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# Algebraic differential formulas for the shuffle, stuffle and duality relations of iterated integrals

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In this paper, we prove certain algebraic identities, which correspond to differentiation of the shuffle relation, the stuffle relation, and the relations which arise from Möbius transformations of iterated integrals. These formulas provide fundamental and useful tools in the study of iterated integrals on a punctured projective line.



## 1. Introduction

Multiple zeta values (MZVs in short) are real numbers defined by the convergent series

$$\zeta(k_1,\dots,k_d)=\sum_{0<m_1<\cdots<m_d}\frac{1}{m_1^{k_1}\cdots m_d^{k_d}}$$

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where $(k_1,\dots,k_d)$ is a $d$-tuple of positive integers such that $k_d>1$, and the sum is taken over all $d$-tuples of positive integers satisfying the inequality. MZVs are, as obvious from their definition, multiple series generalizations of the Riemann zeta function evaluated at positive integers.

Despite their simple and elementary appearance, the algebraic nature of the space spanned by MZVs is surprisingly rich and mysterious. In fact, MZVs arise in various areas of mathematics and particle physics, such as in the Kontsevich integrals of knots ([12], [13], [14]), or in the evaluation of scattering amplitudes ([2], [16]), and are periods of mixed Tate motives over $\mathbb{Z}$ ([3], [7]).

A particularly important aspect of MZVs is their iterated integral representation due to Kontsevich, i.e.,

$$\begin{aligned}(-1)^d\zeta(k_1,\dots,k_d) &= I(0;\overbrace{1,0,\dots,0}^{k_1},\overbrace{1,0,\dots,0}^{k_2},\dots,\overbrace{1,0,\dots,0}^{k_d};1)\\ &\text{where } I(s;a_1,\dots,a_k;t):=\int\limits_{s<t_1<\cdots<t_k<t}\frac{dt_1}{t_1-a_1}\wedge\cdots\wedge\frac{dt_k}{t_k-a_k},\end{aligned}$$

which (also occurs as the coefficients of the KZ-associator) provides a geometric point of view on MZVs. To describe the algebraic structure of the space spanned by such iterated integrals, Hoffman [10] introduced the non-commutative polynomial algebra generated by two indeterminates $e_0,e_1$ which correspond to the differential one-forms $\frac{dt}{t}$ and $\frac{dt}{t-1}$, respectively. This algebraic setup gives a fundamental tool to study MZVs. In [8], [9], the authors consider more general iterated integrals $I(0;a_1,\dots,a_k;1)$ with $a_1,\dots,a_k\in\{0,1,z\}$, where $z$ is a complex variable. For this purpose, they extended Hoffman's algebraic setup by adding another generator $e_z$ corresponding to the one-form $\frac{dt}{t-z}$ to Hoffman's algebra. More precisely, let $\mathcal{A}_{\{0,1,z\}}=\mathbb{Z}\langle e_0,e_1,e_z\rangle$ (resp. $\mathcal{A}_{\{0,1\}}=\mathbb{Z}\langle e_0,e_1\rangle$) be the non-commutative polynomial algebra over $\mathbb{Z}$ generated by the indeterminates $e_0,e_1$ and $e_z$ (resp. $e_0$ and $e_1$). Let $\mathcal{A}^0_{\{0,1,z\}}$ denote the subspace of admissible elements

$$\mathcal{A}^0_{\{0,1,z\}}:=\mathbb{Z}\oplus\mathbb{Z}e_z\oplus\bigoplus_{\substack{a\in\{1,z\}\\ b\in\{0,z\}}}e_a\mathcal{A}_{\{0,1,z\}}e_b,$$

$\mathcal{A}^0_{\{0,1\}}$ its subspace

$$\mathcal{A}^0_{\{0,1\}}=\mathcal{A}^0_{\{0,1,z\}}\cap\mathcal{A}_{\{0,1\}}=\mathbb{Z}\oplus e_1\mathcal{A}_{\{0,1\}}e_0$$

and $L$ a linear map from $\mathcal{A}^0_{\{0,1,z\}}$ to the space of holomorphic functions of $z\in\mathbb{C}\setminus[0,1]$ defined by

$$L(e_{a_1}\cdots e_{a_n})=\int\limits_{0<t_1<\cdots<t_n<1}\prod_{j=1}^{n}\frac{dt_j}{t_j-a_j}.$$

Then $L$ satisfies the following differential formula [8]

$$\frac{d}{dz}L(w) = \sum_{c\in\{0,1\}} \frac{1}{z-c} L(\partial_{z,c}w), \tag{1.1}$$

where $\partial_{\alpha,\beta} : \mathcal{A}_{\{0,1,z\}} \to \mathcal{A}_{\{0,1,z\}}$ is a linear map defined by

$$\partial_{\alpha,\beta}(e_{a_1}\cdots e_{a_n}) = \sum_{i=1}^{n} \left(\delta_{\{a_i,a_{i+1}\},\{\alpha,\beta\}} - \delta_{\{a_{i-1},a_i\},\{\alpha,\beta\}}\right) e_{a_1}\cdots \widehat{e_{a_i}}\cdots e_{a_n} \tag{1.2}$$

with $a_0 = 0$, $a_{n+1} = 1$. Here, $\delta_{S,T}$ is the Kronecker delta for two sets $S$ and $T$. These $\partial_{\alpha,\beta}$'s are the algebraic counterpart of the usual differentiation $d/dz$, and have fundamental importance in the study of iterated integrals. For example, in [8], the authors proved a "sum formula" for iterated integrals, which generalizes the classical sum formula for MZVs, by using its inductive structure with respect to the algebraic differentiation. Also, in [9], the authors exploited the differential structure of $\mathcal{A}^0_{\{0,1,z\}}$ to construct a class of (presumably exhausting all the) relations among MZVs, which they called the confluence relation. The purpose of this paper is to investigate the relationships between $\partial_{\alpha,\beta}$ and other basic algebraic operations.

We denote the shuffle product, the stuffle product and the duality map by $\sqcup\!\sqcup$, $*$ and $\tau_z$, respectively (here, the precise definition of $*$ and $\tau_z$ will be explained later). Then, $L$ satisfies the "shuffle relation"

$$L(u \sqcup\!\sqcup v) = L(u)L(v) \qquad \left(u, v \in \mathcal{A}^0_{\{0,1,z\}}\right), \tag{1.3}$$

the "stuffle relation" [1, Section 5.2]

$$L(u * v) = L(u)L(v) \qquad \left(u \in \mathcal{A}^0_{\{0,1\}},\, v \in \mathcal{A}^0_{\{0,1,z\}}\right), \tag{1.4}$$

and the "duality relation" [8, Theorem 1.1]

$$L(\tau_z(u)) = L(u) \qquad \left(u \in \mathcal{A}^0_{\{0,1,z\}}\right). \tag{1.5}$$

By differentiating the equalities (1.3), (1.4) and (1.5) with respect to $z$ and applying (1.1), we obtain[1]

[1] One may wonder why there is no such term as $\frac{1}{z-c}L(\partial_{z,c}u)L(v)$ on the right-hand side of (1.7). This is because we have assumed $u \in \mathcal{A}^0_{\{0,1\}}$ and thus $\partial_{z,c}u$ vanishes identically.

For readers who are interested in the regularizations of $L$, we make a few remarks here. By similar arguments as in [11], it can be shown that there exist unique extensions $L^{\sqcup\!\sqcup}$ and $L^*$ of $L$ to $\mathcal{A}_{\{0,1,z\}}$ characterized by the linearity together with the conditions $L^{\sqcup\!\sqcup}|_{\mathcal{A}^0_{\{0,1,z\}}} = L^*|_{\mathcal{A}^0_{\{0,1,z\}}} = L$, $L^{\sqcup\!\sqcup}(e_0 \sqcup\!\sqcup u) = L^{\sqcup\!\sqcup}(e_1 \sqcup\!\sqcup u) = L^*(e_0 * u) = L^*(e_1 * u) = 0$ for $u \in \mathcal{A}_{\{0,1,z\}}$. Then $L^{\sqcup\!\sqcup}$ satisfies (1.1) and (1.6), whereas $L^*$ does not satisfy (1.1) and (1.7) in general. For example, $L^*(e_0e_z) = 0$, while $L^*(\partial_{z,0}(e_0e_z)) = L(e_z) \neq 0$. These facts can be checked by careful applications of Theorem 10.

$$\sum_{c\in\{0,1\}} \frac{1}{z-c} L(\partial_{z,c}(u \sqcup\!\sqcup v)) = \sum_{c\in\{0,1\}} \frac{1}{z-c} (L(\partial_{z,c}u)L(v) + L(u)L(\partial_{z,c}v)), \tag{1.6}$$

$$\sum_{c\in\{0,1\}} \frac{1}{z-c} L(\partial_{z,c}(u * v)) = \sum_{c\in\{0,1\}} \frac{1}{z-c} L(u)L(\partial_{z,c}v), \tag{1.7}$$

$$\sum_{c\in\{0,1\}} \frac{1}{z-c} L(\partial_{z,c}\tau_z(u)) = \sum_{c\in\{0,1\}} \frac{1}{z-c} L(\tau_z(\partial_{z,c}u)). \tag{1.8}$$

Therefore, it is natural to ask whether these equalities in complex numbers lift to the equalities in $\mathcal{A}_{\{0,1,z\}}$. In this paper we shall show that the answers to all these three questions are "Yes". More precisely, we shall prove the following theorem.

