



Analytical solutions of peristatic and peridynamic problems for a 1D infinite rod

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ABSTRACT

Peridynamics is a nonlocal theory of continuum mechanics, which was developed by [Silling \(2000\)](#). Since then peridynamics has been applied to a variety of solid mechanics problems ranging from fracture, damage, failure to wave propagation, buckling, and detonation physics. Since the governing equation of peridynamics is an integro-differential equation, most of the treatment in the literature is often numerical. However, the analytical treatment is very important for the development of the peridynamic theory, which is continually developing at the present time. In this paper, peristatic and peridynamic problems for a 1D infinite rod are analytically investigated. We have developed a method to obtain a valid analytical solution starting from a formal analytical solution, which may be divergent. The primary contribution of the present paper is a systematic analytical treatment of peristatic and peridynamic problems for a 1D infinite rod. Additionally, dispersion curves and group velocities for the materials with three different micromoduli are also studied. It is found from the study that some peridynamic materials can have negative group velocities in certain regions of wavenumber. This indicates that peridynamics can be used for modeling certain types of dispersive media with anomalous dispersion such as the one discussed by [Mobley \(2007\)](#).

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1. Introduction

Peridynamics is a nonlocal theory of continuum mechanics, which was developed by [Silling \(2000\)](#) as a reformulation of elasticity theory for handling discontinuities in solid mechanics. Since then peridynamics has been applied to a variety of problems including 1D peridynamic rod to damage, failure and multiscale material modeling ([Silling et al., 2003](#); [Silling and Askari, 2005](#); [Weckner and Abeyaratne, 2005](#); [Emmrich and Weckner, 2007a,b](#); [Weckner et al., 2007](#); [Askari et al., 2008](#); [Bobaru et al., 2009](#); [Kilic and Madenci, 2009](#); [Weckner et al., 2009](#)). Since the governing equation of peridynamics is an integro-differential equation, it is usually treated numerically, and the number of analytical treatments is relatively few ([Silling et al., 2003](#); [Weckner and Abeyaratne, 2005](#); [Emmrich and Weckner, 2007a,b](#); [Weckner et al., 2009](#)). Peridynamics is a term originally used for both static and dynamic problems ([Silling, 2000](#)). In this paper, however, the additional term “peristatics” is introduced to differentiate the static problems from the dynamic problems, when such a differentiation is needed. The author believes that the introduction of the term “peristatics” makes it easier to discuss a variety of peridynamic problems with less confusion under certain situations.

Peristatic and peridynamic problems of 1D infinite rod are treated analytically (see [Fig. 1](#)). Loadings considered are two concentrated point loads with or without time dependence. In each

peristatic and peridynamic problem, two cases are treated with two corresponding micromoduli. The Case 1 of the peristatic problem and the special case of Case 2 of the peridynamic problem have been treated before by [Silling et al. \(2003\)](#) and [Emmrich and Weckner \(2007a\)](#), respectively, but other cases are believed to be new. Transient dynamics of a bar with two concentrated point loads was analytically treated by [Weckner et al. \(2009\)](#), but the micromodulus used in their paper ([Weckner et al., 2009](#)) was different from the one in this paper. Also, even though the Case 1 of the peristatic problem was treated previously ([Silling et al., 2003](#)), the exact analytical solution which is convergent was never obtained until now. What was obtained before was a formal solution by Fourier transform. But unfortunately this formal solution is divergent as will be seen later. Similarly, the infinite series expression for the displacement in [Silling et al. \(2003\)](#) derived from the residue theorem based on the formal solution by Fourier transform is also divergent, which was confirmed numerically by the present author, but the details are not given in this paper. In peridynamic problems, the case of non-zero body force density is often more difficult to deal with compared to the case of zero body force density with non-zero initial conditions. One notable exception is a separable loading that depends harmonically on time and is spatially a Fourier series. This is the example treated by [Weckner and Abeyaratne \(2005\)](#). It should be pointed out here that in peristatic problems considered in this paper the body force density is always non-zero, since otherwise the problem reduces to a triviality. Dispersion curves and group

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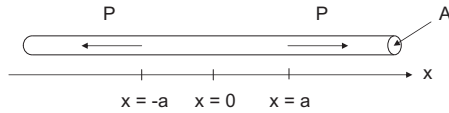


Fig. 1. Infinite rod with two concentrated forces.

velocities for materials with three different micromoduli are also studied. It is interesting to note that in two of the three cases studied there are regions of wavenumber where there is a negative group velocity. This indicates that peridynamics can be used for modeling certain types of dispersive media, which exhibit a negative group velocity.

The primary contribution of this paper is the development of a method to obtain exact analytical solutions for various peristatic and peridynamic problems. One of the main difficulties in treating peridynamic problems analytically is that a formal solution obtained by Fourier transform in the form of an integral is often divergent. We have succeeded to obtain exact analytical solutions by transforming the divergent integrals into singular solutions (generalized functions) plus convergent integrals. (It should be mentioned here that essentially the same method seems to have been also developed by Weckner et al. (2009), which the author was not aware of until the review process of this paper.) Also, by considering the classical limits of both the peristatic and peridynamic solutions as the material horizon $l \rightarrow 0$, it is shown that the corresponding elastostatic and elastodynamic solutions are recovered exactly. Even though only two cases for each peristatic and peridynamic problem have been treated, the method developed in this paper can be applied to other peridynamic problems.

1D peristatic problems are discussed in Section 2, and their classical limits as the material horizon $l \rightarrow 0$ are discussed in Section 3. Similarly, 1D peridynamic problems are discussed in Section 4, and their classical limits are discussed in Section 5. Numerical results and dispersion curves are given in Section 6, and finally the conclusion is stated in Section 7.

2. 1D peristatics

The governing equation for 1D peristatics is given by

$$\int_{-\infty}^{\infty} C(\xi)(u(x-\xi) - u(x))d\xi + b(x) = 0 \quad (-\infty < x < \infty) \quad (1)$$

where $C(\xi)$ is the micromodulus of a peridynamic material, $u(x)$ is a longitudinal displacement, and $b(x)$ is the body force density. The peridynamic material is considered to reduce to the classical elastic material when the material horizon l , which is contained in the micromodulus $C(\xi)$, approaches 0. The micromodulus $C(\xi)$ of the peridynamic material is related to the Young's modulus of the corresponding elastic material by (Silling et al., 2003)

$$E = \int_0^{\infty} \xi^2 C(\xi) d\xi \quad (2)$$

We solve the Eq. (1) by Fourier transform. Let us define the Fourier pair as

$$\begin{aligned} \bar{u}(k) &= \int_{-\infty}^{\infty} u(x)e^{-ikx} dx \\ u(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{u}(k)e^{ikx} dk \end{aligned} \quad (3)$$

The solution to Eq. (1) is given by

$$u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{u}(k)e^{ikx} dk \quad (4)$$

where

$$\bar{u}(k) = \frac{\bar{b}(k)}{M(k)}, \quad M(k) = \bar{C}(0) - \bar{C}(k) \quad (5)$$

Let us consider the following cases:

Case 1:

$$\begin{aligned} b(x) &= \frac{P}{A} [\delta(x-a) - \delta(x+a)] \\ C(\xi) &= \begin{cases} \frac{3E}{l^3}, & |\xi| \leq l \\ 0, & |\xi| > l \end{cases} \end{aligned} \quad (6)$$

