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Analytical solutions for the thick-walled cylinder problem modeled with an isotropic elastic second gradient constitutive equation

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ABSTRACT

Numerical modeling of localization phenomena shows that constitutive equations with internal length scale are necessary to properly model the post-localization behavior. Moreover, these models allow an accurate description of the scale effects observed in some phenomena like micro-indentation. This paper proposes some analytical results concerning a boundary value problem in a medium with microstructure. In addition to their own usefulness, such analytical solutions can be used in benchmark exercises for the validation of numerical codes. The paper focuses on the thick-walled cylinder problem, using a general small strain isotropic elastic second gradient model. The most general isotropic elastic model involving seven different constants is used and the expression of the analytical solutions is explicitly given. The influence of the microstructure is controlled by the internal length scale parameter. The classical macrostress is no more in equilibrium with the classical forces at the boundary. Double stresses are indeed also generated by the classical boundary conditions and, as far as the microstructure effects become predominant (i.e. the internal length scale is much larger than the thickness of the cylinder), the macrostresses become negligible. This leads to solutions completely different from classical elastic ones.

1. Introduction

Since many years, the interest for enhanced models is increasing more and more. Many reasons explain this renewal. The advanced analysis of localization phenomena has shown that constitutive equations with internal length are necessary to properly model the experimental results involving some localized patterns (see, for instance, the pioneering works of Aifantis (1984), Bazant et al. (1984), de Borst and Muhlhaus (1992), Vardoulakis and Sulem (1995)). Moreover, these models allow to properly describe scale effects observed in some phenomena like micro-indentation as described in Nix and Gao (1997), Sulem and Cerrolaza (2002) and Rashid et al. (2004), or more generally in micro- and nanomechanics as detailed in Fleck et al. (1994) and Fleck and Hutchinson (1997).

Many enhanced models have been proposed in the literature, especially within the framework of plastic or damage theories. In this paper, we focus first on general theories not closely related to specific behaviors. These theories based on an enhancement of the kinematic itself can be traced back to the pioneering works of Toupin (1962), Mindlin (1964) and Germain (1973b). The starting point of this paper are the materials with microstructure, as defined by Mindlin (1964) and Germain (1973b). On the contrary to the theories involving a gradient of internal variables, only valid for some specific behavior modeling, the latter are able to generate several kinds of constitutive equations like elasticity (Mindlin, 1965) elastoplasticity (Fleck and Hutchinson, 1997; Chambon et al., 1996, 1998), viscoplasticity (Forest and Sievert, 2006) and also hypoplasticity and continuum damage (Chambon et al., 1998). Adding some mathematical constraints to the most general materials with microstructure yields a large set of models. Among all these models, the first and most famous one is the Cosserat model (see Cosserat and Cosserat, 1909). Some of these models have been extensively studied and it has been demonstrated that general plastic (see Chambon et al., 2001) or viscoplastic models (see Forest and Sievert, 2006) can be derived within this framework. Even large elastoplasticity with multiplicative decomposition of the deformation gradient can be developed at least in the particular case of second gradient model (see Chambon et al., 2004). These models have been used in numerical codes, especially once more for the particular case of second gradient model (see, for instance, Chambon et al. (1998) in the one-dimensional case or Shu et al. (1999) for the elastic two-dimensional case and Matsushima et al. (2002) for the elastoplastic two-dimensional case).

However, as far as we know, there are very few analytical results concerning boundary value problems involving media with

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microstructure. This is a problem because, in addition to their own usefulness, such solutions can be used as benchmark exercises in order to assess the validation of numerical codes. In one-dimensional cases, analytical solutions for a second gradient medium are provided in Chambon et al. (1998, 2001). In true bi-dimensional cases, Eshel and Rosenfeld (1970) and Bleustein (1966) propose the analytical solution of the stress concentration, respectively, at a cylindrical hole in a field of uniaxial tension and at a spherical cavity in a field of isotropic tension. Eshel and Rosenfeld (1975) have also developed the general equations of axi-symmetric problems in second gradient elastic materials, without giving explicit analytical solutions. Seeking the solution to the plastic expansion prior to the fully plastic stage, Zhao et al. (2007) provide a semi-analytical solution for a thick-walled cylinder. In this paper, the analytical solutions of the thick-walled cylinder problem in the case of a general small strain isotropic elastic second gradient model are given. With respect to the work of Zhao et al. (2007), it has to be emphasized that there is no restriction as far as the model is concerned. The more general isotropic elastic model with seven constants is used. Moreover, in the present paper, the analytical solutions are given explicitly, with the constants of integration depending on the prescribed boundary conditions. This allows to generate all the solutions of this problem by solving a set of four algebraic equations in four unknowns.

The sequence of the paper is as follows. The Section 2 is a presentation of the notations. Enhanced models require the use of unusual tensor. Moreover, the problem studied here has to be written in cylindrical coordinate system. In order to avoid any confusion, this section devoted to notation is necessary. The Section 3 is a presentation of the media with microstructure and relatives. This allows us to detail the link between the general media with microstructure and the second gradient model coming from the previous one by adding a mathematical constraint on the kinematic description. We follow mainly the works of Germain (see e.g. Germain (1973a,b)). The names of the models used in this paper is our choice. This is necessary since there is no general agreement concerning this point. In Section 4, the equations to be solved in the case of the thick-walled cylinder are derived for a second gradient model. The method uses extensively the virtual work principle. Both balance equations and boundary conditions are obtained. The fifth part deals with the resolution of the ordinary differential equation obtained in the previous section. The general method for the determination of the integration constants (from the prescribed boundary conditions) is presented. In a sixth part, some particular solutions corresponding to different boundary conditions are exhibited. The influence of the internal length scales introduced either by the model or by the boundary conditions are also exemplified. Some concluding remarks end up this paper.

2. Notations

Since all along this paper only orthonormal basis are used, it is not necessary to distinguish between covariant and contravariant components. We use then lower case subscripts to denote components of vectors and tensors. The other indices, namely superscripts, have other meanings and cannot be confused with the power operation because this later operation has a specific notation (see the first item in the following list). Vectors are denoted with arrows. The summation convention with tensorial indices is used. Let us emphasize that, using cylindrical coordinates, summation is meaningless for indices r, θ and z, even if they are in lower position.

- $(\alpha)^n$ means α to the power n.
- δ_{ij} are the components of the identity tensor (i.e. the Kronecker symbol).
- x_j are the components of the coordinates with respect to an orthonormal Cartesian basis.
- n_j are the components of the unit outward normal of a bounded domain.
- u_i are the components of the displacement field $\vec{u}$.
- $\partial_i a$ or $\partial_i(a)$ in some cases to avoid some ambiguities denotes the partial derivative of any quantity a with respect to the coordinate i.
- ϵ_{ij} are the components of the gradient of the displacement field i.e. $\epsilon_{ij} = \partial_j u_i$.
- ε_{ijk} are the components of the second gradient of the displacement field i.e. $\varepsilon_{ijk} = \partial_k(\partial_j u_i)$.
- χ_{ijk} are the components of the double stress denoted χ.
- σ_{ij} are the components of the macrostress denoted $\sigma, \sigma_{ij} = \sigma_{ji}$.
- τ_{ij} are the components is the microstress denoted τ.
- Ω is a given regular bounded domain.
- $\partial\Omega$ is the boundary of Ω assumed to enjoy the $C1$-continuity property.
- $*$ denotes virtual kinematical quantities.
- . denotes the scalar product of two vectors, for instance, $\vec{a}.\vec{b} = a_i b_i$.
- : denotes the scalar product of two second order tensors, for instance, $A : B = A_{ij}B_{ij}$.
- $\therefore$ denotes the scalar product of two third order tensors, for instance, $C \therefore D = C_{ijk}D_{ijk}$.
- ∇ is the gradient operator, which applies either to scalar or vector fields.
- $D(q)$ is the normal derivative of the quantity q, being a scalar or a vector. This means that: $D(q) = \partial_j(q)n_j = \nabla q.\vec{n}$.
- $D_j(q)$ are the components of the tangential derivative of the quantity q, being a scalar or a vector. $D_j(q) = \partial_j(q) - D(q) n_j = (\nabla q - (\nabla q.\vec{n})\vec{n})_j$.
- $\otimes$ denotes the dyadic product.
- $\vec{e}$ denotes generically a basis vector. The basis vectors of the orthonormal cartesian coordinates are denoted generically $\vec{e}_i$. The basis vectors of the cylindrical coordinates are denoted: $\vec{e}_r, \vec{e}_\theta$ and $\vec{e}_z$. This means that for any vector $\vec{a}$ we have $\vec{a} = a_i\vec{e}_i = a_r\vec{e}_r + a_\theta\vec{e}_\theta + a_z\vec{e}_z$. It may be worth reminding that the vectors $\vec{e}_r, \vec{e}_\theta$ depend on θ and that $\partial_\theta\vec{e}_r = \vec{e}_\theta$ and $\partial_\theta\vec{e}_\theta = -\vec{e}_r$.
- R^i and R^e denote, respectively, the inner and outer radii of the studied thick-walled cylinder.
- For a scalar function u depending only on r, the function v is defined such that: $v = \partial_r u + \frac{u}{r} = \frac{1}{r}\partial_r(ru)$.