**Theorem.** *For* $c \in \{0,1\}$,

$$\begin{aligned}
\partial_{z,c}(u \sqcup\!\sqcup v) &= (\partial_{z,c}u) \sqcup\!\sqcup v + u \sqcup\!\sqcup (\partial_{z,c}v) && \left(u, v \in \mathcal{A}^0_{\{0,1,z\}}\right)\\
\partial_{z,c}(u * v) &= u * (\partial_{z,c}v) && \left(u \in \mathcal{A}^0_{\{0,1\}},\ v \in \mathcal{A}^0_{\{0,1,z\}}\right)\\
\partial_{z,c}\tau_z(u) &= \tau_z(\partial_{z,c}u) && \left(u \in \mathcal{A}^0_{\{0,1,z\}}\right).
\end{aligned}$$

The first two equalities state that the linear operator $\partial_{z,c}$ is a derivation with respect to both the shuffle and the stuffle products (the second equality can also be written as $\partial_{z,c}(u * v) = (\partial_{z,c}u) * v + u * (\partial_{z,c}v)$ since $\partial_{z,c}u = 0$ for $u \in \mathcal{A}^0_{\{0,1\}}$).

These formulas provide fundamental and useful tools in the study of MZVs and iterated integrals. Let us discuss some significance of them. First of all, the formulas are purely algebraic and therefore have wide applications. One of such important applications is given in [9], where the authors have proved that the confluence relation implies the regularized double shuffle and the duality relations. Recently, the confluence relation was proved to be equivalent to the associator relation by Furusho [5], and so this result can also be viewed as an alternative proof of the main result of [4]. Furthermore, let $\gamma$ be a general path from 0 to 1, and $L_\gamma$ denote the map similarly defined as $L$, where the defining iterated integral is replaced with that along $\gamma$. By applying $L_\gamma$ to the algebraic differential formula of the stuffle product, one can obtain

$$\frac{d}{dz} L_\gamma(u * v) = \sum_{c\in\{0,1\}} \frac{1}{z-c} L_\gamma(u * \partial_{z,c}v). \tag{1.9}$$

Note that for the trivial path $\gamma(t) = t$, this identity gives an alternative proof of the stuffle relation (1.4). Thus, we may naturally expect that (1.9) would serve as a first step to discover and prove the counterpart of (1.4) for $L_\gamma$ with more general $\gamma$. We also define a natural generalization of the stuffle product and prove the three formulas above in far more general forms (see Theorems 3, 8, 9). It is quite likely that these formulas have wide applications in general iterated integrals, for instance, for proving analogous results as in [9] (i.e., the implication of the regularized double shuffle and the duality relations by the confluence relation) for Euler sums, multiple L-values and even more general iterated integrals.

This paper is organized as follows. In Section 2, we introduce some basic settings and state a useful lemma used in the proof of the theorem. In Section 3, we prove the algebraic differential formula for the shuffle product (Theorem 3). In Section 4, we prove the algebraic differential formula for the stuffle product (Theorem 8). In Section 5, we introduce algebraic Möbius transformations as a generalization of the duality map, and prove their algebraic differential formula (Theorem 9). In Section 6, we derive the theorem above as special cases of Theorems 3, 8, 9.

## 2. Basic settings

Let $F$ be a field and $\mathcal{A} = \mathcal{A}_F$ be the non-commutative free algebra over $\mathbb{Z}$ generated by the indeterminates $\{e_p \mid p \in F\}$. For $s, t \in F$, we denote by $\mathcal{A}^0_{(s,t)} = \mathcal{A}^0_{F,(s,t)}$ the subalgebra

$$\mathcal{A}^0_{F,(s,t)} = \mathbb{Z} \oplus \bigoplus_{z \in F\setminus\{s,t\}} \mathbb{Z}e_z \oplus \bigoplus_{\substack{x \in F\setminus\{s\} \\ y \in F\setminus\{t\}}} e_x \mathcal{A}_F e_y \subset \mathcal{A}_F.$$

In particular, we put $\mathcal{A}^0_F = \mathcal{A}^0_{F,(0,1)}$.

We fix a homomorphism $\mathfrak{F} : F^\times \to \mathbb{Z}$. For $x \in F$, define $[x] \in \mathbb{Z}$ by

$$[x] = \begin{cases} \mathfrak{F}(x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Note that since $\mathbb{Z}$ is torsion-free, $[-x] = [x]$ for $x \in F^\times$. We define an algebraic differential operator $\partial^{s,t} = \partial^{s,t}_{\mathfrak{F}} : \mathcal{A} \to \mathcal{A}$ by[2]

$$\partial^{s,t}(e_{a_1} \cdots e_{a_n}) := \sum_{i=1}^{n} ([a_{i+1} - a_i] - [a_i - a_{i-1}]) e_{a_1} \cdots \widehat{e_{a_i}} \cdots e_{a_n} \qquad ((a_0, a_{n+1}) = (s,t)).$$

[2] The motivation to consider such an operator comes from the following differential formula (Goncharov [6, Theorem 2.1], Panzer [15, Lemma 3.3.30])

$$\begin{aligned} dI(s; a_1, \dots, a_n; t) = &\sum_{\substack{1 \le i \le n \\ a_i \neq a_{i+1}}} I(s; a_1, \dots, \widehat{a_i}, \dots, a_n; t)\, d\log(a_{i+1} - a_i) \\ &- \sum_{\substack{1 \le i \le n \\ a_i \neq a_{i-1}}} I(s; a_1, \dots, \widehat{a_i}, \dots, a_n; t)\, d\log(a_i - a_{i-1}) \quad ((a_0, a_{n+1}) = (s,t)), \end{aligned}$$

where $I$ is the iterated integral symbol by Goncharov [6].

In particular, we put $\partial = \partial^{0,1}$. The following lemma provides a useful technique in later sections.

**Lemma 1.** *Let $a_0, \ldots, a_{n+1} \in F$ (with $(a_0, a_{n+1}) = (s,t)$) and $f : \{0, \ldots, n\} \to \mathbb{Z}$ be any map such that*

$$f(i) = [a_{i+1} - a_i] \qquad \text{for all } i \in \{0 \leq j \leq n \mid a_j \neq a_{j+1}\}.$$

*Then, we have*

$$\begin{aligned}\partial^{s,t}(e_{a_1} \cdots e_{a_n}) = &\sum_{i=1}^{n} (f(i) - f(i-1))\, e_{a_1} \cdots \widehat{e_{a_i}} \cdots e_{a_n} \\ &+ \delta_{s,a_1} f(0)\, e_{a_2} \cdots e_{a_n} - \delta_{a_n,t} f(n)\, e_{a_1} \cdots e_{a_{n-1}}.\end{aligned}$$

The proof follows directly from the definition of $\partial^{s,t}$. Note that the right-hand side in the above equality does not depend on the choice of $f$ and so we choose suitable $f$ which is convenient for the situation.

As a general notation, for a given function $f : S \to \mathcal{A}$ on a finite set $S$, we define

$$f(T) := \sum_{x \in T} f(x)$$

for $T \subset S$. We often use this notation in the following sections.

## 3. Differential formula for the shuffle product

In this section, we shall prove the derivation property of the algebraic differential operator with respect to the shuffle product.

