



A damage model with evolving nonlocal interactions

Giang D. Nguyen*

School of Civil Engineering, The University of Sydney, Sydney, NSW 2006, Australia

ARTICLE INFO

Article history:

Received 2 August 2010

Received in revised form 24 January 2011

Available online 26 February 2011

Keywords:

Constitutive modelling

Nonlocal

Thermodynamics

Damage

Evolving nonlocal interaction

Evolving internal length

Material stability

Localisation analysis

ABSTRACT

We present a damage model for softening materials with evolving nonlocal interactions. The thermodynamic implications and the material stability issue are addressed. The proposed nonlocal averaging scheme provides the obtained constitutive models with an evolving nonlocal interaction which is activated only when damage occurs. In the analysis of structures made of quasi-brittle materials, this feature helps not only to overcome some issues with the incorrect initiation of damage but also to better control the evolving size of the active fracture process zone. This is an essential feature that is usually not considered in depth in many existing nonlocal approaches to the continuum modelling of quasi-brittle fracture. Numerical examples are given to demonstrate features of the proposed modelling approach.

© 2011 Elsevier Ltd. All rights reserved.

1. Introduction

Material stability is an issue one should always consider when developing constitutive models for softening materials. One of the effective techniques to deal with the issue of material stability is the nonlocal regularisation technique (Pijaudier-Cabot and Bazant, 1987). The key idea of this regularisation is to introduce into constitutive models a characteristic length representing the heterogeneity of the evolving material micro-structure. Considering the richness of the evolving micro-structure of quasi-brittle materials, the introduction of a single length parameter can be seen as an over simplified process aiming at reducing the complexity of the constitutive models and also the numerical implementation. This however has been proved to be sufficient for several practical applications.

Due to the heterogeneity of quasi-brittle material (e.g. concrete, rocks), diffuse micro-cracking over a certain volume is usually observed at early stages of the fracturing process (Haidar et al., 2005; Grassl and Jirásek, 2010). This volume, where micro-cracks develop, is called fracture process zone (FPZ) in the literature. It has been experimentally observed that the size of the active FPZ is evolving with the fracturing process (Jankowski and Styś, 1990; Otsuka and Date, 2000). This zone expands once micro-cracks start to grow, and then reduces to a zero-volume (macro) crack. During this fracturing process, many micro-cracks in the FPZ are deactivated,

while some continuously develop, propagate, and coalesce with other to form a final macro crack.

In the modelling of quasi-brittle fracture using nonlocal damage models, several outstanding issues in the development and application of nonlocal regularisation techniques have not been given adequate treatments. The source of nonlocality, as proved by Bazant (1991, 1994), are due to micro-crack interactions. Since the FPZ, where micro-cracking occurs, is evolving, the nonlocal interaction of the nonlocal continuum models cannot be fixed. However, the issue of evolving FPZ is usually not discussed at length in the literature of constitutive modelling, since most nonlocal damage models are formulated on the basis of fixed nonlocal interaction, e.g. fixed nonlocal weight scheme. Although these models can characterise the evolving (active) damage zone to some extent, they still suffer from the incorrect initiation and propagation of damage (Simone et al., 2004).

The motivation of this paper is to enhance existing nonlocal models with features and capabilities that allow capturing the evolving nature of the FPZ more faithfully. This will hopefully help overcome unwanted issues (e.g. incorrect initiation of damage) associated with models employing fixed nonlocal interactions. We also note here that although the issue of continuum–discontinuum transition has been tackled in some research (e.g., Simone et al., 2003; Comi et al., 2007; Jirasek and Marfia, 2008), it is not covered within the scope of this study. In the literature, the enhancement to fixed internal length nonlocal models may lie in employing an evolving internal length (Geers et al., 1998, 2000; Pijaudier-Cabot et al., 2004), or using a variable mixing between local and nonlocal quantities (Bui, 2010). Both types of

* Tel.: +61 2 9351 3721; fax: +61 2 9351 3343.

E-mail addresses: giang.nguyen@trinity.oxon.org, giang.nguyen@sydney.edu.au

enhancement result in an evolving nonlocal interaction (or evolving nonlocal weight scheme) for the nonlocal continuum model. Geers et al. (1998, 2000) explored the effects of an evolving internal length in a gradient enriched continuum on the regularisation and numerical results. From his study, the FPZ was numerically observed not to spread widely during final stages of the damaging process, when the appearance of macro cracks is expected. This is an enhancement over fixed interaction models. It is noted that attempts to use an internal length decreasing towards the critical value of damage was numerically observed not to effectively regularise the constitutive models (Geers et al., 2000). In other words, localisation to zero-volume FPZ still occurred using such gradient enriched models. Recently Bui (2010) used a mixed local-nonlocal formulation in the context of gradient enriched continuum and vary the ratio between local and nonlocal contribution to obtain the evolving nonlocal effects. His study, although proved to be effective in overcoming issues with the initiation of damage (Simone et al., 2003), did not elaborate much on the effects of nonlocal models with evolving nonlocal interactions on the development of damage during subsequent stages of the fracture process.

In some previous attempts (e.g. Geers et al., 1998, 2000), the effects of the parameter/function governing the evolving nonlocal interaction on the size of the FPZ and material stability were, in the author's view, not always explored at length. In these studies, numerical simulations were used to examine the localisation features of nonlocal/gradient models. In this study, we deliberately want to earn at first some understanding of the proposed nonlocal model through analytical work, before embarking on any numerical simulations. In the literature the localisation analysis based on the propagation of acceleration waves has been employed by several researchers (Pijaudier-Cabot and Benallal, 1993; Comi, 2001; Borino et al., 2003; Luzio and Bazant, 2005) to investigate the stability of a nonlocal continuum. Study by Pijaudier-Cabot et al. (2004) showed the usefulness of this kind of localisation analysis for a simple one-dimensional (1D) bar in exploring the localisation characteristics of damage models with an evolving internal length. In their study the internal length in a Gaussian-type nonlocal weight function was assumed to increase from zero to a certain maximum value, which results in a stronger effect on the initial evolution of the FPZ, compared to that from fixed internal length models. We will show in this study that such a simple localisation analysis in 1D can be a valuable tool to examine the localisation features of nonlocal model with evolving nonlocal interactions.

In the above discussed models, thermodynamic implications and the issue of consistency when comparing fixed and evolving nonlocal interaction models were not addressed at length. Models are usually not compared on the same basis, e.g. changing the internal length (e.g. Simone et al., 2004) should be viewed as changing the response of constitutive model. In this study, whenever possible, we deliberately compare models on the basis that they produce the same fracture energy at least in mode I fracture. This rule may also be considered arbitrary, but helps to put different models in the same scale for a fair comparison.

The outline of this paper is as follows. We start by presenting the thermo-mechanical formulation of a class of constitutive models with evolving nonlocal interaction. The mixed local-nonlocal averaging (Stromberg and Ristinmaa, 1996) is used in this case. However the parameter controlling the mixing is varying with the damage level to have the desired nonlocal effects (Bui, 2010). The spatial nonlocal weight scheme therefore evolves with the damage level. This is different from several existing studies based on this mixed local-nonlocal formulation (e.g. Stromberg and Ristinmaa, 1996; Luzio and Bazant, 2005; Luzio, 2007), where this mixing parameter is usually kept constant. A simple nonlocal damage model is then introduced, with options for different evolving nonlocal interactions, and used throughout this study. We then

briefly address the non-standard stress return algorithm associated with this kind of nonlocal models. The stability analysis in one-dimensional setting will then be presented, helping to assess the localisation characteristics, followed by numerical examples in 1D and 2D to demonstrate features of the new approach.

2. A thermo-mechanical formulation

2.1. The basis

We adopt here the thermo-mechanical approach to nonlocal constitutive modelling pioneered and developed by several Italian researchers (e.g. Polizzotto and Borino, 1998; Polizzotto et al., 1998; Borino et al., 1999, 2003; Comi and Perego, 2001; Benvenuti et al., 2002). A formulation for damage constitutive modelling derived from this approach and employing explicitly defined energy and dissipation potentials (Nguyen, 2008) is used and modified in this study. The details of the formulation are reproduced here only for the sake of completeness, serving the introduction of mixed local-nonlocal energy terms. One should bear in mind that this thermo-mechanical basis is casted in a general format without considering the insights of micro-cracking processes leading to damage. This micromechanical analysis is useful to provide additional information on the underlying physics of the material failure.

The Helmholtz specific free energy for isothermal cases can be written as (Benvenuti et al., 2002; Borino et al., 2003):

$$\dot{f} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - d + P \quad (1)$$

with the insulation condition applied to the nonlocality residual P (Polizzotto and Borino, 1998):

$$\int_V P dV = 0 \quad (2)$$

In the above expressions, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are the Cauchy stress and strain tensor, respectively; d is the dissipation potential.

The combination of local and nonlocal terms is used to define the nonlocal counterpart $\tilde{\alpha}$ of the local damage variable α :

$$\dot{\tilde{\alpha}}(\mathbf{x}) = [1 - m(\tilde{\alpha}(\mathbf{x}))]\dot{\alpha}(\mathbf{x}) + m(\tilde{\alpha}(\mathbf{x}))L(\dot{\alpha}(\mathbf{x})) \quad (3)$$

where m is a function controlling the local-nonlocal mixing, and L the nonlocal operator defined as:

$$L(\dot{\alpha}(\mathbf{x})) = \int_V w(\mathbf{x}, \mathbf{y}) \dot{\alpha}(\mathbf{y}) dV(\mathbf{y}) = \int_V \frac{g(\mathbf{x}, \mathbf{y})}{G(\mathbf{x})} \dot{\alpha}(\mathbf{y}) dV(\mathbf{y}) \quad (4)$$

The weight scheme in (4) is of the form:

$$w(\mathbf{x}, \mathbf{y}) = \frac{g(\mathbf{x}, \mathbf{y})}{G(\mathbf{x})}; G(\mathbf{x}) = \int_V g(\mathbf{x}, \mathbf{y}) dV(\mathbf{y}) \quad (5)$$

in which $\mathbf{x}$ and $\mathbf{y}$ are coordinate vectors; $g(\mathbf{x}, \mathbf{y}) \geq 0$ is a certain weight function.

The form of local-nonlocal mixing in (3) has been used by several researchers, with m being a constant (Stromberg and Ristinmaa, 1996; Luzio and Bazant, 2005) or a function of the damage state (Bui, 2010). It can be seen that the nonlocal variable in (3) has an implicit form. The reason to have m as a function of the nonlocal, instead of local variable, is purely numerical, helping to reduce the effort in the implementation and localisation analysis. It will be discussed in Sections 2.3 and 3.

Remark 2.1

1. The nonlocal operator L defined in (4) cannot reproduce a uniform field in bounded media due to boundary effects (Comi and Perego, 2001). However it is in fact not directly used in

our nonlocal model, which employs the adjoint operator L^* (see (11)) for the spatial averaging of the energy thermo-mechanically associated with the nonlocal damage variable. This adjoint operator L^* allows the reproduction of a nonlocal uniform field from a local uniform field.

- The operator (4) in this study is performed over the entire domain V , using an unbounded support weight function $g(\mathbf{x}, \mathbf{y})$ with a non-zero internal length (see Eq. (28)). In general (4) cannot reproduce local behaviour for the special case of zero internal length. This deficiency is avoided in this study by the use of non-zero internal length and the mixing function m in (3), which can admit values between zero (for local-like behaviour) and 1 (for purely nonlocal behaviour). For bounded support (e.g. using bell-shaped weight function (Borino et al., 2003)), it is easy to adopt other nonlocal operators (e.g. that given in Remark 2.1 as an example) in the present thermo-mechanical formulation that allow the retrieval of local field for zero internal length. In any cases, it is good to have the forms of nonlocal operators and their adjoints supported by a micromechanical analysis giving details on the underlying physics of nonlocality. Such a detailed micromechanical analysis is out of the scope of this study.