To make these notations more clear, let us give some examples. Using orthonormal cartesian coordinates, we have, for instance, for a vector: $\nabla\vec{a} = \partial_j\vec{a} \otimes \vec{e}_j = \partial_j(a_i)\vec{e}_i \otimes \vec{e}_j$ and for a second order tensor $\nabla A = \partial_k A \otimes \vec{e}_k = \partial_k A_{ij}\vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_k$.

Using cylindrical coordinates yields the following well-known results, useful in the following:

$$\nabla\vec{a} = \partial_r\vec{a} \otimes \vec{e}_r + \frac{1}{r}\partial_\theta\vec{a} \otimes \vec{e}_\theta + \partial_z\vec{a} \otimes \vec{e}_z \tag{1}$$

and

$$\nabla A = \partial_r A \otimes \vec{e}_r + \frac{1}{r}\partial_\theta A \otimes \vec{e}_\theta + \partial_z A \otimes \vec{e}_z. \tag{2}$$

3. Media with microstructure and some relatives

3.1. Media with microstructure

The kinematic of a classical continuum is defined by a displacement field denoted $\vec{u}$, function of the coordinates. For media with microstructure, a field of a (not necessarily symmetric) second order tensor denoted E is added in the kinematic description.

Following Germain (1973b), it is necessary to define then the following dual static quantities: σ the macrostress, τ the microstress and χ the double stress. Neglecting all the body forces (i.e. the classical ones and the one which can be added due to the enriched kinematics), the only external forces are the boundary ones, which are in this case the classical forces $\vec{F}$ and the double forces M. The double force M is a second order tensor defined on the boundaries. In the case of a quasi-static problem, the virtual work principle corresponding to the equilibrium of a body Ω, the boundary of which is denoted $\partial\Omega$ reads:

For any kinematically admissible fields $\vec{u}^*$ and E^*, the following equation has to hold:

$$-\int_\Omega (\sigma : \nabla\vec{u}^* + \tau : (E^* - \nabla\vec{u}^*) + \chi \therefore \nabla E^*) d\Omega + \int_{\partial\Omega} (\vec{F}.\vec{u}^* + M : E^*) ds = 0, \tag{3}$$

where σ is the symmetric macrostress tensor.

Eq. (3) can be recast as:

$$-\int_\Omega ((\sigma - \tau) : \nabla\vec{u}^* + \tau : E^* + \chi \therefore \nabla E^*) d\Omega + \int_{\partial\Omega} (\vec{F}.\vec{u}^* + M : E^*) ds = 0. \tag{4}$$

Using the integration by part and Green formulae, it is classical to show that the virtual work principle yields the following balance equations:

$$\partial_j(\sigma_{ij} - \tau_{ij}) = 0, \tag{5}$$
$$\partial_k \chi_{ijk} - \tau_{ij} = 0 \tag{6}$$

and the boundary conditions:

$$(\sigma_{ij} - \tau_{ij}) n_j = F_i, \tag{7}$$
$$\chi_{ijk} n_k = M_{ij}. \tag{8}$$

In a medium with microstructure, Eqs. (7) and (8) are the relevant boundary conditions, with the classical forces $\vec{F}$ and the double forces M, which can be prescribed independently.

3.2. Second gradient model

Among all the constitutive equations suitable for a medium with microstructure, a second gradient model is a model for which the following constraint holds:

$$E = \nabla\vec{u}. \tag{9}$$

Assuming the same constraint for the corresponding virtual quantities, namely

$$E^* = \nabla\vec{u}^*, \tag{10}$$

the Eq. (4) of the virtual work principle becomes:

For any kinematically admissible field $\vec{u}^*$,

$$-\int_\Omega (\sigma : \nabla\vec{u}^* + \chi \therefore \nabla\nabla\vec{u}^*) dv + \int_{\partial\Omega} (\vec{p}.\vec{u}^* + \vec{\Phi}.D(\vec{u}^*)) ds = 0. \tag{11}$$

In Eq. (11), due to the constraint on the virtual quantities, it exists a dependence between the tangential derivatives of $\vec{u}^*$ and $\vec{u}^*$ itself. This means that only the normal derivative of $\vec{u}^*$ can be chosen independently from $\vec{u}^*$ itself. This introduces two new forces, $\vec{p}$ and $\vec{\Phi}$, respectively, the dual quantities of $\vec{u}^*$ and $D(\vec{u}^*)$. The surface traction $\vec{p}$ and the higher order surface traction $\vec{\Phi}$ are thus the relevant variables, which can be prescribed independently on the boundary.

The application of the virtual work principle (Eq. (11)) and two integrations by part give the balance equation and the boundary conditions. The balance equation reads:

$$\partial_j \sigma_{ij} - \partial_j \partial_k \chi_{ijk} = 0. \tag{12}$$

In second gradient media, the expression of the boundary conditions is more complex due to the above-mentioned dependency between the tangential derivatives of $\vec{u}^*$ and $\vec{u}^*$ itself. Finally, assuming that the boundary is regular (which means existence and uniqueness of the normal for every point on the boundary $\partial\Omega$ of the studied domain), we get:

$$\sigma_{ij} n_j - n_k n_j D(\chi_{ijk}) - D_k(\chi_{ijk}) n_j - D_j(\chi_{ijk}) n_k + D_l(n_l) \chi_{ijk} n_j n_k - D_k(n_j) \chi_{ijk} = p_i \tag{13}$$

and

$$\chi_{ijk} n_j n_k = \Phi_i. \tag{14}$$

3.3. Constitutive equations

We use here an isotropic elastic model for which the classical terms and the ones related to the second gradient are decoupled. This means that the stress σ_{ij} is following the classical Hooke's law defined by the Lamé constants λ and μ:

$$\sigma_{ij} = \lambda \delta_{ij} \epsilon_{ll} + 2\mu \epsilon_{ij}. \tag{15}$$

The double stress χ_{ijk} is obeying the elastic gradient law established by Mindlin (1965) with five parameters a^i (let us notice that we use different notations as the ones used in the quoted paper).

$$\chi_{pqs} = a^1(\varepsilon_{jjs}\delta_{pq} + \varepsilon_{jjq}\delta_{ps}) + \frac{1}{2}a^2(\varepsilon_{sii}\delta_{pq} + 2\varepsilon_{jjp}\delta_{qs} + \varepsilon_{qii}\delta_{ps}) + 2a^3\varepsilon_{pii}\delta_{qs} + 2a^4\varepsilon_{pqs} + a^5(\varepsilon_{qsp} + \varepsilon_{spq}). \tag{16}$$

The five parameters a^i are elastic constants associated with gradient terms in the material. These constants have the dimension of a force. Let us remark that Eq. (16) is an equality between tensors, which means that it is valid in any system of orthogonal coordinates, in particular in cylindrical coordinates.

4. Equation of the thick-walled cylinder in cylindrical coordinates

4.1. The thick-walled cylinder problem

We want to find the axi-symmetric solutions of the thick-walled cylinder problem (Fig. 1) for a second gradient medium. This means that Ω is the domain situated in between two cylinders sharing the same axis denoted as usual z (Fig. 1a). The problem is assumed to be a plane problem, which means that the component of the displacement along the z-axis is assumed to be equal to 0.

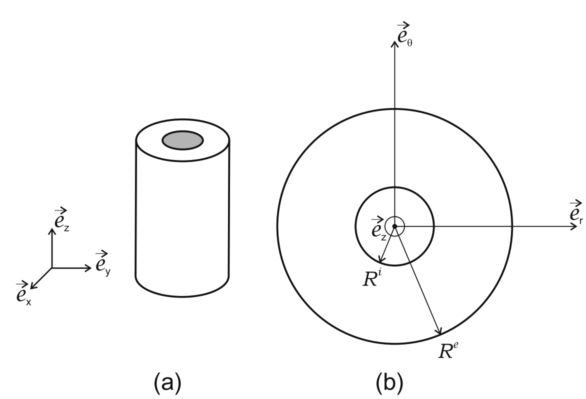

Fig. 1. Thick-walled cylinder problem: (a) definition of the domain in cartesian coordinates; (b) definition of the domain in cylindrical coordinates.

The domain can now be defined in the plane orthogonal to the z-axis. It is the area in between two circles sharing the same center (Fig. 1b). The inner radius is denoted R^i and the outer radius R^e. We use now the classical cylindrical coordinates r, θ and z and the corresponding orthonormal basis, whose vectors are $\vec{e}_r$, $\vec{e}_\theta$ and $\vec{e}_z$. (For more details on the use of strain gradient theory in orthogonal curvilinear coordinates, see Zhao and Pedroso (2008).) In fact only the first two are used in the following. In order to find axi-symmetric solutions, the displacement is assumed to be only radial:

$$\vec{u} = u\vec{e}_r, \tag{17}$$

where u is a function of r only.

