partial, \mu) = 0$. Hence, $b = 0$, since $\phi_{i,j}(\partial, \mu) \neq 0$ for some $i, j \in \mathbb{Z}_+$. Then, taking $\mu = 0$ in (2.2), we obtain

$$\begin{aligned}(i+a-p)\phi_{i,j}(\partial,\lambda) = p\partial\frac{\phi_{i,j}(\partial+\lambda,0)-\phi_{i,j}(\partial,0)}{\lambda} \\ + (i+j+2p)\phi_{i,j}(\partial+\lambda,0)-(j+a)\phi_{i,j}(\partial,0).\end{aligned}$$

Taking $\lambda \to 0$, we obtain $(\frac{2a}{p}-3)\phi_{i,j}(\partial,0) = \partial\frac{d}{d\partial}(\phi_{i,j}(\partial,0))$. This differential equation has solution if and only if $a = \frac{(n+3)p}{2}$, and in this case the solution is $\phi_{i,j}(\partial,0) = e_{i,j}\partial^n$, where $e_{i,j} \in \mathbb{C}$. Note here that the degree of all $\phi_{i,j}(\partial,0)$ are equal to some fixed $n \in \mathbb{Z}_+$.

Using the Jacobi identity for triple (G_0, G_i, G_j), we have

$$\begin{aligned}((a-p)\partial+(i+j+a)\lambda)\phi_{i,j}(\partial+\lambda,\mu) = ((i+j+a)(\lambda+\mu)+(i+p)\partial)\phi_{0,i}(-\lambda-\mu,\lambda) \\ +((p-a-i)\partial-(i+j+a)\mu)\phi_{0,j}(\partial+\mu,\lambda).\end{aligned}$$

In particular, taking $\lambda = 0$, we obtain

$$\begin{aligned}(a-p)\partial\phi_{i,j}(\partial,\mu) = ((i+j+a)\mu+(i+p)\partial)\phi_{0,i}(-\mu,0) \\ +((p-a-i)\partial-(i+j+a)\mu)\phi_{0,j}(\partial+\mu,0).\end{aligned} \tag{2.3}$$

Furthermore, taking $\mu = i = 0$, we have

$$2(a-p)\phi_{0,j}(\partial,0) = p\phi_{0,0}(0,0). \tag{2.4}$$

If $n \geq 1$, then $a-p = \frac{(n+1)p}{2} \neq 0$ and $\phi_{0,0}(0,0) = e_{0,0}0^n = 0$. By (2.4) and then by (2.3), we obtain $\phi_{i,j}(\partial,\mu) = 0$ for all $i,j \in \mathbb{Z}_+$. This contradicts to the nontrivial assumption that $\phi_{i,j}(\partial,\mu) \neq 0$ for some $i,j \in \mathbb{Z}_+$. Hence $n = 0$, and so $a = \frac{3}{2}p$. Denote $\phi_{0,0}(0,0) = c$ $(\neq 0)$. By (2.4) and then by (2.3), we have $\phi_{i,j}(\partial,\mu) = c$ for any $i,j \in \mathbb{Z}_+$.

The last statement can be easily understood by the fact that if we replace G_i by $\sqrt{\frac{c}{2}}G_i$ in the Lie conformal superalgebra $R(\frac{3}{2}p, 0, \{\phi_{i,j}\})$ with $\phi_{i,j}(\partial,\lambda) = c$ $(\neq 0)$ for all $i,j \in \mathbb{Z}_+$, we obtain exactly $\mathfrak{S}(p)$, given by (1.1)–(1.3). $\square$

2.3. Quotient algebras of $\mathfrak{S}(p)$

One can construct many interesting finite Lie conformal superalgebras by considering the quotient algebras of $\mathfrak{S}(p)$. For example, for $n \in \mathbb{Z}_+$, define a subspace $\mathfrak{S}(p)_{\langle n\rangle}$ of $\mathfrak{S}(p)$ by

$$\mathfrak{S}(p)_{\langle n\rangle} = (\oplus_{i\geq n}\mathbb{C}[\partial]L_i) \oplus (\oplus_{j\geq n}\mathbb{C}[\partial]G_j).$$

Clearly, $\mathfrak{S}(p)_{\langle n\rangle}$ is a (Lie conformal superalgebra) ideal of $\mathfrak{S}(p)$. For $n \in \mathbb{Z}_+$, define $\mathfrak{S}(p)_{[n]}$ by

$$\mathfrak{S}(p)_{[n]} = \mathfrak{S}(p)/\mathfrak{S}(p)_{\langle n+1\rangle}. \tag{2.5}$$

Then, $\mathfrak{S}(p)_{[0]} \cong \mathfrak{NS}$. The special cases $p = -n$ with $1 \leq n \in \mathbb{Z}$ supply a series of new finite non-simple Lie conformal superalgebras:

$$\mathfrak{s}(n) = \mathfrak{S}(-n)_{[n]} = \mathfrak{S}(-n)/\mathfrak{S}(-n)_{\langle n+1 \rangle}. \tag{2.6}$$

Here, we write out the explicit conformal structures of $\mathfrak{s}(1)$ and $\mathfrak{s}(2)$.

Example 1. Set $L = -\bar{L}_0$, $M = \bar{L}_1$, $G = \sqrt{-1}\bar{G}_0$, $H = \sqrt{-1}\bar{G}_1 \in \mathfrak{s}(1)$. We have

$$\begin{aligned}
&[L\,{}_\lambda\, L] = (\partial + 2\lambda)L, \quad [L\,{}_\lambda\, M] = (\partial + \lambda)M, \quad [M\,{}_\lambda\, G] = -\frac{1}{2}\lambda H, \;\; [G\,{}_\lambda\, G] = 2L, \\
&[L\,{}_\lambda\, G] = (\partial + \frac{3}{2}\lambda)G, \;\; [L\,{}_\lambda\, H] = (\partial + \frac{1}{2}\lambda)H, \;\; [G\,{}_\lambda\, H] = -2M.
\end{aligned}$$

Other components vanish or are given by skew-symmetry. One can see that $\mathbb{C}[\partial]L \oplus \mathbb{C}[\partial]G$ is a Neveu-Schwarz conformal subalgebra [7], and $\mathbb{C}[\partial]L \oplus \mathbb{C}[\partial]M$ is a Heisenberg-Virasoro conformal subalgebra [13]. Thus we refer to $\mathfrak{s}(1)$ as *Heisenberg-Neveu-Schwarz conformal algebra*.

Example 2. Set $L = -\frac{1}{2}\bar{L}_0$, $Y = \bar{L}_1$, $M = -\bar{L}_2$, $G = \frac{\sqrt{-2}}{2}\bar{G}_0$, $Z = \frac{\sqrt{-2}}{2}\bar{G}_1$, $H = -\sqrt{-2}\bar{G}_2 \in \mathfrak{s}(2)$. We have

$$\begin{aligned}
&[L\,{}_\lambda\, L] = (\partial + 2\lambda)L, \quad [L\,{}_\lambda\, Y] = (\partial + \frac{3}{2}\lambda)Y, \quad [L\,{}_\lambda\, M] = (\partial + \lambda)M, \quad [M\,{}_\lambda\, G] = -\frac{1}{2}\lambda H, \\
&[L\,{}_\lambda\, G] = (\partial + \frac{3}{2}\lambda)G, \;\; [L\,{}_\lambda\, H] = (\partial + \frac{1}{2}\lambda)H, \quad [L\,{}_\lambda\, Z] = (\partial + \lambda)Z, \quad\;\; [G\,{}_\lambda\, G] = 2L, \\
&[Y\,{}_\lambda\, Y] = (\partial + 2\lambda)M, \;\; [Y\,{}_\lambda\, G] = -(\partial + 2\lambda)Z, \;\; [Y\,{}_\lambda\, Z] = \frac{1}{2}(\partial + \lambda)H, \;\; [G\,{}_\lambda\, Z] = -Y, \\
&[G\,{}_\lambda\, H] = -2M, \qquad\quad [Z\,{}_\lambda\, Z] = M.
\end{aligned}$$

Other components vanish or are given by skew-symmetry. One can see that $\mathbb{C}[\partial]L \oplus \mathbb{C}[\partial]G$ is a Neveu-Schwarz conformal subalgebra [7], and $\mathbb{C}[\partial]L \oplus \mathbb{C}[\partial]Y \oplus \mathbb{C}[\partial]M$ is a Schrödinger-Virasoro conformal subalgebra [13]. Thus we refer to $\mathfrak{s}(2)$ as *Schrödinger-Neveu-Schwarz conformal algebra*.

3. Annihilation superalgebra and related representations

In this section, we classify the irreducible modules over a subquotient algebra of the annihilation superalgebra $\mathcal{A}(\mathfrak{S}(p))$ of $\mathfrak{S}(p)$. Let us first give the explicit super-brackets of $\mathcal{A}(\mathfrak{S}(p))$.