**Definition 2.** We define the shuffle product $ш : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ inductively by $w ш 1 = 1 ш w = w$ for $w \in \mathcal{A}$ and

$$e_a u ш e_b v = e_a(u ш e_b v) + e_b(e_a u ш v) \qquad (u, v \in \mathcal{A}).$$

As is well known, the shuffle product can also be defined in a more combinatorial way as

$$a_1 \cdots a_n ш a_{n+1} \cdots a_{n+m} = \sum_{\sigma \in S(n,m)} a_{\sigma(1)} \cdots a_{\sigma(n+m)} \qquad (a_1, \ldots, a_{n+m} \in \{e_p \mid p \in F\}),$$

where $S(n,m)$ is a set of all permutations $\sigma$'s of $\{1, 2, \ldots, n+m\}$ such that $\sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(n)$ and $\sigma^{-1}(n+1) < \sigma^{-1}(n+2) < \cdots < \sigma^{-1}(n+m)$.

**Theorem 3.** *For $u, v \in \mathcal{A}$, $\partial(u ш v) = (\partial u) ш v + u ш (\partial v)$.*

**Proof.** It is sufficient to show the equality for monomials $u$ and $v$. Put $u = a_1 \cdots a_n$, $v = a_{n+1} \cdots a_{n+m}$ $(a_1, \dots, a_{n+m} \in \{e_p \mid p \in F\})$ and $X = S(n,m) \times \{0, 1, \dots, n+m\}$. Let $a_i = e_{p_i}$ for $i = 1, \dots, n+m$. Then, by definition,

$$\partial(u \sqcup\!\sqcup v) = f(X) := \sum_{(\sigma,i) \in X} f(\sigma, i),$$

where $f(\sigma, i) := f^+(\sigma, i) - f^-(\sigma, i)$ with

$$f^+(\sigma,i) := \begin{cases} [p_{\sigma(i+1)} - p_{\sigma(i)}] a_{\sigma(1)} \cdots \widehat{a_{\sigma(i)}} \cdots a_{\sigma(n+m)} & (i \neq 0) \\ 0 & (i = 0), \end{cases}$$

$$f^-(\sigma,i) := \begin{cases} [p_{\sigma(i+1)} - p_{\sigma(i)}] a_{\sigma(1)} \cdots \widehat{a_{\sigma(i+1)}} \cdots a_{\sigma(n+m)} & (i \neq n+m) \\ 0 & (i = n+m), \end{cases}$$

where we put $p_{\sigma(0)} = 0$ and $p_{\sigma(n+m+1)} = 1$. Put $I := \{1, \dots, n\}$, $J := \{n+1, \dots, n+m\}$ and

$$X_{AB} := \{(\sigma, i) \in X \mid 0 < i < n+m,\, \sigma(i) \in A,\, \sigma(i+1) \in B\}$$

for $A, B \in \{I, J\}$. Further, we put

$$\widetilde{X_{II}} := X_{II} \sqcup \{(\sigma, 0) \in X \mid \sigma(1) \in I\} \sqcup \{(\sigma, n+m) \in X \mid \sigma(n+m) \in I\}$$
$$\widetilde{X_{JJ}} := X_{JJ} \sqcup \{(\sigma, 0) \in X \mid \sigma(1) \in J\} \sqcup \{(\sigma, n+m) \in X \mid \sigma(n+m) \in J\}.$$

Then we have

$$X = X_{IJ} \sqcup X_{JI} \sqcup \widetilde{X_{II}} \sqcup \widetilde{X_{JJ}}.$$

Define a bijection $\tau : X_{IJ} \to X_{JI}$ by $\tau(\sigma, i) = (\sigma \circ \tau_i, i)$, where $\tau_i$ denotes the transposition of $i$ and $i+1$. Since $f^{\pm} \circ \tau = f^{\mp}$, we have $f \circ \tau = -f$. Thus

$$f(X_{IJ}) + f(X_{JI}) = 0.$$

Let us further decompose $X_{II}$ as $X_{II} = \bigsqcup_{j=1}^{n-1} X_{II,j}$ where

$$X_{II,j} := \{(\sigma, i) \in X \mid 0 < i < n+m,\, \sigma(i) = j,\, \sigma(i+1) = j+1\}.$$

Then

$$f^+(X_{II,j}) = [p_{j+1} - p_j] a_1 \cdots \widehat{a_j} \cdots a_n \sqcup\!\sqcup a_{n+1} \cdots a_{n+m},$$
$$f^-(X_{II,j}) = [p_{j+1} - p_j] a_1 \cdots \widehat{a_{j+1}} \cdots a_n \sqcup\!\sqcup a_{n+1} \cdots a_{n+m},$$

$$f(\{(\sigma,0)\in X \mid \sigma(1)\in I\}) = -[p_1-0]a_2\cdots a_n ⧢ a_{n+1}\cdots a_{n+m},$$
$$f(\{(\sigma,n+m)\in X \mid \sigma(n+m)\in I\}) = [1-p_n]a_1\cdots a_{n-1} ⧢ a_{n+1}\cdots a_{n+m}.$$

Thus

$$f(\widetilde{X_{II}}) = (\partial u) ⧢ v,$$

and similarly

$$f(\widetilde{X_{JJ}}) = u ⧢ (\partial v).$$

This completes the proof. □

## 4. Differential formula for the stuffle product

In this section, we shall prove the derivation property of the algebraic differential operator with respect to the stuffle product.

### *4.1. Definition of the generalized stuffle product*

**Definition 4.** We define the (generalized) stuffle product $* : \mathcal{A}\times\mathcal{A}\to\mathcal{A}$ inductively by $w*1=1*w=w$ and

$$e_a u * e_b v = e_{ab}(u*e_b v + e_a u * v - e_0(u*v)) \qquad (u,v\in\mathcal{A}).$$

Note that in our definition of the stuffle product, $e_0$ plays a particular role. Thus, we check the compatibility of our definition with the usual one ([1, Section 5], [11, Section 1]) in the first place. Let $\mathfrak{h}^1$ be the non-commutative free algebra generated by infinitely many formal indeterminates $\{z_{k,a} \mid k\in\mathbb{Z}_{\ge 1}, a\in F^\times\}$. Define a binary operator $\bar{*} : \mathfrak{h}^1\times\mathfrak{h}^1\to\mathfrak{h}^1$ by $1 \,\bar{*}\, w = w \,\bar{*}\, 1 = w$ for $w\in\mathfrak{h}^1$ and

$$z_{k,a}u \,\bar{*}\, z_{l,b}v = z_{k,ab}(u \,\bar{*}\, z_{l,b}v) + z_{l,ab}(z_{k,a}u \,\bar{*}\, v) + z_{k+l,ab}(u \,\bar{*}\, v) \quad (u,v\in\mathfrak{h}^1).$$

We define an embedding $i:\mathfrak{h}^1\hookrightarrow\mathcal{A}$ of algebra by $i(uv)=i(u)i(v)$ and $i(z_{k,a}) = -e_a e_0^{k-1}$. By our notation, the multiple polylogarithms defined in [1] and [6] are expressed as

$$l\begin{pmatrix} k_d,\dots,k_1 \\ a_d,\dots,a_1 \end{pmatrix} = L(i(z_{k_1,a_1}\cdots z_{k_d,a_d}))$$

and

$$\mathrm{Li}_{k_1,\dots,k_d}(a_1,\dots,a_d) = L(i(z_{k_1,(a_1\cdots a_d)^{-1}} z_{k_2,(a_2\cdots a_d)^{-1}} \cdots z_{k_d,a_d^{-1}})),$$

respectively. The following proposition assures that our stuffle product is a generalization of the usual stuffle products.

