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On the existence of complements of residuals of finite group

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ABSTRACT

L.A. Shemetkov's theorem on the complementability of the $\mathfrak{F}$-residual of a finite group is developed in the article. For an ω-local Fitting formation $\mathfrak{F}$, it is proved that, if G is a finite group generated by subnormal subgroups $A_1, ..., A_n$, the subgroups $A_1^{\mathfrak{F}}, ..., A_n^{\mathfrak{F}}$ are ω-soluble, and Sylow p-subgroups of $A_1^{\mathfrak{F}}, ..., A_n^{\mathfrak{F}}$ are abelian for every $p \in \omega$, then each $\omega\mathfrak{F}$-normalizer of G is an ω-complement of $G^{\mathfrak{F}}$ in G.

1. Introduction

All groups considered in the paper are finite.

One of the interesting applications of the theory of formations is the finding of sufficient conditions for a group to split over its $\mathfrak{F}$-residual subgroup. The most significant result in this direction is the following theorem of Shemetkov [1] (see also [2, Theorems 4.2.19 and 4.2.20]).

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Theorem 1.1. *Let $\mathfrak{F}$ be a local formation, and G is a group. Suppose that Sylow p-subgroups of $G^{\mathfrak{F}}$ are abelian for every prime p dividing $|G : G^{\mathfrak{F}}|$. Then $G^{\mathfrak{F}}$ has a complement in G.*

The condition in Theorem 1.1 that Sylow p-subgroups of $G^{\mathfrak{F}}$ are abelian is essential (the corresponding example see in [3, p. 135]). Therefore, one of the approaches aimed to weaken abelianity can be given by introducing additional restrictions either on the group G or the formation $\mathfrak{F}$.

In [4] Theorem 1.1 receives further development in two directions. Firstly, in [4, Corollary 3.7] this theorem is generalized for an ω-local formation $\mathfrak{F}$, where $\omega \subseteq \pi(\mathfrak{F})$ (if $\omega = \pi(\mathfrak{F})$, the result implies Theorem 1.1). Secondly, in [4, Theorem 4.3] was obtained a weakening of the condition of Sylow p-subgroups of the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ being abelian in Theorem 1.1. Namely, for an ω-local Fitting formation $\mathfrak{F}$, was established the existence of an ω-complement of the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ in group G if G can be represented as a product of n subnormal subgroups whose $\mathfrak{F}$-residuals are ω-soluble and whose Sylow p-subgroups are abelian for any $p \in \omega$.

The aforementioned results of [4] are developed in the article. The main purpose of our article is to prove the following theorem.

Theorem 1.2. *Let $\mathfrak{F}$ be a non-empty ω-local Fitting formation. Assume that:*

1) *$G = <A_1, A_2, ..., A_n>$, where A_i is a subnormal subgroup of G, for every $i \in \{1, 2, ..., n\}$;*
2) *the $\mathfrak{F}$-residual $A_i^{\mathfrak{F}}$ is ω-soluble, for any $i \in \{1, 2, ..., n\}$;*
3) *for any prime $p \in \omega$ a Sylow p-subgroup of the group $A_i^{\mathfrak{F}}$ is abelian, for any $i \in \{1, 2, ..., n\}$.*

Then every $\omega\mathfrak{F}$-normalizer of G is an ω-complement in G of the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$.

2. Definitions and preliminary results

The notation and terminology correspond to the books [5,6]. We refer the reader to these books for the results on formations and Fitting classes.

Our aim in this section is to collect some definitions and results that are needed subsequently.

Let's recall that a *formation* is a class of groups which is closed under taking homomorphic images and finite subdirect products. If $\mathfrak{F}$ is a non-empty formation, then each group has the *$\mathfrak{F}$-residual* $G^{\mathfrak{F}}$, the smallest normal subgroup whose quotient belongs to $\mathfrak{F}$.

We write $s_n\mathfrak{F}$ to denote the class of all groups G such that $G \trianglelefteq H \in \mathfrak{F}$. If $s_n\mathfrak{F} \subseteq \mathfrak{F}$, then the class $\mathfrak{F}$ is called *s_n-closed.*

A class $\mathfrak{F}$ of finite groups is called a *Fitting class* if it satisfies the following conditions:

1) $\mathfrak{F}$ is a s_n-closed class;
2) any group $G = AB$, where A and B are normal $\mathfrak{F}$-subgroups of G, belongs to $\mathfrak{F}$.