In order to solve the thick-walled cylinder problem, we need the expressions of the strain tensor ϵ and the second gradient tensor ε. Under the assumption (17), the first and second derivatives of the displacement field are obtained (see Appendix A, for the details) and the expressions of ϵ and ε are found:

$$\begin{aligned}\epsilon_{rr} &= \partial_r u,\\ \epsilon_{\theta\theta} &= \frac{u}{r},\end{aligned} \tag{18}$$

$$\begin{aligned}\varepsilon_{rrr} &= \partial_r^2 u,\\ \varepsilon_{\theta\theta r} &= \partial_r\left(\frac{u}{r}\right),\\ \varepsilon_{r\theta\theta} &= \varepsilon_{\theta r\theta} = \frac{1}{r}\left(\partial_r u - \frac{u}{r}\right) = \partial_r\left(\frac{u}{r}\right),\end{aligned} \tag{19}$$

all the other components of the two tensors are equal to 0. It can be checked that $\varepsilon_{\theta\theta r} = \varepsilon_{\theta r\theta}$.

In the following, we use extensively the scalar function v such that:

$$v = \partial_r u + \frac{u}{r}, \tag{20}$$

which means also that:

$$v = \frac{1}{r}\partial_r(ru). \tag{21}$$

Using this notation, ε_{rrr} can be rewritten as:

$$\varepsilon_{rrr} = \partial_r v - \partial_r \frac{u}{r}. \tag{22}$$

Knowing the expressions of the gradient and second gradient of the displacement field, the stress and double stress tensor can be obtained after some calculations using the constitutive Eqs. (15) and (16):

$$\begin{aligned}\sigma_{rr} &= (\lambda + 2\mu)\partial_r u + \lambda\frac{u}{r},\\ \sigma_{\theta\theta} &= \lambda\partial_r u + (\lambda + 2\mu)\frac{u}{r},\\ \sigma_{r\theta} &= \sigma_{\theta r} = 0\end{aligned} \tag{23}$$

and

$$\begin{aligned}\chi_{rrr} &= 2(a^1 + a^2 + a^3 + a^4 + a^5)\partial_r v - 2(a^4 + a^5)\partial_r\frac{u}{r},\\ \chi_{r\theta\theta} &= (a^2 + 2a^3)\partial_r v + 2(a^4 + a^5)\partial_r\frac{u}{r},\\ \chi_{\theta r\theta} &= \chi_{\theta\theta r} = \left(a^1 + \frac{1}{2}a^2\right)\partial_r v + 2(a^4 + a^5)\partial_r\frac{u}{r},\\ \chi_{rr\theta} &= \chi_{r\theta r} = \chi_{\theta rr} = \chi_{\theta\theta\theta} = 0.\end{aligned} \tag{24}$$

4.2. Virtual work principle

The virtual work principle is completely equivalent to balance equations and boundary conditions. Using on one hand a second gradient model or on the other hand a media with microstructure model under the constraints (9) and (10) is also completely equivalent.

The differential equation satisfied by u can be obtained by transferring the constitutive Eqs. (23) and (24) into the balance Eq. (12) and the boundary conditions (13) and (14), written in cylindrical coordinates. It is in fact easier to derive the balance differential equation as well as the boundary conditions from the virtual formulation (4) under the constraint (10) ($E^* = \nabla\vec{u}^*$) which reads for the thick-walled cylinder:

$$\begin{aligned}&-\int_{R^i}^{R^e}\int_0^{2\pi}(\sigma : \nabla\vec{u}^* + \chi\therefore\nabla E^*)rd\theta dr + \int_0^{2\pi}(\vec{F}^i.\vec{u}^* + M^i : E^*)R^i d\theta\\ &+ \int_0^{2\pi}(\vec{F}^e.\vec{u}^* + M^e : E^*)R^e d\theta = 0.\end{aligned} \tag{25}$$

It is usual to consider virtual fields $\vec{u}^*$ and E^* having the same features as the unknown ones. This means that $\vec{u}^* = u^*\vec{e}_r$, where u^* is a function only of $r, E^*_{r\theta} = E^*_{\theta r} = 0, E^*_{rr} = \partial_r u^*$ and $E^*_{\theta\theta} = \frac{u^*}{r}$ and finally $\varepsilon^*_{rrr} = \partial_r^2 u^*, \varepsilon^*_{\theta\theta r} = \varepsilon^*_{r\theta\theta} = \varepsilon^*_{\theta r\theta} = \partial_r\left(\frac{u^*}{r}\right)$. Let us point out that the previous relations rely once more upon the constraint (10).

Since all the previous quantities do not depend on θ, dividing by 2π yields the virtual work equation in our particular case:

$$\begin{aligned}&-\int_{R^i}^{R^e}\left(\sigma_{rr}\partial_r u^* + \sigma_{\theta\theta}\frac{u^*}{r} + \chi_{rrr}\partial_r^2 u^* + (\chi_{\theta\theta r} + \chi_{r\theta\theta} + \chi_{\theta r\theta})\partial_r\left(\frac{u^*}{r}\right)\right)rdr\\ &+\left(F_r^i u^*(R^i) + M_{rr}^i\partial_r u^*(R^i) + M_{\theta\theta}^i\frac{u^*(R^i)}{R^i}\right)R^i\\ &+\left(F_r^e u^*(R^e) + M_{rr}^e\partial_r u^*(R^e) + M_{\theta\theta}^e\frac{u^*(R^e)}{R^e}\right)R^e\\ &= 0.\end{aligned} \tag{26}$$

After two integrations by part, Eq. (26) yields:

$$\begin{aligned}&\int_{R^i}^{R^e}\left(\left(\partial_r(r\sigma_{rr}) - \sigma_{\theta\theta} + \frac{1}{r}\partial_r(r(\chi_{\theta\theta r} + \chi_{r\theta\theta} + \chi_{\theta r\theta})) - \partial_r^2(r\chi_{rrr})\right)u^*\right)dr\\ &+\left(R^i\sigma_{rr}(R^i) + \chi_{\theta\theta r}(R^i) + \chi_{r\theta\theta}(R^i) + \chi_{\theta r\theta}(R^i) - R^i\partial_r\chi_{rrr}(R^i) - \chi_{rrr}(R^i)\right)u^*(R^i)\\ &+R^i\chi_{rrr}(R^i)\partial_r u^*(R^i) + \left(F_r^i u^*(R^i) + M_{rr}^i\partial_r u^*(R^i) + M_{\theta\theta}^i\frac{u^*(R^i)}{R^i}\right)R^i\\ &-\left(R^e\sigma_{rr}(R^e) + \chi_{\theta\theta r}(R^e) + \chi_{r\theta\theta}(R^e) + \chi_{\theta r\theta}(R^e) - R^e\partial_r\chi_{rrr}(R^e) - \chi_{rrr}(R^e)\right)u^*(R^e)\\ &-R^e\chi_{rrr}(R^e)\partial_r u^*(R^e) + \left(F_r^e u^*(R^e) + M_{rr}^e\partial_r u^*(R^e) + M_{\theta\theta}^e\frac{u^*(R^e)}{R^e}\right)R^e = 0.\end{aligned} \tag{27}$$

Using the fact that Eq. (27) has to hold for any kinematically admissible virtual field, we obtain easily the balance equation for the thick-walled cylinder problem, as well as the boundary conditions.

4.3. Balance equations

From Eq. (27), we get the balance equation of the problem:

$$\partial_r(r\sigma_{rr}) - \sigma_{\theta\theta} + \frac{1}{r}\partial_r(r(\chi_{\theta\theta r} + \chi_{r\theta\theta} + \chi_{\theta r\theta})) - \partial_r^2(r\chi_{rrr}) = 0. \tag{28}$$

Substituting the stresses and the double stresses given by Eqs. (23) and (24) in the previous equation yields:

$$\begin{aligned}&(\lambda + 2\mu)\left(\partial_r(r\partial_r u) - \frac{u}{r}\right) + \frac{1}{r}\partial_r\left(r(A - B)\partial_r v + 3rB\partial_r\frac{u}{r}\right)\\ &-\partial_r^2\left(Ar\partial_r v - Br\frac{u}{r}\right) = 0,\end{aligned} \tag{29}$$

where

$$A = 2(a^1 + a^2 + a^3 + a^4 + a^5) \tag{30}$$

and

$$B = 2(a^4 + a^5). \tag{31}$$

The constants A and B have the same dimension as the parameters a^i. Eq. (29) can be recast and yields after some simplifications:

$$(\lambda + 2\mu) r \partial_r v - A r \partial_r \left(\frac{1}{r} \partial_r (r \partial_r v) \right) = 0. \tag{32}$$

It is noteworthy that, even if Eq. (32) is completely general, it involves only one parameter for the classical part, namely $\lambda + 2\mu$, and one parameter for the second gradient part, namely A as defined in Eq. (30). This means that, for the thick-walled problem, only one internal length governs the behavior of the medium:

$$\alpha = \sqrt{\frac{A}{\lambda + 2\mu}}. \tag{33}$$

Using this notation, the ordinary differential equation to solve reads:

$$\partial_r \left(v - (\alpha)^2 \frac{1}{r} \partial_r (r \partial_r v) \right) = 0. \tag{34}$$

4.4. Boundary conditions

The boundary conditions allow us to obtain the constants of integration, which appear in the solution of Eq. (34). These latter conditions can be the prescription of $u(R^i), \partial_r u(R^i), u(R^e), \partial_r u(R^e)$, or the corresponding dual quantities. In order to obtain the expression of these dual quantities, we use once more the Eq. (27) and, $u^*(R^i)$ and $\partial_r u^*(R^i)$ being independent, we get:

$$\begin{aligned} &-R^i \sigma_{rr}(R^i) - \chi_{\theta\theta r}(R^i) - \chi_{r\theta\theta}(R^i) - \chi_{\theta r\theta}(R^i) + R^i \partial_r \chi_{rrr}(R^i) \\ &\quad + \chi_{rrr}(R^i) = F_r^i R^i + M_{\theta\theta}^i \end{aligned} \tag{35}$$

and

$$-\chi_{rrr}(R^i) = M_{rr}^i. \tag{36}$$

Similar relations are obtained for the outer boundary:

$$\begin{aligned} &R^e \sigma_{rr}(R^e) + \chi_{\theta\theta r}(R^e) + \chi_{r\theta\theta}(R^e) + \chi_{\theta r\theta}(R^e) - R^e \partial_r \chi_{rrr}(R^e) \\ &\quad - \chi_{rrr}(R^e) = F_r^e R^e + M_{\theta\theta}^e \end{aligned} \tag{37}$$

and

$$\chi_{rrr}(R^e) = M_{rr}^e. \tag{38}$$

It is noteworthy that it is not necessary to eliminate the tangential derivative of u^*, since in the particular case studied, that derivative is equal to $\frac{u^*(R^e)}{R^e} \vec{e}_\theta \otimes \vec{e}_\theta$ on the outer radius (and to a similar expression on the inner radius) and does not involve any derivation with respect to θ but only the value $u^*(R^e)$.