Lemma 3.1. *The Lie superalgebra $\mathcal{A}(\mathfrak{S}(p))$ is isomorphic to the Lie superalgebra which has a basis over $\mathbb{C}$:*

$$\{L_{i,m},\, G_{j,n} \,|\, i,j \in \mathbb{Z}_+,\, m \in \mathbb{Z}_{\geq -1},\, n \in \frac{1}{2} + \mathbb{Z}_{\geq -1}\}$$

with super-brackets

$$\begin{aligned}
&[L_{i,m}, L_{j,n}] = ((j+p)(m+1) - (i+p)(n+1))L_{i+j,m+n},\\
&[L_{i,m}, G_{j,n}] = ((j+\frac{p}{2})(m+1) - (i+p)(n+\frac{1}{2}))G_{i+j,m+n},\\
&[G_{i,m}, G_{j,n}] = 2L_{i+j,m+n}.
\end{aligned}$$

Proof. By the inversion formula in Definition 2.3, one can first transfer the λ-brackets of $\mathfrak{S}(p)$ to k-products:

$$L_{i(k)}L_j = \begin{cases} (i+p)\partial L_{i+j}, & \text{if } \ k=0,\\ (i+j+2p)L_{i+j}, & \text{if } \ k=1,\\ 0, & \text{if } \ k \geq 2. \end{cases}$$

$$L_{i(k)}G_j = \begin{cases} (i+p)\partial G_{i+j}, & \text{if } \ k=0,\\ (i+j+\frac{3}{2}p)G_{i+j}, & \text{if } \ k=1,\\ 0, & \text{if } \ k \geq 2. \end{cases} \qquad G_{i(k)}G_j = \begin{cases} 2L_{i+j}, & \text{if } \ k=0,\\ 0, & \text{if } \ k \geq 1. \end{cases}$$

Then, by the first formula in Definition 2.3, we essentially obtain the super-brackets of $\mathcal{A}(\mathfrak{S}(p))$:

$$\begin{aligned}
&[(L_i)_m, (L_j)_n] = (m(j+p) - n(i+p))(L_{i+j})_{m+n-1},\\
&[(L_i)_m, (G_j)_n] = (m(j+\frac{1}{2}p) - n(i+p))(G_{i+j})_{m+n-1},\\
&[(G_i)_m, (G_j)_n] = 2(L_{i+j})_{m+n}.
\end{aligned}$$

Finally, making the shift $L_{i,m} = (L_i)_{m+1}$, $G_{j,n} = (G_j)_{n+\frac{1}{2}}$ for $i,j \in \mathbb{Z}$, $m \in \mathbb{Z}_{\geq -1}$, $n \in \frac{1}{2} + \mathbb{Z}_{\geq -1}$, we complete the proof of this lemma. $\square$

Next, we construct a subquotient algebra of $\mathcal{A}(\mathfrak{S}(p))$. Clearly,

$$\mathcal{A}(\mathfrak{S}(p))_+ = \mathrm{span}_{\mathbb{C}}\{L_{i,m},\, G_{j,n} \in \mathcal{A}(\mathfrak{S}(p)) \,|\, i,j,m \in \mathbb{Z}_+,\, n \in \frac{1}{2} + \mathbb{Z}_+\}$$

is a subalgebra of $\mathcal{A}(\mathfrak{S}(p))$. For any fixed $k, N \in \mathbb{Z}_+$,

$$\mathcal{I}(k,N) = \mathrm{span}_{\mathbb{C}}\{L_{i,m},\, G_{j,n} \in \mathcal{A}(\mathfrak{S}(p))_+ \,|\, i,j > k,\, m > N,\, n > N + \frac{1}{2}\}$$

is an ideal of $\mathcal{A}(\mathfrak{S}(p))_+$. Let

$$\mathfrak{g}(k,N) = \mathcal{A}(\mathfrak{S}(p))_+/\mathcal{I}(k,N).$$

We will classify the finite-dimensional irreducible modules over $\mathfrak{g}(k,N)$. To conceptualize the proof, we construct the following ideals of $\mathfrak{g}(k,N)$ for $k,\ N\geq 1$:

$$\mathfrak{r}(k,N)=\mathrm{span}_{\mathbb{C}}\{\bar{L}_{k,m},\ \bar{G}_{k,n}\in\mathfrak{g}(k,N)\,|\,m\leq N,\ n\leq N+\frac{1}{2}\},\tag{3.1}$$

$$\mathfrak{c}(k,N)=\mathrm{span}_{\mathbb{C}}\{\bar{L}_{i,N},\ \bar{G}_{j,N+\frac{1}{2}}\in\mathfrak{g}(k,N)\,|\,i,j\leq k\},\tag{3.2}$$

$$\begin{aligned}\mathfrak{h}(k,N)=\mathrm{span}_{\mathbb{C}}\{&\bar{L}_{k,m},\bar{G}_{k,n},\bar{L}_{i,N},\bar{G}_{j,N+\frac{1}{2}}\in\mathfrak{g}(k,N)\,|\,m\leq N,\ n\leq N+\frac{1}{2},\\ &i,j\leq k-1\},\end{aligned}\tag{3.3}$$

which will be referred to as the *row, column* and *hook ideals* of $\mathfrak{g}(k,N)$, respectively. Let $V=V_{\bar{0}}\oplus V_{\bar{1}}$ be a nontrivial finite-dimensional irreducible module over $\mathfrak{g}(k,N)$.

Theorem 3.2. *We have* $\dim V_{\bar{0}}=\dim V_{\bar{1}}=1$.

To prove this theorem, let us first introduce some auxiliary sets:

$$\begin{aligned}\Omega&=\{(j,n)\,|\,\bar{L}_{j,n},\ \bar{G}_{j,n+\frac{1}{2}}\in\mathfrak{g}(k,N)\}\backslash\{(0,0)\},\\ \Omega_0&=\{(j,n)\in\Omega\,|\,j-pn=0\},\\ \Omega_1&=\{(j,n)\in\Omega\,|\,j-p(n+\frac{1}{2})=0\}.\end{aligned}$$

We remark here that if $k,\ N\geq 1$ and $\Omega_1\neq\emptyset$, then $p\in 2\mathbb{Z}_{\geq 1}$ and $\Omega_0\neq\emptyset$.

Lemma 3.3. *If* $\Omega_0=\Omega_1=\emptyset$, *then* $\dim V_{\bar{0}}=\dim V_{\bar{1}}=1$.

Proof. Let us consider the following decomposition of $\mathfrak{g}(k,N)$:

$$\mathfrak{g}(k,N)=\mathbb{C}\bar{L}_{0,0}+\check{\mathfrak{g}}(k,N),\quad\text{where}\quad\check{\mathfrak{g}}(k,N)=\mathfrak{g}(k,N)\backslash\mathbb{C}\bar{L}_{0,0}.$$

Clearly, $\check{\mathfrak{g}}(k,N)$ is a nilpotent ideal of $\mathfrak{g}(k,N)$. Consider the action of $\bar{L}_{0,0}$ on $\check{\mathfrak{g}}(k,N)$:

$$[\bar{L}_{0,0},\bar{L}_{j,n}]=(j-pn)\bar{L}_{j,n},\quad[\bar{L}_{0,0},\bar{G}_{j,n'+\frac{1}{2}}]=(j-p(n'+\frac{1}{2}))\bar{G}_{j,n'+\frac{1}{2}},$$

where $0\leq j\leq k$, $0\leq n,n'\leq N$. Since $\Omega_0=\Omega_1=\emptyset$, it follows from the above two formulas that $\check{\mathfrak{g}}(k,N)$ is a completely reducible $\mathbb{C}\bar{L}_{0,0}$-module with no trivial summand. By [4, Lemma 1], $\check{\mathfrak{g}}(k,N)$ acts trivially on V. Hence, V can be viewed as a finite-dimensional $\mathbb{C}\bar{L}_{0,0}$-module, and so $\dim V_{\bar{0}}=\dim V_{\bar{1}}=1$. $\square$

Lemma 3.4. *Suppose* $k,\ N\geq 1$ *and* $\Omega_0\neq\emptyset$, $\Omega_1=\emptyset$. *Let*

$$j_0=\max\{j\,|\,(j,n)\in\Omega_0\},\quad n_0=\max\{n\,|\,(j,n)\in\Omega_0\}.$$

(1) *If $j_0 < k$, then the row ideal $\mathfrak{r}(k,N)$ of $\mathfrak{g}(k,N)$ acts trivially on V;*
(2) *If $n_0 < N$, then the column ideal $\mathfrak{c}(k,N)$ of $\mathfrak{g}(k,N)$ acts trivially on V;*
(3) *If $j_0 = k$, $n_0 = N$, then the hook ideal $\mathfrak{h}(k,N)$ of $\mathfrak{g}(k,N)$ acts trivially on V.*

Proof. (1) Assume that $\mathfrak{r}(k,N)$ acts non-trivially on V. By the irreducibility of V, we have $V = \mathfrak{r}(k,N)V$. Consider the action of $\bar{L}_{0,0}$ on $\mathfrak{r}(k,N)$:

$$[\bar{L}_{0,0}, \bar{L}_{k,n}] = (k - pn)\bar{L}_{k,n}, \quad [\bar{L}_{0,0}, \bar{G}_{k,n'+\frac{1}{2}}] = (k - p(n' + \frac{1}{2}))\bar{G}_{k,n'+\frac{1}{2}}, \tag{3.4}$$

where $0 \le n, n' \le N$. Note that $k - pn \neq 0$, since $k > j_0$. Note also that $k - p(n' + \frac{1}{2}) \neq 0$, since $\Omega_1 = \emptyset$. Hence, both $\bar{L}_{k,n}$ and $\bar{G}_{k,n'+\frac{1}{2}}$ acts nilpotently on V. Since $\mathfrak{r}(k,N)$ is abelian, $\mathfrak{r}(k,N)$ acts nilpotently on V, which contradicts to $V = \mathfrak{r}(k,N)V$.

(2) Assume that $\mathfrak{c}(k,N)$ acts non-trivially on V. By the irreducibility of V, we have $V = \mathfrak{c}(k,N)V$. Consider the action of $\bar{L}_{0,0}$ on $\mathfrak{c}(k,N)$:

$$[\bar{L}_{0,0}, \bar{L}_{j,N}] = (j - pN)\bar{L}_{j,N}, \quad [\bar{L}_{0,0}, \bar{G}_{j,N+\frac{1}{2}}] = (j - p(N + \frac{1}{2}))\bar{G}_{j,N+\frac{1}{2}}, \tag{3.5}$$

where $0 \le j \le k$. Note that $j - pN \neq 0$, since $N > n_0$. Note also that $j - p(N + \frac{1}{2}) \neq 0$, since $\Omega_1 = \emptyset$. Hence, both $\bar{L}_{j,N}$ and $\bar{G}_{j,N+\frac{1}{2}}$ acts nilpotently on V. Since $\mathfrak{c}(k,N)$ is abelian, $\mathfrak{c}(k,N)$ acts nilpotently on V, which contradicts to $V = \mathfrak{c}(k,N)V$.