The Helmholtz free energy f can be assumed to be a function of the total strain $\boldsymbol{\varepsilon}$, and nonlocal internal variable $\tilde{\alpha}$, resulting in:

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \frac{\partial f}{\partial \tilde{\alpha}} \dot{\tilde{\alpha}} \quad (6)$$

Using the state law $\boldsymbol{\sigma} = \partial f / \partial \boldsymbol{\varepsilon}$ obtained by comparing (1) and (6), we can write the dissipation as:

$$d = \dot{\tilde{\chi}} \tilde{\alpha} + P \quad (7)$$

where $\tilde{\chi} = -\partial f / \partial \tilde{\alpha}$ is termed generalized stress (Houlsby and Puzrin, 2000) associated with the nonlocal internal variable $\tilde{\alpha}$. For rate-independent materials, the dissipation is of linear form (Borino et al., 1999; Houlsby and Puzrin, 2000):

$$d = \chi \dot{\alpha} \quad (8)$$

where $\chi = \partial d / \partial \dot{\alpha}$ is the dissipative generalized stresses (Houlsby and Puzrin, 2000). Using (2), (3), (7) and (8), one gets:

$$\int_V \{ [\chi(\mathbf{x}) - \tilde{\chi}(\mathbf{x})(1 - m(\tilde{\alpha}(\mathbf{x})))] \dot{\alpha}(\mathbf{x}) - \tilde{\chi}(\mathbf{x}) m(\tilde{\alpha}(\mathbf{x})) L(\dot{\alpha}(\mathbf{x})) \} dV(\mathbf{x}) = 0 \quad (9)$$

The 2nd term in the above expression can be written as:

$$\begin{aligned} \int_V \tilde{\chi} m L(\dot{\alpha}) dV &= \int_V \dot{\alpha}(\mathbf{x}) \frac{1}{G(\mathbf{x})} \int_V \tilde{\chi}(\mathbf{y}) m(\mathbf{y}) g(\mathbf{x}, \mathbf{y}) dV(\mathbf{y}) dV(\mathbf{x}) \\ &= \int_V \dot{\alpha}(\mathbf{x}) L^* (\tilde{\chi}(\mathbf{x}) m(\mathbf{x})) dV(\mathbf{x}) \end{aligned} \quad (10)$$

where the adjoint operator L^* is defined by:

$$L^* (\dot{\alpha}(\mathbf{x})) = \int_V w^*(\mathbf{x}, \mathbf{y}) \dot{\alpha}(\mathbf{y}) dV(\mathbf{y}) = \frac{1}{G(\mathbf{x})} \int_V g(\mathbf{x}, \mathbf{y}) \dot{\alpha}(\mathbf{y}) dV(\mathbf{y}) \quad (11)$$

Substituting (10) in (9) results in the following nonlocal form of the orthogonality condition:

$$\chi(\mathbf{x}) = \tilde{\chi}(\mathbf{x})(1 - m(\tilde{\alpha}(\mathbf{x}))) + L^* (\tilde{\chi}(\mathbf{x}) m(\tilde{\alpha}(\mathbf{x}))) \quad (12)$$

It is seen in (12) that the proposed model in this study is based on nonlocal damage energy. The form of the nonlocal operator with mixing function m is however similar to that proposed in Bui (2010), in which nonlocal strain is used. In both cases (this approach and Bui's), m appears inside the nonlocal integrals (see Eq. (12) above, and Eq. (22) in Bui (2010)). We note that this approach is motivated from thermodynamics, while Bui (2010) used

micromechanical assumptions, with micromechanical support from the study by Pijaudier-Cabot et al. (2004), to arrive at his mixed local-nonlocal averaging applied to strain. Besides, we note that in the context of the above thermo-mechanical formulation, one can also use L^* in the definition (3) of nonlocal damage variable, and then mathematically ends up having L in (12) (e.g. Benvenuti et al. (2002)). This issue has also been mentioned in Comi and Perego (2001). Both ways (starting with either L or L^* in (3)) are thermo-mechanically consistent and it is addressed here that they both have their own (numerical and physical) advantages and disadvantages, as explained in Appendix A. The motivation to use L in (3) in this study is purely for numerical convenience, to avoid any possible interference of the physical boundary in the stability features.

Remark 2.2. Due to boundary effects the use of L^* will result in a non-symmetric stiffness matrix (Comi and Perego, 2001; Borino et al., 2003). The adoption of the nonlocal weight that preserves the symmetry of the stiffness matrix, proposed by Borino et al. (2003), can be readily fitted in the above thermo-mechanical formulation. In such a case, the weight in (5) takes the following form:

$$w(\|\mathbf{y} - \mathbf{x}\|) = \left(1 - \frac{G(\mathbf{x})}{G_\infty}\right) \delta(\mathbf{x}, \mathbf{y}) + \frac{g(\|\mathbf{y} - \mathbf{x}\|)}{G_\infty}$$

where G_∞ is the value of $G(\mathbf{x})$ for infinite body (e.g. without boundary effects), and $\delta(\mathbf{x}, \mathbf{y})$ is the Dirac delta function. This results in the dissipative generalised stress χ of the form:

$$\chi(\mathbf{x}) = \tilde{\chi}(\mathbf{x}) \left[1 - \frac{G(\mathbf{x})}{G_\infty} m(\tilde{\alpha}(\mathbf{x}))\right] + \frac{1}{G_\infty} \int_{V_d} g(\mathbf{x}, \mathbf{y}) \tilde{\chi}(\mathbf{y}) m(\tilde{\alpha}(\mathbf{y})) dV(\mathbf{y})$$

This attempt, as experienced in this study, however led to strong effects of the weight scheme on the localisation characteristics of the mixed local-nonlocal model for material points near the boundary (see the above equation). These effects are due to the appearance of the mixing function m in the averaging of the energy quantity $\tilde{\chi}$ (see also Appendix A). This is why the “traditional” approach (Pijaudier-Cabot and Bazant, 1987) in normalizing the nonlocal weight function was adopted in this study.

2.2. A simple isotropic damage model with evolving nonlocal interaction

We introduce here a simple nonlocal damage model based on the above thermodynamic formulation. This model will be used for the analytical and numerical study in subsequent sections of this paper. For isothermal condition, the following Helmholtz free energy function is assumed:

$$f = \frac{1}{2} (1 - \tilde{\alpha}) \boldsymbol{\varepsilon} : \mathbf{a} : \boldsymbol{\varepsilon} \quad (13)$$

in which $\tilde{\alpha}$ is a nonlocal internal variable characterizing the damage processes; a_{ijkl} is the elasticity stiffness tensor expressed in terms of elasticity modulus E and Poisson's ratio ν

$$a_{ijkl} = \frac{E}{2(1+\nu)} \left[\frac{2\nu}{1-2\nu} \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right]$$

The dissipation potential is assumed in the following form:

$$d = F(\tilde{\alpha}) \dot{\tilde{\alpha}} \quad (14)$$

where $F(\tilde{\alpha})$ is a positive and increasing function associated with the damage process; this function in fact controls the rate of damage dissipation.

Following standard procedures established in Houlsby and Puzrin (2000), the stress, generalized stress $\tilde{\chi}$ (the energy associated with the nonlocal damage variable), and dissipative

generalized stress χ are derived from the energy and dissipation potentials as (assuming damage loading state):

$$\sigma = \frac{\partial f}{\partial \epsilon} = (1 - \tilde{\alpha}) \mathbf{a} : \epsilon \quad (15)$$

$$\bar{\chi} = -\frac{\partial f}{\partial \tilde{\alpha}} = \frac{1}{2} \epsilon : \mathbf{a} : \epsilon \quad (16)$$

$$\chi = \frac{\partial d}{\partial \tilde{\alpha}} = F(\tilde{\alpha}) \quad (17)$$

In general χ in Eq. (17) should be written as the sub-differential of the dissipation potential d , with the help of convex analysis. The readers can also refer to [Houlsby and Puzrin \(2006\)](#) for further details on how to apply convex analysis to the thermo-mechanical formulation.

The degenerate Legendre transformation of the dissipation function directly results in a damage loading function of the form:

$$y = \chi - F(\tilde{\alpha}) \leq 0; \quad \text{with } \dot{\tilde{\alpha}} \geq 0 \quad (18)$$

Imposing the nonlocal form (12) of the orthogonality condition results in the following nonlocal damage loading function:

$$y = \bar{\chi}(\mathbf{x})(1 - m(\tilde{\alpha}(\mathbf{x}))) + \frac{1}{G(\mathbf{x})} \int_V g(\mathbf{x}, \mathbf{y}) \bar{\chi}(\mathbf{y}) m(\tilde{\alpha}(\mathbf{y})) dV(\mathbf{y}) - F(\tilde{\alpha}(\mathbf{x})) \leq 0 \quad (19)$$

In the above expression, $F(\tilde{\alpha})$ takes the following form ([Nguyen, 2005](#)):

$$F(\tilde{\alpha}) = \frac{f_t^2}{2E} \left[\frac{E + E_p(1 - \tilde{\alpha})^n}{E(1 - \tilde{\alpha}) + E_p(1 - \tilde{\alpha})^n} \right]^2 \quad (20)$$

in which f_t is the uniaxial tensile strength; E_p and n are two parameters governing the damage evolution.

In the above model, we have not yet specify function $m(\tilde{\alpha})$ mixing the local and nonlocal terms. Different types of $m(\tilde{\alpha})$ will be later introduced and examined in Section 4. The specification of $m(\tilde{\alpha})$ is governed by the stability of the nonlocal model, and also the physics of the problem. It might be straightforward to use the stability condition based on a simple localisation analysis (Section 3) to examine any given choices of $m(\tilde{\alpha})$. A physically meaningful function $m(\tilde{\alpha})$ however can only be obtained from micromechanics. Such an elaborate analysis however cannot be covered within the scope of this paper. Instead, as already mentioned in the introduction, we enforce a constraint on the calibration of model parameters, so that all variants of a nonlocal model employing different types of function $m(\tilde{\alpha})$ produce the same mode I fracture energy in 1D setting (see [Nguyen and Houlsby \(2007\)](#) for details). This constraint will be used in all numerical examples (involving mode I fracture) in Section 4.

2.3. Implementation

We rewrite the equations describing the model:

$$\sigma = (1 - \tilde{\alpha}) \mathbf{a} : \epsilon \quad (21)$$

$$y = \frac{1}{2} [1 - m(\tilde{\alpha}(\mathbf{x}))] \epsilon(\mathbf{x}) : \mathbf{a} : \epsilon(\mathbf{x}) + \frac{1}{2G(\mathbf{x})} \int_V g(\mathbf{x}, \mathbf{y}) m(\tilde{\alpha}(\mathbf{y})) \epsilon(\mathbf{y}) : \mathbf{a} : \epsilon(\mathbf{y}) dV(\mathbf{y}) - F(\tilde{\alpha}(\mathbf{x})) \leq 0 \quad (22)$$

Due to the presence of function m controlling the local-nonlocal mixing, the implementation of this kind of model deviates from traditional ways of implementation. In other words, the integration of incremental constitutive equations cannot be carried out point-wise, as usually is the case with local models, or some specific

forms of nonlocal models (e.g. [Grassl and Jirásek, 2006](#)). For this model, the stress update process becomes non-standard, and requires solving a system of differential equations for all integration points undergoing damage. We briefly describe the stress return algorithm here, with some references to further details in the literature.

The spatial discretisation of the damage loading function (22), using integration points of the finite element discretisation, leads to:

$$y_i = \frac{1}{2} \epsilon_i : \mathbf{a} : \epsilon_i [1 - m(\tilde{\alpha}_i)] + \frac{1}{2} \sum_j w_{ij} m(\tilde{\alpha}_j) \epsilon_j : \mathbf{a} : \epsilon_j - F(\tilde{\alpha}_i) = 0 \quad (23)$$

where the nonlocal weight w_{ij} governing the spatial interaction between integration points i and j is defined as:

$$w_{ij} = \frac{\omega_j^e g(\|\mathbf{y}_j^e - \mathbf{x}_i\|) \det \mathbf{J}_j^e}{\sum_{e=1}^n \sum_{j=1}^{m_e} \omega_j^e g(\|\mathbf{y}_j^e - \mathbf{x}_i\|) \det \mathbf{J}_j^e} \quad (24)$$

In the above approximation j is the j 'th Gauss point of element e ; m_e is the number of Gauss points of this element inside the interaction volume; ω_j^e and $\mathbf{J}_j^e$ are, respectively, the weight and Jacobian matrix at Gauss point j of element e .

