



Bifurcation in granular materials: An attempt for a unified framework

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ABSTRACT

Sudden collapse mechanisms strictly inside the Mohr–Coulomb plastic limit condition have been observed in granular materials in the laboratory as well as in the field. The purpose of this paper is to show that the theoretical framework of loss of sustainability is convenient to describe such mechanisms. In this context, the notions of both loading and response variables, which characterize the loading path applied to the specimen considered and its response path, are fundamental. Moreover, by investigating the relation between loading and response parameters, it is established that this framework also embeds the notions of loss of constitutive uniqueness and loss of controllability. Therefore, a unified approach is attempted. Finally, by highlighting the basic role played by the loading parameters, the vanishing of the second-order work is shown to be a proper criterion to detect the occurrence of a bifurcation from a quasi-static regime to a dynamic regime leading to the collapse of the material.

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1. Introduction

The failure of a given material is one of the basic issues that engineers must consider when designing materials. The notion of failure is very broad and encompasses a number of aspects related to various failure modes. In geomechanics, for instance, it is well known that different failure modes can occur before the plastic limit condition for nonassociated geomaterials (see for instance Lade et al., 1987; Vardoulakis and Sulem, 1995; Darve and Vardoulakis, 2005), which can be ascertained by both experimental and numerical investigations. This paper will not treat the different failure modes that can be encountered but will instead focus on a specific failure mode that corresponds to the collapse of the specimen. When this failure mode is associated with no visible specific localization pattern within the kinematical field, it is referred to as a diffuse failure mode (Darve et al., 2004; Darve and Vardoulakis, 2005). It is worth noting that the theoretical framework developed in this paper also encompasses localized failure modes. The occurrence of these failure modes also requires that the Rice criterion (vanishing of the acoustic tensor's determinant; Rudnicki and Rice, 1975) be met. Both diffuse and localized failure modes are known to be antagonist, as experimental tests show that depending on the density of the material, the loading conditions, the manner in which the loading is applied, the unavoidable existence of defects, etc., one of the two failure modes will prevail.

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The loss of controllability approach (Nova, 1994) is often presented as a convenient framework for describing the occurrence of diffuse failures. The notion of controllability was developed by examining the conditions in which a prescribed loading program involving specific loading parameters (also referred to as control parameters) can be applied to a system, from a given mechanical state after a given loading history. If the considered loading program can be applied, then this program will be said to be controllable. Following the definition given by Nova (Nova, 1994), starting from a given mechanical state, a loading program can be applied if and only if at each step it produces a unique incremental response to the incremental loading governed by the control parameters. Finally, the controllability of a loading program means that the incremental response exists and is unique, which has been investigated in a number of studies (see for instance Vardoulakis et al., 1978; Vardoulakis and Sulem, 1995; Petryk, 1993; Darve et al., 1995; Bigoni and Hueckel, 1991; Bigoni, 2000; Chambon and Caillerie, 1999).

Nevertheless, the loss of controllability, or loss of uniqueness, approach does not predict what happens from a physical point of view when the uniqueness of the incremental response to an incremental loading is lost. Uniqueness is first and foremost a mathematical problem. On the other hand, the loss of sustainability approach (Nicot et al., 2007; Nicot and Darve, 2007) was developed to describe the collapse mechanism, starting from the physical evidence that such a mechanism is related to a sudden increase in kinetic energy. When the sustainability of a mechanical state is lost, if a certain infinitesimal perturbation is applied, there is a sudden,

dramatic increase in the kinetic energy. Thus, in the context of loss of sustainability, loss of uniqueness receives its proper physical meaning, by considering that the mathematical equations used in the quasi-static regime are no longer valid, that is, there is a transition from a quasi-static problem to a dynamical one. This transition corresponds to a bifurcation mode; there is a discontinuous change in the response of the system (from a quasi-static mode to a dynamic mode), without any external change. The notion of loss of sustainability is intimately related to this notion of bifurcation.

For geomaterials including soft and hard soils, rocks, concrete, etc., experimental tests have shown that in certain conditions, a sudden change in the response of the system can suddenly occur even under continuous variations of the loading parameters. A famous example is the liquefaction of loose sands under axisymmetric isochoric triaxial paths. This experiment shows that the curve giving the changes in the deviatoric stress q (defined as the difference between the axial stress σ_1 and the lateral stress σ_3) against the mean effective pressure p' passes through a maximum. When this maximum is reached, if an infinitesimal axial load (regarded as an infinitesimal perturbation) is added, then a sudden failure occurs. The specimen merely collapses. If the test is strain-controlled (by imposing a constant axial strain rate), the test can be pursued beyond the deviatoric peak: both deviatoric (q) and hydrostatic (p') stresses decrease and tend to zero. This is the well-known liquefaction phenomenon (Fig. 1). Clearly, such experimental evidence highlights the existence of failure modes that cannot be described with the classical tools (failure as a plastic limit state) since the collapse is observed well before the failure Mohr–Coulomb line is reached.

In this paper, examples of numerical simulations (based on a discrete element method) will also be given, showing that sudden collapses of a numerical granular specimen can occur depending on the loading applied. These simulations demonstrate that, for certain loading directions that will be specified below, a very small perturbation is sufficient to collapse the specimen, corresponding to a proper bifurcation mode from a quasi-static regime toward a dynamic regime.

Loss of sustainability will be briefly reviewed to show how the vanishing of the second-order work, through directional analysis, provides a proper criterion to detect the occurrence of this type of failure mode. Finally, it will be established that the loss of sustainability framework encompasses the notions of loss of constitutive uniqueness and loss of controllability, thus providing them with a clear physical meaning, suggesting that a single framework such as loss of sustainability could be relevant to materials engineering.

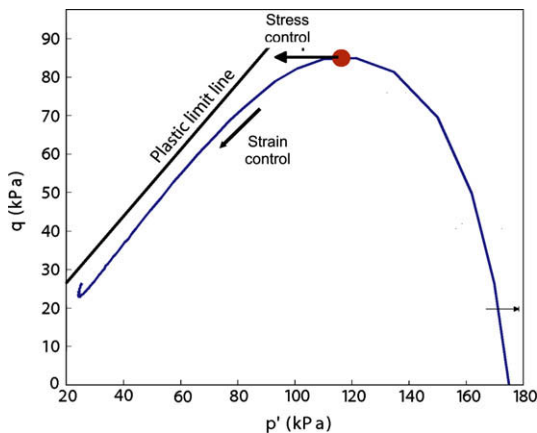


Fig. 1. Typical undrained triaxial behavior of a loose sand. At the q -peak, following the axial control parameters, the specimen can liquefy (strain control) or can give rise to a sudden failure (stress control). After Darve et al. (2004).

It must be emphasized that the framework presented in this paper was developed in the general case of nonviscous (rate-independent) materials. No particular constitutive model is considered to infer the main results, therefore making them general.

Time and spatial derivatives of any variable ψ will be distinguished by denoting $\delta\psi$ the time derivative of ψ (defined as the product of the particulate derivative $\dot{\psi}$ by the infinitesimal time increment δt) with respect to a given frame, and by denoting $d\psi$ the spatial derivative of ψ . For any (one- or two-order) tensor A , tA denotes the transpose tensor.

2. The collapse mechanism

2.1. The fundamental equation

The collapse of a granular material corresponds to a brutal change in the microstructure of the specimen, associated with a sharp decrease in the number of contacts between granules. The material is no longer able to sustain any deviatoric loading. The rapid relative displacements between grains directs a large number of opening contacts. The collapse mechanism is therefore related to an abrupt increase in kinetic energy. As a consequence, describing the collapse mechanism requires describing how the kinetic energy of a given material system can increase.

For this purpose, a system made up of a volume V_o of a given material, initially in a configuration C_o (initial configuration) is considered. After a loading history, the system is in a strained configuration C and occupies a volume V , in equilibrium under a prescribed external loading. The current boundary (Γ) of the material can be resolved into one part (Γ_σ) controlled by static parameters and a complementary part (Γ_ε) controlled by kinematic parameters. \vec{f}^{Γ_σ} denotes the surface density of force applied to (Γ_σ) and $\vec{u}^{\Gamma_\varepsilon}$ represents the displacement field imposed at each point of (Γ_ε).