If $\mathfrak{F}$ is a Fitting class and G is a group, then the subgroup

$$G_{\mathfrak{F}} = <S \mid S \text{ is subnormal } \mathfrak{F}\text{-subgroup of } G>$$

is a normal $\mathfrak{F}$-subgroup of G, and it is called the $\mathfrak{F}$*-radical* of G.

A Fitting class which is also a formation is called a *Fitting formation*.

Lemma 2.1. *[7, Lemma 2] Let $\mathfrak{F}$ be a non-empty formation. Then $\mathfrak{F}$ is a Fitting formation if and only if $G^{\mathfrak{F}} = A^{\mathfrak{F}}B^{\mathfrak{F}}$ for every subnormal subgroups A and B of each group G with $G = AB$.*

Local formations occupy a central place in the theory of formations of finite groups. In the description of the structure of local formations, a substantial role is played by their satellites (see [2,5,6]). As a generalization of local formations, ω-local formations were introduced in [8] (the notion of ω-local formation was initially proposed by Shemetkov in [9] for p-local formations, and further developed in [10]).

Let ω be a non-empty set of primes, and ω' is the complement of ω in the set of all primes. Following [4] to define an ω-local formation it is convenient to consider $\{\omega'\}$, the one-element set consisting of the element ω', that is in this case the symbol ω' is used as one element in the domain of definition of a function f. Every function of the form

$$f : \omega \cup \{\omega'\} \rightarrow \{\mathit{formations}\}$$

is called an ω*-local satellite*.

If f is an ω-local satellite, define the class

$$LF_{\omega}(f) = (G \mid G/O_{\omega}(G) \in f(\omega') \text{ and } G/F_p(G) \in f(p) \text{ for all } p \in \omega \cap \pi(G)).$$

The class $LF_{\omega}(f)$ is a formation. This formation is called an ω*-local formation*, and f is called its ω-satellite.

In general, the ω-local formation may have several ω-local satellites. Let f be an ω-local satellite, and $\mathfrak{F} = LF_{\omega}(f)$. Then f is called *integrated* if $f(\omega') \subseteq \mathfrak{F}$ and $f(p) \subseteq \mathfrak{F}$, for every $p \in \omega$. We say that an integrated ω-local satellite f of $\mathfrak{F}$ is a *maximal integrated* ω*-satellite* if $g \leq f$ for each integrated ω-satellite g such that $\mathfrak{F} = LF_{\omega}(g)$. As shown in [8], every ω-local formation can be defined by a maximal integrated ω-satellite.

Let f be an ω-local satellite. A chief ωd-factor A/B of a group G is called f_{ω}*-central* in G if $G/C_G(A/B) \in f(p)$, for any $p \in \omega \cap \pi(A/B)$. Otherwise it is called f_{ω}*-eccentric.*

Lemma 2.2. *[4, Lemma 2.8] Let $\mathfrak{F} = LF_\omega(f)$ be an ω-local formation with integrated ω-satellite f. If for some prime $p \in \omega$ a Sylow p-subgroup of $G^{\mathfrak{F}}$ is abelian, then G does not have chief f_ω-central pd-factors below $G^{\mathfrak{F}}$.*

Let $\mathfrak{F}$ be a non-empty formation and G be a group. A maximal subgroup M of G is called $\mathfrak{F}$-*critical* in G if $G = MR$ for some normal subgroup $R \subseteq G^{\mathfrak{F}}$ of G such that $R/R \cap \Phi(G)$ is a chief factor of G.

Let $\mathfrak{F}$ be an ω-local formation. Following [4, Definition 4.3], a subgroup F of G is said to be an $\omega\mathfrak{F}$-*normalizer* of G if $F/\Phi(F) \cap O_{\omega'}(F) \in \mathfrak{F}$ and there exists a maximal chain of G of the form

$$F = F_n \subset F_{n-1} \subset ... \subset F_1 \subset F_0 = G$$

such that F_i is an $\mathfrak{F}$-critical subgroup of F_{i-1} for each $i \in \{1, 2, ..., n\}$. As shown in [4, Lemma 4.9], for every ω-local formation $\mathfrak{F}$, a group G contains at least one $\omega\mathfrak{F}$-normalizer F and $G = G^{\mathfrak{F}}F$.