The first order Taylor expansion of the discretised form of the damage function (23), at a trial point B (Fig. 1), results in (for simplicity, we use the notation $\partial X_i / \partial x$ for $(\partial X / \partial x)|_i$):

$$y_i|_C = y_i|_B + \epsilon_i : \mathbf{a} : d\epsilon_i (1 - m_i) - \frac{1}{2} \epsilon_i : \mathbf{a} : \frac{\partial m_i}{\partial \tilde{\alpha}} d\tilde{\alpha}_i + \sum_j w_{ij} m_i \epsilon_j : \mathbf{a} : d\epsilon_j + \frac{1}{2} \sum_j w_{ij} \epsilon_j : \mathbf{a} : \frac{\partial m_j}{\partial \tilde{\alpha}} d\tilde{\alpha}_j - \frac{\partial F_i}{\partial \tilde{\alpha}} d\tilde{\alpha}_i \quad (25)$$

Since there is no strain increment from B to C, we have:

$$y_i|_C = y_i|_B - \frac{1}{2} \epsilon_i : \mathbf{a} : \epsilon_i \frac{\partial m_i}{\partial \tilde{\alpha}} d\tilde{\alpha}_i + \frac{1}{2} \sum_j w_{ij} \epsilon_j : \mathbf{a} : \epsilon_j \frac{\partial m_j}{\partial \tilde{\alpha}} d\tilde{\alpha}_j - \frac{\partial F_i}{\partial \tilde{\alpha}} d\tilde{\alpha}_i \quad (26)$$

Enforcing the yield condition $y_i|_C = 0$ results in:

$$\left(\frac{1}{2} \epsilon_i : \mathbf{a} : \epsilon_i \frac{\partial m_i}{\partial \tilde{\alpha}} + \frac{\partial F_i}{\partial \tilde{\alpha}} \right) d\tilde{\alpha}_i - \frac{1}{2} \sum_j w_{ij} \epsilon_j : \mathbf{a} : \epsilon_j \frac{\partial m_j}{\partial \tilde{\alpha}} d\tilde{\alpha}_j = y_i|_B \quad (27)$$

This is an equation coupling the behaviour of all integration points in the FPZ. Solving this linear algebraic equation gives the nonlocal damage increments for all integration points undergoing cracking. The stress increments can then be computed. Iterative algorithms for the solution of this equation in the context of finite

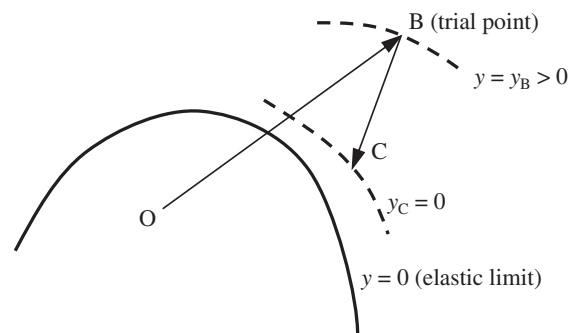


Fig. 1. Schematic representation of the stress update algorithm.

element analysis can be found in Stromberg and Ristinmaa (1996), & Benvenuti and Tralli (2003). In this study, a direct solver taking advantage of the sparsity of the system (27) will be employed.

In the implementation, the secant elastic-damage stiffness is at first used to reach the trial states of all integration points. Based on these trial states and the damage loading function (22), a set of integration points potentially undergoing damage is determined. The system (27) is then formed only for integration points having $y_B > 0$. For a given strain increment vector, an iterative process can be performed to enforce the satisfaction of the yield criterion (27) (within a given tolerance). In such a case, the above process is repeated, but with point B replaced with C, noting that there are no elastic stress increments in the subsequent iterations. During this iterative process, the set of integration points undergoing damage are kept unchanged. Combination of this with sub-incrementation to improve the performance of the algorithm can also be applied. It can be expected in nonlocal analysis that certain integration points can experience negative damage increments at the end of the iterative process, due to elastic unloading. This indicates that the load increment might be too big and that those integration points might not be undergoing damaging. This situation is taken care of in the analysis by reverting to the last converged position and reducing the load increments.

We can see that there is no local damage variable in Eq. (27), as a consequence of making m a function of the nonlocal damage variable. In other words, the stress return algorithm involves nonlocal damage only. If it is not the case, e.g. m is expressed in term of local damage, both local and nonlocal damage variables will appear in (27), which makes the algorithm more complicated. In addition, regarding the presented stress update process, there is no need to compute local damage increments. We however should keep in mind that the local damage zone is smaller than its nonlocal counterpart, due to the averaging process. To be precise, although nonlocal damage increment is present at a material point, it is not always true, especially for material points near the boundary of the nonlocal damage zone, that material point also experiences local damage increment. The local damage increments are in fact needed should one wish to compute the dissipation from integration points in the active FPZ. In such a case, the spatially discretised form of (3) can be used and the dissipated energy can then be obtained directly from (14). This is one of the advantages of having a thermodynamically consistent formulation.

3. Localisation analysis

A localisation analysis based on the propagation of acceleration waves (Pijaudier-Cabot and Benallal, 1993; Comi, 2001; Borino et al., 2003; Luzio and Bazant, 2005) is presented in this section. The 1D version of the model presented in Section 2.2 is examined, and Gaussian-type weight function (with internal length l) is assumed:

$$g(x, y) = g(|x - y|) = g(\eta) = e^{-\pi\eta^2/l^2} \quad (28)$$

We consider the bifurcation from a homogeneous state of strain and damage, and assume that the boundary effects are negligible. From the momentum equation (ρ is the density):

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2} \quad (29)$$

and stress–strain relationship in rate form:

$$\dot{\sigma} = (1 - \tilde{\alpha})E\dot{\varepsilon} - E\tilde{\alpha}\dot{\alpha} \quad (30)$$

one gets:

$$(1 - \tilde{\alpha})E \frac{\partial \dot{\varepsilon}}{\partial x} - E\tilde{\alpha} \frac{\partial \dot{\alpha}}{\partial x} - \rho \frac{\partial^2 u}{\partial t^2} = 0 \quad (31)$$

For a linear comparison solid (with a homogeneous initial condition), damage loading condition is assumed here. The consistency condition reads:

$$\dot{y} = (1 - m)E\dot{\varepsilon}\dot{\varepsilon} - \frac{1}{2}E\dot{\varepsilon}^2 \frac{\partial m}{\partial \tilde{\alpha}} \dot{\tilde{\alpha}} + \frac{mE\dot{\varepsilon}}{G} \int_{-\infty}^{+\infty} g(\xi - x)\dot{\varepsilon}(\xi)d\xi + \frac{E\dot{\varepsilon}^2}{2G} \frac{\partial m}{\partial \tilde{\alpha}} \int_{-\infty}^{+\infty} g(\xi - x)\dot{\tilde{\alpha}}(\xi)d\xi - \frac{\partial F}{\partial \tilde{\alpha}} \dot{\tilde{\alpha}} = 0 \quad (32)$$

In the above expression, the weight associated with the material point x (see Eq. (5)) is constant for a homogeneous infinite medium (e.g. no boundary effects): $G(x) = G = \text{const}$; besides, only the rates are non-homogeneous. We note here that the use of an implicit definition (Eq. (3)) of nonlocal damage leads to the appearance of only nonlocal damage rate in (32). Otherwise, for an explicit definition of nonlocal damage, (32) will involve both local and nonlocal rates of damage, which will then result in the appearance of $A(k)^2$ (see the definition in Eq. (35)) in (38). This will make the analysis more painful without gaining further insights into the localisation characteristics of the model.

We assume a harmonic wave solution of the following form (with k being the wave number, ω the angular frequency, $\dot{\alpha}_0$ and $\dot{u}_0$ are corresponding amplitudes of the velocity and damage of the propagating wave):

$$\dot{\tilde{\alpha}}(x, t) = \dot{\alpha}_0 e^{i(kx - \omega t)} \quad (33)$$

$$\dot{u}(x, t) = \dot{u}_0 e^{i(kx - \omega t)} \quad (34)$$

Substituting (33), (34) and their required time and spatial derivatives into (31) and (32), and using the Fourier transform of the weight function $g(\eta)$:

$$A(k) = \int_{-\infty}^{+\infty} g(\eta) e^{ik\eta} d\eta = l e^{-\frac{k^2 l^2}{4\pi}} \quad (35)$$

after some manipulations, we arrive at the following system of equations:

$$\begin{bmatrix} [(1 - m)E\dot{\varepsilon} + \frac{mE\dot{\varepsilon}}{G}A(k)]ik & [\frac{E\dot{\varepsilon}^2}{2G} \frac{\partial m}{\partial \tilde{\alpha}} A(k) - \frac{1}{2}E\dot{\varepsilon}^2 \frac{\partial m}{\partial \tilde{\alpha}} - \frac{\partial F}{\partial \tilde{\alpha}}] \\ \rho\omega^2 - (1 - \tilde{\alpha})Ek^2 & -E\dot{\varepsilon}ik \end{bmatrix} \begin{pmatrix} \dot{u}_0 \\ \dot{\alpha}_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (36)$$

For a nontrivial solution, the determinant of the coefficient matrix in (36) must vanish, resulting in the following angular frequency:

$$\omega = kc_e \sqrt{(1 - \tilde{\alpha}) - \frac{[(1 - m) + \frac{m}{G}A(k)]E\dot{\varepsilon}^2}{\frac{1}{2}E\dot{\varepsilon}^2 \frac{\partial m}{\partial \tilde{\alpha}} + \frac{\partial F}{\partial \tilde{\alpha}} - \frac{E\dot{\varepsilon}^2}{2G} \frac{\partial m}{\partial \tilde{\alpha}} A(k)}}, \quad \text{where } c_e = \sqrt{\frac{E}{\rho}} \quad (37)$$

from which we obtain the phase velocity:

$$c_p = \frac{\omega}{k} = c_e \sqrt{(1 - \tilde{\alpha}) - \frac{[(1 - m) + \frac{m}{G}A(k)]E\dot{\varepsilon}^2}{\frac{1}{2}E\dot{\varepsilon}^2 \frac{\partial m}{\partial \tilde{\alpha}} + \frac{\partial F}{\partial \tilde{\alpha}} - \frac{E\dot{\varepsilon}^2}{2G} \frac{\partial m}{\partial \tilde{\alpha}} A(k)}} \quad (38)$$

Eq. (38) shows that the acceleration wave is dispersive, with phase velocity being a function of the wave number k . Therefore a general stress wave will compose of different harmonic components, each of which travels at its own velocity. The shape of this stress wave therefore evolves during its propagation along the bar.

Since the solution bifurcates from a homogeneous state with uniform strain and damage distributions, local and nonlocal damage criteria coincide. In a 1D context, we have:

$$y = \frac{1}{2}E\dot{\varepsilon}^2 - F(\tilde{\alpha}) = 0 \quad \text{or} \quad E\dot{\varepsilon}^2 = 2F(\tilde{\alpha}) \quad (39)$$

Using (35), (38) and (39), we can obtain wave numbers that always guarantee real phase velocities:

$$k \geq \frac{2}{l} \sqrt{-\pi \ln \left[\frac{(1-\tilde{\alpha}) \frac{\partial F}{\partial \tilde{\alpha}} + F(\tilde{\alpha}) [(1-\tilde{\alpha}) \frac{\partial m}{\partial \tilde{\alpha}} - 2(1-m)]}{2mF(\tilde{\alpha}) + (1-\tilde{\alpha})F(\tilde{\alpha}) \frac{\partial m}{\partial \tilde{\alpha}}} \right]} \quad (40)$$

From (40), the corresponding wave lengths are then:

$$\lambda = \frac{2\pi}{k} \leq \lambda_{cr} = \frac{l\sqrt{\pi}}{\sqrt{-\ln \left[\frac{(1-\tilde{\alpha}) \frac{\partial F}{\partial \tilde{\alpha}} + F(\tilde{\alpha}) [(1-\tilde{\alpha}) \frac{\partial m}{\partial \tilde{\alpha}} - 2(1-m)]}{2mF(\tilde{\alpha}) + (1-\tilde{\alpha})F(\tilde{\alpha}) \frac{\partial m}{\partial \tilde{\alpha}}} \right]}} \quad (41)$$

It can be seen that waves with wave lengths smaller than the critical wave length λ_{cr} cannot propagate in the bar. This critical wave length is in fact a measure of the width of the localisation zone (Borino et al., 2003; Comi et al., 2007) of the idealised continuum (unbounded linear comparison solid with homogeneous initial states of strain and damage). Mathematically, it may be straightforward to enforce some critical wave length with desired features, and use (41) to work out a corresponding function $m(\tilde{\alpha})$. We note that such a “mathematically rigorous” manipulation may result in a function $m(\tilde{\alpha})$ not supported by any physical basis, and a stable solution does not always mean that it is physically meaningful. As an example, despite its stability the case $m(\tilde{\alpha}) = 1$ for fixed nonlocal interaction produces incorrect initiation of damage (Simone et al., 2004), as will also be seen in the numerical example on a single notch specimen in tension. Therefore (41) should only be considered as a check for a given $m(\tilde{\alpha})$ obtained from micro-mechanics. Besides, one should also bear in mind that the critical wave length in (41) is associated with an idealised linear comparison solid with homogeneous initial states of strain and damage. The fact that the critical wave length in (41) is always positive only guarantees the well-posedness of the linear comparison solid. This however does not always indicate that the associated nonlinear problem is also well-posed (Benallal and Marigo, 2007). An example is given below to illustrate the addressed point, in which the critical wave length of the idealised continuum is proved to be always positive, but material instability in the form of mesh-dependent solutions of the nonlinear system is still numerically observed.