Case 2:

$$\begin{aligned} b(x) &= \frac{P}{A} [\delta(x-a) - \delta(x+a)] \\ C(\xi) &= \frac{4E}{l^3 \sqrt{\pi}} e^{-\left(\frac{\xi}{l}\right)^2} \end{aligned} \quad (7)$$

It should be noted here that both micromoduli given in (6) and (7) satisfy the relation (2). Substituting (6) into (5), and then the resulting equation into (4), we obtain

$$u(x) = \frac{Pl^2}{3\pi EA} \int_0^{\infty} \frac{\sin ka \sin kx}{1 - \frac{\sin kl}{kl}} dk \quad (\text{Case 1}) \quad (8)$$

Similarly, substituting (7) into (5), and then the resulting equation into (4), we obtain

$$u(x) = \frac{Pl^2}{2\pi EA} \int_0^{\infty} \frac{\sin ka \sin kx}{1 - e^{-\frac{k^2 l^2}{4}}} dk \quad (\text{Case 2}) \quad (9)$$

The Case 1 was treated in a paper by Silling et al. (2003), but the Case 2 is new. In Silling et al. (2003), Case 1 was also analytically investigated, but the results were produced by a numerical method (most likely by a dynamic relaxation method (Otter, 1965; Silling et al., 2003; Bobaru et al., 2009)). The analytical solution obtained in Silling et al. (2003) was a formal solution by Fourier transform, which was unfortunately divergent. Therefore, it could not be used for obtaining the numerical results. It has been noted in Bobaru et al. (2009) that it is difficult to produce a numerical solution for Case 1 by solving a static equation (i.e., peristatics) numerically, even though this is a static problem. In this paper, both Case 1 and Case 2 are treated analytically, and the results will be also produced analytically.

The main problem with the solution (8) is that the integral is divergent. Therefore, we rewrite the solution (8) as

$$u(x) = \frac{Pl^2}{3\pi EA} \left[\int_0^{\infty} \sin ka \sin kx dk + \int_0^{\infty} \frac{\sin ka \sin kx \sin kl}{kl - \sin kl} dk \right] \quad (\text{Case 1}) \quad (10)$$

From (10), we finally obtain the non-dimensionalized displacement as

$$\begin{aligned} \frac{u_{ps}(x)}{\frac{Pa}{EA}} &= \frac{l^2}{6a} [\delta(x-a) - \delta(x+a)] \\ &+ \frac{l^2}{3\pi a} \int_0^{\infty} \frac{\sin ka \sin kx \sin kl}{kl - \sin kl} dk \quad (\text{Case 1}) \end{aligned} \quad (11)$$

where we have used the subscript PS (i.e., $u_{ps}(x)$) to indicate the peristatic solution, and also the following result is used:

$$\delta(x) = \frac{1}{\pi} \int_0^{\infty} \cos kx dk \quad (12)$$

What is accomplished in (10) and (11) is that the divergent part of the integral is separated out as Dirac delta functions. Now the integral in (11) is convergent, and it is relatively easy to evaluate numerically. Similarly for Case 2, the non-dimensionalized displacement is obtained as

$$\frac{u_P(x)}{\frac{Pa}{EA}} = \frac{l^2}{4a} [\delta(x-a) - \delta(x+a)] + \frac{l^2}{2\pi a} \int_0^\infty \frac{e^{-\frac{k^2 l^2}{4}} \sin k a \sin k x}{1 - e^{-\frac{k^2 l^2}{4}}} dk \quad (\text{Case 2}) \quad (13)$$

Again the integral in (13) is convergent, and it is easy to evaluate numerically. The numerical results for both cases are shown in Section 6.

3. Classical limit of 1D peristatics, and 1D elastostatics

We would like to consider the classical limit ($l \rightarrow 0$) of the 1D peristatic solutions given by (11) and (13). But let us first consider 1D elastostatics, whose governing equation is given by

$$E \frac{d^2 u}{dx^2} + b(x) = 0 \quad (-\infty < x < \infty) \quad (14)$$

where E is the Young's modulus, and the body force density $b(x)$ is given by

$$b(x) = \frac{P}{A} [\delta(x-a) - \delta(x+a)] \quad (15)$$

The most general solution for (14) with the body force density (15) is given by

$$u(x) = \begin{cases} \frac{Pa}{EA} + c_1 x + c_2, & x \geq a \\ \frac{Px}{EA} + c_1 x + c_2, & -a \leq x \leq a \\ -\frac{Pa}{EA} + c_1 x + c_2, & x \leq -a \end{cases} \quad (16)$$

where c_1 and c_2 are arbitrary constants. Thus the solution to (14) with (15) is unique up to an addition of an arbitrary linear function. The situation is actually the same for the peristatic equation (1) (see also Silling et al., 2003). We had already assumed that these constants are zero for the peristatic Eq. (1). If we demand the conditions $|u(-\infty)| < \infty$, $|u(+\infty)| < \infty$, and also $u(0) = 0$ (without loss of generality) for the solution (16), we obtain

$$\frac{u_E(x)}{\frac{Pa}{EA}} = \begin{cases} 1, & x \geq a \\ \frac{x}{a}, & -a \leq x \leq a \\ -1, & x \leq -a \end{cases} \quad (17)$$

where we have used the subscript E (i.e., $u_E(x)$) to indicate the elastostatic solution. For the peristatic solution (11), let us first note that

$$\frac{l^2 \sin kl}{kl - \sin kl} = \frac{1}{k^2} \left[6 - \frac{7}{10}(kl)^2 + \frac{11}{1400}(kl)^4 + \frac{17}{126000}(kl)^6 + O(k^8 l^8) \right] \quad (18)$$

Substituting (18) into (11), we obtain

$$\frac{u_{PS}(x)}{\frac{Pa}{EA}} = \frac{l^2}{6a} [\delta(x-a) - \delta(x+a)] + \frac{2}{\pi} \int_0^\infty \frac{\sin y \sin(\frac{x}{a} y)}{y^2} dy + O(l^2) \quad (\text{Case 1}) \quad (19)$$

It can be clearly seen from (19) that the singular part (Dirac delta functions) of the solution diminishes with the rate of l^2 when $l \rightarrow 0$. From (19), we readily obtain

$$\lim_{l \rightarrow 0} \frac{u_{PS}(x)}{\frac{Pa}{EA}} = \frac{2}{\pi} \int_0^\infty \frac{\sin y \sin(\frac{x}{a} y)}{y^2} dy = \begin{cases} 1, & x \geq a \\ \frac{x}{a}, & -a \leq x \leq a \\ -1, & x \leq -a \end{cases} \quad (\text{Case 1}) \quad (20)$$

Thus we have exactly recovered the elastostatic solution (17) in the classical limit of peristatics as $l \rightarrow 0$.