**Proposition 5.**

(i) *For* $u, v \in \mathfrak{h}^1$, $i(u \bar{*} v) = i(u) * i(v)$.
(ii) *The stuffle product* $*$ *is commutative and associative.*

**Proof of (i).** It is sufficient to show the equality for monomials $u$ and $v$. First of all, one can easily check that

$$e_0 u * v = e_0(u * v)$$

holds for any monomials $u, v$ by induction on the degree of $v$. Now we shall prove the proposition by induction. Set $u = z_{k,a}u'$, $v = z_{l,b}v'$. Then we get

$$\begin{aligned}
i(u) * i(v) =& (e_a e_0^{k-1} i(u')) * (e_b e_0^{l-1} i(v')) \\
=& e_{ab}(e_0^{k-1} i(u') * e_b e_0^{l-1} i(v')) + e_{ab}(e_a e_0^{k-1} i(u') * e_0^{l-1} i(v')) \\
&- e_{ab} e_0 (e_0^{k-1} i(u') * e_0^{l-1} i(v')) \\
=& - e_{ab} e_0^{k-1}(i(u') * i(v)) - e_{ab} e_0^{l-1}(i(u) * i(v')) \\
&- e_{ab} e_0^{k+l-1}(i(u') * i(v')).
\end{aligned}$$

Using the induction hypothesis, the last quantity is equal to

$$\begin{aligned}
&- e_{ab} e_0^{k-1} i(u' \bar{*} v) - e_{ab} e_0^{l-1} i(u \bar{*} v') - e_{ab} e_0^{k+l-1} i(u' \bar{*} v') \\
&= i(z_{k,ab}(u' \bar{*} v) + z_{l,ab}(u \bar{*} v') + z_{k+l,ab}(u' \bar{*} v')) \\
&= i(u \bar{*} v)
\end{aligned}$$

which proves the proposition. □

**Proof of (ii).** The commutativity of the stuffle product is obvious from the definition. Let us prove the associativity. We define a trilinear map $f : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ inductively by

$$f(u, v, 1) = f(u, 1, v) = f(1, u, v) = u * v$$

and

$$\begin{aligned}
f(u, v, w) :=& e_{abc}\left(f(u', v, w) + f(u, v', w) + f(u, v, w')\right) \\
&- e_{abc} e_0 \left(f(u', v', w) + f(u', v, w') + f(u, v', w')\right) \\
&+ e_{abc} e_0^2 f(u', v', w') \qquad (u = e_a u', v = e_b v', w = e_c w').
\end{aligned}$$

Then we can show $(u*v)*w = f(u,v,w)$ by the induction since

$$\begin{aligned}(u*v)*w =& e_{ab}(u*v' + u'*v - e_0(u'*v'))*w\\ =& e_{abc}((u*v)*w') + e_{abc}\left((u*v' + u'*v - e_0(u'*v'))*w\right)\\ & - e_{abc}e_0((u*v' + u'*v - e_0(u'*v'))*w')\\ =& e_{abc}((u*v)*w' + (u*v')*w + (u'*v)*w)\\ & - e_{abc}e_0((u'*v')*w + (u*v')*w' + (u'*v)*w')\\ & + e_{abc}e_0^2(u'*v'*w') \quad (u = e_a u', v = e_b v', w = e_c w').\end{aligned}$$

Since $f(u,v,w)$ is a symmetric function by definition, we have

$$(u*v)*w = f(u,v,w) = f(v,w,u) = (v*w)*u = u*(v*w)$$

for $u,v,w \in \mathcal{A}$. This is the associativity of $*$. $\square$

*4.2. Combinatorial description of the stuffle product*

The stuffle product as well as the shuffle product enjoys an interesting combinatorial structure. In this section, we give such a description of the stuffle product. This description will be useful in the proof of the algebraic differential formula stated in the next section.

Fix positive integers $m$ and $n$. Let $G = (V, E)$ be a directed graph whose vertex set $V$ and edge set $E$ are given by

$$V = \left\{ (x,y) \in \frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z} \,\middle|\, \begin{array}{l} 1 \le x \le n+1, \\ 1 \le y \le m+1, \end{array} \ x - y \in \mathbb{Z} \right\} \sqcup \left\{ \left(\frac{1}{2}, \frac{1}{2}\right) \right\}$$

and by

$$\begin{aligned} E =& \{((x,y),(x+1,y)) \in V^2 \mid x,y \in \mathbb{Z}\}\\ & \sqcup \{((x,y),(x,y+1)) \in V^2 \mid x,y \in \mathbb{Z}\}\\ & \sqcup \left\{ \left((x,y),(x+\frac{1}{2}, y+\frac{1}{2})\right) \in V^2 \,\middle|\, x,y \in \frac{1}{2}\mathbb{Z} \right\},\end{aligned}$$

respectively (see Fig. 4.1). We denote by $P$ the set of paths from $(1/2,1/2)$ to $(n+1, m+1)$, i.e.,

$$P := \left\{ \underline{p} = (p_0, p_1, \ldots, p_{n+m+1}) \,\middle|\, \begin{array}{ll} p_0 = (1/2,1/2), & (p_i, p_{i+1}) \in E \\ p_{n+m+1} = (n+1, m+1), & \text{for } 0 \le i \le n+m \end{array} \right\}.$$

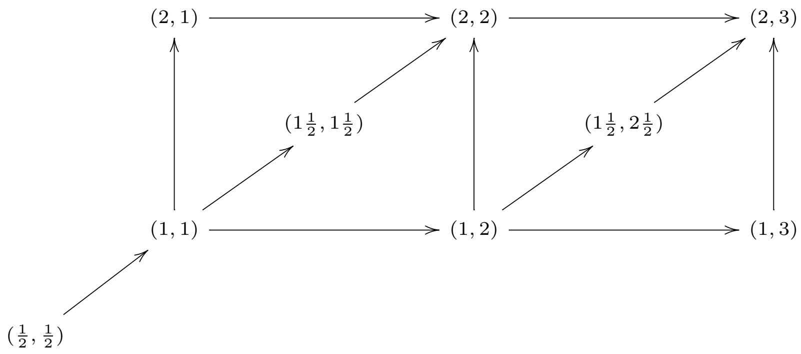


**Fig. 4.1.** The graph $G$ for the case $(n,m)=(1,2)$.

Hereafter, for a path $\underline{p}\in P$, we denote by $p_i$ the $(i+1)$-th entry of $\underline{p}$, i.e., $\underline{p}=(p_0,p_1,\ldots,p_{n+m+1})$. Furthermore we define $\mathrm{sgn}:P\to\{\pm1\}$ by

$$\mathrm{sgn}\left(\underline{p}\right)=\prod_{\substack{1\le i\le n+m\\ p_i\notin\mathbb{Z}^2}}(-1).$$

For monomials $u=e_{a_1}\cdots e_{a_n}$ and $v=e_{b_1}\cdots e_{b_m}$ of degree $n$ and $m$ respectively, define $f_{u,v}:V\to F$ by

$$f_{u,v}(x,y)=\begin{cases}a_xb_y & (x,y)\in\mathbb{Z}^2\\ 0 & (x,y)\notin\mathbb{Z}^2.\end{cases}$$

Here, we set $a_{n+1}=b_{m+1}=1$. Then, we have the following combinatorial expression for the stuffle product.

**Proposition 6.** *For monomials $u$ and $v$ of degree $n$ and $m$ respectively,*

$$u*v=\sum_{\underline{p}\in P}\mathrm{sgn}(\underline{p})\,e_{f_{u,v}(p_1)}\cdots e_{f_{u,v}(p_{n+m})}$$

One can easily check that this expression is equivalent to Definition 4.

**Remark 7.** The vertices $p_0=(\frac{1}{2},\frac{1}{2})$ and $p_{n+m+1}=(n+1,m+1)$ do not appear in Proposition 6, but it is convenient in the description of the proof of Theorem 8.

*4.3. An algebraic differential formula of the stuffle product*

Put

$$\mathcal{A}^1=\mathcal{A}^1_F=\mathbb{Z}\oplus\bigoplus_{z\in F^\times}e_z\mathcal{A}_F\subset\mathcal{A}_F$$

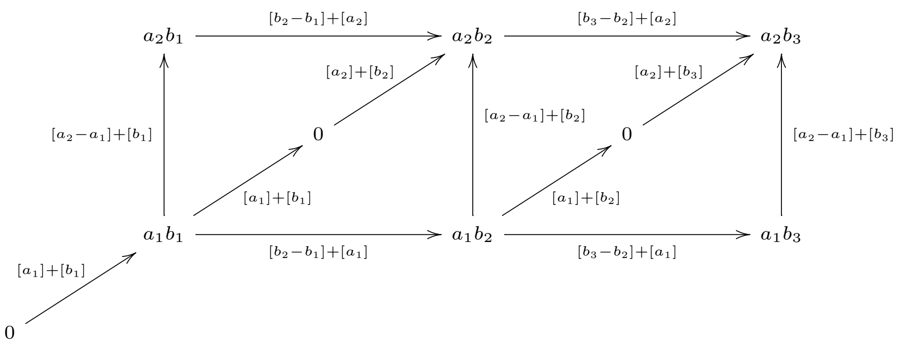


**Fig. 4.2.** Values of $f : V \to \mathbb{Z}$ and $h : E \to \mathbb{Z}$ ($a_2 = b_3 = 1$).

(note that $\mathcal{A}^0 \subset \mathcal{A}^1$). In this section, we prove the following identity:

**Theorem 8.** *For non-constant monomials $u, v \in \mathcal{A}$, we have*

$$\partial(u * v) = (\partial u) * v + u * \partial v + \delta_{a,0}[b](u' * v) + \delta_{b,0}[a](u * v'),$$

*where $u = e_a u', v = e_b v'$. In particular, for $u, v \in \mathcal{A}^1$,*

$$\partial(u * v) = (\partial u) * v + u * \partial v.$$

**Proof.** It is sufficient to show the identity for non-constant monomials $u$ and $v$. Put $u = e_{a_1} \cdots e_{a_n}$ and $v = e_{b_1} \cdots e_{b_m}$. Let $P$ be the set of paths as defined in Section 4.2. Put $f = f_{u,v}$ and $Q = P \times \{1, \cdots, n+m\}$. Define $h : E \to \mathbb{Z}$ by

$$h(p, p') = \begin{cases} [a_{x+1} - a_x] + [b_y] & \text{if } (p, p') = ((x, y), (x+1, y)) \\ [b_{y+1} - b_y] + [a_x] & \text{if } (p, p') = ((x, y), (x, y+1)) \\ [a_x] + [b_y] & \text{if } (p, p') = \begin{cases} ((x, y), (x + \frac{1}{2}, y + \frac{1}{2})) \\ \text{or } ((x - \frac{1}{2}, y - \frac{1}{2}), (x, y)) \end{cases} \end{cases}$$

for $x, y \in \mathbb{Z}$. Then $[f(p) - f(p')] = h(p, p')$ for all $(p, p') \in E$ such that $f(p) \neq f(p')$ (see Fig. 4.2).

We define $s : Q \to \mathcal{A}_F$ by

$$s(\underline{p}, i) = \operatorname{sgn}(\underline{p}) \left( h(p_i, p_{i+1}) - h(p_{i-1}, p_i) \right) e_{f(p_1)} \cdots \widehat{e_{f(p_i)}} \cdots e_{f(p_{n+m})}$$

for $\underline{p} \in P$ and $i \in \{1, \ldots, n+m\}$. Then, by Lemma 1, we have

$$\partial(u * v) = s(Q) + \sum_{\underline{p} \in P} \delta_{f(p_0), f(p_1)} h(p_0, p_1) \operatorname{sgn}(\underline{p}) e_{f(p_2)} \cdots e_{f(p_{n+m})}$$

$$-\sum_{\underline{p}\in P}\delta_{f(p_{n+m}),f(p_{n+m+1})}h(p_{n+m},p_{n+m+1})\mathrm{sgn}(\underline{p})e_{f(p_1)}\cdots e_{f(p_{n+m-1})},$$

where $s(Q) := \sum_{(\underline{p},i)\in Q} s(\underline{p},i)$. Since

$$\delta_{f(p_0),f(p_1)}h(p_0,p_1) = \delta_{0,a_1b_1}h((\frac{1}{2},\frac{1}{2}),(1,1)) = \delta_{0,a_1b_1}([a_1]+[b_1])$$

and

$$\begin{aligned}
&\delta_{f(p_{n+m}),f(p_{n+m+1})}h(p_{n+m},p_{n+m+1})\\
&\quad=\begin{cases}\delta_{a_n,1}([1-a_n]+[1]) & \text{if } p_{n+m}=(n,m+1)\\ \delta_{b_m,1}([1-b_m]+[1]) & \text{if } p_{n+m}=(n+1,m)\\ 0 & \text{if } p_{n+m}=(n+\frac{1}{2},m+\frac{1}{2})\end{cases}\\
&\quad=0,
\end{aligned}$$

we have

$$\partial(u*v) = s(Q)+\Lambda, \tag{4.1}$$

where

$$\Lambda := \delta_{a_1b_1,0}([a_1]+[b_1])\sum_{\underline{p}\in P}\mathrm{sgn}(\underline{p})e_{f(p_2)}\cdots e_{f(p_{n+m})}.$$

Since $f(p_1)=a_1b_1$, $\delta_{a_1b_1,0}e_0=\delta_{a_1b_1,0}e_{f(p_1)}$ and thus

$$e_0\Lambda = \delta_{a_1b_1,0}([a_1]+[b_1])\,u*v.$$

Noting $\delta_{ab,0}[a]=\delta_{b,0}[a]$,

$$\begin{aligned}
e_0\Lambda &= \delta_{b_1,0}[a_1]\,u*v+\delta_{a_1,0}[b_1]\,u*v\\
&= \delta_{b_1,0}[a_1]\,u*e_0v'+\delta_{a_1,0}[b_1]\,e_0u'*v\\
&= \delta_{b_1,0}[a_1]\,e_0(u*v')+\delta_{a_1,0}[b_1]\,e_0(u'*v)
\end{aligned}$$

where $u'=e_{a_2}\cdots e_{a_n}$ and $v'=e_{b_2}\cdots e_{b_m}$. Therefore,

$$\Lambda = \delta_{b_1,0}[a_1]\,u*v'+\delta_{a_1,0}[b_1]\,u'*v. \tag{4.2}$$

We decompose $Q$ as $Q=Q_{\mathbb{Z}}\sqcup Q_{\#}$, where

$$\begin{aligned}
Q_{\mathbb{Z}} &:= \{(\underline{p},i)\in Q \mid p_i\in\mathbb{Z}^2\},\\
Q_{\#} &:= \{(\underline{p},i)\in Q \mid p_i\notin\mathbb{Z}^2\}.
\end{aligned}$$

Moreover, by putting

$$V_\alpha := \begin{cases} (1,0) & \text{if } \alpha = \uparrow \\ (\frac{1}{2},\frac{1}{2}) & \text{if } \alpha = \nearrow \\ (0,1) & \text{if } \alpha = \rightarrow \end{cases}$$

and defining the sets

$$Q_{\alpha,\beta} := \{(\underline{p},i) \in Q \mid i > 0, p_i \in \mathbb{Z}^2,\, p_i = p_{i-1} + V_\alpha,\, p_{i+1} = p_i + V_\beta\}$$

for $\alpha,\beta,\gamma \in \{\uparrow,\nearrow,\rightarrow\}$, we can further decompose $Q_{\mathbb{Z}}$ as

$$Q_{\mathbb{Z}} = \bigsqcup_{\alpha,\beta\in\{\uparrow,\nearrow,\rightarrow\}} Q_{\alpha,\beta}.$$

Now we have

$$s(Q_{\nearrow\nearrow}) = 0 \tag{4.3}$$

$$s(Q_{\uparrow\rightarrow}) + s(Q_{\rightarrow\uparrow}) + s(Q_{\#}) = 0 \tag{4.4}$$

$$s(Q_{\rightarrow\rightarrow}) + s(Q_{\rightarrow\nearrow}) + s(Q_{\nearrow\rightarrow}) = u * \partial v \tag{4.5}$$

$$s(Q_{\uparrow\uparrow}) + s(Q_{\uparrow\nearrow}) + s(Q_{\nearrow\uparrow}) = (\partial u) * v. \tag{4.6}$$

The identity (4.3) is obvious since $h(p, p + (\frac{1}{2},\frac{1}{2})) = h(p - (\frac{1}{2},\frac{1}{2}), p)$ if $p \in \mathbb{Z}^2$.

To check the identity (4.4), define the bijections $\omega^+ : Q_\# \longrightarrow Q_{\uparrow\rightarrow}$ and $\omega^- : Q_\# \longrightarrow Q_{\rightarrow\uparrow}$ by

$$\omega^\pm(\underline{p},i) := (\omega_i^\pm(\underline{p}),i),$$

where

$$\omega_i^\pm(p_0,\dots,p_{n+m+1}) := \left(p_0,\dots,p_i \pm (\frac{1}{2},-\frac{1}{2}),\dots,p_{n+m+1}\right).$$

Then

$$s(\omega^+(\underline{p},i)) + s(\omega^-(\underline{p},i)) + s(\underline{p},i) = C(\underline{p},i)\, e_{f(p_1)}\cdots \widehat{e_{f(p_i)}}\cdots e_{f(p_{n+m})}$$

with

$$\begin{aligned} C(\underline{p},i) = {} & \operatorname{sgn}(\omega_i^+(\underline{p}))\,(h((x+1,y),(x+1,y+1)) - h((x,y),(x+1,y))) \\ & + \operatorname{sgn}(\omega_i^-(\underline{p}))\,(h((x,y+1),(x+1,y+1)) - h((x,y),(x,y+1))) \\ & + \operatorname{sgn}(\underline{p})\left(h((x+\frac{1}{2},y+\frac{1}{2}),(x+1,y+1)) - h((x,y),(x+\frac{1}{2},y+\frac{1}{2}))\right) \end{aligned}$$