The instantaneous evolution of the system, in the equilibrium configuration C at time t , is governed by the following energy conservation equation that includes dynamical effects:

$$\delta E_c(t) = \int_{\Gamma} \sigma_{ij} n_j \delta u_i dS - \int_V \sigma_{ij} \frac{\partial(\delta u_i)}{\partial X_j} dV \quad (1)$$

where δE_c represents the system's current change in kinetic energy related to the incremental displacement field $\delta \vec{u}$. $\vec{\sigma}$ is the Cauchy stress tensor, and \vec{n} is the current normal to the boundary (Γ) at the point considered. Eq. (1) represents the Euler form of the energy conservation, since all variables are given with respect to the current evolving configuration. In this configuration, any material point is described by the coordinate vector \vec{x} . It is convenient to express the integrals in Eq. (1) with respect to the initial configuration, which yields:

$$\delta E_c(t) = \int_{\Gamma_o} \Pi_{ij} N_j \delta u_i dS_o - \int_{V_o} \Pi_{ij} \frac{\partial(\delta u_i)}{\partial X_j} dV_o \quad (2)$$

where $\bar{\Pi}$ denotes the Piola–Kirchhoff stress tensor of the first type, and Γ_o is the V_o boundary. $\bar{\Pi}$ and \bar{N} are, respectively, the transformed quantities of $\vec{\sigma}$ and \vec{n} through the bijection ϑ mapping the material points from the reference configuration to the current configuration: $\vec{x} = \vartheta(\vec{X})$. This bijective transformation is convenient so as to obtain all integrals given with respect to a fixed domain, that is Γ_o and V_o . Thus, the time differentiation of Eq. (2) can be performed in a straightforward manner, without referring to a Reynolds transform. Taking into account Green's formula, differentiating Eq. (2) gives (Nicot et al., 2007):

$$\delta^2 E_c(t) = \int_{\Gamma_o} \delta \Pi_{ij} N_j \delta u_i dS_o - \int_{V_o} \delta \Pi_{ij} \frac{\partial(\delta u_i)}{\partial X_j} dV_o \quad (3)$$

Following Hill's definition (Hill, 1958), $W_2 = \int_{V_0} \delta \Pi_{ij} \frac{\partial(\delta u_i)}{\partial x_j} dV_0$ denotes the second-order work of the system, associated with the incremental evolution $(\delta \Pi_{ij}, \delta(\frac{\partial u_i}{\partial x_j}))$. Both incremental quantities $\delta(\frac{\partial u_i}{\partial x_j})$ and $\delta \Pi_{ij}$ are related through the constitutive equation. Moreover, for the first integral of Eq. (3), at each point of the boundary Γ_o , $\delta f_i^{T_o} = \delta \Pi_{ij} N_j$. Thus, Eq. (3) also reads:

$$\delta^2 E_c(t) = \int_{\Gamma_o} \delta f_i^{T_o} \delta u_i dS_o + \int_{\Gamma_e} \delta f_i \delta u_i^{T_e} dS_o - W_2 \quad (4)$$

In addition, the two-order Taylor expansion of kinetic energy reads:

$$E_c(t + \Delta t) = E_c(t) + \Delta t \dot{E}_c(t) + \frac{(\Delta t)^2}{2} \ddot{E}_c(t) + o(\Delta t)^3 (\forall \Delta t) \quad (5)$$

Noting that $E_c(t) = \int_{V_0} \rho_0 \|\dot{\mathbf{u}}\|^2 dV_0$, where ρ_0 is the density of the material in the initial configuration at point $M(\vec{X})$, since the system is in an equilibrium state at time t , then $E_c(t) = 0$. Furthermore, $\dot{E}_c(t) = \int_{V_0} \rho_0 \dot{\mathbf{u}} \cdot \ddot{\mathbf{u}} dV_0$, and at time t , $\dot{E}_c(t) = 0$. Eq. (5) therefore reads:

$$\delta^2 E_c(t) = \frac{2E_c(t + \Delta t)}{(\Delta t)^2} \delta t^2 + o(\Delta t) \delta t^2 \quad (6)$$

Thus, Eq. (6) establishes that the kinetic energy of the system at the subsequent time $t + \Delta t$ is a second-order term. Ignoring third-order terms and making $\Delta t \rightarrow \delta t$, then $\delta^2 E_c(t) = 2E_c(t + \delta t)$; in combination with Eq. (4), it follows that:

$$2E_c(t + \delta t) = \int_{\Gamma_o} \delta f_i^{T_o} \delta u_i dS_o + \int_{\Gamma_e} \delta f_i \delta u_i^{T_e} dS_o - W_2 \quad (7)$$

Eq. (7) is the fundamental equation that relates the kinetic energy of the system to the second-order work. It should be emphasized that Eq. (7) holds true only when the system is in an equilibrium state at time t . It becomes clear that the second-order work should play a fundamental role with respect to the collapse mechanism of a material system. However, to go further, the role of both boundary integrals $\int_{\Gamma_o} \delta f_i^{T_o} \delta u_i dS_o$ and $\int_{\Gamma_e} \delta f_i \delta u_i^{T_e} dS_o$ has to be specified. For this purpose, it is convenient to consider the notion of the loss of sustainability.

2.2. Loss of sustainability

Loss of sustainability is briefly reviewed here. Greater detail can be found in (Nicot and Darve, 2007; Nicot et al., 2007). At time t , the system is in an equilibrium configuration, under prescribed control parameters acting on the boundary. First, the control parameters are assumed to be given by the distributions of \vec{f}^{T_o} and \vec{u}^{T_e} over the domains (Γ_o) and (Γ_e) . The analysis is specialized by prescribing the control parameters to remain constant: $\delta f_i^{T_o} = 0$ on (Γ_o) and $\delta u_i^{T_e} = 0$ on (Γ_e) . In that case, Eq. (7) simplifies to:

$$2E_c(t + \delta t) = -W_2 \quad (8)$$

As a consequence, the kinetic energy of the system may increase (from a zero value to a strictly positive value) if and only if an incremental evolution $(\delta \vec{\Pi}, \overline{\text{grad}}(\delta \vec{u}))$ exists, which is compatible with the prescribed boundary conditions, and such that the second-order work is strictly negative. An increase in kinetic energy also means that the equilibrium state defined by both stress and strain states cannot be sustained. A class of infinitesimal perturbations exists such that, when applied at time t , the system reaches the mechanical state defined by both stress $\vec{\Pi} + \delta \vec{\Pi}$ and displacement $\vec{u} + \delta \vec{u}$ fields at the subsequent time $t + \delta t$. This is related to a negative value of the second-order work, defined as the inner product of $\delta \vec{\Pi}$ and $\overline{\text{grad}}(\delta \vec{u})$. Thus, when both incremental fields $\delta \vec{\Pi}$ and $\overline{\text{grad}}(\delta \vec{u})$ exist, related by the constitutive equation, compatible with the prescribed boundary conditions and such that the second-order work is

strictly negative, the equilibrium state of the system $(\vec{\Pi}, \overline{\text{grad}}(\vec{u}))$ is said to be unsustainable.

The notion of loss of sustainability is therefore a theoretical framework to describe the occurrence of the collapse of a material system under constant control parameters. The next section considers this theoretical framework in the particular case of a material point.

3. The material point scale

The above theoretical framework is now considered on the material point scale. Investigating this elementary scale can be useful in, for example, standard homogeneous laboratory tests and the interpretation of the derived experimental results, where (cubic) specimens subjected on each wall to a prescribed force or displacement directing both homogeneous stress and strain fields can be studied.

3.1. Formulation in homogeneous conditions

Let us consider a cubic specimen (the area of each side is denoted S and the length of each side is denoted L) subjected to axisymmetric loading paths. Index '1' refers to the axial direction (major principal direction), whereas indices '2' and '3' refer to the two lateral directions perpendicular to the axial direction (Fig. 2). Restricting the analysis to axisymmetric conditions, we denote F_1 (resp. u_1) the force (resp. displacement) acting on the lower (and upper) sides, and F_3 (resp. u_3) the force (resp. displacement) acting on each lateral side. In what follows and for the sake of simplicity, $\delta(\overline{\text{grad}}(\vec{u}))$ will be denoted by $\delta \vec{E}$.

In these conditions, Eq. (7) is expressed as:

$$2E_c(t + \delta t) = \delta F_1 \delta u_1 + 2\delta F_3 \delta u_3 - W_2 \quad (9)$$

with $W_2 = \int_V (\delta \Pi_1 \delta E_1 + 2\delta \Pi_3 \delta E_3) dV$. As far as both stress and strain fields within the specimen can be regarded as homogeneous, the second-order work reads $W_2 = V(\delta \Pi_1 \delta E_1 + 2\delta \Pi_3 \delta E_3)$.