(3) Assume that $\mathfrak{h}(k,N)$ acts non-trivially on V. By the irreducibility of V, we have $V = \mathfrak{h}(k,N)V$. Consider the decomposition of $\mathfrak{h}(k,N)$:

$$\begin{aligned} &\mathfrak{h}(k,N) = \mathrm{span}_{\mathbb{C}}\{\bar{L}_{j_0,n_0}, \bar{G}_{j_0,n_0+\frac{1}{2}}\} + \check{\mathfrak{h}}(k,N), \quad \text{where} \\ &\check{\mathfrak{h}}(k,N) = \mathfrak{h}(k,N) \backslash \mathrm{span}_{\mathbb{C}}\{\bar{L}_{j_0,n_0}, \bar{G}_{j_0,n_0+\frac{1}{2}}\}. \end{aligned}$$

As in (1) and (2), by considering the action of $\bar{L}_{0,0}$ on $\check{\mathfrak{h}}(k,N)$, we see that every element in $\check{\mathfrak{h}}(k,N)$ acts nilpotently on V. Note that $\check{\mathfrak{h}}(k,N)$ is almost abelian, except (note that $p > 0$, since $\Omega_0 \neq \emptyset$)

$$[\bar{L}_{j_0,0}, \bar{L}_{0,n_0}] = b_1\bar{L}_{j_0,n_0}, \quad \text{where} \quad b_1 = -(j_0 + p)n_0 - j_0 < 0, \tag{3.6}$$

$$[\bar{L}_{j_0,0}, \bar{G}_{0,n_0+\frac{1}{2}}] = b_2\bar{G}_{j_0,n_0+\frac{1}{2}}, \quad \text{where} \quad b_2 = -(j_0 + p)n_0 - j_0 - \frac{p}{2} < 0, \tag{3.7}$$

$$[\bar{L}_{0,n_0}, \bar{G}_{j_0,\frac{1}{2}}] = b_3\bar{G}_{j_0,n_0+\frac{1}{2}}, \quad \text{where} \quad b_3 = j_0(n_0 + 1) + \frac{p}{2}(n_0 - 1) > 0. \tag{3.8}$$

Thus, to show that $\mathfrak{h}(k,N)$ acts nilpotently on V (and then arrive at a contradiction to $V = \mathfrak{h}(k,N)V$), we only need to show that the actions of $\bar{L}_{j_0,n_0}$ and $\bar{G}_{j_0,n_0+\frac{1}{2}}$ on V are trivial. One can derive the triviality of the action of $\bar{L}_{j_0,n_0}$ as in the non-super setting [13] by comparing traces: Consider the action of (3.6) on V, and then compare the traces of the matrices of both sides with respect to a basis of V. The right hand side equals $cb_1(\dim V)$, where c is a scalar (since $\bar{L}_{j_0,n_0}$ is an even central element of $\mathfrak{g}(k,N)$). While the left hand side equals zero, since the corresponding matrix has the form $AB - BA$. Hence, $c = 0$, i.e., the action of $\bar{L}_{j_0,n_0}$ is trivial. For the action of $\bar{G}_{j_0,n_0+\frac{1}{2}}$, by relation

$$[\bar{L}_{0,0}, \bar{G}_{j_0,n_0+\frac{1}{2}}] = b_4\bar{G}_{j_0,n_0+\frac{1}{2}}, \quad \text{where} \quad b_4 = j_0 - p(n_0 + \frac{1}{2}) \neq 0,$$

we first see that $\bar{G}_{j_0,n_0+\frac{1}{2}}$ acts nilpotently on V. Here the reason for $b_4 \neq 0$ is that $\Omega_1 = \emptyset$. Further, since $\bar{G}_{j_0,n_0+\frac{1}{2}}$ is an odd central element of $\mathfrak{g}(k, N)$, by Schur lemma for Lie superalgebras, $\bar{G}_{j_0,n_0+\frac{1}{2}}$ must act trivially on V. $\square$

Lemma 3.5. *Suppose k, $N \geq 1$ and $\Omega_1 \neq \emptyset$. Let*

$$j_1 = \max\{j \mid (j, n) \in \Omega_1\}, \quad n_1 = \max\{n \mid (j, n) \in \Omega_1\}.$$

(1) *If $j_1 < k$, then the row ideal $\mathfrak{r}(k, N)$ of $\mathfrak{g}(k, N)$ acts trivially on V;*
(2) *If $n_1 < N$, then the column ideal $\mathfrak{c}(k, N)$ of $\mathfrak{g}(k, N)$ acts trivially on V;*
(3) *If $j_1 = k$, $n_1 = N$, then the row ideal $\mathfrak{r}(k, N)$ of $\mathfrak{g}(k, N)$ acts trivially on V.*

Proof. First, recalling the remark before Lemma 3.3, we have $p \in 2\mathbb{Z}_{\geq 1}$ and $\Omega_0 \neq \emptyset$. Furthermore, if we denote j_0 and n_0 as in Lemma 3.4, then we have $j_0 = j_1 - \frac{p}{2}$ and $n_0 = n_1$.

The statements (1) and (2) can be proved in a similar way as Lemmas 3.4(1) and (2). The differences lie in the reasons for the non-trivialities of the actions of $\bar{L}_{0,0}$ on $\mathfrak{r}(k, N)$ and $\mathfrak{c}(k, N)$. For the statement (1), we still have (3.4), the reason for $k - pn \neq 0$ is $k > j_1 = j_0 + \frac{p}{2} > j_0$; while the reason for $k - p(n' + \frac{1}{2}) \neq 0$ is $k > j_1$. For the statement (2), we still have (3.5), the reason for $j - pN \neq 0$ is $N > n_1 = n_0$; while the reason for $j - p(N + \frac{1}{2}) \neq 0$ is $N > n_1$.

Next, we prove the statement (3). Assume that $\mathfrak{r}(k, N)$ acts non-trivially on V. By the irreducibility of V, we have $V = \mathfrak{r}(k, N)V$. Consider the decomposition of $\mathfrak{r}(k, N)$:

$$\begin{aligned}
&\mathfrak{r}(k, N) = \mathrm{span}_{\mathbb{C}}\{\bar{L}_{j_1,n_1}, \bar{G}_{j_1,n_1+\frac{1}{2}}\} + \check{\mathfrak{r}}(k, N), \quad \text{where} \\
&\check{\mathfrak{r}}(k, N) = \mathfrak{r}(k, N) \backslash \mathrm{span}_{\mathbb{C}}\{\bar{L}_{j_1,n_1}, \bar{G}_{j_1,n_1+\frac{1}{2}}\}.
\end{aligned}$$

Note that $\bar{L}_{j_1,n_1}$ is an even central element of $\mathfrak{g}(k, N)$. By comparing traces, one can prove the triviality of its action as in Lemma 3.4(3). Furthermore, the following relation (note that $j_1 = p(n_1 + \frac{1}{2})$, since $(j_1, n_1) \in \Omega_1$)

$$[\bar{L}_{j_1,n_1}, \bar{G}_{0,\frac{1}{2}}] = b_5\bar{G}_{j_1,n_1+\frac{1}{2}}, \quad \text{where} \quad b_5 = -p(1 + \frac{n_1}{2}) < 0,$$

implies that the action of $\bar{G}_{j_1,n_1+\frac{1}{2}}$ is also trivial. Hence, $V = \check{\mathfrak{r}}(k, N)V$. Consider the action of $\bar{L}_{0,0}$ on $\check{\mathfrak{r}}(k, N)$:

$$[\bar{L}_{0,0}, \bar{L}_{k,n}] = (k - pn)\bar{L}_{k,n}, \quad [\bar{L}_{0,0}, \bar{G}_{k,n'+\frac{1}{2}}] = (k - p(n' + \frac{1}{2}))\bar{G}_{k,n'+\frac{1}{2}},$$

where $0 \le n, n' \le N-1$. Note that $k = j_1 = p(n_1 + \frac{1}{2})$ and $n, n' \le N-1 < N = n_1$, we have $k - pn > 0$ and $k - p(n' + \frac{1}{2}) > 0$. Hence, both $\bar{L}_{k,n}$ and $\bar{G}_{k,n'+\frac{1}{2}}$ act nilpotently on V. Since $\check{\mathfrak{r}}(k, N)$ is abelian, $\check{\mathfrak{r}}(k, N)$ acts nilpotently on V, which contradicts to $V = \check{\mathfrak{r}}(k, N)V$. $\square$

Now, we can give the proof of Theorem 3.2.

Proof of Theorem 3.2. If $k = 0$, then $\Omega_0 = \Omega_1 = \emptyset$, and the result follows from Lemma 3.3.

If $k \ge 1$ and $N = 0$, then $\Omega_0 = \emptyset$ and

$$\Omega_1 \ne \emptyset \Longleftrightarrow p \in 2\mathbb{Z}_{\ge 1}.$$

So, if $p \notin 2\mathbb{Z}_{\ge 1}$, then the result follows from Lemma 3.3. In case $p \in 2\mathbb{Z}_{\ge 1}$, let us consider the action of the row ideal $\mathfrak{r}(k, 0)$ of $\mathfrak{g}(k, 0)$ on V (note that $\mathfrak{r}(k, 0)$ given by (3.1) still make sense). The two generators of $\mathfrak{r}(k, 0)$ are $\bar{L}_{k,0}$ and $\bar{G}_{k,\frac{1}{2}}$. On one hand, since $\bar{L}_{k,0}$ is an even central element of $\mathfrak{g}(k, 0)$, its action is a scalar, say c. On the other hand, the relation

$$[\bar{L}_{0,0}, \bar{L}_{k,0}] = k\bar{L}_{k,0}, \quad \text{where} \quad k > 0,$$

implies that the action of $\bar{L}_{k,0}$ is nilpotent. Hence, $c = 0$, i.e., the action of $\bar{L}_{k,0}$ is trivial. Furthermore, the relation

$$[\bar{L}_{k,0}, \bar{G}_{0,\frac{1}{2}}] = -(k + \frac{p}{2})\bar{G}_{k,\frac{1}{2}}, \quad \text{where} \quad k + \frac{p}{2} > 0,$$

implies that the action of $\bar{G}_{k,\frac{1}{2}}$ is also trivial. Therefore, V is simply an irreducible module over $\mathfrak{g}(k-1, 0)$. By induction on k, we must have $\dim V_{\bar{0}} = \dim V_{\bar{1}} = 1$.

Next, we assume that $k, N \ge 1$. Note that if the row ideal $\mathfrak{r}(k, N)$ (respectively, column ideal $\mathfrak{c}(k, N)$, hook ideal $\mathfrak{h}(k, N)$) of $\mathfrak{g}(k, N)$ acts trivially on V, then V can be viewed as an irreducible module over $\mathfrak{g}(k-1, N)$ (respectively, $\mathfrak{g}(k, N-1)$, $\mathfrak{g}(k-1, N-1)$). By simultaneous induction on k and N, using Lemmas 3.3–3.5, we must have $\dim V_{\bar{0}} = \dim V_{\bar{1}} = 1$. $\square$

4. Equivalence of representations

In this section, we prove the equivalence between the finite conformal modules over $\mathfrak{S}(p)$ and those over its quotient algebra $\mathfrak{S}(p)_{[n]}$ for some $n \in \mathbb{Z}_+$. The following classification of FICMs over $\mathfrak{N}\mathfrak{S}$ will be used.