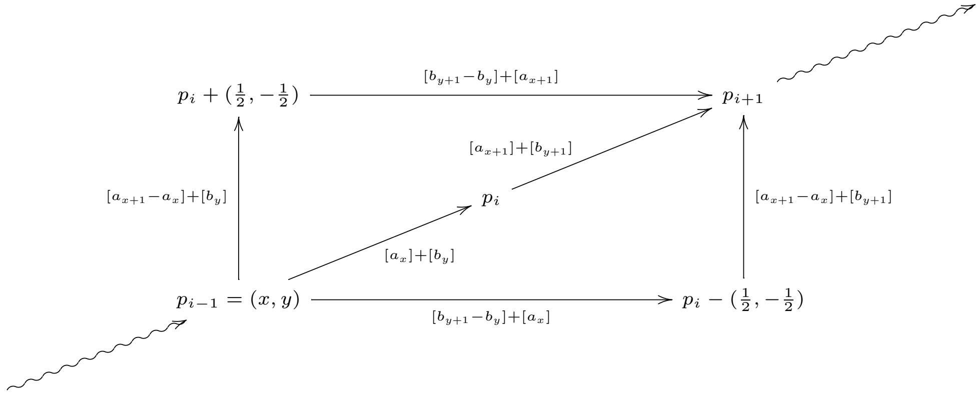


**Fig. 4.3.** The paths $\underline{p}$ (diagonal), $\omega_i^+(\underline{p})$ (upper), $\omega_i^-(\underline{p})$ (lower).

$$
\begin{aligned}
&= -\mathrm{sgn}(\underline{p})\left([b_{y+1} - b_y] + [a_{x+1}] - [a_{x+1} - a_x] - [b_y]\right)\\
&\quad - \mathrm{sgn}(\underline{p})\left([a_{x+1} - a_x] + [b_{y+1}] - [b_{y+1} - b_y] - [a_x]\right)\\
&\quad + \mathrm{sgn}(\underline{p})\left([a_{x+1}] + [b_{y+1}] - [a_x] - [b_y]\right)\\
&= 0
\end{aligned}
$$

where we put $p_{i-1} = (x, y)$ (see Fig. 4.3). Hence we obtain the identity (4.4).

To check the identity (4.5), we first decompose $Q_{\to\to}$, $Q_{\to\nearrow}$ and $Q_{\nearrow\to}$ as

$$Q_{\alpha,\beta} = \bigsqcup_{y=1}^{m} Q_{\alpha,\beta}^{(y)}$$

where $(\alpha, \beta) \in \{(\to, \to), (\to, \nearrow), (\nearrow, \to)\}$ and

$$Q_{\alpha,\beta}^{(y)} = \{(\underline{p}, i) \in Q_{\alpha,\beta} \mid \exists x, p_i = (x, y)\}.$$

On the other hand, by definition, we have

$$u * \partial v = \sum_{y=1}^{m} \left([b_{y+1} - b_y] - [b_y - b_{y-1}]\right) u * e_{b_1} \cdots \widehat{e_{b_y}} \cdots e_{b_m}$$

where we set $b_0 = 0$. By similar methods as for the identity (4.4), we can show

$$s(Q_{\to\to}^{(y)}) + s(Q_{\to\nearrow}^{(y)}) + s(Q_{\nearrow\to}^{(y)}) = \left([b_{y+1} - b_y] - [b_y - b_{y-1}]\right) u * e_{b_1} \cdots \widehat{e_{b_y}} \cdots e_{b_m}$$

(see Figs. 4.4 and 4.5). Hence we have the identity (4.5).

The identity (4.6) is also checked in completely the same way.

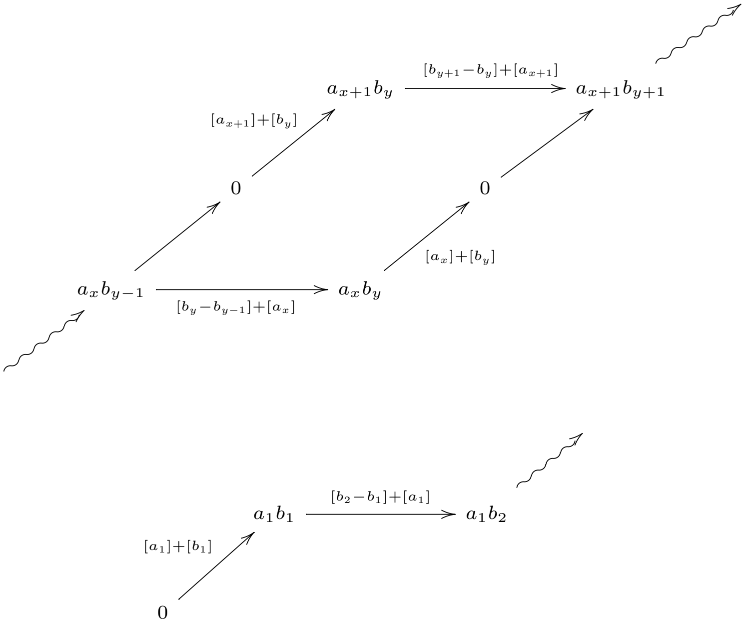


**Fig. 4.4.** A visual explanation of the reason why the terms in $s(Q^{(y)}_{\rightarrow\nearrow}) + s(Q^{(y)}_{\nearrow\rightarrow})$ corresponds to the terms of the form $([b_{y+1} - b_y] - [b_y - b_{y-1}]) \times (\cdots e_{a_x b_{y-1}} e_0 e_{a_{x+1} b_{y+1}} \cdots)$ which appear in $u * \partial v$. The first diagram is for the case $y > 1$ and the second diagram is for the case $y = 1$. Note that $([b_{y+1} - b_y] + [a_{x+1}] - [a_{x+1}] - [b_y]) + ([a_x] + [b_y] - [b_y - b_{y-1}] - [a_x]) = [b_{y+1} - b_y] - [b_y - b_{y-1}]$ for the first diagram, and $[b_2 - b_1] + [a_1] - [a_1] - [b_1] = [b_2 - b_1] - [b_1]$ for the second diagram.

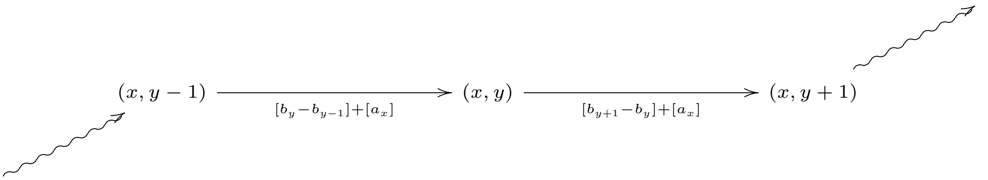


**Fig. 4.5.** A visual explanation of the reason why the terms in $s(Q^{(y)}_{\rightarrow\rightarrow})$ corresponds to the terms of the form $([b_{y+1} - b_y] - [b_y - b_{y-1}])\,(\cdots e_{a_x b_{y-1}} e_{a_x b_{y+1}} \cdots)$ which appear in $u * \partial v$. Note that $[b_{y+1} - b_y] + [a_x] - [b_y - b_{y-1}] - [a_x] = [b_{y+1} - b_y] - [b_y - b_{y-1}]$.

Combining the identities (4.1), (4.2) together with (4.3), (4.4), (4.5) and (4.6), it readily follows that

$$\partial(u * v) = (\partial u) * v + u * \partial v + \delta_{b_1,0}[a_1]\, u * v' + \delta_{a_1,0}[b_1]\, u' * v. \quad \square$$

## 5. Möbius transformation

In this section, we investigate the relation between the differential operator $\partial^{s,t}$ and the transformation on the algebra $\mathcal{A} = \mathcal{A}_F$ associated to the Möbius transformation on $\mathbb{P}^1$.

Recall that $\mathrm{GL}_2(F)$ naturally acts on $\mathbb{P}^1(F) = F \sqcup \{\infty\}$ by the Möbius transformation

$$\gamma(x) = \frac{ax+b}{cx+d} \quad \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right).$$

Put $e_\infty := 0$. We define an automorphism $\gamma^*$ of $\mathcal{A}$ by[3] $\gamma^*(e_x) = e_{\gamma(x)} - e_{\gamma(\infty)}$.