3.2. The directional analysis

How can one detect a loss of sustainability for a specimen in an equilibrium mechanical state defined by $(\vec{\Pi}, \vec{E})$? To answer this question, it is necessary to check whether both incremental stress ${}^t(\delta \Pi_1, \sqrt{2}\delta \Pi_3)$ and strain ${}^t(\delta E_1, \sqrt{2}\delta E_3)$ vectors exist, related through the constitutive equation, such that $W_2 < 0$. For this purpose, the directional analysis (initially introduced by Gudehus (1979) to build the so-called response-envelopes) is particularly

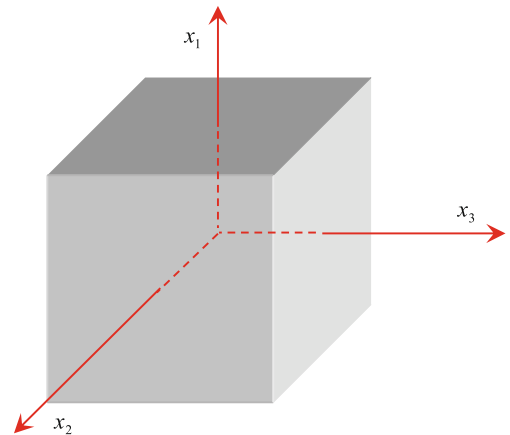


Fig. 2. Cubic specimen and definition of the axes.

convenient. Incremental stress probes are imposed along all the directions within the Rendulic plane (axisymmetric stress plane) and with a fixed (small) norm (namely, 1 kPa), and the conjugate incremental strain is computed through the constitutive equation. Then the normalized second-order work is formed:

$$W_2 = \frac{\delta\Pi_1\delta E_1 + 2\delta\Pi_3\delta E_3}{\sqrt{\delta\Pi_1^2 + 2\delta\Pi_3^2}\sqrt{\delta E_1^2 + 2\delta E_3^2}} \quad (10)$$

The normalized second-order work, which corresponds to the cosine of the angle θ between both incremental strain and stress vectors, is basically a directional quantity. It is therefore suitable to plot the results on a polar diagram, where the normalized second-order work is reported along each stress direction. Examples of such diagrams are given in Fig. 3. These diagrams were obtained from two fundamentally different models: a phenomenological model (Darve's octolinear model; Darve and Labanieh, 1982) on the left and a micromechanically based model (Nicot's microdirectional model; Nicot and Darve, 2005) on the right. The same procedure was applied to both models. After an initial triaxial loading path in drained axisymmetric conditions, a directional analysis was performed at different values of the deviatoric stress ratio $\eta = q/p$ ($q = \Pi_1 - \Pi_3$ and $p = \frac{\Pi_1 + 2\Pi_3}{3}$). In both cases, it is observed that the second-order work takes negative values within a cone of incremental stress directions from a certain value of η . Of course, this critical value of η is not exactly the same for the two models, but it is notable that the negative values of the second-order work are encountered along approximately the same incremental stress directions, within the third quadrant ($\delta\Pi_1 < 0$, $\sqrt{2}\delta\Pi_3 < 0$).

Let us assume that a given stress probe $\delta\vec{\Pi}$ exists such that $\delta\vec{\sigma} \cdot \delta\vec{\varepsilon} < 0$. The basic question that arises is to know whether the system can reach the mechanical state $(\vec{\sigma} + \delta\vec{\sigma}, \vec{\varepsilon} + \delta\vec{\varepsilon})$ without any change in a given set of control parameters. Given the directional nature of the second-order work, the incremental stress direction $\alpha_{\Pi} = A \tan \frac{\delta\Pi_1}{\sqrt{2}\delta\Pi_3}$ (or $\alpha_{\Pi} = A \tan \frac{\delta\Pi_1}{\sqrt{2}\delta\Pi_3} + \pi$ if $\delta\Pi_3 < 0$) leads to a strictly negative value of W_2 , whatever the norm of the stress vector ${}^t(\delta\Pi_1, \sqrt{2}\delta\Pi_3)$. Setting $R = \tan \alpha_{\Pi}$, the control parameter $C_1 = F_1 - \sqrt{2}RF_3$ is introduced. Since:

$$\delta F_1 \delta u_1 + 2\delta F_3 \delta u_3 = (\delta F_1 - \sqrt{2}R\delta F_3)\delta u_1 + R\left(\delta u_1 + \frac{\sqrt{2}}{R}\delta u_3\right)\sqrt{2}\delta F_3 \quad (11)$$

it is quite natural to define $C_2 = u_1 + \frac{\sqrt{2}}{R}u_3$ as the second control parameter. If the control parameters are imposed to remain constant, then $\delta F_1 \delta u_1 + 2\delta F_3 \delta u_3 = 0$, and Eq. (9) yields:

$$2E_c(t + \delta t) = -W_2 \quad (12)$$

Let us assume that $(\vec{\Pi} + \delta\vec{\Pi}, \vec{E} + \delta\vec{E})$ corresponds to an equilibrium state. Then it follows that:

$$\begin{aligned} \delta F_i &= S\delta\Pi_i \\ \delta u_i &= L\delta E_i \end{aligned} \quad \text{with } i = 1, 3 \quad (13)$$

Eq. (13) implies that:

$$\delta F_1 \delta u_1 + 2\delta F_3 \delta u_3 = SL(\delta\Pi_1\delta E_1 + 2\delta\Pi_3\delta E_3) = VW_2 \quad (14)$$

Since the control parameters are imposed to remain constant, $\delta F_1 \delta u_1 + 2\delta F_3 \delta u_3 = 0$, and Eq. (14) yields $W_2 = 0$. The condition $\delta C_1 = 0$ implies that the incremental evolution $(\delta\Pi_1, \sqrt{2}\delta\Pi_3)$ is characterized by the stress direction $\frac{\delta\Pi_1}{\sqrt{2}\delta\Pi_3} = \frac{\delta F_1}{\sqrt{2}\delta F_3} = R$. Thus, this incremental evolution is associated with a strictly negative value of W_2 , which is in contradiction with $W_2 = 0$. As a consequence, the subsequent state $(\vec{\Pi} + \delta\vec{\Pi}, \vec{E} + \delta\vec{E})$ does not correspond to an equilibrium state.

In conclusion, if the control parameters C_1 and C_2 are conveniently chosen and maintained constant, the system's equilibrium is upset. This change to off-equilibrium is associated with an increase in kinetic energy, such that $E_c(t + \delta t) = -\frac{1}{2}W_2$. This increase in kinetic energy results in the specimen collapsing.

This investigation has pointed out the fundamental role played by the vanishing of the second-order work in detecting the occurrence of a collapse mechanism. The vanishing of the second-order work can be checked in a very straightforward manner with (stress) directional analysis. In addition, the choice of the control parameters was highlighted; this choice is directed by the knowledge of the stress directions giving negative values in the second-order work.

The purpose of the next section is to evaluate these theoretical results based on discrete element simulations of particle assemblies.

3.3. Discrete element simulations

The discrete element method (Cundall and Strack, 1979; Cundall and Roger, 1992) is convenient to simulate the response of an assembly of particles subjected to a given loading program. The considerable progress in computation power, including the high memory capacity calculation rate, makes this method a relevant alternative to laboratory testing to investigate the mechanical behavior of granular materials. The same numerical sample can be loaded along a variety of loading programs, and the changes in

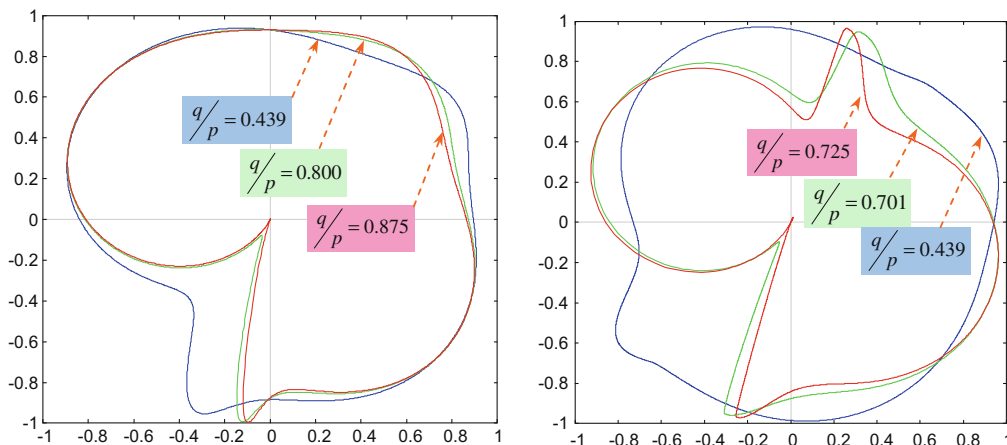


Fig. 3. Polar representation of the second-order work along incremental stress directions (octolinear model on the left side, micro-directional model on the right side) for different deviatoric ratios $\eta = 3(\Pi_1 - \Pi_3)/(\Pi_1 + 2\Pi_3)$ (after Nicot and Darve (2006)).