Finally, looking back to Eq. (11), we can write:

$$p_r^i = F_r^i + \frac{M_{\theta\theta}^i}{R^i} \tag{39}$$

and

$$\Phi_r^i = M_{rr}^i \tag{40}$$

and similar formulae for the outer radius. These two Eqs. (39) and (40) exhibit that the force $\vec{F}^i$ and the double force M^i are not the relevant quantities for the second gradient model, as far as it is only possible to prescribe one component of the double force and a combination of the force and the other component of the double force.

5. Solving the ordinary differential equation

The field equation (Eq. (34)) of the problem can be written as:

$$(\alpha)^2 \left(\partial_r^3 v + \frac{1}{r} \partial_r^2 v - \frac{1}{(r)^2} \partial_r v \right) - \partial_r v = 0. \tag{41}$$

It must be pointed out that Eq. (34) is a third order differential equation for v and, from Eq. (21), a fourth order one for u – as expected for a second gradient problem. The use of the auxiliary function v enables to reduce the differential order from 4 to 3. Due to the form of the Eq. (34), it is moreover clear that a simple integration can decrease the order of the ordinary differential equation to 2, which simplifies the research of the solution u. In fact, before the integration of Eq. (34), it is useful to reduce its differential order by rewriting it in term of $w = \partial_r v$, which yields:

$$(\alpha)^2 \partial_r^2 w + (\alpha)^2 \frac{1}{r} \partial_r w - \left(\frac{(\alpha)^2}{(r)^2} + 1 \right) w = 0. \tag{42}$$

Denoting x as the ratio of r over α, which means $x = \frac{r}{\alpha}$, yields a modified Bessel equation (see Abramowitz and Stegun, 1972):

$$(x)^2 d_x^2 w + x d_x w - (1 + (x)^2) w = 0. \tag{43}$$

The independent solutions of this equation are the modified Bessel functions $B^I(1, x)$ and $B^K(1, x)$ (see Abramowitz and Stegun, 1972), consequently:

$$\partial_r v = w(r) = C_1' B^I \left(1, \frac{r}{\alpha}\right) + C_2' B^K \left(1, \frac{r}{\alpha}\right), \tag{44}$$

where C_1' and C_2' are two constants.

In order to find $u(r)$, the integration of Eq. (34) yields:

$$(\alpha)^2 \frac{1}{r} \partial_r (r \partial_r v) - v = C_3', \tag{45}$$

where C_3' is a third constant.

Knowing that $\partial_r (ru) = r v$ (see Eq. (21)), Eq. (45) gives us the value of rv and hence:

$$\partial_r (ru) = (\alpha)^2 \partial_r (r \partial_r v) - C_3' r. \tag{46}$$

Integrating Eq. (46) yields:

$$u = (\alpha)^2 \partial_r v - \frac{C_3'}{2} r + \frac{C_4'}{r}, \tag{47}$$

where C_4' is a fourth constant.

Finally, using Eqs. (44) and (47), the general solution of the problem reads:

$$u = (\alpha)^2 \left(C_1' B^I \left(1, \frac{r}{\alpha}\right) + C_2' B^K \left(1, \frac{r}{\alpha}\right) \right) - \frac{C_3'}{2} r + \frac{C_4'}{r}. \tag{48}$$

This general solution (Eq. (48)) can be rewritten in order to simplify the forthcoming developments, by introducing new constants of integration:

$$u = C_1 B^I \left(1, \frac{r}{\alpha}\right) + C_2 B^K \left(1, \frac{r}{\alpha}\right) + C_3 r + \frac{C_4}{r}. \tag{49}$$

The boundary conditions allow us to find the values of the constants of integration. These latter conditions can be the prescription of $u(R^i), \partial_r(u(R^i)), u(R^e), \partial_r(u(R^e))$, or the corresponding dual quantities (Eqs. (35)–(37), (3), (39) and (40)) at the inner or outer boundary. The four constants of the problem are thus obtained using

- two conditions at the inner radius:

one of the two following equations:

$$u(R^i) = C_1 B^I \left(1, \frac{R^i}{\alpha}\right) + C_2 B^K \left(1, \frac{R^i}{\alpha}\right) + C_3 R^i + \frac{C_4}{R^i}, \tag{50}$$

$$
\begin{aligned}
p_r^i = &-\Bigg(2C_4(\lambda+\mu) - \frac{2C_3\mu}{(R^i)^2} + \frac{1}{\alpha R^i}\Bigg(C_1(\lambda+2\mu)R^iB^I\left(0,\frac{R^i}{\alpha}\right) \\
&-2\alpha C_1\mu B^I\left(1,\frac{R^i}{\alpha}\right) - C_2(\lambda+2\mu)R^iB^K\left(0,\frac{R^i}{\alpha}\right) - 2\alpha C_2\mu B^K\left(1,\frac{R^i}{\alpha}\right)\Bigg) \\
&-\frac{1}{(\alpha)^2}\Bigg(-\frac{2(a^4+a^5)\left(C_1B^I\left(1,\frac{R^i}{\alpha}\right)+C_2B^K\left(1,\frac{R^i}{\alpha}\right)\right)}{R^i} \\
&+\Bigg((a^1+a^2+a^3+a^4+a^5)\Bigg(C_1\left(B^I\left(0,\frac{R^i}{\alpha}\right)+B^I\left(2,\frac{R^i}{\alpha}\right)\right) \\
&-C_2\left(B^K\left(0,\frac{R^i}{\alpha}\right)+B^K\left(2,\frac{R^i}{\alpha}\right)\right)\Bigg)\Bigg)\Big/\alpha \\
&+\frac{6\alpha(a^4+a^5)\left(-2\alpha C_3+(R^i)^2\left(C_1B^I\left(2,\frac{R^i}{\alpha}\right)-C_2B^K\left(2,\frac{R^i}{\alpha}\right)\right)\right)}{(R^i)^4}\Bigg) \\
&+\frac{1}{(\alpha)^2(R^i)^4}\Bigg((a^4+a^5)\Bigg(-16(\alpha)^2C_3+(R^i)^2\Bigg(-2C_1R^iB^I\left(1,\frac{R^i}{\alpha}\right) \\
&+8\alpha C_1B^I\left(2,\frac{R^i}{\alpha}\right) - 2C_2R^iB^K\left(1,\frac{R^i}{\alpha}\right) - 8\alpha C_2B^K\left(2,\frac{R^i}{\alpha}\right)\Bigg)\Bigg)\Bigg)\Bigg)
\end{aligned} \tag{51}
$$

and one of the equations corresponding to the prescription of $\partial_r(u(R^i))$ or the double stress M_{rr}:

$$
\begin{aligned}
\partial_r(u(R^i)) = C_4 - \frac{C_3}{(R^i)^2} + C_1\left(B^I\left(0,\frac{R^i}{\alpha}\right)+B^I\left(2,\frac{R^i}{\alpha}\right)\right) \\
- C_2\left(B^K\left(0,\frac{R^i}{\alpha}\right)+B^K\left(2,\frac{R^i}{\alpha}\right)\right)\Big/(2a),
\end{aligned} \tag{52}
$$

$$
\begin{aligned}
\Phi_r^i = -\Bigg(2\Bigg((a^1+a^2+a^3+a^4+a^5)\left(C_1B^I\left(1,\frac{R^i}{\alpha}\right)+C_2B^K\left(1,\frac{R^i}{\alpha}\right)\right) \\
+\frac{\alpha(a^4+a^5)\left(2\alpha C_3+(R^i)^2\left(-C_1B^I\left(2,\frac{R^i}{\alpha}\right)+C_2B^K\left(2,\frac{R^i}{\alpha}\right)\right)\right)}{(R^i)^3}\Bigg)\Bigg)\Big/(\alpha)^2
\end{aligned} \tag{53}
$$