Hereafter, we fix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F)$, a starting point $s \in F$ and an endpoint $t \in F$. We assume that $\gamma(s) \neq \infty$ and $\gamma(t) \neq \infty$. We set $\varepsilon_\gamma(z) := [\det \gamma] - 2[cz+d] \in \mathbb{Z}$. We shall prove the following identity[4]:

**Theorem 9.** *For a non-constant monomial $w \in \mathcal{A}$,*

$$\left(\gamma^{-1}\right)^* \partial^{\gamma(s),\gamma(t)} \gamma^*(w) = \partial^{s,t} w + \delta_{x,s}\, \varepsilon_\gamma(s) w' - \delta_{y,t}\, \varepsilon_\gamma(t) w'',$$

*where $w = e_x w' = w'' e_y$. In particular, for $w \in \mathcal{A}^0_{(s,t)}$,*

$$\left(\gamma^{-1}\right)^* \partial^{\gamma(s),\gamma(t)} \gamma^*(w) = \partial^{s,t} w.$$

In the following, we give a proof of this theorem. We put $w = e_{z_1} \cdots e_{z_n} \in \mathcal{A}^0_{(s,t)}$ and $(z_0, z_{n+1}) = (s,t)$. We define a directed graph $G = (V, E)$ whose vertex set $V$ is given by

$$V = \{(0,0)\} \sqcup \{1, \dots, n\} \times \{0,1\} \sqcup \{(n+1, 0)\}$$

and whose edge set $E$ is given by

$$E = \{((m_1, i_1), (m_2, i_2)) \in V^2 \mid m_2 = m_1 + 1\}.$$

Define a labeling $f$ and a sign $\mathrm{sgn} : V \to \{\pm 1\}$ of vertices by

$$f : V \to \mathbb{P}^1(F) \ ; \ (n,i) \mapsto \begin{cases} \gamma(z_n) & i = 0 \\ \gamma(\infty) & i = 1, \end{cases}$$

and by $\mathrm{sgn}((m,i)) = (-1)^i$. For example, if $n = 3$, the labeling and the sign of the graph is as follows:

[3] The motivation of the definition of $\gamma^*$ comes from the formula $\frac{dt}{t-x} = \frac{ds}{s-\gamma(x)} - \frac{ds}{s-\gamma(\infty)}$, where $s = \gamma(t)$.

[4] The motivation of this identity comes from the total differential of the identity

$$\int_{s<t_1<\cdots<t_n<t} \prod_{j=1}^{n} \frac{dt_j}{t_j - z_j} = \int_{\gamma(s)<t'_1<\cdots<t'_n<\gamma(t)} \prod_{j=1}^{n} \left( \frac{dt'_j}{t'_j - \gamma(z_j)} - \frac{dt'_j}{t'_j - \gamma(\infty)} \right)$$

which is obtained by the Möbius transformation $t'_j = \gamma(t_j)$.

$$\begin{tikzcd}
(\gamma(s),+1) \arrow[r] \arrow[dr] & (\gamma(z_1),+1) \arrow[r] \arrow[dr] & (\gamma(z_2),+1) \arrow[r] \arrow[dr] & (\gamma(z_3),+1) \arrow[r] & (\gamma(t),+1) \\
 & (\gamma(\infty),-1) \arrow[r] \arrow[ur] & (\gamma(\infty),-1) \arrow[r] \arrow[ur] & (\gamma(\infty),-1) \arrow[ur] &
\end{tikzcd}$$

As before, we define the set of paths by

$$P := \left\{ \underline{p} = (p_0, \dots, p_{n+1}) \in V^{n+2} \;\middle|\; \begin{array}{ll} p_0 = (0,0)\,, & p_m \in \{(m,0),(m,1)\} \\ p_{n+1} = (n+1,0), & \text{for } 1 \le m \le n \end{array} \right\},$$

the subset of "effective" paths by

$$P^{\mathrm{eff}} := \left\{ \underline{p} \in P \,|\, f(p_m) \neq \infty \ \text{ for } 1 \le m \le n \right\}$$

and $\mathrm{sgn} : P \to \{\pm 1\}$ by $\mathrm{sgn}(\underline{p}) = \prod_{i=1}^{n} \mathrm{sgn}(p_i)$. Then, $\gamma^*(w)$ is expressed as

$$\gamma^*(w) = \sum_{\underline{p} \in P} \mathrm{sgn}(\underline{p})\, e_{f(p_1)} \cdots e_{f(p_n)} = \sum_{\underline{p} \in P^{\mathrm{eff}}} \mathrm{sgn}(\underline{p})\, e_{f(p_1)} \cdots e_{f(p_n)}.$$

Now, define a map $\lambda : E \to \mathbb{Z}$ by

$$\lambda(v_1, v_2) :=$$

$$\begin{cases} [ad-bc] + [z_{m+1} - z_m] - [cz_{m+1} + d] - [cz_m + d] & \text{if } (v_1,v_2) = ((m,0),(m+1,0)) \\ [ad-bc] - [c] - [cz_m + d] & \text{if } (v_1,v_2) = ((m,0),(m+1,1)) \\ [ad-bc] - [c] - [cz_{m+1} + d] & \text{if } (v_1,v_2) = ((m,1),(m+1,0)) \\ [ad-bc] - 2[c] - [z_{m+1} - z_m] & \text{if } (v_1,v_2) = ((m,1),(m+1,1)) \end{cases}$$

for $m \in \{0, \dots, n\}$. Then, by direct calculations, we have

$$[f(v_1) - f(v_2)] = \lambda(v_1, v_2) \tag{5.1}$$

for $(v_1, v_2) \in E$ such that $f(v_1) \neq \infty$, $f(v_2) \neq \infty$ and $f(v_1) \neq f(v_2)$. Set $Q = P \times \{1, \dots, n\}$ and $Q^{\mathrm{eff}} = P^{\mathrm{eff}} \times \{1, \dots, n\}$ and

$$s(\underline{p}, m) := (\lambda(p_m, p_{m+1}) - \lambda(p_{m-1}, p_m))\Big(\prod_{1 \le i \le n} \mathrm{sgn}(p_i)\Big)\, e_{f(p_1)} \cdots \widehat{e_{f(p_m)}} \cdots e_{f(p_n)}$$

for $(\underline{p}, m) \in Q$. Then, by (5.1) and Lemma 1, we get

$$\partial^{\gamma(s),\gamma(t)}(\gamma^*(w)) = s(Q^{\mathrm{eff}}) + \Lambda,$$

where

$$\begin{aligned}\Lambda &:= \delta_{z_1,s}\,\varepsilon_\gamma(s) \sum_{\substack{\underline{p}\in P^{\mathrm{eff}}\\ p_1=(1,0)}} \mathrm{sgn}(\underline{p})\, e_{f(p_2)}\cdots e_{f(p_n)} \\ &\quad - \delta_{z_n,t}\,\varepsilon_\gamma(t) \sum_{\substack{\underline{p}\in P^{\mathrm{eff}}\\ p_n=(n,0)}} \mathrm{sgn}(\underline{p})\, e_{f(p_1)}\cdots e_{f(p_{n-1})} \\ &= \gamma^*\left(\delta_{z_1,s}\,\varepsilon_\gamma(s)w' - \delta_{z_n,t}\,\varepsilon_\gamma(t)w''\right) \quad (w = e_{z_1}w' = w''e_{z_n}).\end{aligned}$$

By setting

$$\begin{aligned}Q' &= \left\{ (\underline{p},m)\in Q \,\middle|\, \exists i\neq m \text{ such that } f(p_i)=\infty \right\}, \\ Q'' &= \left\{ (\underline{p},m)\in Q \,\middle|\, f(p_m)=\infty \text{ and } f(p_i)\neq\infty \text{ for all } i\neq m \right\},\end{aligned}$$

we have $Q = Q^{\mathrm{eff}} \sqcup Q' \sqcup Q''$. Also, we can check by direct calculations[5] that

$$\lambda(v_1,v_2) = \lambda(v_2,v_3)$$

for $v_1,v_2,v_3\in V$ such that $(v_1,v_2),(v_2,v_3)\in E$ and $f(v_1),f(v_3)\neq\infty$, $f(v_2)=\infty$. Thus, $s(\underline{p},m)=0$ for $(\underline{p},m)\in Q''$. Together with the trivial fact that $e_{f(p_1)}\cdots\widehat{e_{f(p_m)}}\cdots e_{f(p_n)} = 0$ for $(\underline{p},m)\in Q'$, we find that $s(\underline{p},m)=0$ for $(\underline{p},m)\in Q'\sqcup Q''$ and hence

$$s(Q^{\mathrm{eff}}) = s(Q).$$

Again, by direct calculations,

$$\begin{aligned}&\sum_{j\in\{0,1\}} (-1)^j\left(\lambda((m,j),(m+1,k)) - \lambda((m-1,i),(m,j))\right) \\ &= [z_{m+1}-z_m] - [z_m - z_{m-1}] \qquad (i,k\in\{0,1\}).\end{aligned}$$

Hence, we have

$$s(Q) = \sum_{m=1}^{n} \left([z_{m+1}-z_m] - [z_m-z_{m-1}]\right) \sum_{\underline{p}\in P_m} e_{f(p_1)}\cdots\widehat{e_{f(p_m)}}\cdots e_{f(p_n)} \prod_{\substack{1\le i\le n\\ i\neq m}} \mathrm{sgn}(p_i), \tag{5.2}$$

where $P_m$ is a coset of $P$ by the equivalence relation $(p_0,\dots,(m,0),\dots,p_{n+1}) \sim (p_0,\dots,(m,1),\dots,p_{n+1})$. Since the right-hand side of (5.2) is $\gamma^*(\partial^{s,t}w)$ by definition, this proves Theorem 9.