Let A/B a factor of a group G, i.e., A and B are subgroups of G and B is a normal subgroup of A. Let X be any subgroup of G. Then $B(A \cap X)$ is a subgroup of G between B and A. We say that X *covers* A/B if $A = B(A \cap X)$ and X *avoids* A/B if $B = B(A \cap X)$.

Lemma 2.3. *[4, Lemma 4.11] Let $\mathfrak{F}$ be an ω-local formation with an integrated ω-satellite f, and G is a group with ω-soluble $\mathfrak{F}$-residual $G^{\mathfrak{F}}$. If H is an $\omega\mathfrak{F}$-normalizer of G, then H covers every f_ω-central and avoids every f_ω-eccentric chief ωd-factor of G.*

One of the most important subgroup properties is the subnormality, i.e., transitive closure of the relation of normality. This property was extensively studied by H. Wielandt (see [11]).

A subgroup H of a group G is said to be *subnormal* in G if there are non-negative integer m and the series $H = H_0 \trianglelefteq H_1 \trianglelefteq ... \trianglelefteq H_m = G$ of subgroups of G.

We need the following information on the properties of subnormal subgroups. The set of all subnormal subgroups of group G is denoted by $sn(G)$.

Lemma 2.4. *[11]*

1) *Let $H \in sn(G)$ and K be a subgroup of G. Then $H \cap K \in sn(K)$. In particular, $H \in sn(L)$ whenever L is a subgroup of G containing H.*
2) *If $H_i \in sn(G)$, for all $i \in I$, then $\bigcap_{i \in I} H_i \in sn(G)$.*
3) *Let $H \in sn(G)$ and N be a normal subgroup of G. Then $HN/N \in sn(G/N)$.*
4) *Let $H \in sn(G)$ and $K \in sn(G)$. Then $<H, K> \in sn(G)$.*

Lemma 2.5. *[2, Theorem 7.1] Let $H \in sn(G)$, but H is non-normal subgroup of G. Then exists an element $x \in G$ such that $H^x \neq H$, $H^x \subseteq N_G(H)$ and $H \subseteq N_G(H^x)$.*

Let ω be a non-empty set of prime numbers. We say that a subgroup K of a group G has an *ω-complement* H in G if $G = HK$ and $|H \cap K|$ is not divisible by numbers in ω ([2], Definition 11.1).

Lemma 2.6. *Let $\mathfrak{F} = LF_\omega(f)$ be an ω-local Fitting formation with maximal integrated ω-satellite f. Assume that:*

1) *$G = <A, B>$, where A and B are subnormal subgroups of G;*
2) *for some prime $p \in \pi(\mathfrak{F}) \cap \omega$, $A^{\mathfrak{F}}$ does not contain A-chief f_ω-central pd-factors and $B^{\mathfrak{F}}$ does not contain B-chief f_ω-central pd-factors.*

Then $G^{\mathfrak{F}}$ does not contain G-chief f_ω-central pd-factors.

Proof. Let's assume, arguing by contradiction, that Lemma 2.6 is false. Suppose that G does not satisfy for Lemma 2.6, and the number $t = |G| + |G : A| + |G : B|$ is minimal. If $t = 3$, then $G = 1$ and $G^{\mathfrak{F}} = 1$ does not contain G-chief f_ω-central *pd*-factors, which contradicts the choice of G.

Therefore $t > 3$ and $G^{\mathfrak{F}} \neq 1$. If $G = A$, then, by hypothesis, $G^{\mathfrak{F}} = A^{\mathfrak{F}}$ does not contain G-chief f_ω-central *pd*-factors, a contradiction. Thus we have that $G \neq A$. Analogously $G \neq B$.

Let N be a minimal normal subgroup of G. If N is not contained in $G^{\mathfrak{F}}$, then $(G/N)^{\mathfrak{F}} = G^{\mathfrak{F}}N/N \simeq G^{\mathfrak{F}}$. Since, by Lemma 2.4, subgroups AN/N and BN/N are subnormal in G/N and $|G/N| + |G/N : AN/N| + |G/N : BN/N| < t$, we have that the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of G does not contain G-chief f_ω-central *pd*-factors, a contradiction. Hence every minimal normal subgroup N of G is contained in $G^{\mathfrak{F}}$.