– similarly two conditions at the outer radius:

one of the two following equations:

$$
u(R^e) = C_1B^I\left(1,\frac{R^e}{\alpha}\right) + C_eB^K\left(1,\frac{R^e}{\alpha}\right) + C_3R^e + \frac{C_4}{R^e}, \tag{54}
$$

$$
\begin{aligned}
p_r^e = &\,2C_4(\lambda+\mu) - \frac{2C_3\mu}{(R^e)^2} + \frac{1}{\alpha R^e}\Bigg(C_1(\lambda+2\mu)R^eB^I\left(0,\frac{R^e}{\alpha}\right) \\
&-2\alpha C_1\mu B^I\left(1,\frac{R^e}{\alpha}\right) - C_2(\lambda+2\mu)R^eB^K\left(0,\frac{R^e}{\alpha}\right) - 2\alpha C_2\mu B^K\left(1,\frac{R^e}{\alpha}\right)\Bigg) \\
&-\frac{1}{(\alpha)^2}\Bigg(-\frac{2(a^4+a^5)\left(C_1B^I\left(1,\frac{R^e}{\alpha}\right)+C_2B^K\left(1,\frac{R^e}{\alpha}\right)\right)}{R^e} \\
&+\Bigg((a^1+a^2+a^3+a^4+a^5)\Bigg(C_1\left(B^I\left(0,\frac{R^e}{\alpha}\right)+B^I\left(2,\frac{R^e}{\alpha}\right)\right) \\
&-C_2\left(B^K\left(0,\frac{R^e}{\alpha}\right)+B^K\left(2,\frac{R^e}{\alpha}\right)\right)\Bigg)\Bigg)\Big/\alpha \\
&+\frac{6\alpha(a^4+a^5)\left(-2\alpha C_3+(R^e)^2\left(C_1B^I\left(2,\frac{R^e}{\alpha}\right)-C_2B^K\left(2,\frac{R^e}{\alpha}\right)\right)\right)}{(R^e)^4}\Bigg) \\
&+\frac{1}{(\alpha)^2(R^e)^4}\Bigg((a^4+a^5)\Bigg(-16(\alpha)^2C_3+(R^e)^2\Bigg(-2C_1R^eB^I\left(1,\frac{R^e}{\alpha}\right) \\
&+8\alpha C_1B^I\left(2,\frac{R^e}{\alpha}\right) - 2C_2R^eB^K\left(1,\frac{R^e}{\alpha}\right) - 8\alpha C_2B^K\left(2,\frac{R^e}{\alpha}\right)\Bigg)\Bigg)\Bigg)
\end{aligned} \tag{55}
$$

and one of the equations corresponding to the prescription of $\partial_r(u(R^e))$ or the double stress M_{rr}:

$$
\begin{aligned}
\partial_r(u(R^e)) = C_4 - \frac{C_3}{(R^e)^2} + C_1\left(B^I\left(0,\frac{R^e}{\alpha}\right)+B^I\left(2,\frac{R^e}{\alpha}\right)\right) \\
- C_2\left(B^K\left(0,\frac{R^e}{\alpha}\right)+B^K\left(2,\frac{R^e}{\alpha}\right)\right)\Big/(2a),
\end{aligned} \tag{56}
$$

$$
\begin{aligned}
\Phi_r^e = 2\Bigg((a^1+a^2+a^3+a^4+a^5)\left(C_1B^I\left(1,\frac{R^e}{\alpha}\right)+C_2B^K\left(1,\frac{R^e}{\alpha}\right)\right) \\
+\frac{\alpha(a^4+a^5)\left(2\alpha C_3+(R^e)^2\left(-C_1B^I\left(2,\frac{R^e}{\alpha}\right)+C_2B^K\left(2,\frac{R^e}{\alpha}\right)\right)\right)}{(R^e)^3}\Bigg)\Big/(\alpha)^2
\end{aligned} \tag{57}
$$

These equations yields a linear system of four equations in four unknowns C_i. The next section shows some examples of particular solutions of this set of equations. On the contrary to the general solution of the problem (Eq. (49)), it is noteworthy that the boundary conditions involve other parameters than the modulus A. This means that some boundary layer effects can be related to an internal length different from α.

6. Some examples of particular solutions

All the following solutions are obtained with a ratio $\frac{R^e}{R^i} = 10$ and an unique value of $\lambda + 2\mu$. Concerning the second gradient constitutive equation, we use the general isotropic elastic model proposed by Mindlin (1965). In the reference case, the modeling is performed with a particular case of Mindlin's model, in which the second gradient part only depends on one parameter. This one-parameter model has been used by Bésuelle et al. (2006), for regularization purpose in elastoplastic strain localization computations. Let us first recall the general isotropic relation between the double stress and the second gradient of the displacements. In the bi-dimensional case, Eq. (24) can be written following Mindlin (1965) as:

$$
\begin{bmatrix} \chi_{rrr} \\ \chi_{rr\theta} \\ \chi_{r\theta r} \\ \chi_{r\theta\theta} \\ \chi_{\theta rr} \\ \chi_{\theta r\theta} \\ \chi_{\theta\theta r} \\ \chi_{\theta\theta\theta} \end{bmatrix} = \begin{bmatrix} A & 0 & 0 & a^{23} & 0 & a^{12} & a^{12} & 0 \\ 0 & a^{145} & a^{145} & 0 & a^{25} & 0 & 0 & a^{12} \\ 0 & a^{145} & a^{145} & 0 & a^{25} & 0 & 0 & a^{12} \\ a^{23} & 0 & 0 & a^{34} & 0 & a^{25} & a^{25} & 0 \\ 0 & a^{25} & a^{25} & 0 & a^{34} & 0 & 0 & a^{23} \\ a^{12} & 0 & 0 & a^{25} & 0 & a^{145} & a^{145} & 0 \\ a^{12} & 0 & 0 & a^{25} & 0 & a^{145} & a^{145} & 0 \\ 0 & a^{12} & a^{12} & 0 & a^{23} & 0 & 0 & A \end{bmatrix} \begin{bmatrix} \varepsilon_{rrr} \\ \varepsilon_{rr\theta} \\ \varepsilon_{r\theta r} \\ \varepsilon_{r\theta\theta} \\ \varepsilon_{\theta rr} \\ \varepsilon_{\theta r\theta} \\ \varepsilon_{\theta\theta r} \\ \varepsilon_{\theta\theta\theta} \end{bmatrix}, \tag{58}
$$

where the constants $A, a^{23}, a^{12}, a^{145}, a^{25}$ and a^{34} depend on the a^i introduced in Eq. (24) according to the following formulae:

$$
\left\{\begin{aligned} A &= 2(a^1+a^2+a^3+a^4+a^5) \\ a^{23} &= a^2+2a^3 \\ a^{12} &= a^1+\tfrac{a^2}{2} \\ a^{145} &= \tfrac{a^1}{2}+a^4+\tfrac{a^5}{2} \\ a^{25} &= \tfrac{a^2}{2}+a^5 \\ a^{34} &= 2(a^3+a^4) \end{aligned}\right\}. \tag{59}
$$

In the one-parameter model used in the reference case, we chose the following values for the a^i parameters:

$$\left\{\begin{array}{l} a^1 = 0 \\ a^2 = D \\ a^3 = -D/2 \\ a^4 = D \\ a^5 = -D \end{array}\right\}. \tag{60}$$

As far as the non-classical boundary conditions are concerned, two cases are considered in the following: one with natural boundary conditions for the double forces (Case A) and the other one with prescribed values different from zero (Case B).

6.1. Natural boundary conditions for the double forces

In this section, the prescribed boundary conditions are:

- for the non-dimensional external forces

$$\frac{p_r^i}{\lambda + 2\mu} = 0.002, \quad \frac{p_r^e}{\lambda + 2\mu} = 0, \tag{61}$$

- for the non-dimensional external double forces

$$\frac{\Phi_r^i}{(\lambda + 2\mu)R^i} = 0, \quad \frac{\Phi_r^e}{(\lambda + 2\mu)R^i} = 0. \tag{62}$$

6.1.1. Reference case – Case A.1

In this reference case, the one parameter-model recalled above is used with $\frac{D}{(\lambda+2\mu)R^{i^2}} = 4$. The corresponding non-dimensional internal length scale is $\frac{\alpha}{R^i} = 2$. In order to compute the solutions, the expressions of the constants of integration are found using Eqs. (51), (53), (55), (57). For this reference case, the constants C_1 and C_2 are null and the effect of microstructure disappears in the solution: we find the response of a (conventional) classical elastic medium. Fig. 2 shows the radial displacement $\frac{u_r}{R^i}$ and the macrostresses $\frac{\sigma_{rr}}{\lambda+2\mu}, \frac{\sigma_{\theta\theta}}{\lambda+2\mu}$ as a function of the radial distance $\frac{r}{R^i}$. No double stresses are generated by the application of the boundary conditions and the obtained solution is independent of the internal length scale.