Similarly for Case 2 (see (13)), we have

$$\frac{l^2 e^{-\left(\frac{kl}{2}\right)^2}}{1 - e^{-\left(\frac{kl}{2}\right)^2}} = \frac{1}{k^2} \left[4 - \frac{(kl)^2}{2} + \frac{(kl)^4}{48} - \frac{(kl)^8}{46080} + O(k^{12} l^{12}) \right] \quad (21)$$

Substituting (21) into (13), we obtain

$$\frac{u_{PS}(x)}{\frac{Pa}{EA}} = \frac{l^2}{4a} [\delta(x-a) - \delta(x+a)] + \frac{2}{\pi} \int_0^\infty \frac{\sin y \sin(\frac{x}{a} y)}{y^2} dy + O(l^2) \quad (\text{Case 2}) \quad (22)$$

Similarly to Case 1, it can be clearly seen from (22) that the singular part (Dirac delta functions) of the solution diminishes with the rate of l^2 when $l \rightarrow 0$. It can be seen from (19) and (22) that the only difference between solutions for Case 1 and Case 2 in this asymptotic form is the coefficients for the singular part: $\frac{l^2}{6a}$ vs $\frac{l^2}{4a}$. From (22), we readily obtain as in Case 1

$$\lim_{l \rightarrow 0} \frac{u_{PS}(x)}{\frac{Pa}{EA}} = \frac{2}{\pi} \int_0^\infty \frac{\sin y \sin(\frac{x}{a} y)}{y^2} dy = \begin{cases} 1, & x \geq a \\ \frac{x}{a}, & -a \leq x \leq a \\ -1, & x \leq -a \end{cases} \quad (\text{Case 2}) \quad (23)$$

Thus we have again exactly recovered the elastostatic solution (17) in the classical limit of peristatics as $l \rightarrow 0$.

4. 1D peridynamics

The governing equation for 1D peridynamics is given by

$$\int_{-\infty}^\infty C(\xi)(u(x-\xi, t) - u(x, t)) d\xi + b(x, t) = \rho \ddot{u}(x, t) \quad (-\infty < x < \infty) \quad (24)$$

Let us set the initial conditions as

$$u(x, 0) = f(x), \quad \frac{\partial u}{\partial t}(x, 0) = g(x) \quad (25)$$

Eq. (24) with the initial conditions (25) can be solved by Fourier transform. With appropriate boundary conditions at infinity, the solution can be obtained as

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^\infty \bar{u}(k, t) e^{ikx} dk \quad (26)$$

where

$$\begin{aligned} \bar{u}(k, t) = & \frac{\bar{g}(k)}{\omega(k)} \sin[\omega(k)t] + \bar{f}(k) \cos[\omega(k)t] \\ & + \frac{1}{\rho \omega(k)} \int_0^t \bar{b}(k, \varsigma) \sin[\omega(k)(t - \varsigma)] d\varsigma \end{aligned} \quad (27)$$

and

$$\omega(k) = \sqrt{\frac{M(k)}{\rho}}, \quad M(k) = \bar{C}(0) - \bar{C}(k) \quad (28)$$

Here, ρ is the mass density of the material.

Let us consider the following cases:

Case 1:

$$\begin{aligned} b(x, t) = & \frac{P}{A} [\delta(x-a) - \delta(x+a)] H(t) \\ C(\xi) = & \begin{cases} \frac{3E}{\beta}, & |\xi| \leq l \\ 0, & |\xi| > l \end{cases} \end{aligned} \quad (29)$$

$$f(x) = U e^{-\left(\frac{x}{l_1}\right)^2}, \quad g(x) = V e^{-\left(\frac{x}{l_2}\right)^2}$$

Case 2:

$$\begin{aligned} b(x, t) &= \frac{P}{A} [\delta(x-a) - \delta(x+a)] H(t) \\ C(\xi) &= \frac{4E}{l^3 \sqrt{\pi}} e^{-\left(\frac{\xi}{l}\right)^2} \\ f(x) &= U e^{-\left(\frac{x}{l_1}\right)^2}, \quad g(x) = V e^{-\left(\frac{x}{l_2}\right)^2} \end{aligned} \quad (30)$$

In (29) and (30), $H(t)$ denotes the Heaviside step function. Substituting (29) into (27), and then the resulting equation into (26), we obtain

$$\begin{aligned} u_{PD}(x, t) &= \frac{Pl^2}{3\pi EA} \int_0^\infty \frac{\sin k a \sin k x}{1 - \frac{\sin k l}{k l}} \left(1 - \cos \left[\frac{\sqrt{6} c t}{l} \sqrt{1 - \frac{\sin k l}{k l}} \right] \right) dk \\ &+ \frac{UL_1}{\sqrt{\pi}} \int_0^\infty e^{-\left(\frac{k l_1}{2}\right)^2} \cos k x \cos \left[\frac{\sqrt{6} c t}{l} \sqrt{1 - \frac{\sin k l}{k l}} \right] dk \\ &+ \frac{VL_2 l}{c \sqrt{6\pi}} \int_0^\infty e^{-\left(\frac{k l_2}{2}\right)^2} \frac{\cos k x}{\sqrt{1 - \frac{\sin k l}{k l}}} \sin \left[\frac{\sqrt{6} c t}{l} \sqrt{1 - \frac{\sin k l}{k l}} \right] dk \quad (\text{Case 1}) \end{aligned} \quad (31)$$

where we have used the subscript PD (i.e., $u_{PD}(x)$) to indicate the peridynamic solution, and the wave speed c of the corresponding elastic material is given by

$$c = \sqrt{\frac{E}{\rho}} \quad (32)$$

where ρ is the mass density of the material. Let us note here that the dispersion relation $\omega(k)$ for Case 1 is given by

$$\omega(k) = \frac{\sqrt{6} c}{l} \sqrt{1 - \frac{\sin k l}{k l}} \quad (\text{Case 1}) \quad (33)$$

Similarly for Case 2, substituting (30) into (27), and then the resulting equation into (26), we obtain

$$\begin{aligned} u_{PD}(x, t) &= \frac{Pl^2}{2\pi EA} \int_0^\infty \frac{\sin k a \sin k x}{1 - e^{-\left(\frac{k l}{2}\right)^2}} \left(1 - \cos \left[\frac{2 c t}{l} \sqrt{1 - e^{-\left(\frac{k l}{2}\right)^2}} \right] \right) dk \\ &+ \frac{UL_1}{\sqrt{\pi}} \int_0^\infty e^{-\left(\frac{k l_1}{2}\right)^2} \cos k x \cos \left[\frac{2 c t}{l} \sqrt{1 - e^{-\left(\frac{k l}{2}\right)^2}} \right] dk \\ &+ \frac{VL_2 l}{2 c \sqrt{\pi}} \int_0^\infty e^{-\left(\frac{k l_2}{2}\right)^2} \frac{\cos k x}{\sqrt{1 - e^{-\left(\frac{k l}{2}\right)^2}}} \sin \left[\frac{2 c t}{l} \sqrt{1 - e^{-\left(\frac{k l}{2}\right)^2}} \right] dk \quad (\text{Case 2}) \end{aligned} \quad (34)$$

Also, the dispersion relation $\omega(k)$ for Case 2 is given by

$$\omega(k) = \frac{2c}{l} \sqrt{1 - e^{-\frac{k^2 l^2}{4}}} \quad (\text{Case 2}) \quad (35)$$

The special case of Case 2 has been discussed by Weckner and Abeyaratne (2005) and Emmrich and Weckner (2007a,b), where $b(x) \equiv 0$ and $g(x) \equiv 0$. Case 1 does not seem to have been considered in the past. In both Case 1 and Case 2, the second and third integrals in (31) and (34), which corresponds to the initial conditions, are convergent, and they are relatively easy to evaluate numerically. However, the first integrals in (31) and (34) are both divergent. Therefore, a similar treatment is needed for these integrals as was done for 1D peristatics in Section 2. As revealed from this discussion, in peridynamics, the case of non-zero body force density ($b(x, t) \neq 0$) is often more difficult to deal with when the body force density involves singular functions such as the Dirac delta function.