4. Numerical examples

Numerical examples are used to illustrate features of the proposed model with evolving nonlocal effects. These examples include 1D and 2D problems devoted to exploring the evolving size of the FPZ produced by the new model. The issue of incorrect initiation of damage associated with traditional models with fixed nonlocal interaction, reported in Simone et al. (2004), will be shown to be overcome by the new model. The readers may also

find in Appendix B that the incorrect initiation of damage is in fact not the cause of unwanted migration of shear band described in Simone et al. (2004).

For all numerical examples in this paper, the Newton–Raphson method combined with local arc-length control proposed by May and Duan (1997) was used. Following this method, only relative displacements of nodal points of elements in the FPZ are taken into account. In the numerical implementation, a selection criterion based on the positiveness of the damage variables and their increments at element Gauss points is used to select dominant elements in the FPZ. This control is essential to capture the highly nonlinear structural behaviour, experienced through the snap-through and even snap-back points on the load–displacement responses.

4.1. One-dimensional examples

We use a simple 1D version of the damage model described in Section 2 to investigate some characteristics of the evolving nonlocal interaction in one-dimensional setting. A bar with length $L = 60$ mm was considered, and the following properties were used (Shi et al., 2000): Young modulus $E = 30000$ MPa, Poisson's ratio $\nu = 0.2$, tensile strength $f_t = 2.4$ MPa, fracture energy $G_F = 0.059$ Nmm/mm². Three different uniform meshes (60, 120, and 240 linear elements) were employed to show the effectiveness of the nonlocal regularisation. To trigger localisation, the strength of an element in the middle of the bar was reduced by 1%. For the use of Gaussian weight function (28) in numerical analysis, we introduced a cut-off value of 10^{-4} (e.g. weights less than this cut-off value is set to zero), which is sufficient for accuracy (Pijaudier-Cabot and Huerta, 1991). The internal length for this weight function is $l = 3$ mm.

4.1.1. Function m is non-monotonic

This is to show the mesh-dependency of the numerical solution encountered even when the critical wave length (41) is proved to be always positive. The same 1D bar and elasticity parameters described above was used. For damage process, we used $E_p = 2500$ MPa and $n = 0.3$ (for function $F(\tilde{\alpha})$; see Eq. (20)). The following form of function m (Fig. 2a) controlling the local-nonlocal mixing was employed:

$$m(\tilde{\alpha}) = (1.5p)^{1/p} \tilde{\alpha} e^{1/p - 1.5\tilde{\alpha}^p}, \quad \text{with } p = 1 \text{ and } p = 1.5 \quad (42)$$

Fig. 2b plots the critical wave lengths against the initial damage state. We can see that non-negative wave lengths are observed for both cases of $p = 1$ and $p = 1.5$. Mesh-dependent solution is however numerically observed (Fig. 3) for the case of $p = 1.5$. In such a case, beyond the snap-back point, the damage distribution

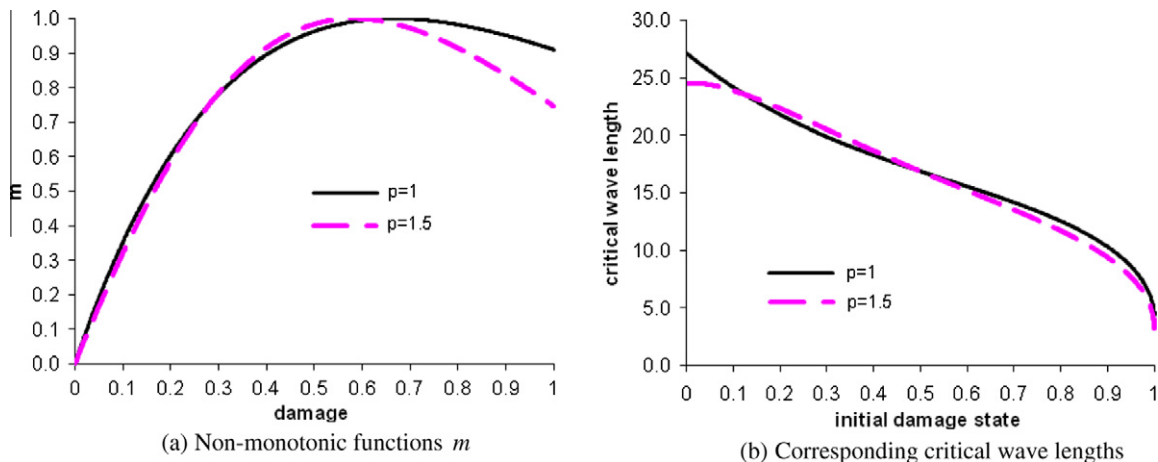


Fig. 2. Functions m and corresponding critical wave lengths.

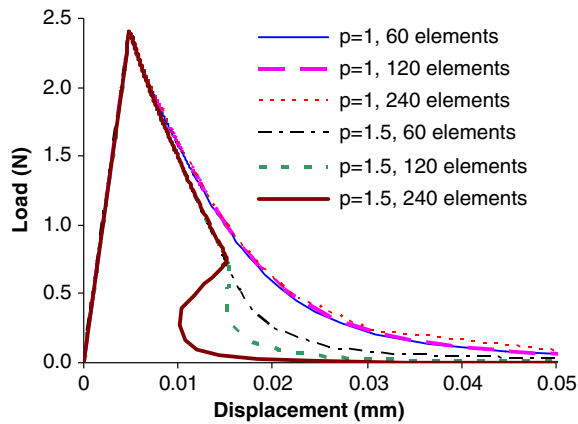


Fig. 3. Numerical solutions.

starts to localise on a single element, with damage on that element keeping increasing, and the rest of the bar unloading elastically. Damage then gradually spreads out again towards the end of the analysis. This is indicated through the more severe snap-back in Fig. 3 if finer mesh is used. In other words, the finer the mesh becomes, the closer to the elastic unloading path the snap-back load-displacement response is, since in such cases damage loading occurs in a smaller area. The above analysis indicates that results from a simple localisation analysis should be treated with care should one wish to use them for any purpose. We however leave this issue to further research.

4.1.2. Function m is monotonically increasing

For the localisation characteristics of models with evolving non-local interactions, we investigate various forms of monotonically increasing function $m(\tilde{\alpha})$ governing the local-nonlocal mixing:

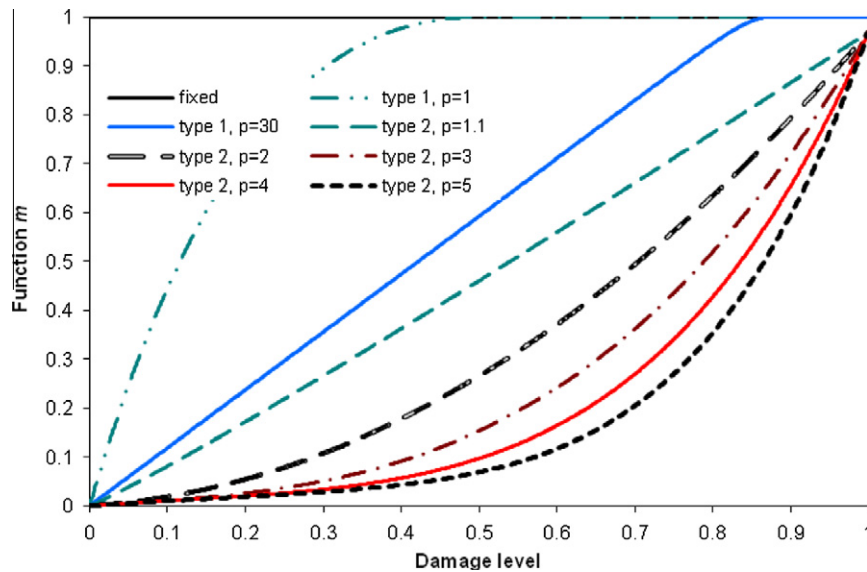
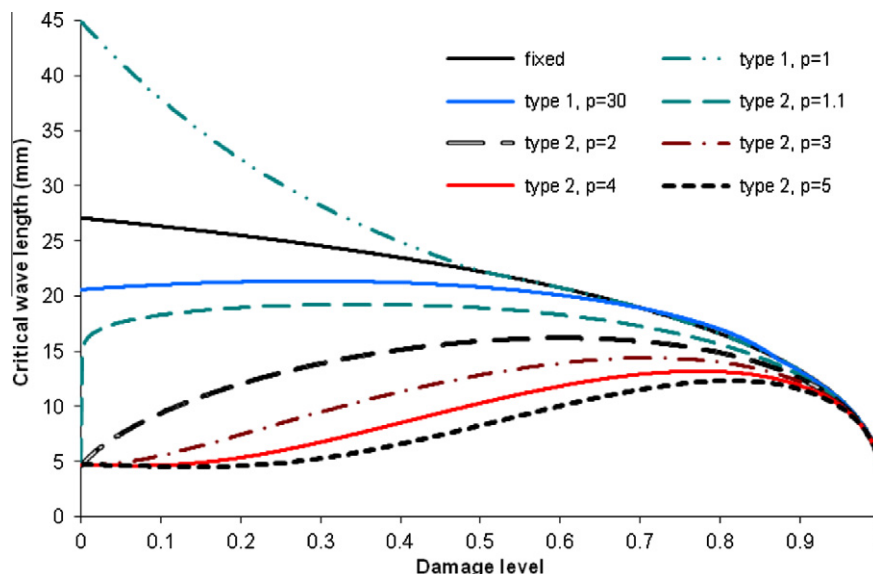
Fig. 4. Function m governing the degree of local-nonlocal mixing.

Fig. 5. Critical wave length vs. damage level.

- Fixed interaction: $m = 1$
- Local-nonlocal mixing type 1:

$$m = \begin{cases} (kp)^{1/p} \tilde{\alpha} e^{1/p - k\tilde{\alpha}^p} & \text{if } \tilde{\alpha} < \tilde{\alpha}_c, \\ 1 & \text{if } \tilde{\alpha} \geq \tilde{\alpha}_c \end{cases}, \quad \text{with } \tilde{\alpha}_c = \left(\frac{1}{kp}\right)^{1/p}, \quad \text{and } k = 2 \quad (43)$$

- Local-nonlocal mixing type 2 (fully under-nonlocal):

$$m = k\tilde{\alpha}^p + q \ln(1 + \tilde{\alpha}), \quad \text{with } k = 0.9, \quad \text{and } q = 0.1 \quad (44)$$

For mixings function m of type 1 and type 2, the parameter p is varied to have different forms of nonlocal interaction (Fig. 4). As mentioned in the introduction, all variants of the damage constitutive model, corresponding to different forms of function $m(\tilde{\alpha})$, are

put in the same scale, described by them producing the same mode I fracture energy $G_F = 0.059 \text{ Nmm/mm}^2$ in 1D setting. The procedure to do that can be found in Nguyen and Houlsby (2007). As a consequence, all load–displacement responses are almost identical (Fig. 6b). Table 1 lists parameters of the corresponding nonlocal models. These parameters just need to vary slightly (less than 12%) to assure the unchanged structural response of the bar.

The variation of the critical wave length of the idealised continuum (e.g. homogeneous distribution of stress, strain and state variables), against the initial damage state, is plotted in Fig. 6. We can see that all nonlocal variations of the damage model described in Section 2.1 are at least theoretically stable. Numerical results (Fig. 6a, for type 2, $p = 3$ as an example) confirm the analytical investigation.