6.1.2. Solutions with $(a^4 + a^5)$ different from zero – Cases A.2/A.4

Considering the expression of the double stresses (Eq. (24)), we notice that the sum $(a^4 + a^5)$ appears in each non-zero component of the double stress tensor. However, with the one-parameter model, this value vanishes and the obtained results in Case 1 are not general. In order to investigate the role of $(a^4 + a^5)$, we solve the problem for different ratios a^5/a^4 and different values of the parameter D (Case A.2: $\frac{D}{(\lambda+2\mu)R^{i^2}} = 0.4$, Case A.3: $\frac{D}{(\lambda+2\mu)R^{i^2}} = 4$ and Case A.4: $\frac{D}{(\lambda+2\mu)R^{i^2}} = 400$). The following set of parameters are used: the values of a^1, a^2, a^3 and a^4 are defined by Eq. (60) as a function of D and the value a^5 is computed using the ratio a^5/a^4.

Fig. 3 shows the influence of $(a^4 + a^5)$ for the Case A.2 ($\frac{\alpha}{R^i} = 0.633$ for $a^5/a^4 = -1$). Actually, varying $(a^4 + a^5)$ modifies the parameter $\frac{\alpha}{R^i}$ as it is shown in Table 1. We recall that the solution for $a^5/a^4 = -1$ corresponds to the response of a (conventional) classical elastic medium and the effect of the microstructure is therefore highlighted by comparison to this reference case. For increasing values of the difference between a^4 and a^5, we observe that the radial macrostress $\frac{\sigma_{rr}}{\lambda+2\mu}$ at the inner radius is no more equal to $\frac{p_r^i}{\lambda+2\mu}$ and even tends to zero. This is due to second gradient effects: double stresses are generated in this case for $a^5/a^4 <> -1$. Indeed, the double stresses $\frac{\chi_{r\theta\theta}}{(\lambda+2\mu)R^i}$ and $\frac{\chi_{\theta r\theta}}{(\lambda+2\mu)R^i}$ become more and more negative at the inner radius as far as $(a^4 + a^5)$ is increasing. The double stress $\frac{\chi_{rrr}}{(\lambda+2\mu)R^i}$ remains equal to zero at the inner and outer radii (as a consequence of the natural boundary conditions for the double forces). This double stress is not equal to zero within the cylinder: it is negative near the inner radius and positive near the outer radius. The values of the negative minimum and the positive maximum of χ_{rrr} are increasing with the difference between a^4 and a^5.

Fig. 4 shows the influence of $(a^4 + a^5)$ for the Case A.3 ($\frac{\alpha}{R^i} = 2$ for $a^5/a^4 = -1$). The results are similar to the previous case, except for the double stress $\frac{\chi_{rrr}}{(\lambda+2\mu)R^i}$. Fig. 4b shows that this latter double stress component is negative all over the cylinder. The minimum is still increasing with the difference between a^4 and a^5. It is also noticeable that, applying a compressive classical force at the inner radius, the corresponding radial macrostress is a low traction stress for $a^5/a^4 = -0.5$. This means that the microstructure effects become predominant.

Fig. 5 shows the influence of $(a^4 + a^5)$ for Case A.4 ($\frac{\alpha}{R^i} = 20$ for $a^5/a^4 = -1$). This internal length is much larger than the thickness of the cylinder and the response should be the same for each value of a^5/a^4 different from -1 ($a^5/a^4 = -1$ being the classical elastic response as explained previously). We observe however a slight influence of this ratio on the double stress components. This is related to the fact that, changing a^4 and a^5, the internal length is also slightly increased. As a consequence the minimum of χ_{rrr} is decreasing with a^5/a^4. We have also to point out the fact that, even if a classical force is applied at the inner radius, no radial macrostress is generated and the external force is only balanced by the double stresses.

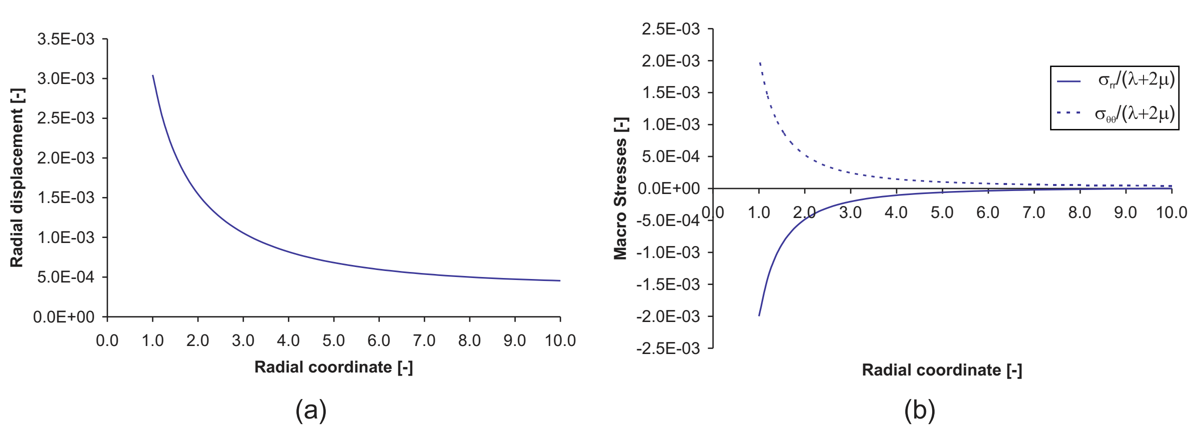

Fig. 2. Case A.1 – classical force: (a) radial displacement; and (b) radial and orthoradial macrostresses.

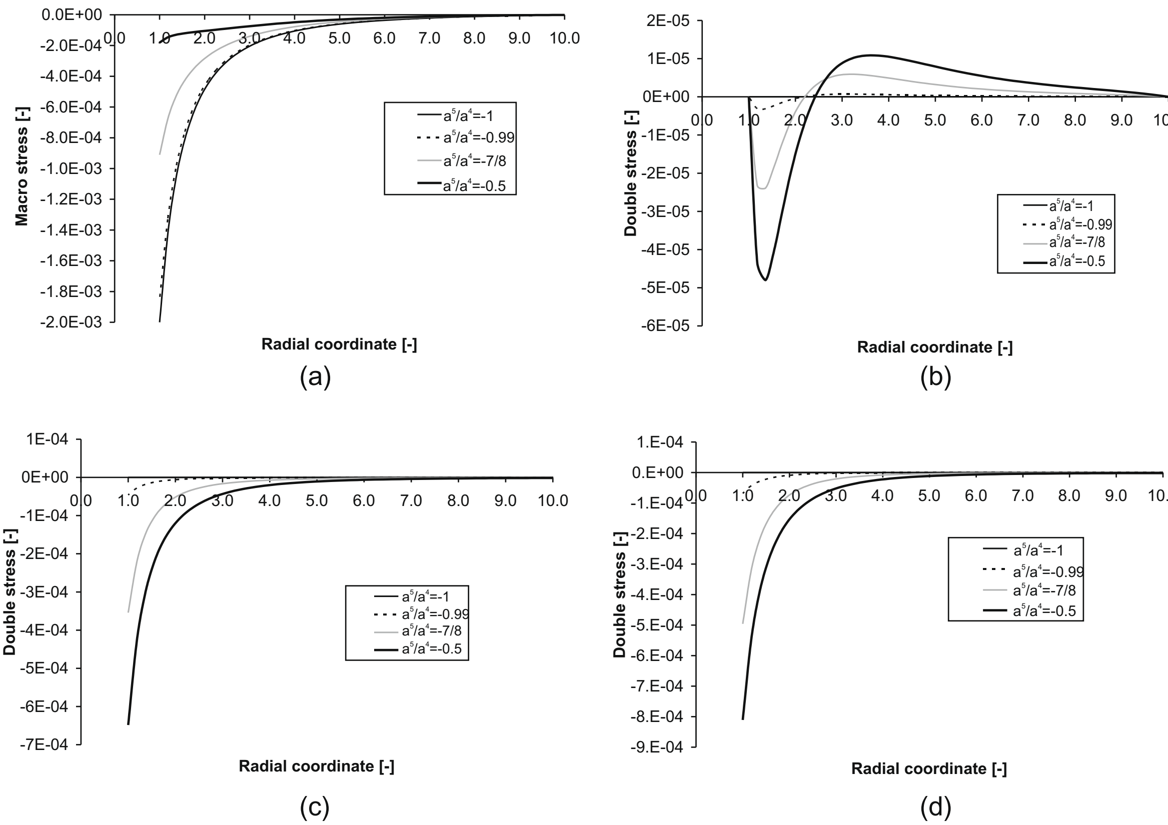

Fig. 3. Case A.2 – classical force: (a) macrostress $\frac{\sigma_{rr}}{\lambda+2\mu}$; (b) double stress $\frac{\chi_{rrr}}{(\lambda+2\mu)R^i}$; (c) double stress $\frac{\chi_{r\theta\theta}}{(\lambda+2\mu)R^i}$; and (d) double stress $\frac{\chi_{\theta r\theta}}{(\lambda+2\mu)R^i}$ $\left(\frac{D}{(\lambda+2\mu)R^{i2}} = 0.4\right)$.

Table 1
Non-dimensional internal length scale $\frac{\alpha}{R^i}$ for the different cases.