By separating the first integral in (31) into a singular part (Dirac delta functions) and a convergent integral, we obtain

$$\begin{aligned} u_{PD}(x, t) &= \frac{Pl^2}{6EA} [\delta(x-a) - \delta(x+a)] \left(1 - \cos \left[\frac{\sqrt{6} c t}{l} \right] \right) \\ &+ \frac{Pl^2}{3\pi EA} \int_0^\infty \sin k a \sin k x \left(\frac{1 - \cos \left[\frac{\sqrt{6} c t}{l} \sqrt{1 - \frac{\sin k l}{k l}} \right]}{1 - \frac{\sin k l}{k l}} - 1 + \cos \left[\frac{\sqrt{6} c t}{l} \right] \right) dk \\ &+ \frac{UL_1}{\sqrt{\pi}} \int_0^\infty e^{-\left(\frac{k l_1}{2}\right)^2} \cos k x \cos \left[\frac{\sqrt{6} c t}{l} \sqrt{1 - \frac{\sin k l}{k l}} \right] dk \\ &+ \frac{VL_2 l}{c \sqrt{6\pi}} \int_0^\infty e^{-\left(\frac{k l_2}{2}\right)^2} \frac{\cos k x}{\sqrt{1 - \frac{\sin k l}{k l}}} \sin \left[\frac{\sqrt{6} c t}{l} \sqrt{1 - \frac{\sin k l}{k l}} \right] dk \quad (\text{Case 1}) \end{aligned} \quad (36)$$

Similarly, for Case2, we obtain

$$\begin{aligned} u_{PD}(x, t) &= \frac{Pl^2}{4EA} [\delta(x-a) - \delta(x+a)] \left(1 - \cos \left[\frac{2 c t}{l} \right] \right) \\ &+ \frac{Pl^2}{2\pi EA} \int_0^\infty \sin k a \sin k x \left(\frac{1 - \cos \left[\frac{2 c t}{l} \sqrt{1 - e^{-\left(\frac{k l}{2}\right)^2}} \right]}{1 - e^{-\left(\frac{k l}{2}\right)^2}} - 1 + \cos \left[\frac{2 c t}{l} \right] \right) dk \\ &+ \frac{UL_1}{\sqrt{\pi}} \int_0^\infty e^{-\left(\frac{k l_1}{2}\right)^2} \cos k x \cos \left[\frac{2 c t}{l} \sqrt{1 - e^{-\left(\frac{k l}{2}\right)^2}} \right] dk \\ &+ \frac{VL_2 l}{2 c \sqrt{\pi}} \int_0^\infty e^{-\left(\frac{k l_2}{2}\right)^2} \frac{\cos k x}{\sqrt{1 - e^{-\left(\frac{k l}{2}\right)^2}}} \sin \left[\frac{2 c t}{l} \sqrt{1 - e^{-\left(\frac{k l}{2}\right)^2}} \right] dk \quad (\text{Case 2}) \end{aligned} \quad (37)$$

In this final form, the singular part is given by the first term (Dirac delta functions), and it can be shown that the integral in the second term is now convergent for both (36) and (37). The proofs of the convergence of these integrals are given in Appendix A. Thus, we have obtained desired expressions for the peridynamic solutions.

Next, it can be shown that the peridynamic solution is related to the corresponding peristatic solution in a relatively simple fashion as follows:

Case 1:

$$\begin{aligned} \frac{u_{PD}(x, t)}{\frac{Pa}{EA}} &= \frac{u_{PS}(x, t)}{\frac{Pa}{EA}} \left(1 - \cos \left[\frac{\sqrt{6} c t}{l} \right] \right) \\ &+ \frac{l^2}{3\pi a} \int_0^\infty \sin k a \sin k x \frac{\cos \left[\frac{\sqrt{6} c t}{l} \right] - \cos \left[\frac{\sqrt{6} c t}{l} \sqrt{1 - \frac{\sin k l}{k l}} \right]}{1 - \frac{\sin k l}{k l}} dk \\ &+ \frac{EA UL_1}{Pa \sqrt{\pi}} \int_0^\infty e^{-\left(\frac{k l_1}{2}\right)^2} \cos k x \cos \left[\frac{\sqrt{6} c t}{l} \sqrt{1 - \frac{\sin k l}{k l}} \right] dk \\ &+ \frac{EA VL_2 l}{Pa c \sqrt{6\pi}} \int_0^\infty e^{-\left(\frac{k l_2}{2}\right)^2} \frac{\cos k x}{\sqrt{1 - \frac{\sin k l}{k l}}} \sin \left[\frac{\sqrt{6} c t}{l} \sqrt{1 - \frac{\sin k l}{k l}} \right] dk \end{aligned} \quad (38)$$

Case 2:

$$\begin{aligned} \frac{u_{PD}(x, t)}{\frac{Pa}{EA}} &= \frac{u_{PS}(x, t)}{\frac{Pa}{EA}} \left(1 - \cos \left[\frac{2 c t}{l} \right] \right) \\ &+ \frac{l^2}{2\pi a} \int_0^\infty \sin k a \sin k x \frac{\cos \left[\frac{2 c t}{l} \right] - \cos \left[\frac{2 c t}{l} \sqrt{1 - e^{-\frac{k^2 l^2}{4}}} \right]}{1 - e^{-\frac{k^2 l^2}{4}}} dk \\ &+ \frac{EA UL_1}{Pa \sqrt{\pi}} \int_0^\infty e^{-\left(\frac{k l_1}{2}\right)^2} \cos k x \cos \left[\frac{2 c t}{l} \sqrt{1 - e^{-\left(\frac{k l}{2}\right)^2}} \right] dk \\ &+ \frac{EA VL_2 l}{Pa 2 c \sqrt{\pi}} \int_0^\infty e^{-\left(\frac{k l_2}{2}\right)^2} \frac{\cos k x}{\sqrt{1 - e^{-\left(\frac{k l}{2}\right)^2}}} \sin \left[\frac{2 c t}{l} \sqrt{1 - e^{-\left(\frac{k l}{2}\right)^2}} \right] dk \end{aligned} \quad (39)$$

Thus, the peridynamic solution is expressed as a product of a peristatic solution and a certain time-dependent function plus other time-dependent integrals. The proofs of the convergence for the integrals in the second term for both cases are given in Appendix A. The numerical results for both cases, which are based on the above results, are given in Section 6.