Although they produce almost identical structural responses (Fig. 7), the localisation features of these nonlocal models are

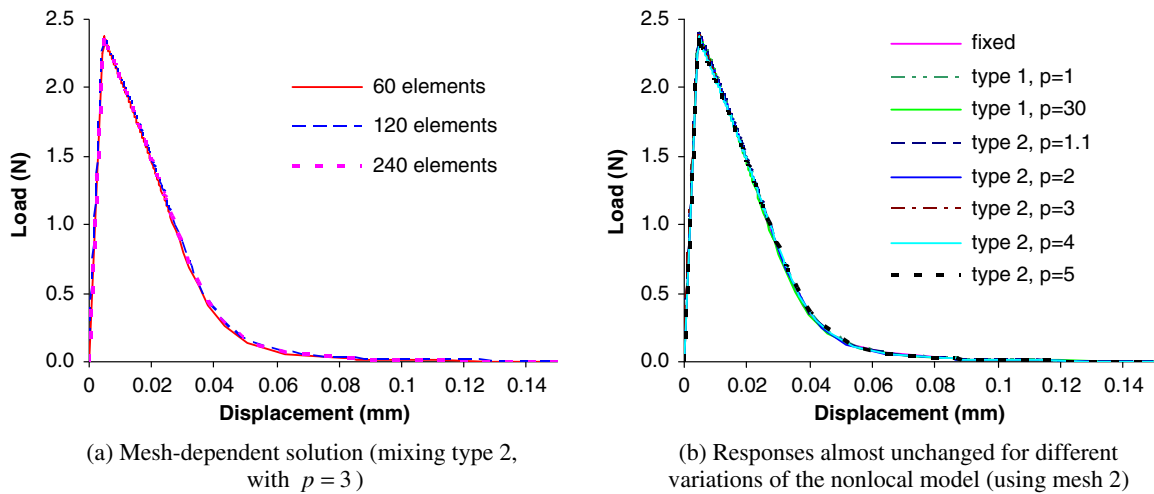


Fig. 6. Load–displacement responses.

Table 1
Model parameters.

	Fixed	Mixing type 1		Mixing type 2				
		$p = 1$	$p = 30$	$p = 1.1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$
E_p (MPa)	1181.12	1181.12	1194.73	1136.38	1129.57	1122.19	1114.75	1107.87
n	0.2336	0.2336	0.2340	0.2322	0.2319	0.2317	0.2314	0.2312

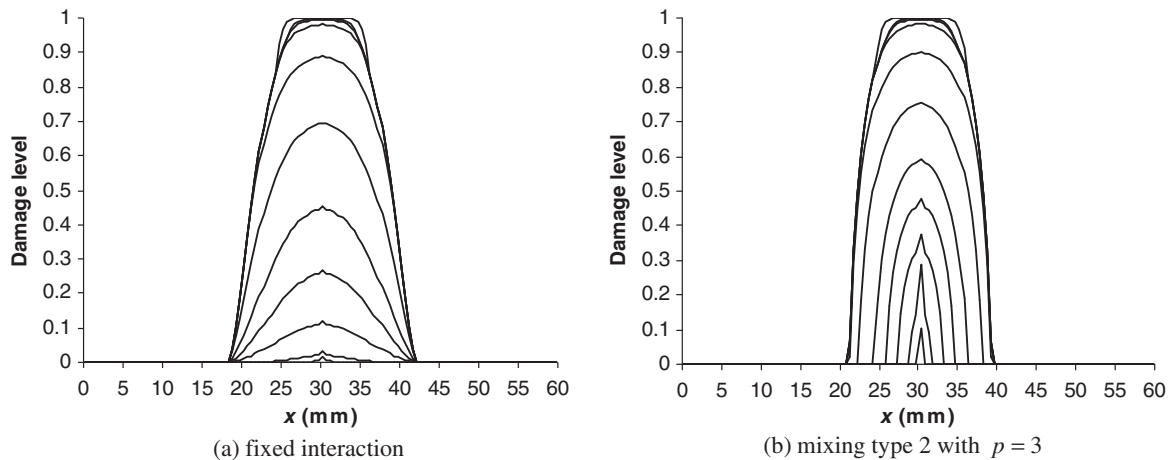


Fig. 7. Evolution of damage profiles (corresponding to elongation of 0.00476, 0.00481, 0.0049, 0.00511, 0.0055, 0.00669, 0.01, 0.02, 0.03, 0.04, 0.15 mm).

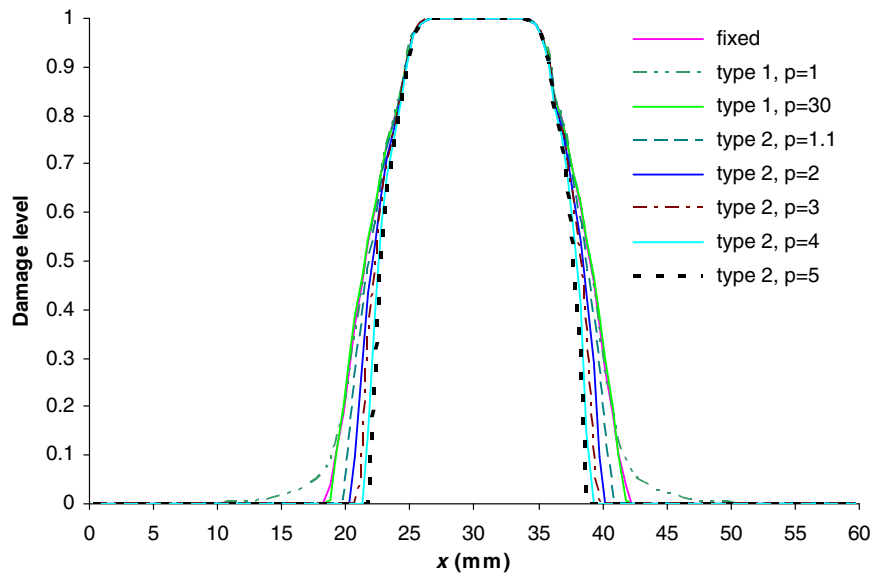


Fig. 8. Damage profile at the end of the analysis.

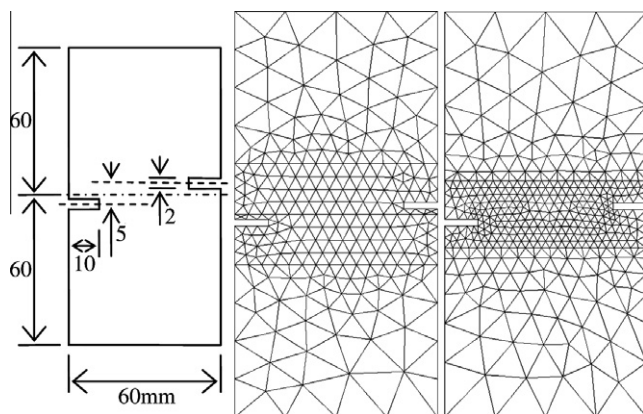


Fig. 9. Double notch specimen in tension: geometry and finite element meshes.

however very different. We present here the evolutions of the damage profiles for 2 nonlocal models: fixed nonlocal interaction and mixing type 2 with $p = 3$ (Fig. 7). The damage zone in model with fixed nonlocal interaction expands very quickly during early stage of the deformation process, reaching the maximum width at damage indicator (in the defect) smaller than 0.15. In contrast, damage of the latter model (mixing type 2 with $p = 3$) localises in small zone at early stage of the damage process, and the FPZ reaches its maximum width at very high damage (~ 0.9 in the defect). These numerically observed features can also be explained by the critical wave length (Fig. 5), physically interpreted as the width of the static active damage zone (Borino et al., 2003; Comi et al., 2007).

We can also see that the numerically observed sizes of the localisation zone vary with the types of nonlocal interaction, again due to their localisation characteristics, e.g. critical wave length (Fig. 8). This is what one should pay attention to when trying to capture both the response and the size of the FPZ. The slope of function $m(\tilde{x})$, governing the local-nonlocal mixing, has strong effects on the localisation feature of the nonlocal model. High slope (e.g. type 1, $p = 1$; Fig. 4) leads to wider damage zone, compared to fixed interaction model, even at an early stage of the damage process when the value of function $m(\tilde{x})$ is still very small, e.g. with more

weight to local term than nonlocal term. Previous studies with constant m (Luzio and Bazant, 2005; Luzio, 2007) indicated that m should not be less than 1, to obtain the desired regularisation effects (e.g. localisation of deformation onto finite zone). We can see here that all nonlocal models employing function $m(\tilde{x})$ of mixing type 2 in this study are under-nonlocal, as the maximum values of m in those models are below 1. These models are however capable of capturing a finite size localisation zone (Fig. 8), and produce converging numerical solution upon the refinement of the spatial discretisation (Fig. 6).

4.2. Two-dimensional examples

4.2.1. Double notch in tension

In this section, a simple structural problem is analysed using the models with fixed and evolving nonlocal interaction (type 2, with $p = 5$; see Eq. (44) in previous example). The specimen geometry (Fig. 9) and experimental results are from Shi et al. (2000), with material properties already calibrated and listed in the previous one-dimensional example. Plane stress condition is assumed. The bottom edge of the specimen is fixed in both directions, and the top edge in horizontal direction. Prescribed vertical displacement is applied on the top edge. We used 2 finite element meshes with different spatial resolutions (Fig. 9). GiD version 9.0.2 (2009) was used for post-processing the results.

The trend in the structural behaviour of the specimen is captured well, and numerical results (Fig. 11) show the mesh-independence of the obtained numerical solutions. Difference in the evolution of the FPZ can also be seen (Fig. 10). All these again confirm the analytical prediction for the material stability in the previous section.

4.2.2. Initiation of damage: single notch in tension

This example is to show how the wrong initiation of damage (Simone et al., 2004) is overcome using models with evolving non-local effect. For the physically correct initiation of damage at the notch tip, $m(\tilde{x} = 0) = 0$ is required, and all types of varying function m in this paper meet this condition. This condition is motivated by the physics of crack initiation. Again, we use the evolving nonlocal interaction model of mixing type 2 with $p = 5$, and also the fixed interaction model with $m = 1$ for comparison.

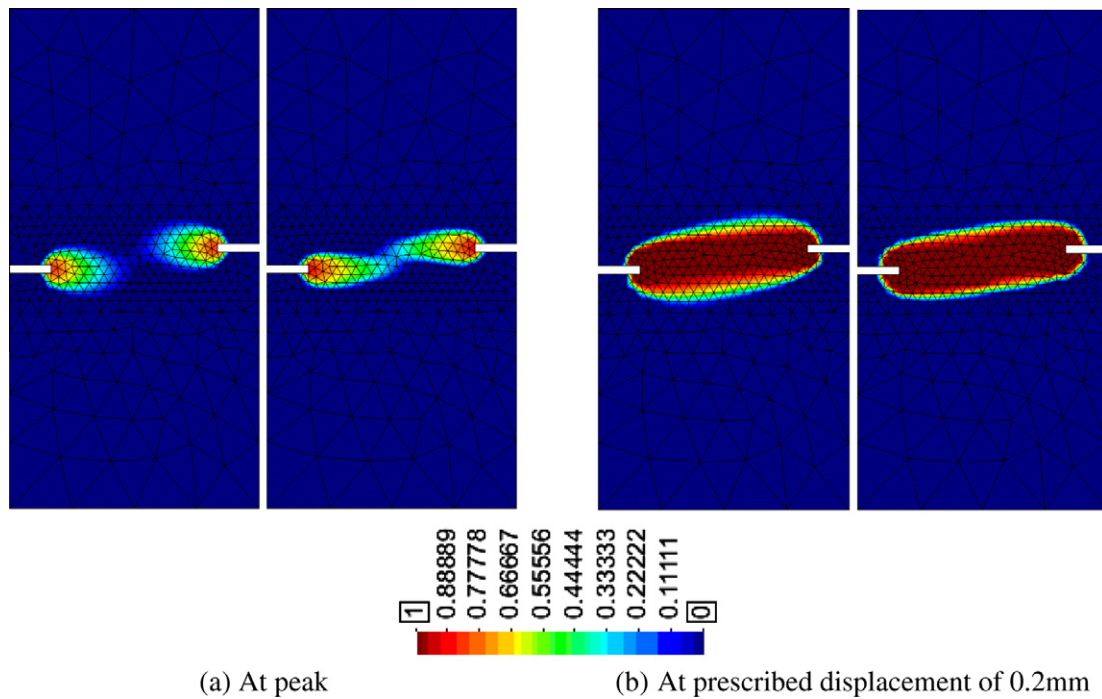


Fig. 10. Damage zones for models with fixed (left) and evolving nonlocal interactions (right).