By properties of an $\mathfrak{F}$-residual, we have $(AN/N)^{\mathfrak{F}} = A^{\mathfrak{F}}N/N$ and $(BN/N)^{\mathfrak{F}} = B^{\mathfrak{F}}N/N$. Moreover, the group G/N is represented in the form $G/N = <AN/N, BN/N>$, where AN/N and BN/N are subnormal subgroups of G/N. Consequently, by the choice of G/N, the $\mathfrak{F}$-residual $G^{\mathfrak{F}}/N$ of G/N does not contain G-chief f_ω-central *pd*-factors.

If L is a minimal normal subgroup of G different from N, then it is proved similarly that the $\mathfrak{F}$-residual $G^{\mathfrak{F}}/L$ of G/L does not contain G-chief f_ω-central *pd*-factors. But then, by $G^{\mathfrak{F}} \simeq G^{\mathfrak{F}}/N \cap L$, it follows that the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ does not contain G-chief f_ω-central *pd*-factors. We get a contradiction with the choice of G.

Now we can conclude that N is unique minimal normal subgroup of G and every chief *pd*-factor of G in interval $[N, G^{\mathfrak{F}}]$ is f_ω-eccentric. Since for G Lemma 2.6 is false, the minimal normal subgroup N of G is an f_ω-central *pd*-factor in G. Since f is an integrated ω-satellite of the formation $\mathfrak{F}$, $G/C_G(N) \in f(p) \subseteq \mathfrak{F}$. Therefore $G^{\mathfrak{F}} \subseteq C_G(N)$, and $N \subseteq Z(G^{\mathfrak{F}})$. Thus, it follows, in particular, that N is an elementary abelian p-group.

Let's suppose that $A^{\mathfrak{F}} = 1$. Since $\mathfrak{F}$ is a Fitting formation, by subnormality of A in G, we get $A \subseteq G_{\mathfrak{F}}$ and $G = <A, B> = <G_{\mathfrak{F}}, B> = G_{\mathfrak{F}}B$. Thus, by Lemma 2.1, $G^{\mathfrak{F}} = (G_{\mathfrak{F}})^{\mathfrak{F}}B^{\mathfrak{F}} = B^{\mathfrak{F}}$. Then, by hypothesis, $G^{\mathfrak{F}}$ does not contain G-chief f_ω-central *pd*-factors, a contradiction. So $A^{\mathfrak{F}} \neq 1$. Analogously $B^{\mathfrak{F}} \neq 1$.

Denote $A^{\mathfrak{F}} \cap N$ by D. Assume that $D \neq 1$. Since f is a maximal integrated ω-satellite of the formation $\mathfrak{F}$, by [4, Lemma 2.10], $f(p)$ is a s_n-closed formation. Thus, by $G/C_G(N) \in f(p)$ and subnormality of $AC_G(N)/C_G(N)$ in $G/C_G(N)$, we have $AC_G(N)/C_G(N) \in f(p)$. Then, by isomorphism

$$AC_G(N)/C_G(N) \simeq A/A \cap C_G(N) = A/C_A(N),$$

it follows that $A/C_A(N) \in f(p)$. Since $C_A(N) \subseteq C_A(D)$, $A/C_A(D) \in f(p)$. Therefore all A-chief factors of $A^{\mathfrak{F}}$ in interval $[1, N]$ are f_ω-central in A, a contradiction. So $A^{\mathfrak{F}} \cap N = 1$. Analogously $B^{\mathfrak{F}} \cap N = 1$.

Let's assume that A is a normal subgroup of G. Since $A^{\mathfrak{F}}$ is characteristic in A, it follows that $A^{\mathfrak{F}} \trianglelefteq G$. Since N is an unique minimal normal subgroup of G and $A^{\mathfrak{F}} \neq 1$, $N \subseteq A^{\mathfrak{F}}$. This contradicts the condition that $A^{\mathfrak{F}} \cap N = 1$. Consequently, A is a not normal subgroup of G. Analogously B is a not normal subgroup of G.