5. Classical limit of 1D peridynamics, and 1D elastodynamics

As in Section 3, let us first consider 1D elastodynamics, whose governing equation is given by

$$E \frac{\partial^2 u}{\partial x^2} + b(x, t) = \rho \frac{\partial^2 u}{\partial t^2} \quad (-\infty < x < \infty) \quad (40)$$

where E is the Young's modulus, ρ is the mass density, and $b(x, t)$ is the body force density. The initial conditions can be given as

$$u(x, 0) = f(x), \quad \frac{\partial u}{\partial t}(x, 0) = g(x) \quad (41)$$

With appropriate boundary conditions at infinity, Eq. (40) can be solved by Fourier transform, and the solution is given by

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{u}(k, t) e^{ikx} dk \quad (42)$$

where

$$\begin{aligned} \bar{u}(k, t) &= \frac{\bar{g}(k)}{\omega(k)} \sin[\omega_E(k)t] + \bar{f}(k) \cos[\omega_E(k)t] \\ &+ \frac{1}{\rho \omega_E(k)} \int_0^t \bar{b}(k, \varsigma) \sin[\omega_E(k)(t - \varsigma)] d\varsigma \end{aligned} \quad (43)$$

and

$$\omega_E(k) = kc = k \sqrt{\frac{E}{\rho}} \quad (44)$$

Let us note the close parallel between peridynamics ((26) and (27)) and elastodynamics ((42) and (43)).

Eq. (40) can be also solved in a slightly different way. First, the solution of (40) can be split into two as

$$u = u_1 + u_2 \quad (45)$$

where

$$c^2 \frac{\partial^2 u_1}{\partial x^2} = \frac{\partial^2 u_1}{\partial t^2} \quad (46)$$

$$c^2 \frac{\partial^2 u_2}{\partial x^2} + \frac{b(x, t)}{\rho} = \frac{\partial^2 u_2}{\partial t^2} \quad (47)$$

and the initial conditions are given by

$$u_1(x, 0) = f(x), \quad \frac{\partial u_1}{\partial t}(x, 0) = g(x) \quad (48)$$

$$u_2(x, 0) = 0, \quad \frac{\partial u_2}{\partial t}(x, 0) = 0 \quad (49)$$

With appropriate boundary conditions, (46) and (47) can be solved with the initial conditions (48) and (49), respectively, as

$$u_1(x, t) = \frac{1}{2} [f(x + ct) + f(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\xi) d\xi \quad (50)$$

$$u_2(x, t) = \frac{c}{2E} \int_0^t d\varsigma \int_{-\infty}^{\infty} d\xi b(\xi, \varsigma) [H(x - \xi + c(t - \varsigma)) - H(x - \xi - c(t - \varsigma))] \quad (51)$$

where $H(t)$ denotes the Heaviside step function. $u_1(x, t)$ in (50) is the D'Alembert's solution, and $u_2(x, t)$ in (51) corresponds to a particular solution for the body force density (loading). Thus the solution to the elastodynamic Eq. (40) with the initial conditions (41) is given by the sum of these two solutions.

Let us consider the following case:

$$\begin{aligned} b(x, t) &= \frac{P}{A} [\delta(x - a) - \delta(x + a)] H(t) \\ f(x) &= U e^{-\left(\frac{x}{l_1}\right)^2}, \quad g(x) = V e^{-\left(\frac{x}{l_2}\right)^2} \end{aligned} \quad (52)$$

The above case given by (52) is the elastodynamic counterpart of peridynamic cases given by (29) and (30). By using (52) in (43), and substituting the resulting expression into (42), we obtain

$$\begin{aligned} u_{ED}(x, t) &= \frac{2P}{\pi EA} \int_0^\infty \frac{\sin ka \sin kx}{k^2} (1 - \cos kct) dk \\ &+ \frac{UL_1}{\sqrt{\pi}} \int_0^\infty e^{-\left(\frac{kl_1}{2}\right)^2} \cos kx \cos kct dk \\ &+ \frac{VL_2}{c\sqrt{\pi}} \int_0^\infty e^{-\left(\frac{kl_2}{2}\right)^2} \cos kx \frac{\sin kct}{k} dk \end{aligned} \quad (53)$$

where we have used the subscript ED (i.e., $u_{ED}(x, t)$) to indicate the elastodynamic solution. Eq. (53) is the solution to the elastodynamic problem defined by (40), (41), and (52). From (29), (30) and (52), the solutions to the both peridynamic cases (Cases 1 and 2) are expected to asymptotically approach to the solution (53) in the classical limit of peridynamics as $l \rightarrow 0$.

For the peridynamic solution (36) in Case1, let us first note that

$$\begin{aligned} \frac{1 - \cos \left[\frac{\sqrt{6}ct}{l} \sqrt{1 - \frac{\sin kl}{kl}} \right]}{1 - \frac{\sin kl}{kl}} &= \frac{6(1 - \cos(kct))}{(kl)^2} + \frac{3}{10}(1 - \cos(kct)) - \frac{3}{20}kct \sin(kct) + O(k^2 l^2) \\ \cos \left[\frac{\sqrt{6}ct}{l} \sqrt{1 - \frac{\sin kl}{kl}} \right] &= \cos(kct) + \frac{(kl)^2}{40}kct \sin(kct) + O(k^4 l^4) \\ \frac{\sin \left[\frac{\sqrt{6}ct}{l} \sqrt{1 - \frac{\sin kl}{kl}} \right]}{\sqrt{1 - \frac{\sin kl}{kl}}} &= \frac{\sqrt{6} \sin(kct)}{kl} + \frac{\sqrt{6}}{40}(\sin(kct) - kct \cos(kct))kl + O(k^3 l^3) \end{aligned} \quad (54)$$

Substituting (54) into (36), we finally obtain

$$\begin{aligned} u_{PD}(x, t) &= \frac{Pl^2}{6EA} [\delta(x - a) - \delta(x + a)] \left(1 - \cos \left[\frac{\sqrt{6}ct}{l} \right] \right) \\ &+ \frac{2P}{\pi EA} \int_0^\infty \frac{\sin ka \sin kx}{k^2} (1 - \cos(kct)) dk + O(l^2) \\ &+ \frac{UL_1}{\sqrt{\pi}} \int_0^\infty e^{-\left(\frac{kl_1}{2}\right)^2} \cos kx \cos(kct) dk + O(l^2) \\ &+ \frac{VL_2}{c\sqrt{\pi}} \int_0^\infty e^{-\left(\frac{kl_2}{2}\right)^2} \cos kx \frac{\sin(kct)}{k} dk + O(l^2) \quad (\text{Case 1}) \end{aligned} \quad (55)$$

For the peridynamic solution (37) in Case2, let us first note that

$$\begin{aligned} \frac{1 - \cos \left[\frac{2ct}{l} \sqrt{1 - e^{-\frac{k^2 l^2}{4}}} \right]}{1 - e^{-\frac{k^2 l^2}{4}}} &= \frac{4(1 - \cos(kct))}{(kl)^2} + \frac{1}{2}(1 - \cos(kct)) - \frac{kct}{4} \sin(kct) + O(k^2 l^2) \\ \cos \left[\frac{2ct}{l} \sqrt{1 - e^{-\frac{k^2 l^2}{4}}} \right] &= \cos(kct) + \frac{(kl)^2}{16}kct \sin(kct) + O(k^4 l^4) \\ \frac{\sin \left[\frac{2ct}{l} \sqrt{1 - e^{-\frac{k^2 l^2}{4}}} \right]}{\sqrt{1 - e^{-\frac{k^2 l^2}{4}}}} &= \frac{2 \sin(kct)}{kl} + \frac{1}{8}(\sin(kct) - kct \cos(kct))kl + O(k^3 l^3) \end{aligned} \quad (56)$$