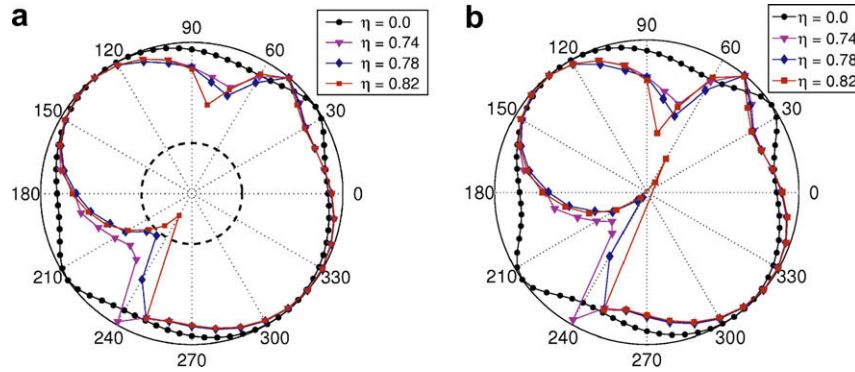


Fig. 4. Discrete element simulations. (a) Circular and (b) polar representations of the second-order work versus stress direction for different deviatoric stress ratios $\eta = q/p$ in axisymmetric conditions. For the circular representation, the second-order work is negative inside the dashed circle (whose radius is r) and is positive outside ($W_2^{\text{circular}} = W_2^{\text{polar}} + r$).

various internal parameters describing contact forces, relative displacements, or packing can be easily tracked. In addition, as inertial mechanisms are taken into account for each particle, the transition from a quasi-static regime to a dynamic regime can be investigated. The introduction of very simple, basic laws on the contact scale is of course advantageous, since only three mechanical parameters and a single numerical parameter are required.

The simulations presented in this paper were carried out with the computational software SDEC (Magnier and Donzé, 1998). Following the molecular dynamic approach popularized for granular media by Cundall (Cundall and Roger, 1992), each element is described as an elastic body subjected to contact forces occurring when two neighboring granules are in contact. The motion of each particle is determined by solving the balance equations, which by time integration yields the location of the particles. The procedure adopted by this software relates the incremental normal and tangential contact forces to the incremental normal and tangential displacements through standard frictional elastic–plastic relations. An outstanding point is that for a given assembly, defined by a given fabric, only three mechanical parameters are required: both normal k_n and tangential k_t elastic stiffness, together with the local friction angle φ_g between contacting grains.

Numerical simulations were carried out with a granular sample contained in a cubic box whose rigid lateral, lower, and upper walls can move without tilting. A fixed frame $\{x_1, x_2, x_3\}$ whose axes are perpendicular to the walls of the box is attached to the physical space, as described in Fig. 1. The sample is composed of a random assembly of approximately 10,000 spherical grains, and the initial porosity was chosen equal to 0.38 (the sample exhibits a dilatant behavior during a drained triaxial loading). The particle diameter distribution ranges continuously from 2.0 to 9.5 mm. The mechanical parameters were fixed as follows: $k_n/d_s = 356$ MPa, where d_s is the mean particle diameter, $k_t/k_n = 0.42$ and $\varphi_g = 35$ deg.

For the simulations presented throughout this paper, nonviscous damping is applied to a given particle by a damping coefficient κ , consisting in reducing driving unbalanced external forces and increasing unbalanced external forces opposed to particle motion (Cundall, 1987). κ ranges between 0 and 1. When $\kappa = 1$, there is no damping, whereas for $\kappa = 0$, particles can no longer move. In the following simulations, $\kappa = 0.95$, corresponding to a very small damping effect. It was shown that this coefficient value alters the quantitative results only very slightly (with respect to a vanishing value of κ), for kinetic energy of the whole granular assembly lower than 0.01 J (corresponding to the range of kinetic energy shown in Fig. 5). Moreover, the qualitative nature of the results was not altered. From a mechanical state within the bifurcation domain, the collapse of the specimen can be observed for any value of the

damping coefficient within the range considered: $0.8 < \kappa < 1$ (depending of course on the loading direction).

After an initial isotropic compression, a drained triaxial compression test is simulated in axisymmetric conditions ($\Pi_2 = \Pi_3 = 100$ kPa); then, at different loading states corresponding to increasing values of the deviatoric ratio ($\eta = 0.0, \eta = 0.74, \eta = 0.78$ and $\eta = 0.82$), a stress directional analysis is performed. A stress increment $\Delta \vec{\Pi}$ in all directions of the Rendulic plane ($\sqrt{2}\Delta \Pi_3, \Delta \Pi_1$) with the same norm (1 kPa) is imposed and the strain response $\Delta \vec{E}$ is computed.

As seen in Fig. 4a, stress loading directions exist for $\eta = 0.82$ that lead to negative values of the second-order work. The great similarity in the general shape of the polar diagrams depicted in Fig. 4b with those given in Fig. 3 should be noted. Even though three fundamentally different constitutive approaches have been considered, the same shape of polar diagrams of the second-order work is observed, and the second-order work is negative within a cone located in the same region of the third stress quadrant ($\Delta \Pi_1 < 0, \sqrt{2}\Delta \Pi_3 < 0$) (Sibille et al., 2007).

In the following, the loading state corresponding to $\eta = 0.82$ is considered. As seen in Fig. 4a, the second-order work is negative for the stress directions such that $223 \text{ deg} < \alpha_{\Pi} < 245 \text{ deg}$, or, as $R = \tan \alpha_{\Pi}$, such that $0.933 < R < 2.14$. As specified in the previous section, the control parameters $C_1 = F_1 - \sqrt{2}RF_3$ and $C_2 = u_1 + \frac{\sqrt{2}}{R}u_3$ are chosen and imposed to remain constant. Different values of the R -parameter are considered, inside or outside the range [0.933, 2.14]: $R_1 = 0.839, R_2 = 1.19, R_3 = 1.43, R_4 = 1.73$ and $R_5 = 2.75$. Starting from the same equilibrium state ($\eta = 0.82$) for each value of R , an infinitesimal perturbation is applied. This infinitesimal perturbation corresponds to an input of kinetic energy (namely, 10^{-5} J) applied to any particle belonging to the weak phase.¹ This input of kinetic energy is small with respect to the magnitude of kinetic energy (10^{-4} J) developed by the sample during fully stress-controlled probes. Then the condition $\delta C_1 = 0$ imposes that $\frac{\delta \Pi_1}{\sqrt{2}\delta \Pi_3} = \frac{\delta F_1}{\sqrt{2}\delta F_3} = R$. Thus, the instantaneous evolution of the system is such that $W_2 < 0$ or $W_2 > 0$, depending on the value of R . Both conditions $\delta C_1 = 0$ and $\delta C_2 = 0$ ensure that $\delta F_1 \delta u_1 + 2\delta F_3 \delta u_3 = 0$, and thus that $E_c(t + \delta t) = -\frac{1}{2}W_2$. An increase in the kinetic energy of the specimen is therefore expected for the R -values corresponding to $W_2 < 0$. Fig. 5 reveals that for the values $R_2 = 1.19$ and $R_3 = 1.43$ (strictly inside the range corresponding to negative values of second-order work), an abrupt increase in kinetic energy develops, until the specimen entirely collapses by an exponential

¹ The weak phase, by opposition to the strong phase, corresponds to the set of contacts transmitting normal forces lower than the mean normal force over the whole assembly.

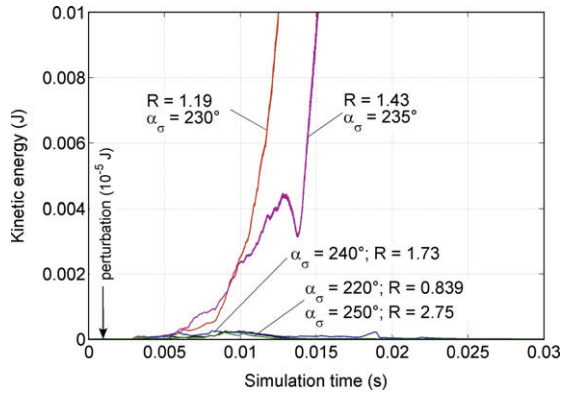


Fig. 5. Discrete simulations of loss of sustainability. A burst in kinetic energy is observed for the values of the R -parameter associated with a negative second-order work.

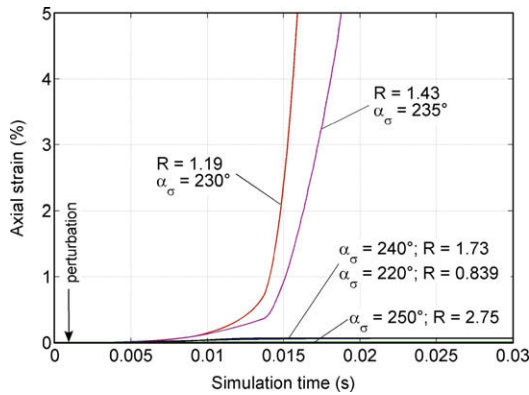


Fig. 6. Exponential growing of strains for the values of the R -parameter associated with a negative second-order work.

increase in strains (Fig. 6). When the value $R_4 = 1.73$ is considered (corresponding to a direction inside the range, close to the upper limit of the cone), the increase in kinetic energy is very low, and the specimen recovers an equilibrium state.² Nevertheless, the sample was close to collapse, with a strong decrease in both axial and lateral stresses. Finally, when values $R_1 = 0.839$ and $R_5 = 2.75$ (strictly outside the range) are considered, the initial increase in kinetic energy is very small and the initial stress-strain state is only slightly perturbed. No visible pattern of collapse is observed.