Lemma 4.1 *([4]). Let V be a finite irreducible conformal module over $\mathfrak{NS}$. Then V is isomorphic to one of the following ($|c_\alpha| = |v_{\bar 0}| = |v'_{\bar 0}| = \bar 0$, $|\epsilon_\alpha| = |v_{\bar 1}| = |v'_{\bar 1}| = \bar 1$):*

(1) *even one-dimensional module $\mathbb{C}c_\alpha$ with $L\,{}_\lambda\, c_\alpha = G\,{}_\lambda\, c_\alpha = 0$, $\partial c_\alpha = \alpha c_\alpha$ for some $\alpha \in \mathbb{C}$;*
(2) *odd one-dimensional module $\mathbb{C}\epsilon_\alpha$ with $L\,{}_\lambda\, \epsilon_\alpha = G\,{}_\lambda\, \epsilon_\alpha = 0$, $\partial \epsilon_\alpha = \alpha \epsilon_\alpha$ for some $\alpha \in \mathbb{C}$;*
(3) *$N_{\Delta,\alpha} = \mathbb{C}[\partial]v_{\bar 0} \oplus \mathbb{C}[\partial]v_{\bar 1}$ with*

$$\begin{cases} L\,{}_\lambda\, v_{\bar 0} = (\partial + \Delta\lambda + \alpha)v_{\bar 0}, \\ L\,{}_\lambda\, v_{\bar 1} = (\partial + (\Delta + \frac{1}{2})\lambda + \alpha)v_{\bar 1}, \\ G\,{}_\lambda\, v_{\bar 0} = v_{\bar 1}, \\ G\,{}_\lambda\, v_{\bar 1} = (\partial + 2\Delta\lambda + \alpha)v_{\bar 0}, \end{cases} \tag{4.1}$$

for some $\Delta \neq 0$ and $\alpha \in \mathbb{C}$;
(4) *$N'_{\Delta,\alpha} = \mathbb{C}[\partial]v'_{\bar 0} \oplus \mathbb{C}[\partial]v'_{\bar 1}$ with*

$$\begin{cases} L\,{}_\lambda\, v'_{\bar 0} = (\partial + (\Delta + \frac{1}{2})\lambda + \alpha)v'_{\bar 0}, \\ L\,{}_\lambda\, v'_{\bar 1} = (\partial + \Delta\lambda + \alpha)v'_{\bar 1}, \\ G\,{}_\lambda\, v'_{\bar 0} = (\partial + 2\Delta\lambda + \alpha)v'_{\bar 1}, \\ G\,{}_\lambda\, v'_{\bar 1} = v'_{\bar 0}, \end{cases} \tag{4.2}$$

for some $\Delta \neq 0$ and $\alpha \in \mathbb{C}$.

Theorem 4.2. *Let M be a nontrivial finite conformal module over $\mathfrak{S}(p)$. Then the λ-actions of $L_i, G_i \in \mathfrak{S}(p)$ on M are trivial for all $i \gg 0$. In particular, a finite conformal module over $\mathfrak{S}(p)$ is simply a finite conformal module over $\mathfrak{S}(p)_{[n]}$ for some big enough integer n, where $\mathfrak{S}(p)_{[n]}$ is defined by (2.5).*

Proof. Since M can be viewed as a finite conformal module over $\mathfrak{S}(p)_{\bar 0}$, it follows from [13] that the λ-action of $L_i \in \mathfrak{S}(p)_{\bar 0}$ on M is trivial for all $i \gg 0$. Next, we only need to show that the λ-action of $G_i \in \mathfrak{S}(p)_{\bar 1}$ on M is also trivial for all $i \gg 0$.

By regarding M as a module over $\mathfrak{NS} \subset \mathfrak{S}(p)$, we can choose a composition series

$$M = M_N \supset M_{N-1} \supset \cdots \supset M_1 \supset M_0 = 0,$$

such that for each $1 \leq k \leq N$, the composition factor $\overline{M}_k = M_k/M_{k-1}$ is one of the modules in Lemma 4.1. Denote by $\bar c_{\alpha_k}$, $\bar\epsilon_{\alpha_k}$ or $\{\bar v_{\bar 0}(k), \bar v_{\bar 1}(k)\}$ a $\mathbb{C}[\partial]$-generating set of $\overline{M}_k$ according to its type, and c_{α_k}, ϵ_{α_k}, $v_{\bar 0}(k)$, $v_{\bar 1}(k)$ the corresponding preimages. Then the set of all c_{α_k}, ϵ_{α_k}, $v_{\bar 0}(k)$, $v_{\bar 1}(k)$, $1 \leq k \leq N$, is a $\mathbb{C}[\partial]$-generating set of M.

We first claim that the λ-action of G_i on M_1 is trivial for all $i \gg 0$. Namely,

$$G_i\,{}_\lambda\, v = 0 \quad \text{for all} \quad i \gg 0,\ v \in M_1. \tag{4.3}$$

Fix $i \gg 0$ and assume that $G_{i\,\lambda}\, v \neq 0$ for some $v \in M_1$. Let $k_i \geq 1$ be the largest integer such that $G_{i\,\lambda}\, v \notin M_{k_i-1}$ for some $v \in M_1$. We proceed to derive a contradiction. According to Lemma 4.1, we need to consider the following cases:

Case 1: Both M_1 and $\overline{M}_{k_i}$ have the form $N_{\Delta,\alpha}$ or $N'_{\Delta,\alpha}$.

We only give the proof for the case $M_1 = N_{\Delta_1,\alpha_1}$ and $\overline{M}_{k_i} = N_{\Delta_{k_i},\alpha_{k_i}}$; other cases can be proved in a similar way. In this case, by assumption, we can write

$$G_{i\,\lambda}\, v_{\bar{0}}(1) \equiv g_i(\partial, \lambda) v_{\bar{1}}(k_i) (\mathrm{mod}\ M_{k_i-1}), \tag{4.4}$$

$$G_{i\,\lambda}\, v_{\bar{1}}(1) \equiv h_i(\partial, \lambda) v_{\bar{0}}(k_i) (\mathrm{mod}\ M_{k_i-1}), \tag{4.5}$$

where $g_i(\partial, \lambda) \neq 0$ or $h_i(\partial, \lambda) \neq 0$. Let us consider the action of the operator $G_{0\,\mu}$ on (4.4). By the definition of conformal module, on one hand, we have (note that $G_0 = \sqrt{p}G$, cf. (1.4))

$$\begin{aligned} G_{0\,\mu}(G_{i\,\lambda}\, v_{\bar{0}}(1)) &\equiv G_{0\,\mu}(g_i(\partial, \lambda) v_{\bar{1}}(k_i)) (\mathrm{mod}\ M_{k_i-1}) \\ &\equiv g_i(\partial + \mu, \lambda)\sqrt{p}(\partial + 2\Delta_{k_i}\mu + \alpha_{k_i}) v_{\bar{0}}(k_i) (\mathrm{mod}\ M_{k_i-1}). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} G_{0\,\mu}(G_{i\,\lambda}\, v_{\bar{0}}(1)) &= [G_{0\,\mu}\, G_i]_{\,\lambda+\mu}\, v_{\bar{0}}(1) - G_{i\,\lambda}(G_{0\,\mu} v_{\bar{0}}(1)) \\ &= 2L_{i\,\lambda+\mu}\, v_{\bar{0}}(1) - G_{i\,\lambda}(\sqrt{p}\, v_{\bar{1}}(1)) \\ &\equiv -\sqrt{p} h_i(\partial, \lambda) v_{\bar{0}}(k_i) (\mathrm{mod}\ M_{k_i-1}). \end{aligned}$$

Then, $h_i(\partial, \lambda) = -(\partial + 2\Delta_{k_i}\mu + \alpha_{k_i}) g_i(\partial + \mu, \lambda)$. In particular, $h_i(\partial, \lambda) = -(\partial + \alpha_{k_i}) g_i(\partial, \lambda)$. Similarly, by considering the action of the operator $G_{0\,\mu}$ on (4.5), one can obtain $h_i(\partial + \mu, \lambda) = -(\partial + \lambda + 2\Delta_1\mu + \alpha_1) g_i(\partial, \lambda)$. In particular, $h_i(\partial, \lambda) = -(\partial + \lambda + \alpha_1) g_i(\partial, \lambda)$. Hence, we must have $g_i(\partial, \lambda) = h_i(\partial, \lambda) = 0$, a contradiction.

Case 2: M_1 has the form $\mathbb{C}c_\alpha$ or $\mathbb{C}\epsilon_\alpha$, and $\overline{M}_{k_i}$ has the form $N_{\Delta,\alpha}$ or $N'_{\Delta,\alpha}$.

We only give the proof for the case $M_1 = \mathbb{C}c_{\alpha_1}$, $\overline{M}_{k_i} = N_{\Delta_{k_i},\alpha_{k_i}}$; other cases can be proved in a similar way. In this case, by assumption, we can write

$$G_{i\,\lambda}\, c_{\alpha_1} \equiv g_i(\partial, \lambda) v_{\bar{1}}(k_i) (\mathrm{mod}\ M_{k_i-1}), \tag{4.6}$$

where $g_i(\partial, \lambda) \neq 0$. Considering the action of the operator $G_{0\,\mu}$ on (4.6), on one hand, we have

$$\begin{aligned} G_{0\,\mu}(G_{i\,\lambda}\, c_{\alpha_1}) &\equiv G_{0\,\mu}(g_i(\partial, \lambda) v_{\bar{1}}(k_i)) (\mathrm{mod}\ M_{k_i-1}) \\ &\equiv g_i(\partial + \mu, \lambda)\sqrt{p}(\partial + 2\Delta_{k_i}\mu + \alpha_{k_i}) v_{\bar{0}}(k_i) (\mathrm{mod}\ M_{k_i-1}). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} G_{0\ \mu}(G_{i\ \lambda}\, c_{\alpha_1}) &= [G_{0\ \mu}\, G_i]_{\ \lambda+\mu}\, c_{\alpha_1} - G_{i\ \lambda}(G_{0\ \mu} c_{\alpha_1}) \\ &= 2L_{i\ \lambda+\mu}\, c_{\alpha_1} = 0. \end{aligned}$$

Then, $(\partial + 2\Delta_{k_i}\mu + \alpha_{k_i})g_i(\partial+\mu,\lambda) = 0$. Taking $\mu = 0$, we see that $g_i(\partial,\lambda) = 0$, a contradiction.