Case	$a^5/a^4 = -1$	$a^5/a^4 = -0.99$	$a^5/a^4 = -7/8$	$a^5/a^4 = -0.5$
Case A.2	0.633	0.639	0.707	0.894
Case A.3	2	2.020	2.236	2.828
Case A.4	20	20.20	22.36	28.28

6.2. Double forces at the boundary – Case B

In this section, the prescribed boundary conditions are:

- for the non-dimensional external forces

$$\frac{p_r^i}{\lambda+2\mu} = 0, \quad \frac{p_r^e}{\lambda+2\mu} = 0. \tag{63}$$

- for the non-dimensional external double forces

$$\frac{\Phi_r^i}{(\lambda+2\mu)R^i} = 0.004, \quad \frac{\Phi_r^e}{(\lambda+2\mu)R^i} = 0.004. \tag{64}$$

For this Case B, the one-parameter model is used and the values of the parameters a^i are defined by Eq. (60). Four different values of parameter $\frac{D}{(\lambda+2\mu)R^{i2}}$ are considered (0.04, 0.4, 4.0 and 400) to investigate the influence of the internal length scale. Using the Eqs. (51), (53), (55), (57) as in the previous cases, the expressions of the constants of integration are found. In this case, the four constants C_i are different from zero, meaning that microstructure effects are generated. It must be pointed out that no comparison with a classical elastic solution is possible as far as the double forces do not exist in the conventional mechanic.

Fig. 6 shows the influence of the non-dimensional internal length scale $\frac{\alpha}{R^i}$ (0.2, 0.632, 2.0, 20). On Fig. 6c, it can be seen that the double stress χ_{rrr} is well balanced by the boundary conditions at the inner and outer radii. Depending on the internal length, starting from the boundaries, this double stress decreases more or less quickly to zero. The conclusions are the same for the other double stress component $\chi_{\theta r\theta}$, which is exactly equal to $\chi_{rrr}/2$. Fig. 6b shows that macrostresses are also generated by double forces and are maximum at the boundaries. The area of influence of the boundaries depends on the internal length scale α. This is clearly a boundary layer effect. The intensity of the radial macrostress at the boundaries is decreasing with respect to the internal length.

7. Concluding remarks

This paper proposes the analytical solutions of the thick-walled cylinder problem for an isotropic elastic second gradient medium. The expressions of the general equation and the boundary conditions have been first established. They are general and, knowing the seven parameters involved in the model, it is possible to find the four constants of integration, depending on the four boundary conditions on the inner radius and outer radii. The constants of integration are obtained by solving a set of four equations in four unknowns (the constants of integration). Some particular solutions are given explicitly for the thick-walled cylinder, loaded by classical forces (Case A) or double forces (Case B). The analytical solutions of Case A show that the influence of the microstructure is controlled by the internal length scale parameter α. The classical macrostress σ_{rr} is no more in equilibrium with the classical forces at the boundary. Double stresses are indeed also generated by the classical boundary conditions and, as far as the microstructure ef-

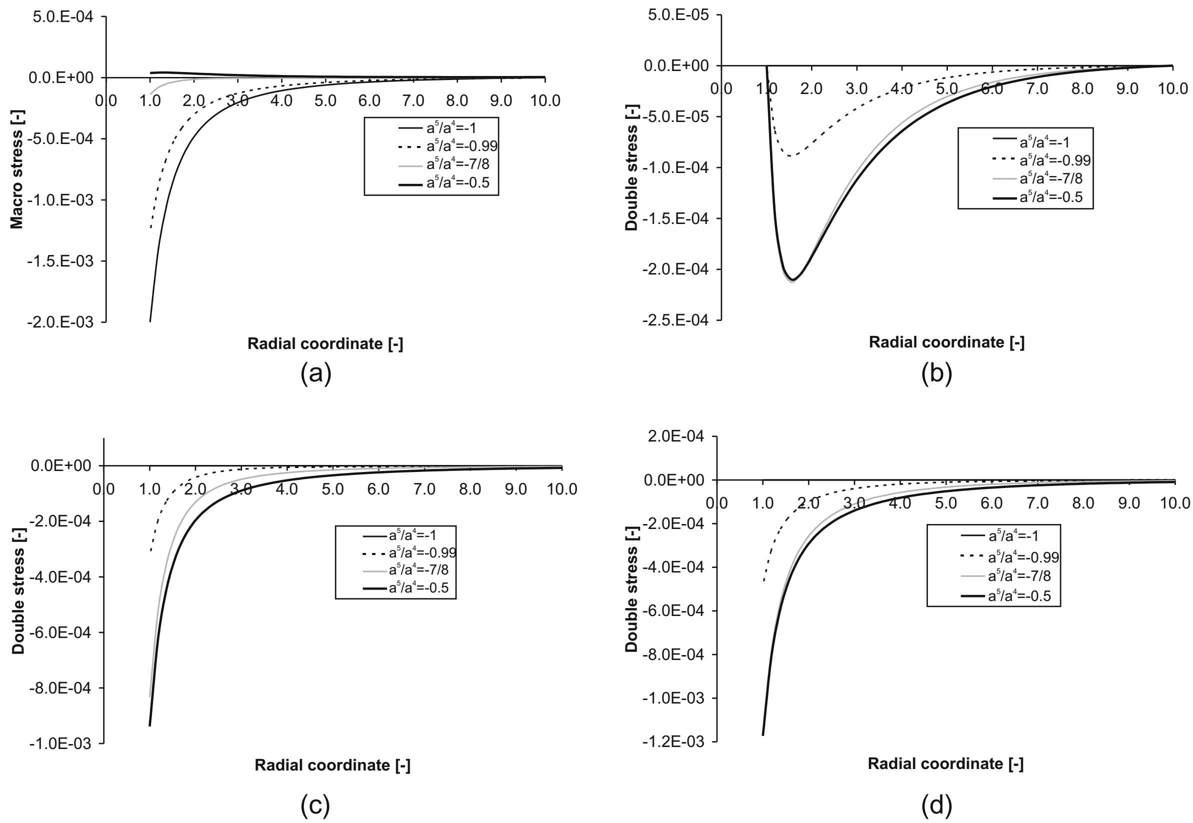

Fig. 4. Case A.3 – classical force: (a) macrostress $\frac{\sigma_{rr}}{\lambda+2\mu}$; (b) double stress $\frac{\chi_{rrr}}{(\lambda+2\mu)R^l}$; (c) double stress $\frac{\chi_{r\theta\theta}}{(\lambda+2\mu)R^l}$; and (d) double stress $\frac{\chi_{\theta r\theta}}{(\lambda+2\mu)R^l}$ $\left(\frac{D}{(\lambda+2\mu)R^{l2}}=4.0\right)$.

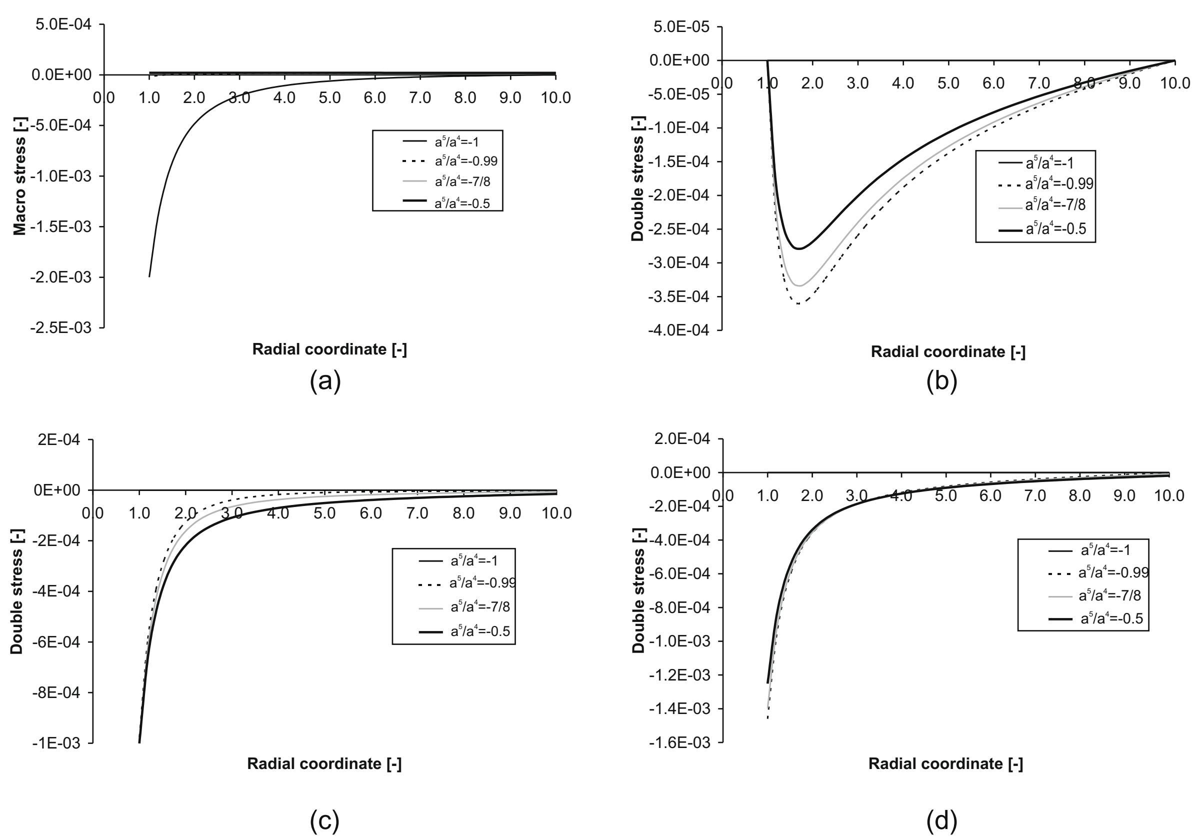

Fig. 5. Case A.4 – classical force: (a) macrostress $\frac{\sigma_{rr}}{\lambda+2\mu}$; (b) double stress $\frac{\chi_{rrr}}{(\lambda+2\mu)R^l}$; (c) double stress $\frac{\chi_{r\theta\theta}}{(\lambda+2\mu)R^l}$; and (d) double stress $\frac{\chi_{\theta r\theta}}{(\lambda+2\mu)R^l}$ $\left(\frac{D}{(\lambda+2\mu)R^{l2}}=400\right)$.