As a conclusion, the results derived from the numerical simulations are perfectly in line with the theoretical framework (of the loss of sustainability) developed in the previous sections. When the control parameters $C_1 = F_1 - \sqrt{2}RF_3$ and $C_2 = u_1 + \frac{\sqrt{2}}{R}u_3$ are chosen with a R -value corresponding to an incremental stress loading direction associated with a negative value of the second-order work, then the application of a very small perturbation under these control parameters maintained constant implies a dramatic increase in kinetic energy, leading to the collapse of the specimen by exponentially increasing strains. On the other hand, when the control parameters are associated with a strictly positive value of the second-order work, the specimen is able to recover an equilibrium state after the application of a perturbation. These results confirm the relevance of the theoretical notion of loss of sustain-

ability to describe collapse mechanisms of a granular material. In the next section, this theoretical framework is presented by adopting an algebraic point of view, making it possible to recover the notions of loss of constitutive uniqueness (Bigoni and Hueckel, 1991; Darve et al., 2004) or loss of controllability (Nova, 1994), and finally to propose a possible unified approach.

3.4. Constitutive uniqueness, controllability and sustainability

In this section, two-dimensional conditions are considered, so that the calculations do not become overly complex. The results can be extended to three-dimensional conditions.

3.4.1. Mathematical investigations on loading and response parameters

The mechanical response of a granular specimen (regarded as a material point) under a given incremental loading can also be analyzed through a relation between the control parameters that define the loading and the response parameters. The control parameters consist of a set of linear combinations of the external forces or displacements applied to the boundary of the specimen. Loading is controlled at the boundary of the specimen. Likewise, for simple materials in Noll's sense (see for instance Noll, 1972), the response of the specimen is characterized by parameters measured on its boundary. In axisymmetry or, broadly speaking, in two-dimensional conditions, two control parameters, C_1 and C_2 , and two response parameters, R_1 and R_2 , are necessary. Parameters C_i and R_i must be conjugate, that is C_i is intensive, R_i is extensive (or conversely), and their product scales to an energy. Namely, if the control parameters are $C_1 = \lambda_1 F_1 + \lambda_2 F_2$ and $C_2 = \lambda_3 u_1 + \lambda_4 u_2$, the response parameters can be chosen, respectively, as $R_1 = \mu_1 u_1 + \mu_2 u_2$ and $R_2 = \mu_3 F_1 + \mu_4 F_2$. It must be emphasized that the same variable cannot be used both as a control and a response parameter. Thus, both parameters C_1 and R_2 , as polynomial functions of F_1 and F_2 , must be independent, which requires that:

$$\lambda_1 \mu_4 \neq \lambda_2 \mu_3 \quad (15a)$$

The same holds true for parameters C_2 and R_1 , which gives:

$$\lambda_3 \mu_2 \neq \lambda_4 \mu_1 \quad (15b)$$

An additional standard constraint is now considered (Nova, 1994). This constraint states that:

$$\delta C_1 \delta R_1 + \delta C_2 \delta R_2 = \delta F_1 \delta u_1 + \delta F_2 \delta u_2 \quad (16)$$

For example, the undrained biaxial test is controlled by imposing a constant deviatoric stress rate under a constant volume. Thus the control parameters are $C_1 = F_1 - F_2$ and $C_2 = u_1 + u_2$, and the response parameters can be conveniently chosen as $R_1 = u_1$ and $R_2 = F_2$, ensuring that $\delta C_1 \delta R_1 + \delta C_2 \delta R_2 = \delta F_1 \delta u_1 + \delta F_2 \delta u_2$.

So as to investigate the consequences of Eq. (16), the hybrid case where C_1 is a linear combination of F_1 and F_2 , and C_2 is a linear combination of u_1 and u_2 , is considered. It is useful to express both control and response parameters in the following form:

$$\begin{aligned} C_1 &= \chi_1 (\cos \alpha_F F_1 + \sin \alpha_F F_2) & \text{and} & & C_2 &= \chi_2 (\cos \alpha_u u_1 + \sin \alpha_u u_2) \\ R_1 &= \rho_1 (\cos \beta_u u_1 + \sin \beta_u u_2) & \text{and} & & R_2 &= \rho_2 (\cos \beta_F F_1 + \sin \beta_F F_2) \end{aligned}$$

where χ_1, χ_2, ρ_1 and ρ_2 are positive. Under a matricial formalism, we have:

$$\begin{bmatrix} C_1 \\ R_2 \end{bmatrix} = \overline{\overline{T}}^F \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \quad \text{with} \quad \overline{\overline{T}}^F = \begin{bmatrix} \chi_1 \cos \alpha_F & \chi_1 \sin \alpha_F \\ \rho_2 \cos \beta_F & \rho_2 \sin \beta_F \end{bmatrix} \quad (17)$$

and

$$\begin{bmatrix} R_1 \\ C_2 \end{bmatrix} = \overline{\overline{T}}^u \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{with} \quad \overline{\overline{T}}^u = \begin{bmatrix} \rho_1 \cos \beta_u & \rho_1 \sin \beta_u \\ \chi_2 \cos \alpha_u & \chi_2 \sin \alpha_u \end{bmatrix} \quad (18)$$

² Since inertial terms are taken into account, the perfectly static equilibrium state (if it exists) is reached only asymptotically. Thus, when diffuse failure did not occur, the simulations were stopped at a quasi-static equilibrium close to the static equilibrium (if it exists) with respect to the kinetic energy.

As $\delta C_1 \delta R_1 + \delta C_2 \delta R_2 = \begin{bmatrix} \delta C_1 \\ \delta C_2 \end{bmatrix} \cdot \begin{bmatrix} \delta R_1 \\ \delta R_2 \end{bmatrix} = \left(\overline{\overline{T^F}} \begin{bmatrix} \delta F_1 \\ \delta F_2 \end{bmatrix} \right) \cdot \left(\overline{\overline{T^u}} \begin{bmatrix} \delta u_1 \\ \delta u_2 \end{bmatrix} \right)$, and recalling that for any matrix $\overline{\overline{A}}$ and vectors $\overline{\overline{a}}$ and $\overline{\overline{b}}$, $(\overline{\overline{Aa}}) \cdot \overline{\overline{b}} = \overline{\overline{A}} : (\overline{\overline{a}} \otimes \overline{\overline{b}})$, then it follows that:

$$\delta C_1 \delta R_1 + \delta C_2 \delta R_2 = \overline{\overline{T^F}} : \left[\begin{bmatrix} \delta F_1 \\ \delta F_2 \end{bmatrix} \otimes \left(\overline{\overline{T^u}} \begin{bmatrix} \delta u_1 \\ \delta u_2 \end{bmatrix} \right) \right] \quad (19)$$

which gives:

$$\delta C_1 \delta R_1 + \delta C_2 \delta R_2 = \overline{\overline{T^F}} : \left(\begin{bmatrix} \delta F_1 \\ \delta F_2 \end{bmatrix} \begin{bmatrix} \delta u_1 & \delta u_2 \end{bmatrix} \right) \overline{\overline{T^u}} \quad (20)$$

Recalling that for any two-order tensors $\overline{\overline{A}}, \overline{\overline{B}}$ and $\overline{\overline{C}}, \overline{\overline{A}} : (\overline{\overline{BC}}) = (\overline{\overline{AC}}) : \overline{\overline{B}} = (\overline{\overline{B^T A}}) : \overline{\overline{C}}$, Eq. (20) can be rewritten as:

$$\delta C_1 \delta R_1 + \delta C_2 \delta R_2 = \left(\overline{\overline{T^F T^u}} \right) : \left(\begin{bmatrix} \delta F_1 \\ \delta F_2 \end{bmatrix} \begin{bmatrix} \delta u_1 & \delta u_2 \end{bmatrix} \right) \quad (21)$$

Condition (16) imposes that $\overline{\overline{T^F T^u}} = I_2$, where I_2 denotes the identity matrix. Thus, both matrices $\overline{\overline{T^F}}$ and $\overline{\overline{T^u}}$ are commuting, and equation $\overline{\overline{T^u T^F}} = I_2$, after transposition, gives:

$$\overline{\overline{T^F T^u}} = I_2 \quad (22)$$

The algebraic form of Eq. (22) is:

$$\begin{bmatrix} \chi_1 \cos \alpha_F & \chi_1 \sin \alpha_F \\ \rho_2 \cos \beta_F & \rho_2 \sin \beta_F \end{bmatrix} \begin{bmatrix} \rho_1 \cos \beta_u & \chi_2 \cos \alpha_u \\ \rho_1 \sin \beta_u & \chi_2 \sin \alpha_u \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (23)$$

which gives:

$$\begin{bmatrix} \chi_1 \rho_1 \cos(\beta_u - \alpha_F) & \chi_1 \chi_2 \cos(\alpha_u - \alpha_F) \\ \rho_1 \rho_2 \cos(\beta_u - \beta_F) & \chi_2 \rho_2 \cos(\alpha_u - \beta_F) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (24)$$