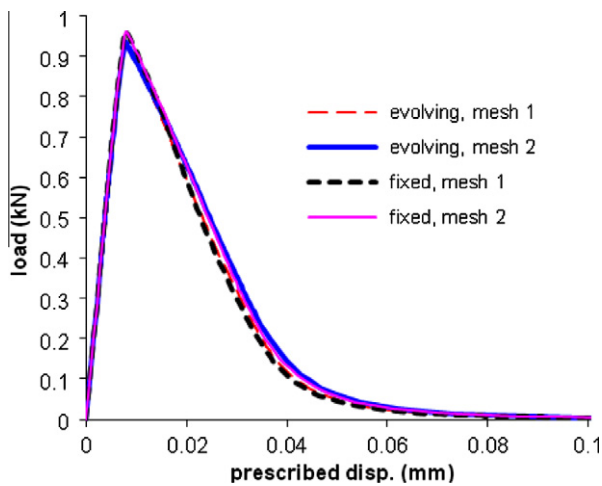


Fig. 11. Double notch specimen in tension: load–displacement curves.

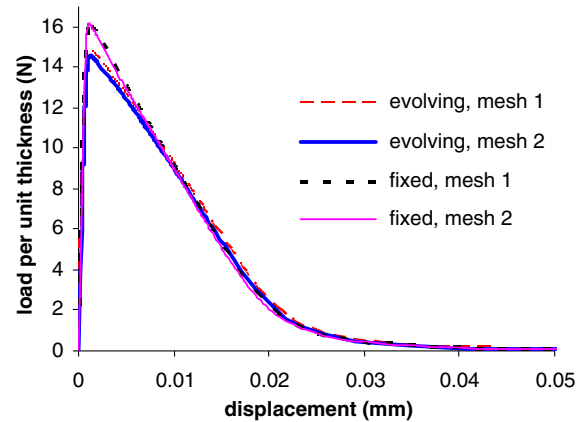


Fig. 13. Single notch specimen: load–displacement responses.

The corresponding model parameters are listed in Table 1. We pick an example of a notched specimen in tension (Fig. 12), which is similar to that used in Simone et al. (2004). For sufficient resolu-

tion of the discretisation, our numerical results show the insensitivity of the results with respect to finite element meshes (Fig. 13).

The damage contour at damage onset, at prescribed displacement of 0.01 mm, and at final state (prescribed displacement of ~ 0.1 mm) corresponding to 2 nonlocal models are shown in

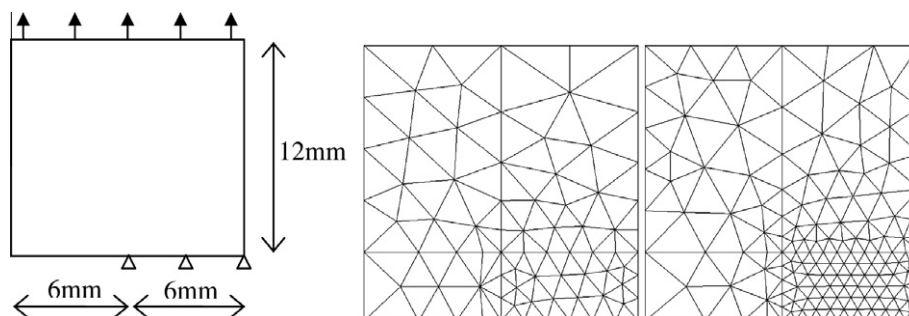


Fig. 12. Single notch specimen: geometry and finite element meshes.

Fig. 14. The corresponding damage distributions along the lower edge, at 3 stages of the deformation process, are also depicted (Fig. 15). It can be seen that the initiation of damage and its evolu-

tion are totally different for the two models under investigation. While the traditional fixed internal length model incorrectly predicts the onset of damage at some distance away from the notch

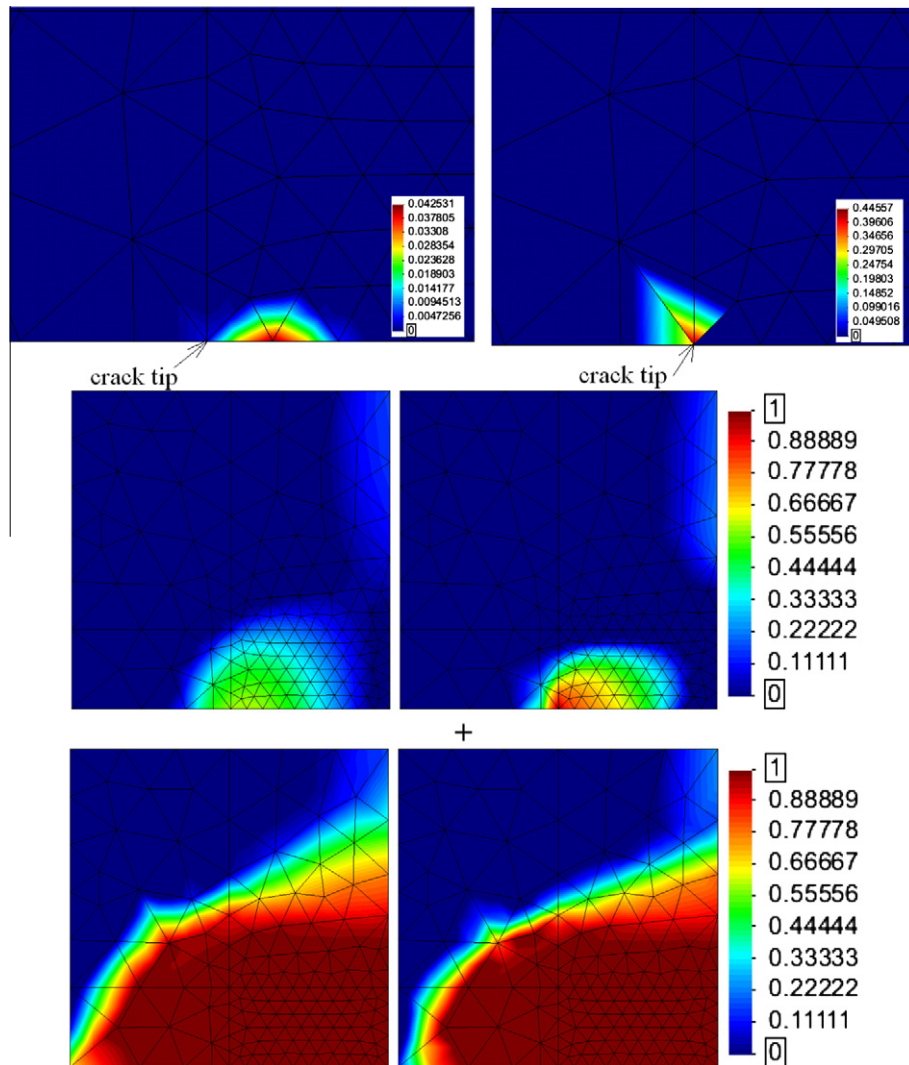


Fig. 14. Damage distribution at different stages of the fracturing process, for models with fixed (left) and evolving nonlocal interactions (right). Top figures: zoomed in of damage initiation around crack tip; middle figures: damage distribution at 0.01 mm prescribed displacement; bottom figures: damage distribution at the end of the analysis.

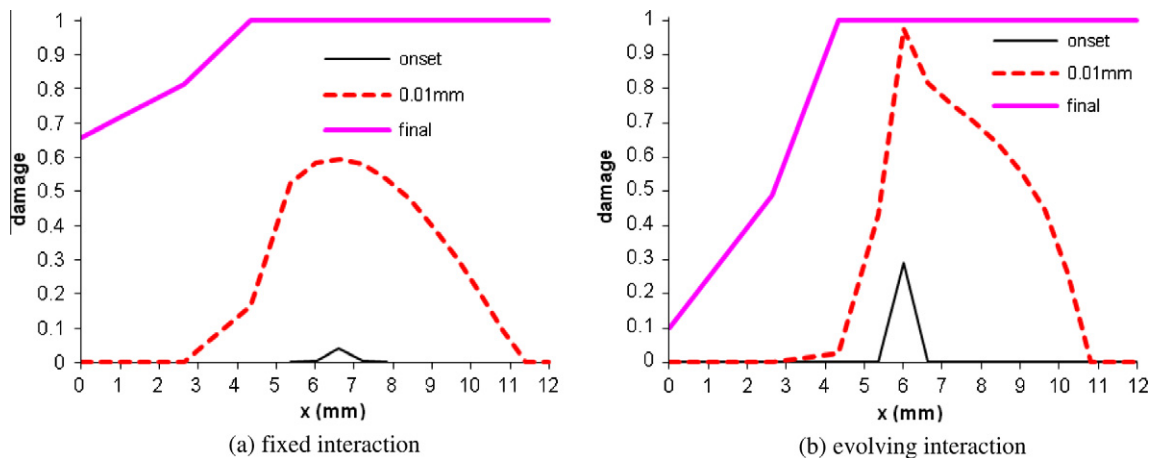


Fig. 15. Damage distribution along the lower edge.

tip (Simone et al., 2004), the introduction of evolving nonlocal interaction, which only activates nonlocal behaviour once damage occurs, has effects in predicting correctly the location of damage onset (Figs. 14 & 15). During early stage of the fracturing process, the damage is more localised in the new model, than it is in model with fixed nonlocal interaction. The final damage distributions are also slightly different, with smaller damage zone for the new model with evolving nonlocal interaction. Again these numerical observations confirm our analytical results in Section 4.1.

5. Conclusions

A model with evolving nonlocal interactions is developed in this study. The thermodynamic admissibility of the model is addressed and localisation analysis based on the propagation of acceleration waves carried out. Numerical examples illustrate the mesh-independence of the numerical solutions, thus confirming the analytical predictions by localisation analysis. We show the capability of this model in overcoming the wrong initiation of damage usually encountered in other models with fixed nonlocal interaction (Simone et al., 2004). Our numerical study also indicates that the migration of shear bands, once thought to be related to the wrong initiation of damage, is in fact associated with the behaviour of the material model. Significant reduction in the size of the FPZ during late stage of the damage process, when a macro crack is expected, is however hard to achieve due to constraint on the stability of the nonlocal model. Attempts with non-monotonic function $m(\tilde{\alpha})$ decreasing towards the critical value of damage lead to material instability observed through mesh-dependent numerical solutions (Section 4.1). This requires further work on the localisation characteristics of the proposed local-nonlocal mixing.

Besides, the difference between the current model and fixed nonlocal interaction models in producing different sizes and evolutions of FPZ is also addressed both analytically (through localisation analysis) and numerically (thorough finite element analysis). The variable interaction gives more flexibility to nonlocal models in capturing faithfully the evolving FPZ observed in experiments (Otsuka and Date, 2000).

The stability issue of the model is also examined. For an idealised continuum (linear comparison solid; unbounded; homogeneous initial state) the critical wave length given by a localisation analysis is a good indicator for the material stability. For real non-linear problems, an example in this study shows the mesh-dependency of the numerical solution even though the critical wave length of the corresponding idealised solid is always positive. Therefore, the use of dispersion analysis as a tool to examine the stability and the localisation characteristics of a model should be treated with care. The companion of numerical study is always essential.

The evolving nonlocal interaction in this research is of phenomenological type, aiming at first to investigate the localisation features of constitutive models, rather than to capture faithfully the richness of micro-crack interactions at lower scales. This calls for further research on the nature of nonlocality in the modelling of quasi-brittle fracture within the framework of continuum mechanics.

Acknowledgements

The author is grateful to A/Professor Itai Einav (School of Civil Engineering, University of Sydney) for many stimulating discussions. Thanks are also to Dr. Angelo Simone (Faculty of Civil Engineering and Geosciences, TU Delft) for his constructive comments on the manuscript. This research was supported under the Australian Research Council's Discovery Projects funding scheme (project

number DP1093485). Support from the University of Sydney to the author, through the USyd Postdoctoral Fellowship scheme, is also gratefully acknowledged.

Appendix A

Advantages and disadvantages, mostly on the basis of numerical implementation and convenience, of two formulations employing the nonlocal operators L and L^* in (3) are detailed in this section. It is noted that both approaches are thermo-mechanically consistent. In other words, for thermo-mechanical requirements, the employment of the nonlocal operator L (L^*) in (3) leads to the appearance of its adjoint L^* (L) in (12).