Case 3: M_1 has the form $N_{\Delta,\alpha}$ or $N'_{\Delta,\alpha}$, and $\overline{M}_{k_i}$ has the form $\mathbb{C}c_\alpha$ or $\mathbb{C}\epsilon_\alpha$.

We only give the proof for the case $M_1 = N_{\Delta_1,\alpha_1}$, $\overline{M}_{k_i} = \mathbb{C}c_{\alpha_{k_i}}$; other cases can be proved in a similar way. In this case, since ∂ acts on $c_{\alpha_{k_i}}$ as the scalar α_{k_i}, by assumption, we can write

$$G_{i\ \lambda}\, v_{\bar{1}}(1) \equiv g_i(\lambda)c_{\alpha_{k_i}} (\mathrm{mod}\ M_{k_i-1}), \tag{4.7}$$

where $g_i(\lambda) \neq 0$. Considering the action of the operator $L_{0\ \mu}$ on (4.7), on one hand, we have

$$L_{0\ \mu}(G_{i\ \lambda}\, v_{\bar{1}}(1)) \equiv L_{0\ \mu}(g_i(\lambda)c_{\alpha_{k_i}})(\mathrm{mod}\ M_{k_i-1}) \equiv 0(\mathrm{mod}\ M_{k_i-1}).$$

On the other hand, we have (note that $L_0 = p\,L$, cf. (1.4))

$$\begin{aligned} L_{0\ \mu}(G_{i\ \lambda}\, v_{\bar{1}}(1)) &= [L_{0\ \mu}\, G_i]_{\ \lambda+\mu}\, v_{\bar{1}}(1) + G_{i\ \lambda}(L_{0\ \mu} v_{\bar{1}}(1)) \\ &= \Big((p\partial + (i+\frac{3}{2}p)\mu)G_i\Big)_{\ \lambda+\mu}\, v_{\bar{1}}(1) \\ &\quad + G_{i\ \lambda}\Big(p(\partial + (\Delta_1 + \frac{1}{2})\mu + \alpha_1)v_{\bar{1}}(1)\Big) \\ &\equiv \Big(((i+\frac{1}{2}p)\mu - p\lambda)g_i(\lambda+\mu) \\ &\quad + p(\partial + \lambda + (\Delta_1 + \frac{1}{2})\mu + \alpha_1)g_i(\lambda)\Big)c_{\alpha_{k_i}} (\mathrm{mod}\ M_{k_i-1}). \end{aligned}$$

Then, $((i+\frac{1}{2}p)\mu - p\lambda)g_i(\lambda+\mu) + p(\partial + \lambda + (\Delta_1+\frac{1}{2})\mu + \alpha_1)g_i(\lambda) = 0$. Equating the coefficients of ∂, we see that $g_i(\lambda) = 0$, a contradiction.

Case 4: Both M_1 and $\overline{M}_{k_i}$ have the form $\mathbb{C}c_\alpha$ or $\mathbb{C}\epsilon_\alpha$.

We only give the proof for the case $M_1 = \mathbb{C}c_{\alpha_1}$, $\overline{M}_{k_i} = \mathbb{C}\epsilon_{\alpha_{k_i}}$; other cases are trivial or can be proved in a similar way. In this case, by assumption, we can write

$$G_{i\ \lambda}\, c_{\alpha_1} \equiv g_i(\lambda)\epsilon_{\alpha_{k_i}} (\mathrm{mod}\ M_{k_i-1}), \tag{4.8}$$

where $g_i(\lambda) \neq 0$. Considering the action of the operator $L_{0\ \mu}$ on (4.8), on one hand, we have

$$L_{0\ \mu}(G_{i\ \lambda}\, c_{\alpha_1}) \equiv L_{0\ \mu}(g_i(\lambda)\epsilon_{\alpha_{k_i}})(\mathrm{mod}\ M_{k_i-1}) \equiv 0(\mathrm{mod}\ M_{k_i-1}).$$

On the other hand, we have

$$\begin{aligned} L_{0\ \mu}(G_{i\ \lambda}\, c_{\alpha_1}) &= [L_{0\ \mu}\, G_i]_{\ \lambda+\mu}\, c_{\alpha_1} + G_{i\ \lambda}(L_{0\ \mu} c_{\alpha_1}) \\ &= \left((p\partial + (i+\frac{3}{2}p)\mu)G_i \right)_{\lambda+\mu}\, c_{\alpha_1} \\ &\equiv ((i+\frac{1}{2}p)\mu - p\lambda)g_i(\lambda+\mu)\epsilon_{\alpha_{k_i}} (\text{mod } M_{k_i-1}). \end{aligned}$$

Then, $((i+\frac{1}{2}p)\mu - p\lambda)g_i(\lambda+\mu) = 0$. Taking $\mu = 0$, we see that $g_i(\lambda) = 0$, a contradiction.

Now, start from (4.3), one can inductively show that the λ-action of G_i on M_k is trivial for $1 \le k \le N$. Hence, the λ-action of G_i on M $(= M_N)$ is trivial. This completes the proof. $\square$

5. Free conformal modules of rank $(1+1)$

In this section, we classify all the free conformal modules of rank $(1+1)$ over $\mathfrak{S}(p)$. Obviously, the following two $\mathbb{C}[\partial]$-modules are such conformal modules.

(1) $T_{\Delta,\alpha} = \mathbb{C}[\partial]v_{\bar{0}} \oplus \mathbb{C}[\partial]v_{\bar{1}}$ with

$$\begin{cases} L_{0\ \lambda}\, v_{\bar{0}} = p(\partial + \Delta\lambda + \alpha)v_{\bar{0}}, \\ L_{0\ \lambda}\, v_{\bar{1}} = p(\partial + (\Delta+\frac{1}{2})\lambda + \alpha)v_{\bar{1}}, \\ G_{0\ \lambda}\, v_{\bar{0}} = \sqrt{p}\, v_{\bar{1}}, \\ G_{0\ \lambda}\, v_{\bar{1}} = \sqrt{p}\,(\partial + 2\Delta\lambda + \alpha)v_{\bar{0}}, \\ L_{i\ \lambda}\, v_s = G_{i\ \lambda}\, v_s = 0,\ i \ge 1,\ s \in \mathbb{Z}/2\mathbb{Z}, \end{cases} \tag{5.1}$$

where $\Delta,\ \alpha \in \mathbb{C}$;

(2) $T'_{\Delta,\alpha} = \mathbb{C}[\partial]v'_{\bar{0}} \oplus \mathbb{C}[\partial]v'_{\bar{1}}$ with

$$\begin{cases} L_{0\ \lambda}\, v'_{\bar{0}} = p(\partial + (\Delta+\frac{1}{2})\lambda + \alpha)v'_{\bar{0}}, \\ L_{0\ \lambda}\, v'_{\bar{1}} = p(\partial + \Delta\lambda + \alpha)v'_{\bar{1}}, \\ G_{0\ \lambda}\, v'_{\bar{0}} = \sqrt{p}\,(\partial + 2\Delta\lambda + \alpha)v'_{\bar{1}}, \\ G_{0\ \lambda}\, v'_{\bar{1}} = \sqrt{p}\, v'_{\bar{0}}, \\ L_{i\ \lambda}\, v'_s = G_{i\ \lambda}\, v'_s = 0,\ i \ge 1,\ s \in \mathbb{Z}/2\mathbb{Z}, \end{cases} \tag{5.2}$$

where $\Delta,\ \alpha \in \mathbb{C}$.

In fact, $T_{\Delta,\alpha}$ and $T'_{\Delta,\alpha}$ are respectively trivial extensions of the conformal $\mathfrak{NS}$-modules $N_{\Delta,\alpha}$ and $N'_{\Delta,\alpha}$ (cf. (4.1), (4.2), and note that $L_0 = pL$, $G_0 = \sqrt{p}\,G$). For $\mathfrak{S}(-1)$, $T_{\Delta,\alpha}$ and $T'_{\Delta,\alpha}$ can be generalized to the following $T_{\Delta,\alpha,\beta}$ and $T'_{\Delta,\alpha,\beta}$, which are respectively non-trivial extensions of the conformal $\mathfrak{NS}$-modules $N_{\Delta,\alpha}$ and $N'_{\Delta,\alpha}$ if $\beta \neq 0$.