Similarly, by substituting (56) into (37), we obtain

$$\begin{aligned} u_{PD}(x, t) &= \frac{Pl^2}{4EA} [\delta(x - a) - \delta(x + a)] \left(1 - \cos \left[\frac{2ct}{l} \right] \right) \\ &+ \frac{2P}{\pi EA} \int_0^\infty \frac{\sin ka \sin kx}{k^2} (1 - \cos(kct)) dk + O(l^2) \\ &+ \frac{UL_1}{\sqrt{\pi}} \int_0^\infty e^{-\left(\frac{kl_1}{2}\right)^2} \cos kx \cos(kct) dk + O(l^2) \\ &+ \frac{VL_2}{c\sqrt{\pi}} \int_0^\infty e^{-\left(\frac{kl_2}{2}\right)^2} \cos kx \frac{\sin(kct)}{k} dk + O(l^2) \quad (\text{Case 2}) \end{aligned} \quad (57)$$

It can be clearly seen from (55) and (57) that, for both Case 1 and Case 2, the singular part (Dirac delta functions) of the solution

diminishes with the rate of l^2 when $l \rightarrow 0$ as in peristatics (see Section 3). From (55) and (57), we readily obtain

$$\lim_{l \rightarrow 0} \frac{u_{PD}(x, t)}{\frac{Pa}{EA}} = \frac{2}{\pi a} \int_0^\infty \frac{\sin k a \sin k x}{k^2} (1 - \cos k c t) dk + \frac{U L_1 EA}{\sqrt{\pi} Pa} \int_0^\infty e^{-\left(\frac{k l_1}{2}\right)^2} \cos k x \cos k c t dk + \frac{V L_2 EA}{c \sqrt{\pi} Pa} \int_0^\infty e^{-\left(\frac{k l_2}{2}\right)^2} \cos k x \frac{\sin k c t}{k} dk \quad (\text{Case 1 and Case 2}) \quad (58)$$

Thus we have exactly recovered the elastodynamic solution (53) in the classical limit of peridynamics as $l \rightarrow 0$ for both Case 1 and Case 2.

6. Numerical results and dispersion curves

6.1. Numerical results for peristatics

Let us consider Case 1 and Case 2 given by (6) and (7). For Case 1, the non-dimensionalized displacement is given by (11). The non-singular part of the solution for the non-dimensionalized displacement is shown in Fig. 2, when $l/a = 1, 2, 4$. For Case 2, the non-dimensionalized displacement is given by (13). The non-singular part of the solution for the non-dimensionalized displacement is shown in Fig. 3, when $l/a = 1, 2, 4$. For both cases, the elastostatic solution is also shown. It is clearly seen from Figs. 2 and 3 that when l/a becomes smaller, the non-singular part of the peristatic solution approaches the elastostatic solution. It should be also mentioned here that the singular part of the peristatic solution

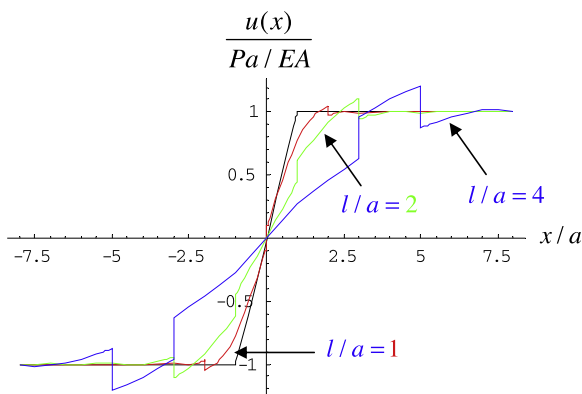


Fig. 2. Non-singular part of the peristatic solution for Case 1 when $l/a = 1, 2, 4$ with the elastostatic solution.

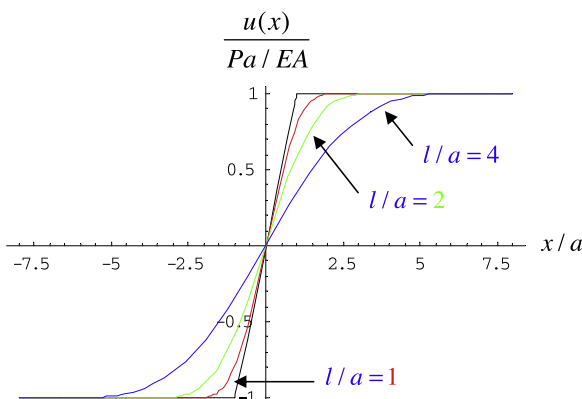


Fig. 3. Non-singular part of the peristatic solution for Case 2 when $l/a = 1, 2, 4$ with the elastostatic solution.

for both cases approaches zero as l/a becomes smaller, due to (19) and (22).

6.2. Numerical results for peridynamics

The case where the body force density is present is more difficult than the case where the initial displacement and the initial velocity are given without the body force density, when the body force density involves singular functions such as the Dirac delta function. Therefore, let us focus on Case 1 and Case 2 given by (29) and (30) with $f(x) = g(x) = 0$. The solutions for these cases are given by setting $U = V = 0$ in (38) and (39). The peristatic parts of the solution in both cases were already treated in Section 2. The integral appearing in Case 2 (i.e., Eq. (39)) is easy to evaluate numerically. However, the integral appearing in Case 1 (i.e., Eq. (38)) is rather difficult to evaluate even though the integral is convergent. Therefore, the treatment of this integral is given in the following. Let us write the integral as

$$I(x, t) = \int_0^\infty \sin k a \sin k x \frac{\cos p t - \cos(p t \sqrt{1 - z(k)})}{1 - z(k)} dk \quad (59)$$

where

$$p = \frac{\sqrt{6} c t}{l}, \quad z(k) = \frac{\sin k l}{k l} \quad (60)$$

We have

$$\cos(p t \sqrt{1 - z}) = \cos p t + \frac{p t}{2} z \sin p t + \frac{p t}{8} z^2 (\sin p t - p t \cos p t) + \frac{p t}{16} z^3 (\sin p t - p t \cos p t - \frac{(p t)^2}{3} \sin p t) + O(z^4) \quad (61)$$

By using (61), the integral I can be rewritten as

$$I = I_1 + I_2 \quad (62)$$

where

$$I_1 = \int_0^\infty \frac{\sin k a \sin k x}{1 - z(k)} \left[\cos p t - \cos(p t \sqrt{1 - z(k)}) \right] dk + \frac{p t}{2} z(k) \sin p t + \frac{p t}{8} z(k)^2 (\sin p t - p t \cos p t) dk \\ I_2 = \int_0^\infty \frac{\sin k a \sin k x}{1 - z(k)} \left[\frac{p t}{2} z(k) \sin p t + \frac{p t}{8} z(k)^2 (\sin p t - p t \cos p t) \right] dk \quad (63)$$

Now the integral I_1 is more rapidly convergent than the original integral I . It should be also mentioned here that in evaluating the integral I_2 , the following results can be used to accelerate the convergence.