We rewrite the keys expressions/equations here. For the use of L in (3), as adopted in this study and identified here as approach #1, we have:

$$\dot{\tilde{\alpha}}(\mathbf{x}) = (1 - m)\dot{\alpha}(\mathbf{x}) + m \int_V \frac{g(\mathbf{x}, \mathbf{y})}{G(\mathbf{y})} \dot{\alpha}(\mathbf{y}) dV(\mathbf{y}) \quad (\text{A1})$$

$$\chi(\mathbf{x}) = \bar{\chi}(\mathbf{x})(1 - m) + \frac{1}{G(\mathbf{x})} \int_V g(\mathbf{x}, \mathbf{y}) \bar{\chi}(\mathbf{y}) m dV(\mathbf{y}) \quad (\text{A2})$$

On the other hand, for approach #2 with L^* employed in (3), one ends up at:

$$\dot{\tilde{\alpha}}(\mathbf{x}) = (1 - m)\dot{\alpha}(\mathbf{x}) + \frac{m}{G(\mathbf{x})} \int_V g(\mathbf{x}, \mathbf{y}) \dot{\alpha}(\mathbf{y}) dV(\mathbf{y}) \quad (\text{A3})$$

$$\chi(\mathbf{x}) = \bar{\chi}(\mathbf{x})(1 - m) + \int_V \frac{g(\mathbf{x}, \mathbf{y})}{G(\mathbf{y})} \bar{\chi}(\mathbf{y}) m dV(\mathbf{y}) \quad (\text{A4})$$

It can be seen from (A1) and (A2) that approach #1 can reproduce a uniform nonlocal energy-like field from its uniform local counterpart, while that is impossible with the nonlocal damage variable. This issue will surface if one wants to compute the dissipation directly from the dissipation potential (14), where local damage increments are required (see the explicitly defined dissipation potential in (14)). In such a situation, from the nonlocal damage increments obtained from numerical analysis (e.g. solving system (27)) a system of equations involving both local and nonlocal damage increments needs to be solved for the local damage increments. Alternatively, (27) can also be restructured (using the discretised form of (3)) to have local damage increments as unknowns. In both cases, as have been found numerically in this study, the boundary effects on the nonlocal operator L can lead to local damage exceeding 1 during late stage of the analysis.

In approach #2, the above issue of local damage larger than 1 can, at least theoretically (not numerically confirmed yet), be avoided, thanks to a proper normalisation of the weight in the nonlocal damage averaging. On the other hand, due to boundary effects, the inability to reproduce a uniform nonlocal energy-like field from its uniform local counterpart, as seen in (A4), is encountered. More severely, strong boundary effects on the regularisation feature of the model, particularly because of the involvement of the mixing function m in (A4) was also numerically observed. In particular, for a uniform local field the contribution from the 2nd term in (A4) is not solely governed by m , but also by the nonlocal averaging when going towards the physical boundary. These boundary effects cannot be detected from the localisation analysis (Section 3), because L and L^* coincide in such a case under the assumption of infinite medium.

Appendix B (on the correct initiation of damage and migration of shear band)

We investigate whether the new model with evolving nonlocal interactions is able to overcome the unwanted migration of shear band described in Simone et al. (2004). A biaxial compression

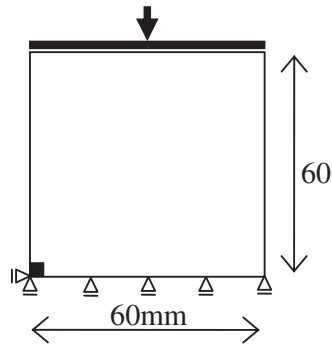


Fig. A1. Geometry and boundary conditions for the biaxial test.

example (Fig. A1) similar to that used in Simone et al. (2004) is employed, and plane stress condition assumed. The following parameters are used: $E = 30000$ MPa, Poisson's ratio $\nu = 0.2$, strength $f_t = 2.4$ MPa, $E_p = 300$ and $n = 0.3$. To trigger localisation, a reduction of 50% is applied to the strength of the corner element (Fig. A1). Two nonlocal models with fixed ($m = 1$) and evolving nonlocal interaction (type 2, $p = 5$; Eq. (44)) are placed under investigation. We vary the length parameter l (with values of 1, 2 and 3 mm) in the Gaussian weight function to see the effects on the shear band inclination. The results are similar for both models (Fig. A2), showing that the migration of shear band may not be due to the (wrong) initiation of damage, but rather the constitutive behaviour. Numerical results obtained from the use of model with evolving nonlocal interactions, on a larger specimen, (Fig. A3) also confirm the observed shear band migration.

An investigation is carried out to see what is really behind the observed numerical results. A one dimensional bar is used to see

the effect of the length parameter on the nonlocal model response. We can show (in 1D and 2D) that despite the difference in the initiation of damage, the constitutive behaviour of the two models (fixed and evolving nonlocal interaction, with the same parameters) should be very similar to produce almost identical structural responses (Fig. A4). This obviously excludes the wrong initiation of damage usually associated with fixed nonlocal interaction models from causing the migration of shear bands. This conclusion however does not exclude the effects of the nonlocal averaging, e.g. isotropic or anisotropic nonlocal weight, on the shear band migration. For nonlocal models, changes in the internal length parameter result in different model responses (Fig. A4a), e.g. different tangent stiffnesses at the same stage of deformation. We believe that this is the main cause for the migration of inclined shear bands.

Similar trend in the model behaviour and shear band migration is also observed with the local version of the model ($m = 0$). From the literature on the localisation characteristics of constitutive models (Runesson et al., 1991; Neilsen and Schreyer, 1993), it has been known that the shear band orientation is governed by the constitutive behaviour, and also boundary conditions. Figs. A5 and A6 illustrate the effects of (local) constitutive behaviour on this migration. It can be seen in those figures that the inclination angle is sensitive to the behaviour of the constitutive model (through a simple 1D model). Inclined shear bands only emerge beyond a certain stage of the deformation process, and that certain stage is dependent on the constitutive behaviour. For the local behaviour, smaller values of parameter n in function $F(\tilde{\alpha})$ (see Eq. 20) result in lower rates of change of the tangent stiffness (Fig. A5; see also details in Nguyen (2005) and Nguyen and Houlsby (2007)), e.g. the tangent stiffness is increasing slowly. This leads to inclined shear bands further away from the weakened spot (Fig. A6).

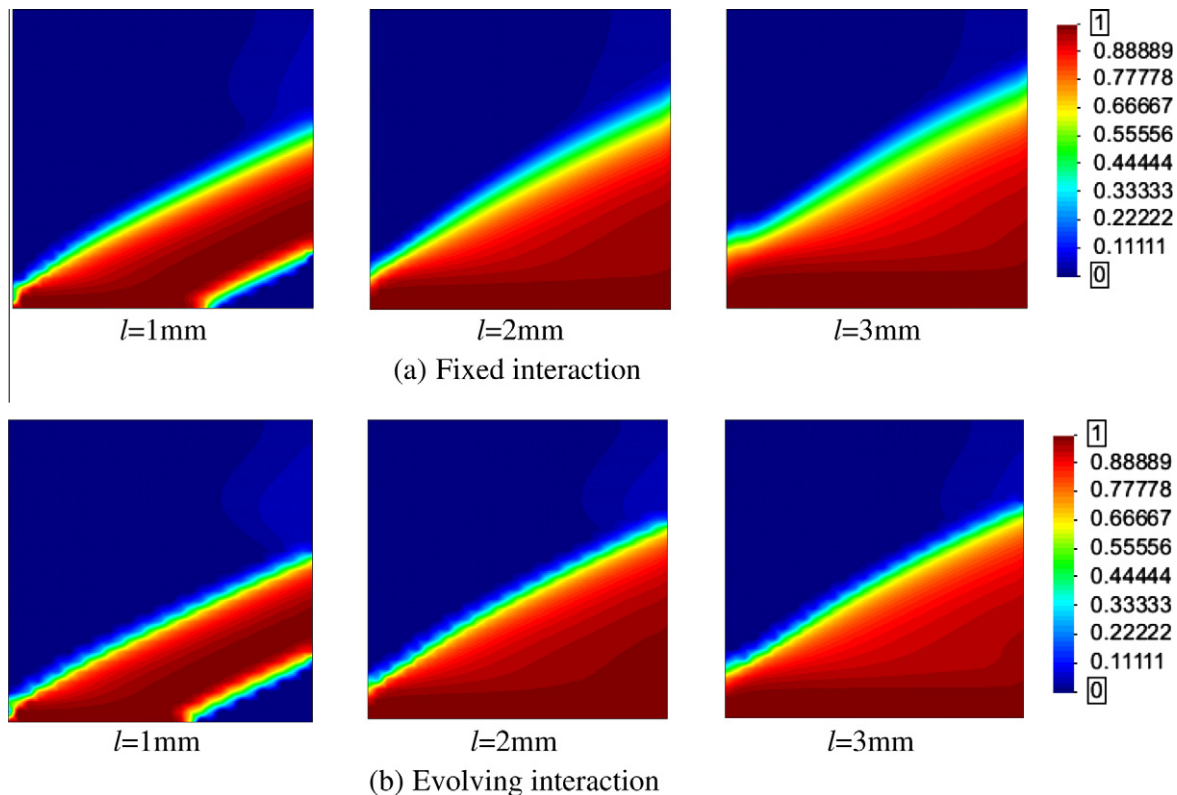


Fig. A2. Effects of the internal length on shear band migration.

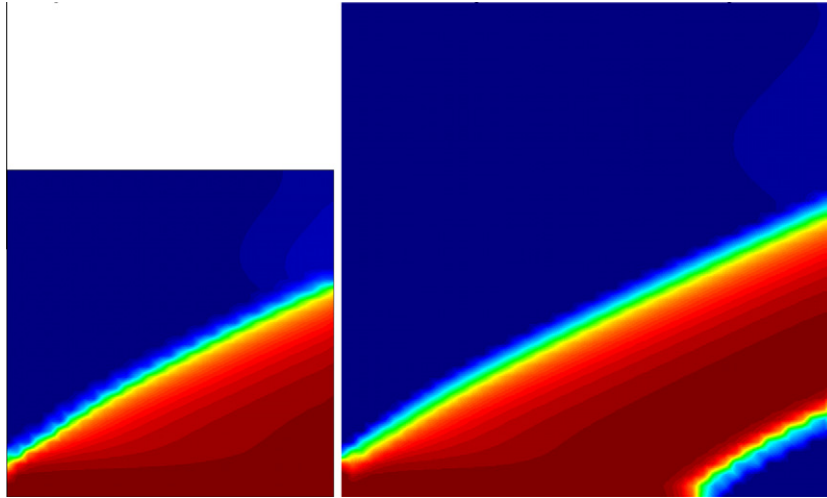


Fig. A3. The migration of shear band for $l=2$ mm, using model with evolving nonlocal interactions (left: smaller specimen ($60 \times 60 \text{ mm}^2$); right: larger specimen ($90 \times 90 \text{ mm}^2$)).

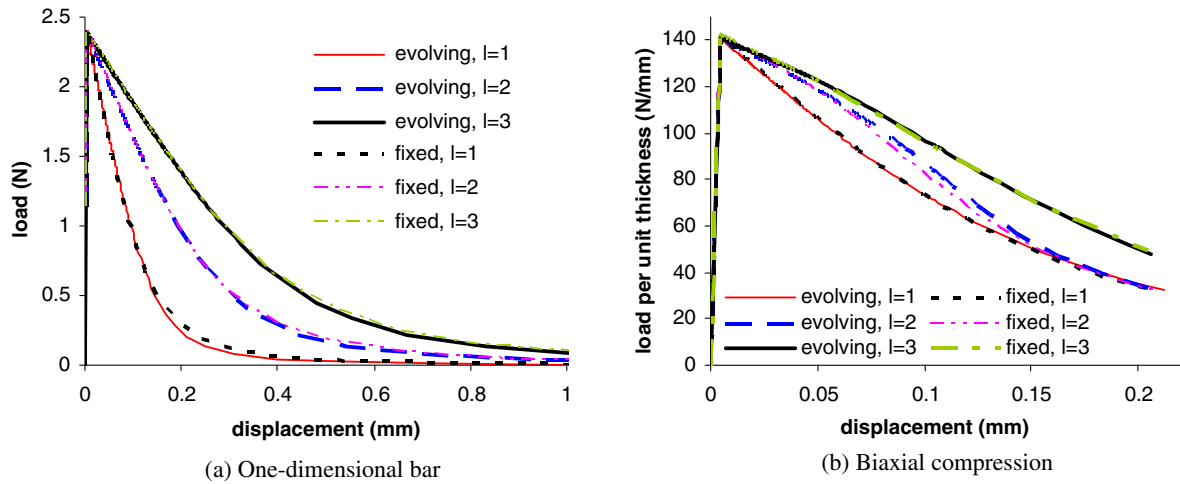


Fig. A4. Load–displacement responses.