(3) $T_{\Delta,\alpha,\beta} = \mathbb{C}[\partial]v_{\bar{0}} \oplus \mathbb{C}[\partial]v_{\bar{1}}$ with

$$\begin{cases} L_0\,{}_\lambda\, v_{\bar{0}} = -(\partial + \Delta\lambda + \alpha)v_{\bar{0}}, \\ L_0\,{}_\lambda\, v_{\bar{1}} = -(\partial + (\Delta + \frac{1}{2})\lambda + \alpha)v_{\bar{1}}, \\ L_1\,{}_\lambda\, v_{\bar{0}} = \beta v_{\bar{0}}, \\ L_1\,{}_\lambda\, v_{\bar{1}} = \beta v_{\bar{1}}, \\ G_0\,{}_\lambda\, v_{\bar{0}} = \sqrt{-1}\, v_{\bar{1}}, \\ G_0\,{}_\lambda\, v_{\bar{1}} = \sqrt{-1}\,(\partial + 2\Delta\lambda + \alpha)v_{\bar{0}}, \\ G_1\,{}_\lambda\, v_{\bar{0}} = 0, \\ G_1\,{}_\lambda\, v_{\bar{1}} = -2\sqrt{-1}\beta v_{\bar{0}}, \\ L_i\,{}_\lambda\, v_s = G_i\,{}_\lambda\, v_s = 0,\ i \geq 2,\ s \in \mathbb{Z}/2\mathbb{Z}, \end{cases} \tag{5.3}$$

where $\Delta,\ \alpha,\ \beta \in \mathbb{C}$;

(4) $T'_{\Delta,\alpha,\beta} = \mathbb{C}[\partial]v'_{\bar{0}} \oplus \mathbb{C}[\partial]v'_{\bar{1}}$ with

$$\begin{cases} L_0\,{}_\lambda\, v'_{\bar{0}} = -(\partial + (\Delta + \frac{1}{2})\lambda + \alpha)v'_{\bar{0}}, \\ L_0\,{}_\lambda\, v'_{\bar{1}} = -(\partial + \Delta\lambda + \alpha)v'_{\bar{1}}, \\ L_1\,{}_\lambda\, v'_{\bar{0}} = \beta v'_{\bar{0}}, \\ L_1\,{}_\lambda\, v'_{\bar{1}} = \beta v'_{\bar{1}}, \\ G_0\,{}_\lambda\, v'_{\bar{0}} = \sqrt{-1}\,(\partial + 2\Delta\lambda + \alpha)v'_{\bar{1}}, \\ G_0\,{}_\lambda\, v'_{\bar{1}} = \sqrt{-1}\, v'_{\bar{0}}, \\ G_1\,{}_\lambda\, v'_{\bar{0}} = -2\sqrt{-1}\beta v'_{\bar{1}}, \\ G_1\,{}_\lambda\, v'_{\bar{1}} = 0, \\ L_i\,{}_\lambda\, v'_s = G_i\,{}_\lambda\, v'_s = 0,\ i \geq 2,\ s \in \mathbb{Z}/2\mathbb{Z}, \end{cases} \tag{5.4}$$

where $\Delta,\ \alpha, \beta \in \mathbb{C}$.

Theorem 5.1. *Let M be a nontrivial free conformal module of rank $(1+1)$ over $\mathfrak{S}(p)$.*

(1) *If $p \neq -1$, then $M \cong T_{\Delta,\alpha}$ or $T'_{\Delta,\alpha}$ defined by (5.1) and (5.2) for some $\Delta, \alpha \in \mathbb{C}$.*
(2) *If $p = -1$, then $M \cong T'_{\Delta,\alpha,\beta}$ or $T'_{\Delta,\alpha,\beta}$ defined by (5.3) and (5.4) for some $\Delta, \alpha, \beta \in \mathbb{C}$.*

Proof. Let $M = \mathbb{C}[\partial]v_{\bar{0}} \oplus \mathbb{C}[\partial]v_{\bar{1}}$. By regarding M as a conformal module over $\mathfrak{N}\mathfrak{S}$, we see that (cf. (4.1) and (4.2))

$$\begin{cases} L_0\,{}_\lambda\, v_{\bar{0}} = p(\partial + \Delta\lambda + \alpha)v_{\bar{0}}, \\ L_0\,{}_\lambda\, v_{\bar{1}} = p(\partial + (\Delta + \frac{1}{2})\lambda + \alpha)v_{\bar{1}}, \\ G_0\,{}_\lambda\, v_{\bar{0}} = \sqrt{p}\, v_{\bar{1}}, \\ G_0\,{}_\lambda\, v_{\bar{1}} = \sqrt{p}\,(\partial + 2\Delta\lambda + \alpha)v_{\bar{0}}, \end{cases} \text{ or } \begin{cases} L_0\,{}_\lambda\, v_{\bar{0}} = p(\partial + (\Delta + \frac{1}{2})\lambda + \alpha)v_{\bar{0}}, \\ L_0\,{}_\lambda\, v_{\bar{1}} = p(\partial + \Delta\lambda + \alpha)v_{\bar{1}}, \\ G_0\,{}_\lambda\, v_{\bar{0}} = \sqrt{p}\,(\partial + 2\Delta\lambda + \alpha)v_{\bar{1}}, \\ G_0\,{}_\lambda\, v_{\bar{1}} = \sqrt{p}\, v_{\bar{0}}, \end{cases} \tag{5.5}$$

where $\Delta, \alpha \in \mathbb{C}$. By Theorem 4.2, $L_i\,{}_\lambda\, v_s = G_i\,{}_\lambda\, v_s = 0$ for $i \gg 0$, $s \in \mathbb{Z}/2\mathbb{Z}$. Note that $\mathfrak{S}(p)$ is $\mathbb{Z}$-graded in the sense that $\mathfrak{S}(p) = \oplus_{i\in\mathbb{Z}_+}\mathfrak{S}(p)_i$, where $\mathfrak{S}(p)_i = \mathbb{C}[\partial]L_i \oplus \mathbb{C}[\partial]G_i$.

Assume that $k \in \mathbb{Z}_+$ is the largest integer such that the action of $\mathfrak{S}(p)_k$ on M is nontrivial.

If $k = 0$, then M is simply a conformal $\mathfrak{NS}$-module. Then,

$$M \cong \begin{cases} T_{\Delta,\alpha} \quad \text{or} \quad T'_{\Delta,\alpha}, & \text{if} \quad p \neq -1; \\ T_{\Delta,\alpha,0} \quad \text{or} \quad T'_{\Delta,\alpha,0}, & \text{if} \quad p = -1. \end{cases}$$

Next, consider the case $k > 0$. Without specification, we always assume that the λ-actions of L_0 and G_0 have the first form of (5.5) (the case for the second form can be treated in a similar way). By the assumption of k, we can suppose

$$L_{k\ \lambda}\, v_{\bar{0}} = a(\partial,\lambda)v_{\bar{0}}, \quad L_{k\ \lambda}\, v_{\bar{1}} = b(\partial,\lambda)v_{\bar{1}}, \quad G_{k\ \lambda}\, v_{\bar{0}} = c(\partial,\lambda)v_{\bar{1}}, \quad G_{k\ \lambda}\, v_{\bar{1}} = d(\partial,\lambda)v_{\bar{0}},$$

where $a(\partial,\lambda)$, $b(\partial,\lambda)$, $c(\partial,\lambda)$, $d(\partial,\lambda) \in \mathbb{C}[\partial,\lambda]$ and at least one of them is nonzero. Considering the actions of the zero operator $[L_{k\ \lambda}\, L_k]_{\lambda+\mu} = 0$ on $v_{\bar{0}}$ and $v_{\bar{1}}$, respectively, we obtain

$$\begin{aligned} a(\partial,\lambda)a(\partial+\lambda,\mu) &= a(\partial,\mu)a(\partial+\mu,\lambda), \\ b(\partial,\lambda)b(\partial+\lambda,\mu) &= b(\partial,\mu)b(\partial+\mu,\lambda). \end{aligned}$$

Comparing the coefficients of λ in the above equations, we see that $a(\partial,\lambda)$ and $b(\partial,\lambda)$ are independent of the variable ∂, and so we can denote $a(\lambda) = a(\partial,\lambda)$, $b(\lambda) = b(\partial,\lambda)$. Then, considering the actions of the operator $[L_{0\ \lambda}\, L_k]_{\lambda+\mu} = ((k+p)\lambda - p\mu)L_{k\ \lambda+\mu}$ on $v_{\bar{0}}$ and $v_{\bar{1}}$, respectively, we obtain

$$(p\mu - (k+p)\lambda)a(\lambda+\mu) = p\mu a(\mu), \tag{5.6}$$

$$(p\mu - (k+p)\lambda)b(\lambda+\mu) = p\mu b(\mu). \tag{5.7}$$

If $k \neq -p$, then $k + p \neq 0$. By (5.6) and (5.7) with $\mu = 0$, we obtain $a(\lambda) = b(\lambda) = 0$. Hence, the action of L_k on M is trivial. Furthermore, since $((k+p)\partial + (k+\frac{3}{2}p)\lambda)G_k = [L_{k\ \lambda}\, G_0]$, we see that the action of G_k on M is also trivial. This contradicts to the assumption that the action of $\mathfrak{S}(p)_k$ is nontrivial.

If $k = -p$, then $p \in \mathbb{Z}_{<0}$ is a negative integer. By (5.6) and (5.7), we see that $a(\lambda)$ and $b(\lambda)$ are independent of the variable λ, and so we can denote $a = a(\lambda)$, $b = b(\lambda)$.

If $p \leq 2$, then $k \geq 2$. Let us prove that the action of $\mathfrak{S}(p)_{k-1}$ is trivial. First, we consider the action of L_{k-1} on $v_{\bar{0}}$. Suppose $L_{k-1\ \lambda}\, v_{\bar{0}} = f(\partial,\ \lambda)v_{\bar{0}}$, where $f(\partial,\ \lambda) \in \mathbb{C}[\partial,\ \lambda]$. On one hand, we have

$$\begin{aligned} [L_{0\ \lambda} L_{k-1}]_{\lambda+\mu}\, v_{\bar{0}} &= L_{0\ \lambda}\, L_{k-1\ \mu}\, v_{\bar{0}} - L_{k-1\ \mu}\, L_{0\ \lambda}\, v_{\bar{0}} \\ &= L_{0\ \lambda}(f(\partial,\ \mu)v_{\bar{0}}) - L_{k-1\ \mu}(p(\partial + \Delta\lambda + \alpha)v_{\bar{0}}) \\ &= (p(\partial + \Delta\lambda + \alpha)f(\partial+\lambda,\ \mu) - p(\partial + \mu + \Delta\lambda + \alpha)f(\partial,\ \mu))v_{\bar{0}}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}[L_0\,{}_\lambda L_{k-1}]\,{}_{\lambda+\mu}\,v_{\bar 0} &= ((p\partial+(p-1)\lambda)L_{k-1})\,{}_{\lambda+\mu}\,v_{\bar 0}\\ &= -(\lambda+p\mu)f(\partial,\,\lambda+\mu)v_{\bar 0}.\end{aligned}$$

Then, $p(\partial+\mu+\Delta\lambda+\alpha)f(\partial,\,\mu)-p(\partial+\Delta\lambda+\alpha)f(\partial+\lambda,\,\mu)=(\lambda+p\mu)f(\partial,\,\lambda+\mu)$. In particular, taking $\mu=0$, we have $p(\partial+\Delta\lambda+\alpha)(f(\partial,\,0)-f(\partial+\lambda,\,0))=\lambda f(\partial,\,\lambda)$, i.e.,

$$f(\partial,\,\lambda)=-p(\partial+\Delta\lambda+\alpha)\frac{f(\partial+\lambda,\,0)-f(\partial,\,0)}{\lambda}. \tag{5.8}$$