Let $|B| \leq |A|$. By Lemma 2.5, exists an element x in G such that $A^x \neq A$, $A^x \subseteq N_G(A)$ and $A \subseteq N_G(A^x)$. Thus we have that $<A, A^x> \subseteq N_G(A) \subset G$. Denote $<A, A^x>$ by T. Since $A^{\mathfrak{F}}$ does not contain A-chief f_ω-central *pd*-factors, $(A^x)^{\mathfrak{F}}$ does not contain A^x-chief f_ω-central *pd*-factors. Moreover, by Lemma 2.4, A and A^x are subnormal subgroups of $<A, A^x>$. Since $|G| + |G : T| + |G : B| < t$, we have that $T^{\mathfrak{F}}$ does not contain T-chief f_ω-central *pd*-factors.

By Lemma 2.4, the subgroup T is subnormal in G. Since $G = <T, B>$ and $|T| + |T : A| + |T : A^x| < t$, it follows that $G^{\mathfrak{F}}$ does not contain G-chief f_ω-central *pd*-factors. A contradiction. The proof of Lemma 2.6 is complete. □

By induction for n, we have

Corollary 2.7. *Let $\mathfrak{F} = LF_\omega(f)$ be an ω-local Fitting formation with maximal integrated ω-satellite f. Assume that:*

1) *$G = <A_1, A_2, ..., A_n>$, where A_i is a subnormal subgroup of G, for every $i \in \{1, 2, ..., n\}$;*
2) *for some prime $p \in \pi(\mathfrak{F}) \cap \omega$ and for every $i \in \{1, 2, ..., n\}$, $A_i^{\mathfrak{F}}$ does not contain A_i-chief f_ω-central pd-factors.*

Then $G^{\mathfrak{F}}$ does not contain G-chief f_ω-central pd-factors.

3. Proof of Theorem 1.2 and corollaries

Obviously, the set $\mathfrak{H}$ of all ω-soluble groups is a Fitting formation. Since $A_i^{\mathfrak{F}}$ is an ω-soluble subgroup for every $i \in \{1, 2, ..., n\}$, $<A_1^{\mathfrak{F}}, A_2^{\mathfrak{F}}, ..., A_n^{\mathfrak{F}}> \subseteq G_{\mathfrak{H}}$. Moreover $A_i^{\mathfrak{F}} \subseteq G^{\mathfrak{F}}$, for every $i \in \{1, 2, ..., n\}$, hence $<A_1^{\mathfrak{F}}, A_2^{\mathfrak{F}}, ..., A_n^{\mathfrak{F}}> \subseteq G^{\mathfrak{F}}$. Therefore $<A_1^{\mathfrak{F}}, A_2^{\mathfrak{F}}, ..., A_n^{\mathfrak{F}}> \subseteq G^{\mathfrak{F}} \cap G_{\mathfrak{H}}$. Denote $G^{\mathfrak{F}} \cap G_{\mathfrak{H}}$ by S. Then, by $A_i^{\mathfrak{F}} \subseteq S$, we have that

$A_iS/S \simeq A_i/A_i \cap S \in \mathfrak{F}$, for every $i \in \{1, 2, ..., n\}$. Since $\mathfrak{F}$ is a Fitting formation, it follows that $G/S = <A_1S/S, A_2S/S, ..., A_nS/S> \in \mathfrak{F}$. This implies that $G^{\mathfrak{F}} \subseteq G_{\mathfrak{H}}$. Thus, $G^{\mathfrak{F}}$ is an ω-soluble subgroup of G.

By [4, Lemma 4.9], G contains at least one $\omega\mathfrak{F}$-normalizer H and $G = G^{\mathfrak{F}}H$. By Lemma 2.3, H covers every f_ω-central and avoids every f_ω-excentric chief factor of G. Consequently, by Lemmas 2.2 and 2.6, H avoids every chief ωd-factor of G below $G^{\mathfrak{F}}$.