Then:

$$\alpha_u - \alpha_F = \frac{\pi}{2}(\pi) \quad \text{and} \quad \beta_u - \beta_F = \frac{\pi}{2}(\pi) \quad (25)$$

and:

$$\chi_1 \rho_1 \cos(\beta_u - \alpha_F) = \chi_2 \rho_2 \cos(\alpha_u - \beta_F) = 1 \quad (26)$$

Eq. (25) give $(\beta_u - \alpha_F) + (\alpha_u - \beta_F) = (\pi)$, and by combination with Eq. (26), it can be obtained that $|\chi_1 \rho_1| = |\chi_2 \rho_2|$. The positiveness of χ_1, χ_2, ρ_1 , and ρ_2 finally yields that:

$$\chi_1 \rho_1 = \chi_2 \rho_2 \quad \text{and} \quad (\beta_u - \alpha_F) = \pm(\alpha_u - \beta_F) \quad (27)$$

It can be shown that necessarily $(\beta_u - \alpha_F) = -(\alpha_u - \beta_F)$, and thus $(\alpha_u - \alpha_F) = -(\beta_u - \beta_F)$. In conclusion, two cases are possible: $\alpha_u - \alpha_F = \frac{\pi}{2}$ and $\beta_F - \beta_u = \frac{\pi}{2}$; or $\alpha_F - \alpha_u = \frac{\pi}{2}$ and $\beta_u - \beta_F = \frac{\pi}{2}$. The geometrical interpretation of this result is given in Fig. 7, where $\alpha_u - \alpha_F = \frac{\pi}{2}$ and $\beta_F - \beta_u = \frac{\pi}{2}$ are illustrated. Both vectors $\overline{\overline{C}}_1$ and $\overline{\overline{C}}_2$,

on the one hand, and both vectors $\overline{\overline{R}}_1$ and $\overline{\overline{R}}_2$, on the other hand, are orthogonal in a frame (F_1, F_2) or (u_1, u_2) . This is therefore a *normality condition*.

However, it is worth noting that there is a priori no reason for having $\alpha_F = \beta_u(\overline{\overline{C}}_1 / \overline{\overline{R}}_1)$ and $\alpha_u = \beta_F(\overline{\overline{C}}_2 / \overline{\overline{R}}_2)$.

3.4.2. An attempt at a unified framework

Let us consider a granular specimen in an equilibrium state defined by $(\overline{\overline{\Pi}}^1, \overline{\overline{E}}^1)$, after a given loading history. The stress state is assumed to be strictly inside the plastic limit surface. The conditions required for the specimen to reach a novel equilibrium state defined by $(\overline{\overline{\Pi}}^2, \overline{\overline{E}}^2)$, after the application of an incremental loading defined by δC_1 and δC_2 are examined. In other words, we investigate in which conditions a regular relation exists between both incremental control $(\delta C_1, \delta C_2)$ and response $(\delta R_1, \delta R_2)$ parameters, which ensures the existence of a unique response $(\delta R_1, \delta R_2)$ to the loading $(\delta C_1, \delta C_2)$. By doing so, following Nova's line, we query the controllability of the loading program with the control parameters C_1 and C_2 .

As previously, the hybrid case where C_1 (resp. R_2) is a linear combination of F_1 and F_2 , and C_2 (resp. R_1) is a linear combination of u_1 and u_2 , is considered. As reported in Nicot et al. (2007), since the stress state considered is strictly inside the plastic limit surface, if only force combinations are considered for the control parameters, a unique displacement response always exists. Thus, both control and response parameters are expressed as $C_1 = \lambda_1 F_1 + \lambda_2 F_2$, $C_2 = \lambda_3 u_1 + \lambda_4 u_2$, $R_1 = \mu_1 u_1 + \mu_2 u_2$ and $R_2 = \mu_3 F_1 + \mu_4 F_2$. Assuming the existence of an equilibrium state makes it possible to write $F_i^j = SI_i^j$ and $u_i^j = LE_i^j$, with $i = 1, 2$ denoting the spatial direction and $j = 1, 2$ referring to the equilibrium state. Thus:

$$\frac{1}{S} \delta C_1 = \lambda_1 (\Pi_1^2 - \Pi_1^1) + \lambda_2 (\Pi_2^2 - \Pi_2^1) = \lambda_1 \delta \Pi_1 + \lambda_2 \delta \Pi_2 \quad (28a)$$

$$\frac{1}{L} \delta C_2 = \lambda_3 (E_1^2 - E_1^1) + \lambda_4 (E_2^2 - E_2^1) = \lambda_3 \delta E_1 + \lambda_4 \delta E_2 \quad (28b)$$

$$\frac{1}{L} \delta R_1 = \mu_1 (E_1^2 - E_1^1) + \mu_2 (E_2^2 - E_2^1) = \mu_1 \delta E_1 + \mu_2 \delta E_2 \quad (29a)$$

$$\frac{1}{S} \delta R_2 = \mu_3 (\Pi_1^2 - \Pi_1^1) + \mu_4 (\Pi_2^2 - \Pi_2^1) = \mu_3 \delta \Pi_1 + \mu_4 \delta \Pi_2 \quad (29b)$$

C_1, C_2, R_1 and R_2 are defined as linear combinations of Π_1, Π_2, E_1 and E_2 , with the given set of parameters $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \mu_1, \mu_2, \mu_3, \mu_4\}$. The transition from an equilibrium state to another equilibrium state is governed by the homogeneous constitutive relation:

$$\begin{bmatrix} \delta \Pi_1 \\ \delta \Pi_2 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} \delta E_1 \\ \delta E_2 \end{bmatrix} \quad (30)$$

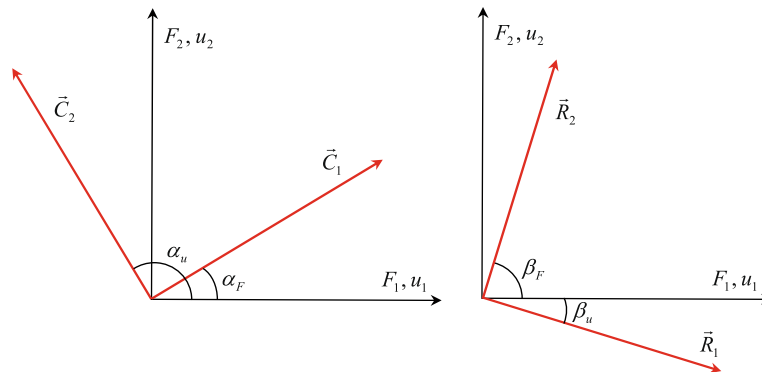


Fig. 7. Orthogonality of both control vectors $\overline{\overline{C}}_1$ and $\overline{\overline{C}}_2$, and response vectors $\overline{\overline{R}}_1$ and $\overline{\overline{R}}_2$.

where $\bar{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$ denotes the tangent stiffness matrix. In the sequel, the material considered is assumed to be nonassociated, and \bar{K} is therefore nonsymmetric. In addition, as the stress state is strictly inside the plastic limit surface, $\det(\bar{K}) > 0$. Taking conditions (15) into account, both matrices $\bar{T}^F = \begin{bmatrix} \lambda_1 & \lambda_2 \\ \mu_3 & \mu_4 \end{bmatrix}$ and $\bar{T}^u = \begin{bmatrix} \mu_1 & \mu_2 \\ \lambda_3 & \lambda_4 \end{bmatrix}$ are nonsingular. Eq. (30) is therefore equivalent to the following equation:

$$\begin{bmatrix} \lambda_1 & \lambda_2 \\ \mu_3 & \mu_4 \end{bmatrix} \begin{bmatrix} \delta\Pi_1 \\ \delta\Pi_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ \mu_3 & \mu_4 \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} \mu_1 & \mu_2 \\ \lambda_3 & \lambda_4 \end{bmatrix}^{-1} \begin{bmatrix} \mu_1 & \mu_2 \\ \lambda_3 & \lambda_4 \end{bmatrix} \begin{bmatrix} \delta E_1 \\ \delta E_2 \end{bmatrix} \quad (31)$$

which yields:

$$\begin{bmatrix} \delta C_1 \\ \delta R_2 \end{bmatrix} = \bar{H} \begin{bmatrix} \delta R_1 \\ \delta C_2 \end{bmatrix} \quad (32)$$

$$\text{with } \bar{H} = \frac{1}{L} \begin{bmatrix} \lambda_1 & \lambda_2 \\ \mu_3 & \mu_4 \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} \mu_1 & \mu_2 \\ \lambda_3 & \lambda_4 \end{bmatrix}^{-1}$$

Assuming the normality condition, Eq. (22) holds. Thus:

$$\bar{H} = \frac{S}{L} \bar{T}^F \bar{K}^t \bar{T}^F \quad (33)$$

It follows that:

$$\bar{H} - {}^t\bar{H} = \frac{S}{L} \bar{T}^F (\bar{K} - {}^t\bar{K}) {}^t\bar{T}^F \quad (34)$$

Since \bar{T}^F is invertible, $\bar{K} - {}^t\bar{K} \neq \bar{0}$ yields that $\bar{H} - {}^t\bar{H} \neq \bar{0}$, indicating that \bar{H} is also nonsymmetric.