Taking $\lambda\to 0$ in (5.8), we obtain

$$f(\partial,\,0)=\lim_{\lambda\to 0}f(\partial,\,\lambda)=-p(\partial+\alpha)\lim_{\lambda\to 0}\frac{f(\partial+\lambda,\,0)-f(\partial,\,0)}{\lambda}=-p(\partial+\alpha)\frac{d}{d\partial}f(\partial,\,0).$$

Since $p\le 2$, the above formula implies that $f(\partial,\,0)=0$. Then, by (5.8), we have $f(\partial,\,\lambda)=0$. Namely, the action of L_{k-1} on $v_{\bar 0}$ is trivial. Similarly, by relation $[L_0\,{}_\lambda L_{k-1}]=(p\partial+(p-1)\lambda)L_{k-1}$, one can prove that the action of L_{k-1} on $v_{\bar 1}$ is also trivial. Furthermore, by relation $[L_{k-1}\,{}_\lambda G_0]=-(\partial+(1-\frac{1}{2}p)\lambda)G_{k-1}$, we see that the action of G_{k-1} on $v_{\bar s}$, $s\in\mathbb{Z}/2\mathbb{Z}$ is also trivial. So, the action of $\mathfrak{S}(p)_{k-1}$ is trivial. Then, we have

$$0=[L_1\,{}_\lambda L_{k-1}]\,{}_{\lambda+\mu}v_s=\begin{cases}-\left(\lambda+(1+p)\mu\right)av_{\bar 0}, \text{ if } & s=\bar 0,\\ -\left(\lambda+(1+p)\mu\right)bv_{\bar 1}, \text{ if } & s=\bar 1,\end{cases}$$

$$0=[L_1\,{}_\lambda G_{k-1}]\,{}_{\lambda+\mu}v_s=\begin{cases}-\left((1+\frac{p}{2})\lambda+(1+p)\mu\right)c(\partial,\lambda)v_{\bar 0}, \text{ if } & s=\bar 0,\\ -\left((1+\frac{p}{2})\lambda+(1+p)\mu\right)d(\partial,\lambda)v_{\bar 1}, \text{ if } & s=\bar 1.\end{cases}$$

These imply that $a=b=c(\partial,\lambda)=d(\partial,\lambda)=0$, a contradiction.

If $p=-1$, then $k=1$. As a conformal $\mathfrak{NS}$-module, if $M\cong N_{\Delta,\alpha}$, then the actions of L_0 and G_0 have the first form of (5.5). Applying the operator $[L_1\,{}_\lambda G_0]\,{}_{\lambda+\mu}=-\frac{1}{2}\lambda G_1\,{}_{\lambda+\mu}$ on $v_{\bar 0}$, we obtain

$$\sqrt{-1}(a-b)v_{\bar 1}=\frac{1}{2}\lambda\, c(\partial,\lambda+\mu)v_{\bar 1},$$

which implies that $a=b$ and $c(\partial,\lambda)=0$. Denote $\beta=a=b$. Furthermore, applying the above operator on $v_{\bar 1}$, we obtain

$$\sqrt{-1}\beta\lambda\, v_{\bar 0}=-\frac{1}{2}\lambda\, d(\partial,\lambda+\mu)v_{\bar 0},$$

which implies that $d(\partial,\lambda)=-2\sqrt{-1}\beta$. Hence, $M\cong T_{\Delta,\alpha,\beta}$. Similarly, if $M\cong N'_{\Delta,\alpha}$ as a conformal $\mathfrak{NS}$-module, then one can show that $M\cong T'_{\Delta,\alpha,\beta}$. This completes the proof. $\square$

The simplicities of conformal $\mathfrak{S}(p)$-modules in Theorem 5.1 can be easily determined.

Proposition 5.2. *Let $M = \mathbb{C}[\partial]v_{\bar{0}} \oplus \mathbb{C}[\partial]v_{\bar{1}}$ be a conformal $\mathfrak{S}(p)$-module in Theorem 5.1.*

(1) *If $M \cong T_{\Delta,\alpha}$, then M is simple if and only if $\Delta \neq 0$. More precisely, $T_{0,\alpha}$ contains a unique nontrivial submodule $\mathbb{C}[\partial](\partial+\alpha)v_{\bar{0}} \oplus \mathbb{C}[\partial]v_{\bar{1}} \cong T'_{\frac{1}{2},\alpha}$.*
(2) *If $M \cong T'_{\Delta,\alpha}$, then M is simple if and only if $\Delta \neq 0$. More precisely, $T'_{0,\alpha}$ contains a unique nontrivial submodule $\mathbb{C}[\partial]v_{\bar{0}} \oplus \mathbb{C}[\partial](\partial+\alpha)v_{\bar{1}} \cong T_{\frac{1}{2},\alpha}$.*
(3) *If $M \cong T_{\Delta,\alpha,\beta}$, then M is simple if and only if $(\Delta,\beta) \neq (0,0)$. More precisely, $T_{0,\alpha,0}$ contains a unique nontrivial submodule $\mathbb{C}[\partial](\partial+\alpha)v_{\bar{0}} \oplus \mathbb{C}[\partial]v_{\bar{1}} \cong T'_{\frac{1}{2},\alpha,0}$.*
(4) *If $M \cong T'_{\Delta,\alpha,\beta}$, then M is simple if and only if $(\Delta,\beta) \neq (0,0)$. More precisely, $T'_{0,\alpha,0}$ contains a unique nontrivial submodule $\mathbb{C}[\partial]v_{\bar{0}} \oplus \mathbb{C}[\partial](\partial+\alpha)v_{\bar{1}} \cong T_{\frac{1}{2},\alpha,0}$.*

6. Classification theorems

6.1. Main result

Our main result in this paper is as follows:

Theorem 6.1. *Let M be a nontrivial finite irreducible conformal module over $\mathfrak{S}(p)$.*

(1) *If $p \neq -1$, then $M \cong T_{\Delta,\alpha}$ or $T'_{\Delta,\alpha}$ defined by (5.1) and (5.2) for some $\Delta, \alpha \in \mathbb{C}$ with $\Delta \neq 0$.*
(2) *If $p = -1$, then $M \cong T_{\Delta,\alpha,\beta}$ or $T'_{\Delta,\alpha,\beta}$ defined by (5.3) and (5.4) for some $\Delta, \alpha, \beta \in \mathbb{C}$ with $(\Delta,\beta) \neq (0,0)$.*

Let M be a nontrivial FICM over $\mathfrak{S}(p)$. The outline of our proof is as follows: First, by Theorem 4.2, we may view M as a FICM over a quotient algebra of $\mathfrak{S}(p)$. Next, by Proposition 2.4, M can be further viewed as certain module over a Lie superalgebra. Then, by Theorem 3.2 and another key lemma (cf. Lemma 6.2), we show that M must be free of rank $(1+1)$ (cf. Lemma 6.3), and so the main result will follow from Theorem 5.1 and Proposition 5.2.

Lemma 6.2 *([4]). Let $\mathcal{L}$ be a Lie superalgebra with a descending sequence of subspaces $\mathcal{L} \supset \mathcal{L}_0 \supset \mathcal{L}_1 \supset \dots$ and an element T satisfying $[T,\mathcal{L}_n] = \mathcal{L}_{n-1}$ for $n \geq 1$. Let V be an $\mathcal{L}$-module and let*

$$V_n = \{v \in V \mid \mathcal{L}_n v = 0\}, \quad n \in \mathbb{Z}_+.$$

Suppose that $V_n \neq 0$ for $n \gg 0$, and that the minimal $N \in \mathbb{Z}_+$ for which $V_N \neq 0$ is positive. Then $\mathbb{C}[T]V_N = \mathbb{C}[T] \otimes_{\mathbb{C}} V_N$. In particular, V_N is finite-dimensional if V is a finitely generated $\mathbb{C}[T]$-module.

Lemma 6.3. *The conformal $\mathfrak{S}(p)$-module M must be free of rank* $(1+1)$.

Proof. First, by Theorem 4.2, the λ-actions of L_i and G_i on M are trivial for all $i \gg 0$. Assume that $k \in \mathbb{Z}_+$ is the largest integer such that the λ-action of $\mathfrak{S}(p)_k$ on M is nontrivial. Then M is simply a nontrivial FICM over $\mathfrak{S}(p)_{[k]}$, where $\mathfrak{S}(p)_{[k]}$ is defined by (2.5). Furthermore, by Proposition 2.4, as a conformal $\mathfrak{S}(p)_{[k]}$-module, M can be viewed as a module over the associated extended annihilation algebra $\mathcal{L} = \mathcal{A}(\mathfrak{S}(p)_{[k]})^e$ satisfying

$$\bar{L}_{i,m}v = \bar{G}_{j,n}v = 0 \quad \text{for} \quad v \in M, \quad 0 \le i, j \le k, \quad 0 \ll m \in \mathbb{Z}, \quad \frac{1}{2} \ll n \in \frac{1}{2} + \mathbb{Z}. \tag{6.1}$$

Let

$$\mathcal{L}_z = \operatorname{span}_{\mathbb{C}}\{\bar{L}_{i,m},\, \bar{G}_{j,n} \in \mathcal{L} \,|\, 0 \le i,\, j \le k,\, z-1 \le m \in \mathbb{Z},\, z-\frac{1}{2} \le n \in \frac{1}{2}+\mathbb{Z}\}, \quad z \in \mathbb{Z}_+.$$

Then $\mathcal{L}_0 = \mathcal{A}(\mathfrak{S}(p)_{[k]})$ and $\mathcal{L} \supset \mathcal{L}_0 \supset \mathcal{L}_1 \supset \ldots$. By the definition of extended annihilation superalgebra, we see that the element $T \in \mathcal{L}$ satisfies $[T, \mathcal{L}_z] = \mathcal{L}_{z-1}$ for $z \ge 1$. Let

$$M_z = \{v \in M \,|\, \mathcal{L}_z v = 0\}, \quad z \in \mathbb{Z}_+.$$

By (6.1), $M_z \neq 0$ for $z \gg 0$. Assume that $N \in \mathbb{Z}_+$ is the smallest integer such that $M_N \neq \emptyset$.