$$\int_0^\infty \cos(b y) \left(\frac{\sin y}{y} \right)^n dy = \begin{cases} \frac{n \pi}{2^n} \sum_{k=0}^{\lfloor \frac{n+b}{2} \rfloor} \frac{(-1)^k (n+b-2k)^{n-1}}{k!(n-k)!}, & n \geq 1, b \geq 0 \text{ except for } n=b=1 \\ \frac{\pi}{4}, & \text{if } n=b=1 \end{cases} \quad (64)$$

where n is an integer, b is real, and $[r]$ is the integer part of r , i.e., the largest integer not exceeding r . Some examples are given below:

$$[7] = 7, \quad [2.3] = 2, \quad [-4.75] = -5 \quad (65)$$

It should be noted that the above formula (64) is more general than the formula 2 of 3.836 in Gradshteyn and Ryzhik (1980), since b seems to be treated as an integer m , and the range of m (i.e., b) is more limited ($0 \leq m < n$) in Gradshteyn and Ryzhik (1980). Also, the second line of the formula 2 of 3.836 (reproduced below)

$$\int_0^\infty \cos(by) \left(\frac{\sin y}{y} \right)^n dy = 0 \text{ for } b \geq n \geq 2 \quad (66)$$

is both superfluous and incomplete, since (66) can be proven from (64) as discussed below, and the condition $b \geq n \geq 2$ should be slightly modified. In other words, (64) is a generalized and more unified version of the formula 2 of 3.836, since we have extended the range of b . However, in order to have this unified version, it is necessary to provide a proof that (64) implies (66), which is provided in Appendix B. The formula (64) itself can be probably proven by analytic continuation, if we start from the original derivation of the formula 2 of 3.836, but this was not pursued here. Instead, the formula (64) has been confirmed analytically and numerically by Mathematica. We believe that this is sufficient for our purpose. Furthermore, the fact that we are able to unify the original formula (formula 2 of 3.836) by proving (66) from (64) with an extended range of b seems to support the correctness of the formula (64).

In light of the above formula (64), the formula 5 of 3.836 (Gradshteyn and Ryzhik, 1980) seems to have a typographical error. A minus sign before b in the formula 5 of 3.836 (Gradshteyn and Ryzhik, 1980) should be plus instead. Thus, the formula (64) is the correct and expanded form of both of the formulas 2 and 5 of 3.836 (Gradshteyn and Ryzhik, 1980). Secondly, according to the formula 6 of 3.836 and the formula 2 of 3.741 (Gradshteyn and Ryzhik, 1980), we have

$$\int_0^\infty \cos(by) \left(\frac{\sin y}{y} \right)^n dy = 0 \text{ for } b \geq n \geq 1 \text{ except for } b = n = 1 \quad (67)$$

which would be the correct form of the second line of the formula 2 of 3.836 as opposed to (66). Therefore, the proof of (67) from (64) is given in Appendix B instead of (66).

For Case 1, the non-dimensionalized displacement is given by (38). The non-singular part of the solution for the non-dimensionalized displacement is shown in Figs. 4 and 5, when $l/a = 1, 2, 3, 4$. For Case 2, the non-dimensionalized displacement is given by (39). The non-singular part of the solution for the non-dimensionalized displacement is shown in Figs. 6 and 7, when $l/a = 1, 2, 3, 4$. For both cases, the elastodynamic solution is also shown. It is clearly seen from Figs. 4–7 that when l/a becomes smaller, the non-singular part of the peridynamic solution approaches the elastodynamic solution. It should be also mentioned here that the singular part of the peridynamic solution for both cases approaches zero as l/a becomes smaller, due to (55) and (57).

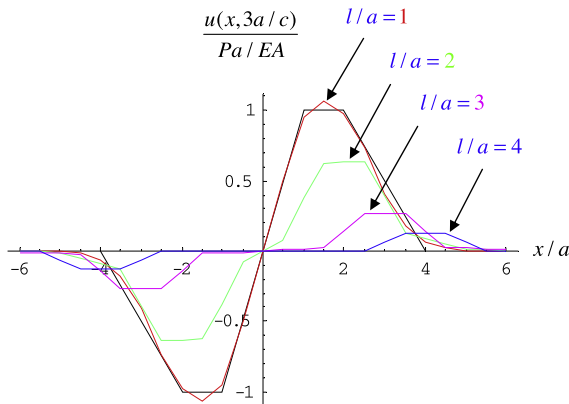


Fig. 4. Non-singular part of the peridynamic solution for Case 1 when $l/a = 1, 2, 3, 4$ as a function of x at $t = 3a/c$ with the elastodynamic solution.

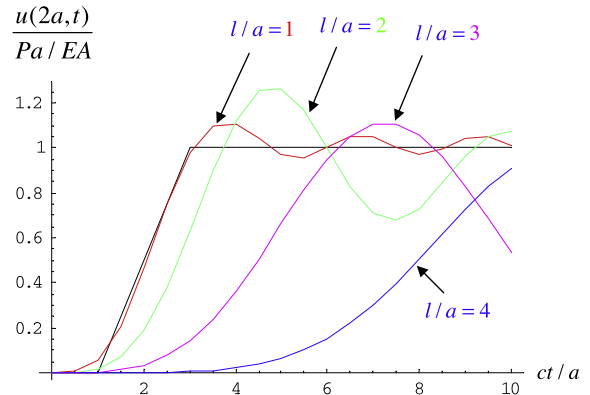


Fig. 5. Non-singular part of the peridynamic solution for Case 1 when $l/a = 1, 2, 3, 4$ as a function of t at $x = 2a$ with the elastodynamic solution.

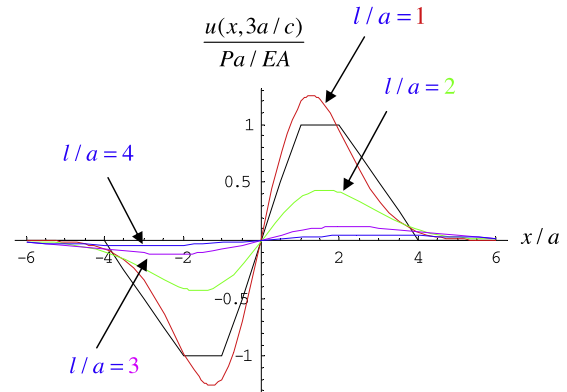


Fig. 6. Non-singular part of the peridynamic solution for Case 2 when $l/a = 1, 2, 3, 4$ as a function of x at $t = 3a/c$ with the elastodynamic solution.

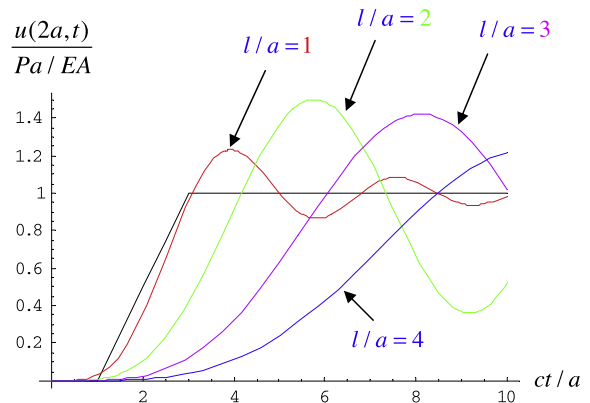


Fig. 7. Non-singular part of the peridynamic solution for Case 2 when $l/a = 1, 2, 3, 4$ as a function of t at $x = 2a$ with the elastodynamic solution.