Supposing that $D = G^{\mathfrak{F}} \cap H$ is an ωd-group, let's consider a chief series of G passing through $G^{\mathfrak{F}}$ of the form

$$1 = G_0 \subset ... \subset G_k = G^{\mathfrak{F}} \subset G_{k+1} \subset ... \subset G_t = G.$$

Then $1 = D \cap G_0 \subseteq ... \subseteq D \cap G_{k-1} \subseteq D \cap G_k = D$ is a normal series of D, and $|D|$ is equal to the product of all indices of this series. Since D is an ωd-group, it follows that $D \cap G_i/D \cap G_{i-1}$ is an ωd-group for some $i \in \{1, 2, ..., k\}$. Since $(D \cap G_i)G_{i-1} \subseteq G_i$, it follows that

$$(D \cap G_i)G_{i-1}/G_{i-1} \subseteq G_i/G_{i-1},$$

and $(D \cap G_i)G_{i-1}/G_{i-1} \simeq D \cap G_i/D \cap G_{i-1}$. Therefore G_i/G_{i-1} is a chief ωd-factor of G below $G^{\mathfrak{F}}$, and therefore H avoids the factor G_i/G_{i-1}. Thus, $H \cap G_i \subseteq G_{i-1}$. Then $D \cap G_i = (H \cap D) \cap G_i = H \cap (D \cap G_i) \subseteq D \cap G_{i-1}$. We have obtained a contradiction to the fact that $D \cap G_i/D \cap G_{i-1}$ is an ωd-group.

Consequently, D is an ω'-group. Since $G = G^{\mathfrak{F}}H$, by definition of ω-complement, we obtain that the $\omega\mathfrak{F}$-normalizer H of G is an ω-complement in G of the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$. The theorem is proved.

Corollary 3.1. *[4, Theorem 4.3] Let $\mathfrak{F}$ be an ω-local Fitting formation, and suppose that a group $G = A_1A_2...A_n$ is a product of subnormal subgroups A_i, $i = 1, 2, ..., n$. If the $\mathfrak{F}$-residual $A_i^{\mathfrak{F}}$ is ω-soluble for any $i = 1, 2, ..., n$, and its Sylow p-subgroups are abelian for any $p \in \omega$, then every $\omega\mathfrak{F}$-normalizer of G is an ω-complement in G of the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$.*

The class of local formations coincides with the class of saturated formations (see [6]). A formation $\mathfrak{F}$ is called *saturated* if $G/\Phi(G) \in \mathfrak{F}$ always implies $G \in \mathfrak{F}$.

Corollary 3.2. *Let $\mathfrak{F}$ be a saturated Fitting formation. Assume that:*

1) *$G = <A_1, A_2, ..., A_n>$, where A_i is a subnormal subgroup of G, for every $i \in \{1, 2, ..., n\}$;*
2) *for every $i \in \{1, 2, ..., n\}$, the $\mathfrak{F}$-residual $A_i^{\mathfrak{F}}$ of A_i is $\pi(\mathfrak{F})$-soluble;*
3) *for every $i \in \{1, 2, ..., n\}$ and for every prime p dividing $|G : G^{\mathfrak{F}}|$, a Sylow p-subgroup of the group $A_i^{\mathfrak{F}}$ is abelian.*

Then each Hall $\pi(|G : G^{\mathfrak{F}}|)$-subgroup of every $\mathfrak{F}$-normalizer of G is a complement for $G^{\mathfrak{F}}$ in G.

Proof. Let $\omega = \pi(|G : G^{\mathfrak{F}}|)$. Then $\mathfrak{F}$ is an ω-local formation. By Theorem 1.2, every $\omega\mathfrak{F}$-normalizer of G is an ω-complement of $G^{\mathfrak{F}}$ in G. Let H be an $\omega\mathfrak{F}$-normalizer of G. Then $G = G^{\mathfrak{F}}H$ and $|H \cap G^{\mathfrak{F}}|$ is an ω'-number.

By Theorem 1.2, the subgroup $G^{\mathfrak{F}}$ is $\pi(\mathfrak{F})$-soluble. Then G has a normal series

$$1 = G_0 \subseteq G_1 \subseteq ... \subseteq G_n = G,$$

such that every quotient factor G_i/G_{i-1} is either an ω-group or an ω'-group. If $2 \in \omega$, then G is an ω'-soluble group. If $2 \in \omega'$, then G is an ω-soluble group. Applying Hall theorem, G (and every subgroup of G) has a Hall ω-subgroup. Then the subgroup H has a Hall ω-subgroup H_ω.