If $N = 0$, we can take $0 \neq v \in M_0$. Then $U(\mathcal{L})v = \mathbb{C}[T]U(\mathcal{L}_0)v = \mathbb{C}[T]v$. So, $M = \mathbb{C}[T]v$ by the irreducibility of M. Since $\mathcal{L}_0$ is an ideal of $\mathcal{L}$, we see that $\mathcal{L}_0$ acts trivially on M. Hence, M is simply an irreducible $\mathbb{C}[T]$-module, and so M is one-dimensional. Equivalently, M is a one-dimensional trivial conformal $\mathfrak{S}(p)$-module, a contradiction.

Next, consider the case $N \ge 1$. By the definition of extended annihilation superalgebra and the shift used in the proof of Lemma 3.1, we have

$$[T, \bar{L}_{i,m}] = -(m+1)\bar{L}_{i,m-1}, \quad [T, \bar{G}_{j,n}] = -(n+\frac{1}{2})\bar{G}_{j,n-1}.$$

It follows that $T - \frac{1}{p}\bar{L}_{0,-1} \in \mathcal{L}$ is an even central element, and so $T - \frac{1}{p}\bar{L}_{0,-1}$ acts on M as a scalar. Therefore, $\mathcal{L}_0$ acts irreducibly on M. Furthermore, by relations

$$\bar{L}_{i,-1} = \frac{1}{p}[\bar{L}_{i,0}, \bar{L}_{0,-1}], \quad \bar{G}_{i,-\frac{1}{2}} = \frac{1}{p}[\bar{G}_{i,\frac{1}{2}}, \bar{L}_{0,-1}],$$

we see that the action of $\mathcal{L}_0$ is determined by $\mathcal{L}_1$ and $\bar{L}_{0,-1}$ (or equivalently, determined by $\mathcal{L}_1$ and T). Note that M_N is $\mathcal{L}_1$-invariant. By the irreducibility of M and Lemma 6.2, we see that $M = \mathbb{C}[T] \otimes_{\mathbb{C}} M_N$ and M_N is a nontrivial irreducible finite-dimensional $\mathcal{L}_1$-module.

If $N = 1$, then by the definition of M_1, we see that M_1 is a trivial $\mathcal{L}_1$-module, a contradiction.

If $N \geq 2$, then by the definition of M_N, we see that M_N is simply a $\mathcal{L}_1/\mathcal{L}_N$-module. Note that $\mathcal{L}_1/\mathcal{L}_N \cong \mathfrak{g}(k, N-2)$. By Theorem 3.2, we have that M_N is (1+1)-dimensional. Equivalently, M is free of rank $(1+1)$ as a conformal $\mathfrak{S}(p)$-module. $\square$

6.2. Applications

By (2.6), we see that $\mathfrak{s}(n)$ has a $\mathbb{C}[\partial]$-basis $\{\bar{L}_i, \bar{G}_j \,|\, 0 \leq i, j \leq n\}$. In case $i + j \leq n$, the λ-brackets are as follows:

$$\begin{aligned}
&[\bar{L}_i\,{}_\lambda\,\bar{L}_j] = ((i-n)\partial + (i+j-2n)\lambda)\bar{L}_{i+j},\\
&[\bar{L}_i\,{}_\lambda\,\bar{G}_j] = ((i-n)\partial + (i+j-\frac{3}{2}n)\lambda)\bar{G}_{i+j},\\
&[\bar{G}_i\,{}_\lambda\,\bar{G}_j] = 2\bar{L}_{i+j}.
\end{aligned}$$

In case $i + j > n$, the above λ-brackets are trivial. The following two $\mathbb{C}[\partial]$-modules are conformal modules over $\mathfrak{s}(n)$ (here, we adopt the same notations as in (5.1) and (5.2) for $\mathfrak{S}(-n)$).

(1) $T_{\Delta,\alpha} = \mathbb{C}[\partial]v_{\bar{0}} \oplus \mathbb{C}[\partial]v_{\bar{1}}$ with

$$\begin{cases}
\bar{L}_0\,{}_\lambda\,v_{\bar{0}} = -n(\partial + \Delta\lambda + \alpha)v_{\bar{0}},\\
\bar{L}_0\,{}_\lambda\,v_{\bar{1}} = -n(\partial + (\Delta + \frac{1}{2})\lambda + \alpha)v_{\bar{1}},\\
\bar{G}_0\,{}_\lambda\,v_{\bar{0}} = \sqrt{-n}\,v_{\bar{1}},\\
\bar{G}_0\,{}_\lambda\,v_{\bar{1}} = \sqrt{-n}\,(\partial + 2\Delta\lambda + \alpha)v_{\bar{0}},\\
\bar{L}_i\,{}_\lambda\,v_s = \bar{G}_i\,{}_\lambda\,v_s = 0,\ 1 \leq i \leq n,\ s \in \mathbb{Z}/2\mathbb{Z},
\end{cases} \tag{6.2}$$

where $\Delta, \alpha \in \mathbb{C}$;

(2) $T'_{\Delta,\alpha} = \mathbb{C}[\partial]v'_{\bar{0}} \oplus \mathbb{C}[\partial]v'_{\bar{1}}$ with

$$\begin{cases}
\bar{L}_0\,{}_\lambda\,v'_{\bar{0}} = -n(\partial + (\Delta + \frac{1}{2})\lambda + \alpha)v'_{\bar{0}},\\
\bar{L}_0\,{}_\lambda\,v'_{\bar{1}} = -n(\partial + \Delta\lambda + \alpha)v'_{\bar{1}},\\
\bar{G}_0\,{}_\lambda\,v'_{\bar{0}} = \sqrt{-n}\,(\partial + 2\Delta\lambda + \alpha)v'_{\bar{1}},\\
\bar{G}_0\,{}_\lambda\,v'_{\bar{1}} = \sqrt{-n}\,v'_{\bar{0}},\\
\bar{L}_i\,{}_\lambda\,v'_s = \bar{G}_i\,{}_\lambda\,v'_s = 0,\ 1 \leq i \leq n,\ s \in \mathbb{Z}/2\mathbb{Z},
\end{cases} \tag{6.3}$$

where $\Delta, \alpha \in \mathbb{C}$.

The following $T_{\Delta,\alpha,\beta}$ and $T'_{\Delta,\alpha,\beta}$ are more general conformal modules over $\mathfrak{s}(1)$ (here, we adopt the same notations as in (5.3) and (5.4) for $\mathfrak{S}(-1)$).

(3) $T_{\Delta,\alpha,\beta} = \mathbb{C}[\partial]v_{\bar{0}} \oplus \mathbb{C}[\partial]v_{\bar{1}}$ with

$$\begin{cases} \bar{L}_{0\ \lambda}\, v_{\bar{0}} = -(\partial + \Delta\lambda + \alpha)v_{\bar{0}}, \\ \bar{L}_{0\ \lambda}\, v_{\bar{1}} = -(\partial + (\Delta + \frac{1}{2})\lambda + \alpha)v_{\bar{1}}, \\ \bar{L}_{1\ \lambda}\, v_{\bar{0}} = \beta v_{\bar{0}}, \\ \bar{L}_{1\ \lambda}\, v_{\bar{1}} = \beta v_{\bar{1}}, \\ \bar{G}_{0\ \lambda}\, v_{\bar{0}} = \sqrt{-1}\, v_{\bar{1}}, \\ \bar{G}_{0\ \lambda}\, v_{\bar{1}} = \sqrt{-1}\,(\partial + 2\Delta\lambda + \alpha)v_{\bar{0}}, \\ \bar{G}_{1\ \lambda}\, v_{\bar{0}} = 0, \\ \bar{G}_{1\ \lambda}\, v_{\bar{1}} = -2\sqrt{-1}\beta v_{\bar{0}}, \end{cases} \tag{6.4}$$

where Δ, α, $\beta \in \mathbb{C}$;

(4) $T'_{\Delta,\alpha,\beta} = \mathbb{C}[\partial]v'_{\bar{0}} \oplus \mathbb{C}[\partial]v'_{\bar{1}}$ with

$$\begin{cases} \bar{L}_{0\ \lambda}\, v'_{\bar{0}} = -(\partial + (\Delta + \frac{1}{2})\lambda + \alpha)v'_{\bar{0}}, \\ \bar{L}_{0\ \lambda}\, v'_{\bar{1}} = -(\partial + \Delta\lambda + \alpha)v'_{\bar{1}}, \\ \bar{L}_{1\ \lambda}\, v'_{\bar{0}} = \beta v'_{\bar{0}}, \\ \bar{L}_{1\ \lambda}\, v'_{\bar{1}} = \beta v'_{\bar{1}}, \\ \bar{G}_{0\ \lambda}\, v'_{\bar{0}} = \sqrt{-1}\,(\partial + 2\Delta\lambda + \alpha)v'_{\bar{1}}, \\ \bar{G}_{0\ \lambda}\, v'_{\bar{1}} = \sqrt{-1}\, v'_{\bar{0}}, \\ \bar{G}_{1\ \lambda}\, v'_{\bar{0}} = -2\sqrt{-1}\beta v'_{\bar{1}}, \\ \bar{G}_{1\ \lambda}\, v'_{\bar{1}} = 0, \end{cases} \tag{6.5}$$

where Δ, $\alpha, \beta \in \mathbb{C}$.

Since $\mathfrak{s}(n)$ is a quotient algebra of $\mathfrak{S}(-n)$ (cf. (1.5)), by Theorem 5.1 and Proposition 5.2, we have that

Corollary 6.4. *Let M be a nontrivial free conformal module of rank $(1 + 1)$ over $\mathfrak{s}(n)$.*

(1) *If $n > 1$, then $M \cong T_{\Delta,\alpha}$ or $T'_{\Delta,\alpha}$ defined by (6.2) and (6.3) for some $\Delta, \alpha \in \mathbb{C}$.*
(2) *If $n = 1$, then $M \cong T_{\Delta,\alpha,\beta}$ or $T'_{\Delta,\alpha,\beta}$ defined by (6.4) and (6.5) for some $\Delta, \alpha, \beta \in \mathbb{C}$.*

Furthermore, for the above modules we have the same simplicity assertions as those for $\mathfrak{S}(-n)$-modules in Proposition 5.2.

Furthermore, by Theorem 6.1, we have that

Corollary 6.5. *The irreducible modules in Corollary 6.4 exhaust all nontrivial finite irreducible conformal modules over $\mathfrak{s}(n)$.*

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