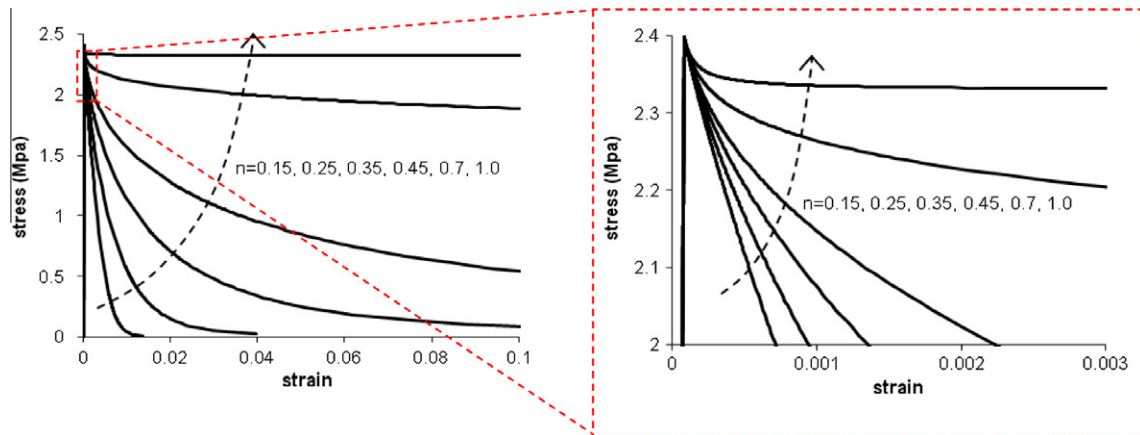


Fig. A5. Effects of parameter n on (local) stress–strain response (in one dimension).

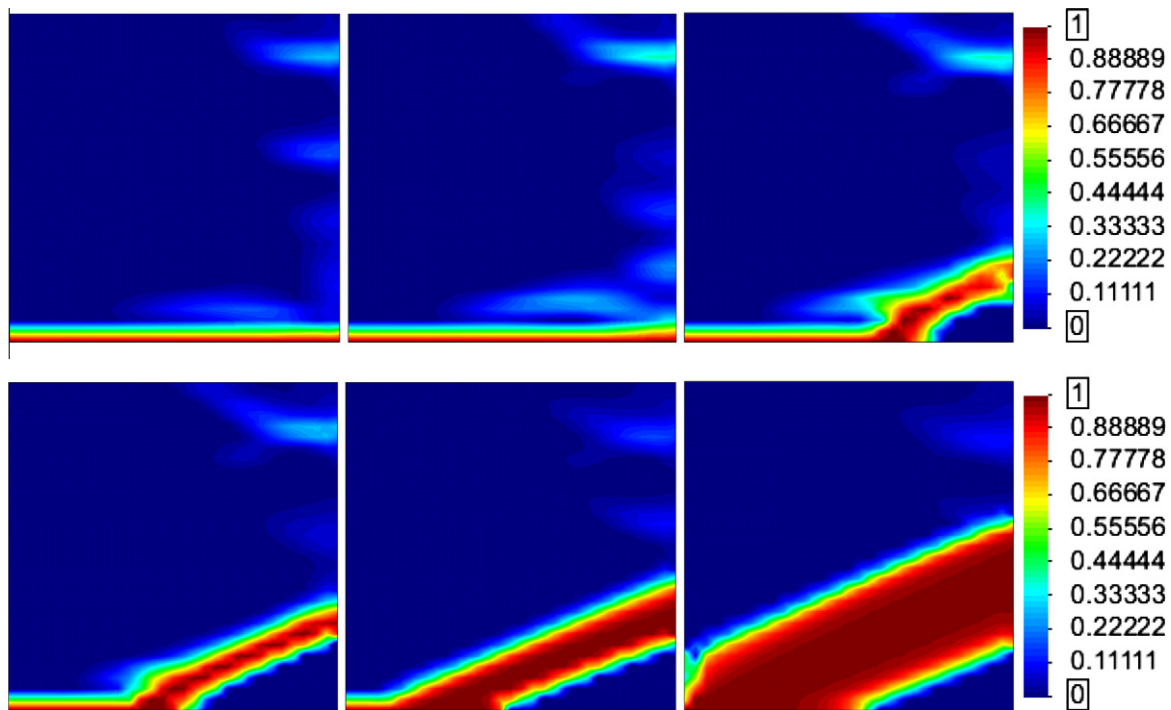


Fig. A6. Effects of the material behaviour on shear band propagation (corresponding to the values of parameter n of 0.15, 0.25, 0.35, 0.45, 0.7, and 1.0, respectively).

References

- Bazant, Z.P., 1991. Why continuum damage is nonlocal: Micromechanics arguments. *ASCE Journal of Engineering Mechanics* 117 (5), 1070–1087.
- Bazant, Z.P., 1994. Nonlocal damage theory based on micromechanics of crack interactions. *ASCE Journal of Engineering Mechanics* 120 (3), 593–617.
- Benallal, A., Marigo, J.-J., 2007. Bifurcation and stability issues in gradient theories with softening. *Modelling and Simulation in Materials Science and Engineering* 15, S283–S295.
- Benvenuti, E., Tralli, A., 2003. Iterative LCP solvers for non-local loading–unloading conditions. *International Journal for Numerical Methods in Engineering* 58, 2343–2370.
- Benvenuti, E., Borino, G., Tralli, A., 2002. A thermodynamically consistent nonlocal formulation for damaging materials. *European Journal of Mechanics A/Solids* 21, 535–553.
- Borino, G., Fuschi, P., Polizzotto, C., 1999. A thermodynamic approach to non-local plasticity and related variational principles. *ASCE Journal of Applied Mechanics* 66, 952–963.
- Borino, G., Failla, B., Parrinello, F., 2003. A symmetric nonlocal damage theory. *International Journal of Solids and Structures* 40, 3621–3645.
- Bui, Q.V., 2010. Initiation of damage with implicit gradient-enhanced damage models. *International Journal of Solids and Structures* 47 (18–19), 2425–2435.
- Comi, C., 2001. A nonlocal model with tension and compression damage mechanisms. *European Journal of Mechanics A/Solids* 20, 1–22.
- Comi, C., Perego, U., 2001. Numerical aspects of nonlocal damage analyses. *European Journal of Finite Elements* 10, 227–242.
- Comi, C., Mariani, S., Perego, U., 2007. An extended FE strategy for transition from continuum damage to mode I cohesive crack propagation. *International Journal for Numerical and Analytical Methods in Geomechanics* 31, 213–238.
- Geers, M.G.D., de Borst, R., Brekelmans, W.A.M., Peerlings, R.H.J., 1998. Strain-based transient-gradient damage model for failure analyses. *Computer Methods in Applied Mechanics and Engineering* 160, 133–153.
- Geers, M.G.D., Peerlings, R.H.J., Brekelmans, W.A.M., de Borst, R., 2000. Phenomenological nonlocal approaches based on implicit gradient-enhanced damage. *Acta Mechanica* 144, 1–15.
- GiD version 9.0.2, 2009. The personal pre- and post-processor. International Center for Numerical Methods in Engineering, Barcelona, Spain.
- Grassl, P., Jirásek, M., 2006. Plastic model with non-local damage applied to concrete. *International Journal for Numerical and Analytical Methods in Geomechanics* 30, 71–90.
- Grassl, P., Jirásek, M., 2010. Meso-scale approach to modelling the fracture process zone of concrete subjected to uniaxial tension. *International Journal of Solids and Structures* 47 (7–8), 957–968.
- Haidar, K., Pijaudier-Cabot, G., Dube, J.F., Loukili, A., 2005. Correlation between the internal length, the fracture process zone and size effect in model materials. *Material and Structures* 38, 201–210.
- Houlsby, G.T., Puzrin, A.M., 2000. A thermomechanical framework for constitutive models for rate-independent dissipative materials. *International Journal of Plasticity* 16, 1017–1047.
- Houlsby, G.T., Puzrin, A.M., 2006. *Principles of Hyperplasticity*. Springer, -Verlag, London.
- Jankowski, L.J., Styś, D.J., 1990. Formation of the fracture process zone in concrete. *Engineering Fracture Mechanics* 36 (2), 245–253.
- Jirasek, M., & Marfia, S. 2008. Nonlocal damage model combined with displacement discontinuity. In: *Proceedings of 8th World Congress on computational mechanics (WCCM8)*, June 30–July 5 Venice, Italy.
- Luzio, G., 2007. A symmetric over-nonlocal microplane model M4 for fracture in concrete. *International Journal of Solids and Structures* 44, 4418–4441.
- Luzio, G., Bazant, Z.P., 2005. Spectral analysis of localization in nonlocal and over-nonlocal materials with softening plasticity or damage. *International Journal of Solids and Structures* 42 (23), 6071–6100.
- May, I.M., Duan, Y., 1997. A local arc-length procedure for strain softening. *Computers and Structures* 64 (1–4), 297–303.
- Neilsen, M.K., Schreyer, H.L., 1993. Bifurcation in elastic-plastic materials. *International Journal of Solids and Structures* 30 (4), 521–544.
- Nguyen, G.D. 2005. A thermodynamic approach to constitutive modelling of concrete using damage mechanics and plasticity theory. PhD dissertation, Department of Engineering Science, University of Oxford.
- Nguyen, G.D., 2008. A thermodynamic approach to nonlocal damage modelling of concrete. *International Journal of Solids and Structures* 45, 1918–1934.
- Nguyen, G.D., Houlsby, G.T., 2007. Nonlocal damage modelling of concrete: a procedure for the determination of parameters. *International Journal for Numerical and Analytical Methods in Geomechanics* 31, 867–891.
- Otsuka, K., Date, H., 2000. Fracture process zone in concrete tension specimen. *Engineering Fracture Mechanics* 65 (2–3), 111–131.
- Pijaudier-Cabot, G., Bazant, Z.P., 1987. Nonlocal damage theory. *ASCE Journal of Engineering Mechanics* 113 (10), 1512–1533.
- Pijaudier-Cabot, G., Benallal, A., 1993. Strain localization and bifurcation in a nonlocal continuum. *International Journal of Solids and Structures* 30 (13), 1761–1775.
- Pijaudier-Cabot, G., Huerta, A., 1991. Finite element analysis of bifurcation strain softening solids. *Computer Methods in Applied Mechanics and Engineering* 90, 905–919.
- Pijaudier-Cabot, G., Haidar, K., Dube, J.-F., 2004. Nonlocal damage model with evolving internal length. *International Journal for Numerical and Analytical Methods in Geomechanics* 28, 633–652.
- Polizzotto, C., Borino, G., 1998. A thermodynamics-based formulation of gradient-dependent plasticity. *European Journal of Mechanics A/Solids* 17, 741–761.
- Polizzotto, C., Borino, G., Fuschi, P., 1998. A thermodynamically consistent formulation of nonlocal and gradient plasticity. *Mechanics Research Communication* 25 (1), 75–82.
- Runesson, K., Ottosen, N.S., Peric, D., 1991. Discontinuous bifurcations of elastic-plastic solutions at plane stress and plane strain. *International Journal of Plasticity* 7, 99–121.

- Shi, C., van Dam, A.G., van Mier, J.G.M., Sluys, L.J., 2000. Crack interaction in concrete. In: Wittmann, F.H. (Ed.), *Materials for Buildings and Structures EUROMAT*, Vol. 6. Wiley-VCH Verlag, Weinheim, Germany, pp. 125–131.
- Simone, A., Wells, G.N., Sluys, L.J., 2003. From continuous to discontinuous failure in a gradient-enhanced continuum damage model. *Computer Methods in Applied Mechanics and Engineering* 192 (41–42), 4581–4607.
- Simone, A., Askes, H., Sluys, L.J., 2004. Incorrect initiation and propagation of failure in non-local and gradient-enhanced media. *International Journal of Solids and Structures* 41, 351–363.
- Stromberg, L., Ristinmaa, M., 1996. FE-formulation of a nonlocal plasticity theory. *Computer Methods in Applied Mechanics and Engineering* 136, 127–144.