By $G = G^{\mathfrak{F}}H$ and $\omega = \pi(|G : G^{\mathfrak{F}}|)$, we obtain that $G = G^{\mathfrak{F}}H_\omega$. Since $|H \cap G^{\mathfrak{F}}|$ is an ω'-number, it follows that $H_\omega \cap G^{\mathfrak{F}} = 1$. Thus we have established that H_ω is a complement of $G^{\mathfrak{F}}$ in G. The corollary is proved. □

Corollary 3.3. *Let $\mathfrak{F}$ be a saturated Fitting formation. Assume that:*

1) *$G = <A_1, A_2, ..., A_n>$, where A_i is a subnormal subgroup of G, for every $i \in \{1, 2, ..., n\}$;*
2) *for every $i \in \{1, 2, ..., n\}$, the $\mathfrak{F}$-residual $A_i^{\mathfrak{F}}$ of A_i is $\pi(\mathfrak{F})$-soluble;*
3) *for every $i \in \{1, 2, ..., n\}$ and for every prime $p \in \pi(\mathfrak{F})$, a Sylow p-subgroup of the group $A_i^{\mathfrak{F}}$ is abelian.*

Then every $\mathfrak{F}$-normalizer of G is a complement for $G^{\mathfrak{F}}$ in G.

Corollary 3.4. *Let $\mathfrak{F}$ be a saturated Fitting formation. Assume that:*

1) *$G = <A_1, A_2, ..., A_n>$, where A_i is a subnormal subgroup of G, for every $i \in \{1, 2, ..., n\}$;*
2) *for every $i \in \{1, 2, ..., n\}$, the $\mathfrak{F}$-residual $A_i^{\mathfrak{F}}$ of A_i is abelian.*

Then every $\mathfrak{F}$-normalizer of G is a complement for $G^{\mathfrak{F}}$ in G.

Corollary 3.5. *[4, Corollary 4.1] Let $\mathfrak{F}$ be a saturated Fitting formation. Assume that:*

1) *$G = A_1A_2...A_n$, where A_i is a subnormal subgroup of G, for every $i \in \{1, 2, ..., n\}$;*
2) *for every $i \in \{1, 2, ..., n\}$, the $\mathfrak{F}$-residual $A_i^{\mathfrak{F}}$ of A_i is $\pi(\mathfrak{F})$-soluble;*
3) *for every $i \in \{1, 2, ..., n\}$ and for every prime $p \in \pi(\mathfrak{F})$, a Sylow p-subgroup of the group $A_i^{\mathfrak{F}}$ is abelian.*

Then every $\mathfrak{F}$-normalizer of G is a complement for $G^{\mathfrak{F}}$ in G.

Corollary 3.6. *[12, Theorem 2.1] Let $\mathfrak{F}$ be a saturated Fitting formation. Assume that:*

1) *$G = A_1A_2...A_n$, where A_i is a normal subgroup of G, for every $i \in \{1, 2, ..., n\}$;*
2) *for every $i \in \{1, 2, ..., n\}$, the $\mathfrak{F}$-residual $A_i^{\mathfrak{F}}$ of A_i is $\pi(\mathfrak{F})$-soluble;*
3) *for every $i \in \{1, 2, ..., n\}$ and for every prime $p \in \pi(\mathfrak{F})$, a Sylow p-subgroup of the group $A_i^{\mathfrak{F}}$ is abelian.*

Then every $\mathfrak{F}$-normalizer of G is a complement for $G^{\mathfrak{F}}$ in G.

Corollary 3.7. *Let $\mathfrak{F}$ be the formation of all p-groups for some prime p. Suppose $G = <A_1, A_2, ..., A_n>$, where A_i is a subnormal subgroup of a group G, for every $i \in \{1, 2, ..., n\}$. If the subgroup $O^p(G)$ is p-soluble with abelian Sylow p-subgroups for any $i \in \{1, 2, ..., n\}$, then every $\mathfrak{F}$-normalizer of G is a complement of $O^p(G)$ in G.*

Corollary 3.8. *Let $\mathfrak{F}$ be a saturated Fitting formation containing all nilpotent groups, and suppose that $G = <A_1, A_2, ..., A_n>$, where A_i is a subnormal subgroup of a group G, for every $i \in \{1, 2, ..., n\}$. If the $\mathfrak{F}$-residual $A_i^{\mathfrak{F}}$ is soluble with abelian Sylow subgroups, for any $i \in \{1, 2, ..., n\}$, then the subgroup $G^{\mathfrak{F}}$ has a complement in G.*

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