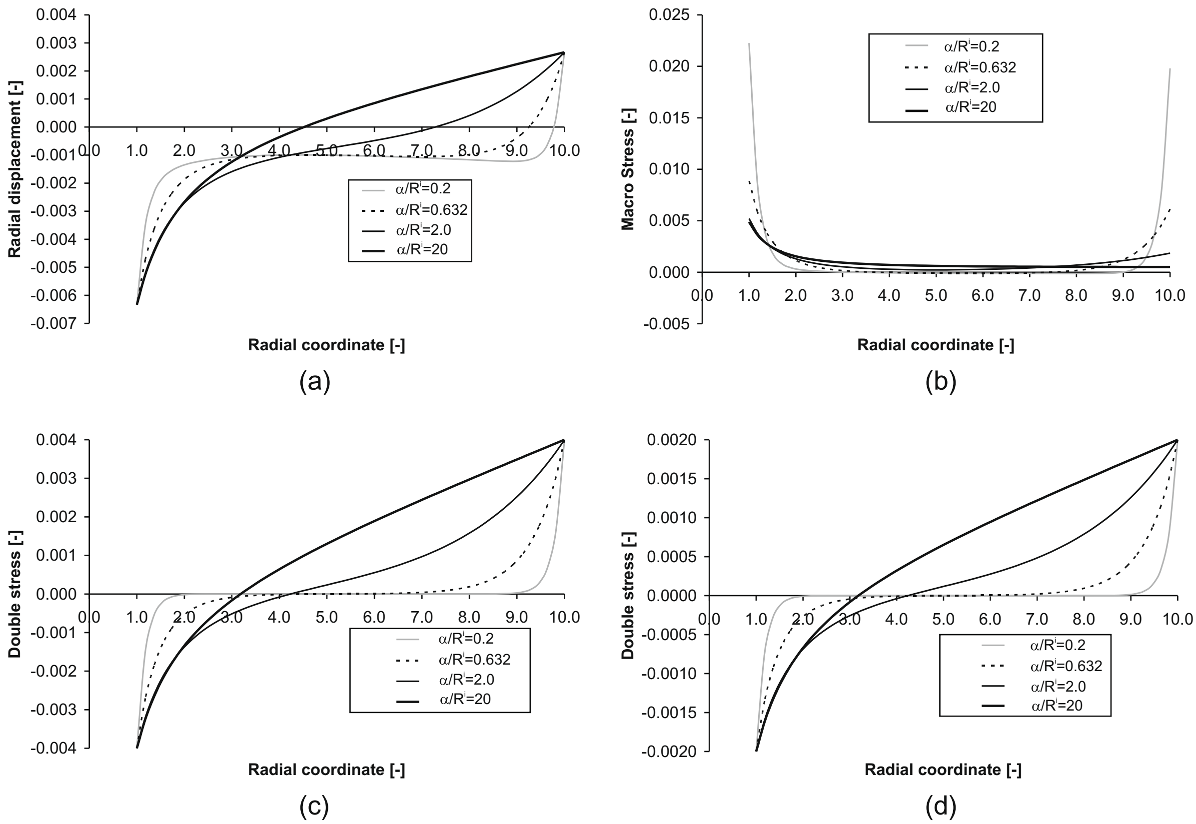

Fig. 6. Case B – double force: (a) radial displacement; (b) macrostress σ_{rr}; (c) double stress χ_{rrr}; and (d) double stress $\chi_{\theta r\theta}$.

fects become predominant (i.e. the internal length scale is much larger than the thickness of the cylinder), the macrostresses become negligible. This leads to solutions completely different from classical elastic ones. The analytical solutions of Case B show that the double forces generate of course double stresses but also macrostresses. The influence of these microstructure effects are again controlled by the internal length scale.

Following the same method, it is likely that general solutions for the general models with microstructure can be obtained, even for different loading conditions.

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Appendix A

Expressions of the strain tensor and the second gradient tensor.

Under the assumption (17), the derivatives of the displacement field are the following: $\partial_r \vec{u} = \partial_r u \vec{e}_r, \partial_\theta \vec{u} = u \vec{e}_\theta$ and $\partial_z \vec{u} = 0$. Using Eq. (1), the gradient of the displacement yields:

$$\nabla \vec{u} = \partial_r u \vec{e}_r \otimes \vec{e}_r + \frac{u}{r} \vec{e}_\theta \otimes \vec{e}_\theta. \tag{65}$$

Clearly $\nabla \vec{u}$ is symmetric and then the strain tensor ϵ is given by:

$$\epsilon = \nabla \vec{u}. \tag{66}$$

In the case of the second gradient model, the constraint $E = \nabla \vec{u}$ is considered and thus the second order tensor E has the following form in axi-symmetric conditions:

$$E = E_{rr} \vec{e}_r \otimes \vec{e}_r + E_{\theta\theta} \vec{e}_\theta \otimes \vec{e}_\theta. \tag{67}$$

Eq. (67) means consequently that $E_{r\theta} = E_{\theta r} = 0, E_{rr} = \partial_r u$ and $E_{\theta\theta} = \frac{u}{r}$. As clearly $\partial_z E = 0$ in axi-symmetric conditions, Eq. (2) yields:

$$\nabla E = (\partial_r E) \otimes \vec{e}_r + \frac{1}{r} (\partial_\theta E) \otimes \vec{e}_\theta. \tag{68}$$

Since vectors $\vec{e}_r$ and $\vec{e}_\theta$ are not depending on r, the partial derivative of E with respect to the radial coordinate r is:

$$\partial_r E = (\partial_r E_{rr}) \vec{e}_r \otimes \vec{e}_r + (\partial_r E_{\theta\theta}) \vec{e}_\theta \otimes \vec{e}_\theta. \tag{69}$$

On the other hand, since E_{rr} and $E_{\theta\theta}$ are not depending on θ, the partial derivative of E with respect to θ is:

$$\begin{aligned} \partial_\theta E &= E_{rr} (\vec{e}_\theta \otimes \vec{e}_r + \vec{e}_r \otimes \vec{e}_\theta) - E_{\theta\theta} (\vec{e}_r \otimes \vec{e}_\theta + \vec{e}_\theta \otimes \vec{e}_r) \\ &= (E_{rr} - E_{\theta\theta}) (\vec{e}_r \otimes \vec{e}_\theta + \vec{e}_\theta \otimes \vec{e}_r). \end{aligned} \tag{70}$$

We obtain then the expression of the gradient of the second order tensor E:

$$\begin{aligned} \nabla E = {} & (\partial_r E_{rr}) (\vec{e}_r \otimes \vec{e}_r) \otimes \vec{e}_r + (\partial_r E_{\theta\theta}) (\vec{e}_\theta \otimes \vec{e}_\theta) \otimes \vec{e}_r \\ & + \frac{1}{r} (E_{rr} - E_{\theta\theta}) [(\vec{e}_r \otimes \vec{e}_\theta) \otimes \vec{e}_\theta + (\vec{e}_\theta \otimes \vec{e}_r) \otimes \vec{e}_\theta]. \end{aligned} \tag{71}$$

Finally, using the definition of the ε, the previous equation can be recast as:

$$\begin{aligned} \nabla E = {} & \varepsilon_{rrr} (\vec{e}_r \otimes \vec{e}_r) \otimes \vec{e}_r + \varepsilon_{\theta\theta r} (\vec{e}_\theta \otimes \vec{e}_\theta) \otimes \vec{e}_r + \varepsilon_{r\theta\theta} (\vec{e}_r \otimes \vec{e}_\theta) \otimes \vec{e}_\theta \\ & + \varepsilon_{\theta r\theta} (\vec{e}_\theta \otimes \vec{e}_r) \otimes \vec{e}_\theta, \end{aligned} \tag{72}$$

with

$$\varepsilon_{rrr} = \partial_r E_{rr} = \partial_r^2 u,$$
$$\varepsilon_{\theta\theta r} = \partial_r E_{\theta\theta} = \partial_r\left(\frac{u}{r}\right),$$
$$\varepsilon_{r\theta\theta} = \varepsilon_{\theta r\theta} = \frac{1}{r}(E_{rr} - E_{\theta\theta}) = \frac{1}{r}\left(\partial_r u - \frac{u}{r}\right) = \partial_r\left(\frac{u}{r}\right), \tag{73}$$

all the other components are equal to 0. It can be checked that $\varepsilon_{\theta\theta r} = \varepsilon_{\theta r\theta}$.

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