[5] Note that if $v_2$ is at the bottom i.e. $\mathrm{sgn}(v_2)=-1$ then $c=0$ since $\gamma(\infty)=\infty$ in this case.

## 6. An application to iterated integrals on $\mathbb{P}^1 \setminus \{0,1,\infty,z\}$

In this section, we consider the special case $F = \mathbb{Q}(z)$. We put $\mathcal{A}_{\{0,1,z\}} = \mathbb{Z}\langle e_0, e_1, e_z\rangle \subset \mathcal{A}_F$ and $\mathcal{A}_{\{0,1\}} = \mathbb{Z}\langle e_0, e_1\rangle \subset \mathcal{A}_F$. Then, by definition, we have

$$\begin{aligned} \partial u \in \mathcal{A}_{\{0,1,z\}} &\quad (u \in \mathcal{A}_{\{0,1,z\}}),\\ u \sqcup\!\sqcup v \in \mathcal{A}_{\{0,1,z\}} &\quad (u, v \in \mathcal{A}_{\{0,1,z\}}),\\ u * v \in \mathcal{A}_{\{0,1,z\}} &\quad (u \in \mathcal{A}_{\{0,1,z\}}, v \in \mathcal{A}_{\{0,1\}}). \end{aligned}$$

Recall the linear operator $\partial_{z,c} : \mathcal{A}_F \to \mathcal{A}_F$ introduced in (1.2) i.e.,

$$\partial_{z,c}(e_{a_1}\cdots e_{a_n}) := \sum_{i=1}^{n} \left(\delta_{\{a_i,a_{i+1}\},\{z,c\}} - \delta_{\{a_{i-1},a_i\},\{z,c\}}\right) e_{a_1}\cdots \widehat{e_{a_i}} \cdots e_{a_n},$$

where $(a_0, a_{n+1}) = (0,1)$. Note that $\partial_{z,c} = \partial^{0,1}_{\mathfrak{F}_c}$, where $\mathfrak{F}_c : F^\times \to \mathbb{Z}$ is any homomorphism satisfying

$$\mathfrak{F}_c(z-c) = 1,\ \mathfrak{F}_c(z-(1-c)) = 0.$$

Let $\tau_z : \mathcal{A}_{\{0,1,z\}} \to \mathcal{A}_{\{0,1,z\}}$ be the anti-automorphism (i.e., $\tau_z(uv) = \tau_z(v)\tau_z(u)$ for $u, v \in \mathcal{A}_{\{0,1,z\}}$) defined by

$$\tau_z(e_0) = e_z - e_1,\ \tau_z(e_1) = e_z - e_0,\ \tau_z(e_z) = e_z.$$

We also put $\mathcal{A}^1_{\{0,1\}} := \mathcal{A}_{\{0,1\}} \cap \mathcal{A}^1_F$ and $\mathcal{A}^0_{\{0,1,z\}} := \mathcal{A}_{\{0,1,z\}} \cap \mathcal{A}^0_F$.

**Theorem 10.** *For $c \in \{0,1\}$, we have the following formulas.*

*(1) For $u, v \in \mathcal{A}_{\{0,1,z\}}$,*

$$\partial_{z,c}(u \sqcup\!\sqcup v) = (\partial_{z,c}u) \sqcup\!\sqcup v + u \sqcup\!\sqcup (\partial_{z,c}v).$$

*(2) For non-constant monomials $u \in \mathcal{A}_{\{0,1\}}$ and $v \in \mathcal{A}_{\{0,1,z\}}$,*

$$\partial_{z,c}(u * v) = u * (\partial_{z,c}v) + \delta_{a,0}\delta_{b,z}\delta_{c,0}\, u' * v,$$

*where $u = e_a u', v = e_b v'$. In particular, for $u \in \mathcal{A}^1_{\{0,1\}}$ and $v \in \mathcal{A}_{\{0,1,z\}}$,*

$$\partial_{z,c}(u * v) = u * (\partial_{z,c}v).$$

*(3) For a non-constant monomial $u \in \mathcal{A}_{\{0,1,z\}}$,*

$$\tau_z^{-1} \circ \partial_{z,c} \circ \tau_z(u) = \partial_{z,c}u + (\delta_{c,1} - \delta_{c,0})(\delta_{a,0}u' + \delta_{b,1}u''),$$

*where* $u = e_a u' = u'' e_b$. *In particular, for* $u \in \mathcal{A}^0_{\{0,1,z\}}$,

$$\tau_z^{-1} \circ \partial_{z,c} \circ \tau_z(u) = \partial_{z,c} u.$$

**Proof.** (1) and (2) of Theorem 10 follow from Theorem 3 and 8. Let $\varphi : \mathcal{A}_F \to \mathcal{A}_F$ be an anti-automorphism defined by

$$\varphi(e_x) = -e_x \quad (x \in F).$$

Then, we have

$$\varphi \circ \partial^{x,y} = \partial^{y,x} \circ \varphi \qquad (x, y \in F).$$

Since $\tau_z = \varphi \circ \gamma_z^*$ with $\gamma_z = \begin{pmatrix} z & -z \\ 1 & -z \end{pmatrix} \in \mathrm{GL}_2(F)$, (3) of Theorem 10 follows from Theorem 9. $\square$

We can further generalize (2) of Theorem 10 to a more symmetric form as an example of Theorem 8. Note that, for $u, v \in \mathcal{A}_{\{0,1,z\}}$, the stuffle product $u * v$ is in $\mathcal{A}_{\{0,1,z,z^2\}}$ where

$$\mathcal{A}_{\{0,1,z,z^2\}} := \mathbb{Z}\left\langle e_0, e_1, e_z, e_{z^2} \right\rangle \subset \mathcal{A}_F.$$

For $c \in \{0, 1, -1\}$, we extend the previous definition of $\partial_{z,c}$ to $\partial_{z,c} : \mathcal{A}_{\{0,1,z,z^2\}} \to \mathcal{A}_{\{0,1,z,z^2\}}$ by

$$\begin{aligned}\partial_{z,c}(e_{a_1} \cdots e_{a_n}) := &\sum_{\substack{1 \le i \le n \\ a_i \ne a_{i+1}}} \mathrm{ord}_{z=c}(a_{i+1} - a_i) e_{a_1} \cdots \widehat{e_{a_i}} \cdots e_{a_n} \\ &- \sum_{\substack{1 \le i \le n \\ a_{i-1} \ne a_i}} \mathrm{ord}_{z=c}(a_i - a_{i-1}) e_{a_1} \cdots \widehat{e_{a_i}} \cdots e_{a_n}\end{aligned}$$

where $(a_0, a_{n+1}) = (0, 1)$, and $\mathrm{ord}_{z=c}(f)$ is the order of zero of $f$ at $z = c$. Then the following theorem follows from Theorem 8.

**Theorem 11.** *For* $c \in \{0, 1, -1\}$ *and non-constant monomials* $u, v \in \mathcal{A}_{\{0,1,z\}}$,

$$\partial_{z,c}(u * v) = u * (\partial_{z,c} v) + (\partial_{z,c} u) * v + \delta_{a,0}\delta_{b,z}\delta_{c,0}\, u' * v + \delta_{b,0}\delta_{a,z}\delta_{c,0}\, u * v',$$

*where* $u = e_a u', v = e_b v'$.

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