The existence of a response $(\delta R_1, \delta R_2)$ to the loading controlled by $(\delta C_1, \delta C_2)$ imposes that a nonsingular tensor \bar{M} exists, such that:

$$\begin{bmatrix} \delta R_1 \\ \delta R_2 \end{bmatrix} = \bar{M} \begin{bmatrix} \delta C_1 \\ \delta C_2 \end{bmatrix} \quad (35)$$

Eq. (35) is equivalent to Eq. (30) if and only if $H_{11} \neq 0$. If this condition is fulfilled, then:

$$\bar{M} = \begin{bmatrix} \frac{1}{H_{11}} & -\frac{H_{12}}{H_{11}} \\ \frac{H_{21}}{H_{11}} & \frac{H_{11}H_{22} - H_{12}H_{21}}{H_{11}} \end{bmatrix} \quad (36)$$

When $H_{11} \neq 0$, the system can be controlled by the parameters C_1 and C_2 . If an incremental loading defined by $(\delta C_1, \delta C_2)$ is imposed, a unique incremental response $(\delta R_1, \delta R_2)$ exists. On the other hand, when $H_{11} = 0$, a unique incremental response $(\delta R_1, \delta R_2)$ to any incremental loading $(\delta C_1, \delta C_2)$ can no longer be found. The system cannot reach an equilibrium state under the prescribed boundary conditions. The controllability of the loading program with the parameters C_1 and C_2 is lost.

As $\bar{H} = \frac{S}{L} \bar{T}^F \bar{K}^t \bar{T}^F$, it follows that:

$$\begin{aligned} \frac{L}{S} H_{11} &= K_{11} \lambda_1^2 + (K_{12} + K_{21}) \lambda_1 \lambda_1 + K_{22} \lambda_2^2 \\ &= K_{11} \lambda_4^2 - (K_{12} + K_{21}) \lambda_3 \lambda_4 + K_{22} \lambda_3^2 \end{aligned} \quad (37)$$

Noting that $\bar{K}^s = \begin{bmatrix} K_{11} & \frac{K_{12}+K_{21}}{2} \\ \frac{K_{12}+K_{21}}{2} & K_{22} \end{bmatrix}$, it follows that:

$$\frac{L}{S} H_{11} = (\lambda_1, \lambda_2) \bar{K}^s \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = (\lambda_4, -\lambda_3) \bar{K}^s \begin{pmatrix} \lambda_4 \\ -\lambda_3 \end{pmatrix} \quad (38)$$

As a consequence, H_{11} is a quadratic form associated with the matrix \bar{K}^s . The existence of vectors $(\lambda_1, \lambda_2) \neq (0, 0)$, or $(\lambda_4, \lambda_3) \neq (0, 0)$, such that $H_{11} = 0$ requires that at least one eigen value of \bar{K}^s be nil. Thus, when the normality condition is assumed, the vanishing of the determinant of \bar{K}^s plays a fundamental role in the existence

of a noncontrollable loading program. Moreover, it must be noted that H_{11} depends only on terms λ_i (associated with control parameters) and not on terms μ_i (associated with response parameters). Eq. (38) gives the terms λ_i leading to the vanishing of H_{11} . For these values of λ_i , $H_{11} = 0$ whatever the choice of response parameters.

Now, let us consider a directional analysis carried out from the equilibrium state defined by $(\bar{\Pi}^1, \bar{E}^1)$. A stress loading $(\delta\Pi_1, \delta\Pi_2)$ is applied, and the strain response $(\delta E_1, \delta E_2)$ is obtained. Then, the second-order work $W_2 = \delta\Pi_1 \delta E_1 + \delta\Pi_2 \delta E_2$ can be formed. Thus:

$$W_2 = \delta\Pi_i \delta E_i = K_{ij} \delta E_i \delta E_j \quad (39)$$

As the skew part \bar{K}^a of \bar{K} satisfies the relation $K_{ij}^a \delta E_i \delta E_j = 0$ for any vectors $\delta \bar{E}$, Eq. (39) yields:

$$W_2 = (\delta E_1, \delta E_2) \bar{K}^s \begin{pmatrix} \delta E_1 \\ \delta E_2 \end{pmatrix} \quad (40)$$

Thus, the expressions of both the term $\frac{L}{S} H_{11}$ and the second-order work W_2 are formally identical. By denoting f the bilinear application defined as $f(x, y) = (x, y) \bar{K}^s \begin{pmatrix} x \\ y \end{pmatrix}$, it follows that:

$$\frac{L}{S} H_{11} = f(\lambda_1, \lambda_2) = f(\lambda_4, -\lambda_3) \quad \text{and} \quad W_2 = f(\delta E_1, \delta E_2) \quad (41)$$

Moreover, recalling that \bar{K} is nonsymmetric, the Bromwich theorem (Willam and lordache, 2001) states that $\det(\bar{K}^s) < \det(\bar{K})$, which allows the vanishing of $\det(\bar{K}^s)$ strictly before the vanishing of $\det(\bar{K})$. Thus, the bilinear form f can be degenerated ($\det(\bar{K}^s) < 0$) strictly inside the plastic surface. Since $\det(\bar{K}) > 0$, \bar{K} is regular. The existence of a non-zero vector $\delta \bar{E}$ such that $W_2 < 0$ thereby ensures the existence of a non-zero stress loading $\delta \bar{\Pi}$ leading to $W_2 < 0$. As shown in Section 2, this proves the unsustainable character of the equilibrium state considered.

Eq. (41) establish that the vanishing of term H_{11} is basically related to the vanishing of the second-order work. From Eq. (41), it becomes clear that the notions of loss of controllability and loss of sustainability are closely related. Starting from a given equilibrium state, a noncontrollable loading program exists if and only if this equilibrium state is unsustainable.

Moreover, if $\det(\bar{K}^s) < 0$, solving the equation $f(x, y) = 0$ provides the values of the ratios λ_1/λ_2 and λ_4/λ_3 corresponding to a noncontrollable program:

$$\lambda_1/\lambda_2 = \frac{-(K_{12} + K_{21}) \mp \sqrt{(K_{12} + K_{21})^2 - 4K_{11}K_{22}}}{2K_{11}} \quad (42)$$

and:

$$\lambda_4/\lambda_3 = \frac{K_{12} + K_{21} \mp \sqrt{(K_{12} + K_{21})^2 - 4K_{11}K_{22}}}{2K_{11}} \quad (43)$$

The loading program (with δC_1 and δC_2 being constant), defined from the parameters:

$$\begin{aligned} C_1 &= \gamma_1 \left(\frac{-(K_{12} + K_{21}) \mp \sqrt{(K_{12} + K_{21})^2 - 4K_{11}K_{22}}}{2K_{11}} \Pi_1 + \Pi_2 \right) \quad \text{and} \\ C_2 &= \gamma_2 \left(E_1 + \frac{K_{12} + K_{21} \mp \sqrt{(K_{12} + K_{21})^2 - 4K_{11}K_{22}}}{2K_{11}} E_2 \right) \end{aligned}$$

where γ_1 and γ_2 are two any constants, is uncontrollable.

In addition, if these parameters C_1 and C_2 are maintained constant ($\delta C_1 = 0$ and $\delta C_2 = 0$), then the application of an infinitesimal perturbation will direct the collapse of the specimen: there is loss of sustainability. It must be noted that imposing $\delta C_1 = \delta a$ and $\delta C_2 = \delta b$ where δa and δb are any two infinitesimal values, corre-

sponds to a special infinitesimal perturbation of the stationary condition $\delta C_1 = 0$ and $\delta C_2 = 0$. Thus, the specimen will collapse. Following Nova's line, this is a loss of controllability; following Nicot and Darve's line, this is a loss of sustainability. Hence, these two notions are related to the same physics: imposing a noncontrollable program leads to the specimen collapsing; imposing a certain infinitesimal perturbation on a material in an unsustainable state, under constant parameters, also leads to the material collapsing. Furthermore, when a loading program is not controllable, at the point of the loading path where the control is lost, the mechanical state is also unsustainable. However, the main difference between these two notions is that the controllability emphasizes the notion of a loading program (i.e., a certain path within the mixed strain–stress space), whereas the notion of sustainability applies to a mechanical state (i.e., a certain point within the stress space).

Let us come back to the classical view of failure. In the context of the classical approach, the failure corresponds to the perfect plasticity regime, characterized by undefined strains under constant stresses: $\delta \bar{H} = \bar{0}$ with $\|\delta \bar{E}\| \neq 0$. The perfect plasticity condition is thus given by:

$$\det(\bar{K}) = 0 \quad (44)$$

and the flow rule by:

$$\bar{K} \delta \bar{E} = \bar{0} \quad (45)$$

Eq. (44) corresponds to the characteristic equation of matrix \bar{K} for the first nil eigen value and Eq. (45) gives the related eigen vector.

In this paper, these notions are generalized by invoking the notions of both control and response parameters. Starting from Eq. (35), the occurrence of failure can also be described as undefined response parameters under constant control parameters. Given that Eq. (35) can also be written as:

$$\begin{bmatrix} \delta C_1 \\ \delta C_2 \end{bmatrix} = \bar{N} \begin{bmatrix} \delta R_1 \\ \delta R_2 \end{bmatrix} \quad (46)$$

with $\bar{N} = \frac{1}{H_{22}} \begin{bmatrix} H_{11}H_{22} - H_{12}H_{21} & H_{12} \\ -H_{21} & 1 \end{bmatrix}$, it follows that the generalized failure condition is expressed as $\det(\bar{N}) = 0$, that is, $H_{11} = 0$. We recognize the criterion (the so-called bifurcation criterion) associated with the notion of loss of controllability and loss of sustainability. The flow rule is thus given by $\bar{N} \begin{bmatrix} \delta R_1 \\ \delta R_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which yields:

$$\delta R_2 - H_{21} \delta R_1 = 0 \quad (47)$$

Eq. (47) corresponds to a generalized flow rule (the so-called failure rule) in the sense that both strain and stress variables are involved (Darve et al., 2004).

In conclusion, assuming the normality condition, the relation between the notions of loss of controllability and loss of sustainability was clarified. These notions correspond to a failure, in a generalized sense: the condition $\delta C_2 = \delta C_1 = 0$ is prescribed as a generalized limit state in the (C_1, C_2) plane, the incremental response $\delta \bar{R}$ remains undefined, but the direction of $\delta \bar{R}$ is determined by the failure rule. In fact, the problem is no longer quasi-static. There is a bifurcation from a quasi-static regime to a dynamic regime, associated with a creation of kinetic energy, as ascertained by the existence of negative values of the second-order work. Hence, the domain of unsustainable strain–stress states (incremental stress (or strain) directions exist such that the second-order work is negative) is denoted the bifurcation domain. When the normality condition is assumed, the equation of the bifurcation domain is $\det(\bar{K}^s) \leq 0$. The main interest of the notion of loss of sustainability is that it has a clear physical meaning: this notion describes the collapse mechanism of a system, associated with a

burst in kinetic energy. Henceforth, this makes it possible to answer the question: what happens from a physical point of view when a loss of constitutive uniqueness or a loss of controllability is detected for a particular choice of control parameters? When any loading program defined from those control parameters is imposed, there is a brutal creation of kinetic energy (with an exponential increase in strains), leading to the collapse of the specimen.

3.4.3. Orientation of unstable stress directions

Throughout this section, it is assumed that both terms K_{11} and K_{22} are positive. In the previous section, it was established that $W_2 = K_{22}(\delta E_2)^2 + (K_{12} + K_{21})\delta E_1\delta E_2 + K_{11}(\delta E_1)^2$. This is a quadratic form that can vanish if and only if the discriminant d is positive. Given that:

$$d = (K_{12} + K_{21})^2 - 4K_{11}K_{22} = -\det(\bar{K}^s) \quad (48)$$

as long as \bar{K}^s is strictly positive (all the eigen values are strictly positive), W_2 has the same sign as K_{11} and K_{22} , that is H_{11} is strictly positive. When $\det(\bar{K}^s) < 0$, a cone gathering incremental strain directions corresponding to negative values of W_2 exists. This cone is delimited by the following two directions, corresponding to $W_2 = 0$:

$$\delta E_1 / \delta E_2 = \frac{-(K_{12} + K_{21}) \mp \sqrt{-\det(\bar{K}^s)}}{K_{11}} \quad (49)$$

Noting that $-\det(\bar{K}^s) = (\frac{K_{12} + K_{21}}{2})^2 - K_{11}K_{22} \leq (\frac{K_{12} + K_{21}}{2})^2$, it follows that both values of $\delta E_1 / \delta E_2$ given in Eq. (49) are negative. These incremental strain directions therefore belong to the second or the fourth quadrant of the incremental strain space. Given that $W_2 = \delta \bar{H} \cdot \delta \bar{E}$, condition $W_2 = 0$ imposes that both incremental strain and stress vectors are orthogonal. Consequently, the related incremental stress direction belongs to the first or the third quadrant of the incremental stress space. This is in line with most results reported in the literature using a variety of constitutive models. It is worth noting that after a compression loading, the first vanishing direction is always observed within the third quadrant; a vanishing direction may also appear within the first quadrant (this is the case, for instance, with the INL model; Darve et al., 2004), but this occurs only once a vanishing cone has first opened within the third quadrant.

Furthermore, after a first biaxial loading path in two-dimensional compression, it is clear that the rigidity along the axial direction ('1') is much greater than the rigidity along the lateral direction ('2'): $K_{11} > K_{22}$. The first stress direction $(\frac{\delta \Pi_1}{\delta \Pi_2})_0$ leading to the vanishing of the second-order work is obtained when $\det(\bar{K}^s) = 0$. Then, Eq. (44) gives $(\frac{\delta E_1}{\delta E_2})_0 = -\sqrt{\frac{K_{22}}{K_{11}}}$, and thus $(\frac{\delta \Pi_1}{\delta \Pi_2})_0 = \sqrt{\frac{K_{11}}{K_{22}}}$. Setting $\tan \alpha_\sigma = (\frac{\delta \Pi_1}{\delta \Pi_2})_0$, since this first direction belongs to the third quadrant, it follows that $\alpha_\Pi \in [225; 270]$ (deg). In axisymmetric three-dimensional conditions, since $\delta \sigma_1 = K_{11}\delta \varepsilon_1 + 2K_{12}\delta \varepsilon_2$ and $\delta \sigma_2 = K_{21}\delta \varepsilon_1 + (K_{22} + K_{23})\delta \varepsilon_2$, we have $\tan \alpha_\Pi = \sqrt{\frac{K_{11}}{K_{22} + K_{23}}}$. Assuming, for example, that $K_{23} = K_{22}$, then $\alpha_\Pi \in [215; 270]$ (deg). This is in line with most results found in the literature (see for instance Nicot and Darve, 2006). In particular, the range found from discrete element simulations in Section 3.3 was $[223; 245]$ (deg), which matches the theoretical predicted interval quite well $[215; 270]$ (deg).

4. Concluding remarks

This paper investigates the mechanism of collapse within a material specimen strictly inside the plastic limit surface. It can

be described as a burst in kinetic energy, from an equilibrium state, under constant control parameters. This definition has made it possible to develop the theoretical framework of loss of sustainability, in which the second-order work plays a fundamental role. As this quantity is basically directional, a directional analysis can be performed to check whether an incremental stress direction exists that corresponds to a negative value of the second-order work. If such directions exist, it is possible to build two control parameters, such that the incremental evolution corresponding to a negative value of the second-order work is associated with two constant control parameters. For practical purposes, if such control parameters are considered and maintained constant, the application of an infinitesimal perturbation is sufficient to induce a dramatic collapse of the whole specimen. These theoretical considerations were very clearly demonstrated from discrete element simulations performed with granular assemblies of spherical particles. It is worth noting that this framework was developed without any assumption regarding the constitutive behavior of the material, except the rate-independent character, providing a general scope to this framework.

Interestingly, this approach was investigated by considering the nature of the relation between both incremental control and response parameters. In other words, given an incremental control vector, does a unique incremental response vector exist? This relation was found to be unique as soon as the second-order work, which is a quadratic form, was no longer elliptic: at least one eigenvalue is negative or nil. In fact, the determinant of the matrix relating both incremental control and response parameters and the second-order work were found to be associated with the same matrix, i.e., the symmetric part of the tangent stiffness matrix. Thus, the relation between both the notions of loss of controllability and of loss of sustainability was derived, and the determinant of the symmetric part of the tangent stiffness matrix was shown to play a fundamental role in the occurrence of a loss of sustainability (or controllability).

Finally, the notions of loss of controllability or of loss of sustainability can also be regarded as a generalization of the classical perfect plastic failure condition (undefined strains under constant stress): the response parameters remain undefined under constant mixed control parameters. The plastic failure criterion is generalized into a bifurcation criterion and the flow rule into a mixed failure rule. The subsequent loss of uniqueness is related to the transition (bifurcation) from a quasi-static regime to a dynamical regime leading to the collapse of the specimen.

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