6.3. Dispersion relation

The dispersion relation can be obtained from (28) for a given micromodulus. In this paper, we have considered two different micromoduli so far. Let us also consider another micromodulus for the sake of comparison. Thus, the following three micromoduli will be considered:

$$\begin{aligned}
C(\xi) &= \begin{cases} \frac{3E}{l^3}, & |\xi| \leq l \\ 0, & |\xi| > l \end{cases} \quad (\text{micromodulus A}) \\
C(\xi) &= \frac{4E}{l^3 \sqrt{\pi}} e^{-\left(\frac{\xi}{l}\right)^2} \quad (\text{micromodulus B}) \\
C(\xi) &= \begin{cases} \frac{12E}{l^3} \left(1 - \frac{|\xi|}{l}\right), & |\xi| \leq l \\ 0, & |\xi| > l \end{cases} \quad (\text{micromodulus C})
\end{aligned} \quad (68)$$

The corresponding dispersion relations are obtained as (see also (33) and (35))

$$\begin{aligned}
\omega(k) &= \frac{\sqrt{6}c}{l} \sqrt{1 - \frac{\sin kl}{kl}} \quad (\text{micromodulus A}) \\
\omega(k) &= \frac{2c}{l} \sqrt{1 - e^{-\frac{k^2 l^2}{4}}} \quad (\text{micromodulus B}) \\
\omega(k) &= \frac{c}{l} \sqrt{24 \left(\frac{1}{2} - \frac{1 - \cos kl}{(kl)^2} \right)} \quad (\text{micromodulus C})
\end{aligned} \quad (69)$$

where c is the wave speed of the corresponding elastic material defined in (32). The so-called group velocity c_g is defined by (see Achenbach, 1973)

$$c_g = \frac{d\omega(k)}{dk} \quad (70)$$

which describes the speed of energy flow (Lamb, 1904; Mandelstam, 1945; Brillouin, 1960; Achenbach, 1973) under normal circumstances. It is also related to the signal speed (speed of information), and they are the same under certain conditions, but they can differ in some cases such as for electromagnetic wave propagation in an absorbing medium (Brillouin, 1960). An extensive account of group velocity, which contains two original separate papers by Sommerfeld and Brillouin both published in 1914, is given by Brillouin (1960). Since the works of Garrett and McCumber (1970), and Chu and Wong (1982), it is now accepted that superluminal and negative group velocities can happen in a relatively simple physical setting (see also Woodley and Mojahedi, 2004; Mobley, 2007 and Robertson et al., 2007). In these settings, the group velocity is different from the signal velocity, but the theory of special relativity (causality) is not violated.

Conditions for the existence of superluminal and negative group velocities can be relatively easily obtained. First, from (70) we obtain

$$\frac{1}{c_g(\omega)} \equiv \frac{dk(\omega)}{d\omega} = \frac{d}{d\omega} \frac{\omega}{c_p(\omega)} = \frac{1}{c_p} \left(1 - \frac{\omega}{c_p} \frac{dc_p}{d\omega} \right) \quad (71)$$

where c_p is the phase velocity of a medium, and for the 1D elastic bar it is the same as c defined in (32). By using (71), conditions for the superluminal and negative group velocities are given by

$$\frac{c_p}{\omega} \left(1 - \frac{c_p}{c_L} \right) < \frac{dc_p}{d\omega} < \frac{c_p}{\omega} \quad \text{for } c_L < c_g(\omega) < \infty \quad (\text{superluminal group velocity}) \quad (72)$$

$$\frac{dc_p}{d\omega} > \frac{c_p}{\omega} \quad \text{for } c_g(\omega) < 0 \quad (\text{negative group velocity}) \quad (73)$$

where c_L is the speed of light. For acoustic waves in steel, the ratio of the phase velocity to the speed of light c_p/c_L is of the order of 1.7×10^{-5} . Thus it can be seen from (72) that the superluminal condition is attained for a very narrow band when expressed in terms of $\frac{dc_p}{d\omega}$. It is also clear from (72) and (73) that the domains for the superluminal and negative group velocities are adjacent to each other when expressed in terms of $\frac{dc_p}{d\omega}$. Taking the inverse of (71), it can be shown that

$$c_g(\omega) = \frac{c_L}{n(\omega) + \omega \frac{dn}{d\omega}} \quad (74)$$

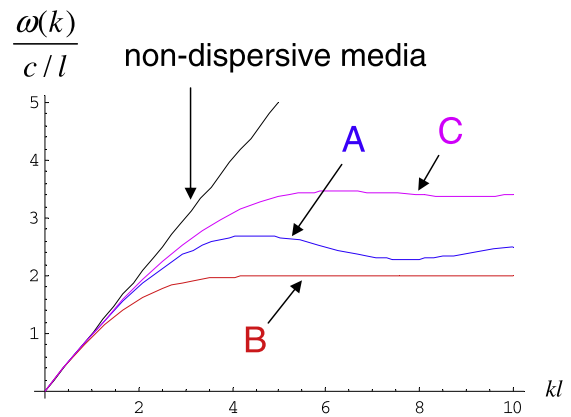


Fig. 8. Dispersion relations for different micromoduli.

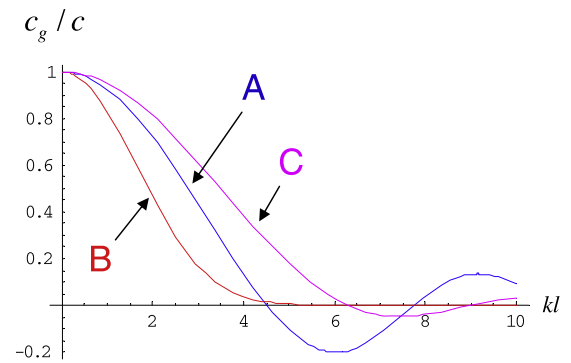


Fig. 9. Group velocities for different micromoduli.

where n is the index of refraction defined as

$$n(\omega) = \frac{c_L}{c_p(\omega)} \quad (75)$$

It is seen from (74) that the group velocity can be larger than the speed of light, or even negative when $\frac{dn}{d\omega} < 0$, which is called the anomalous dispersion in optics. From the definition of the index of refraction (75), the following correspondence is revealed.

$$\begin{aligned}
\frac{dn}{d\omega} > 0 &\iff \frac{dc_p}{d\omega} < 0 \quad (\text{normal dispersion}) \\
\frac{dn}{d\omega} < 0 &\iff \frac{dc_p}{d\omega} > 0 \quad (\text{anomalous dispersion})
\end{aligned} \quad (76)$$

It can be seen from (72), (73), and (76) that the superluminal and negative group velocities happen with the anomalous dispersion.

From (69) and (70), we obtain

$$\begin{aligned}
c_g &= \frac{c \sqrt{\frac{3}{2} \left(-\frac{\cos kl}{kl} + \frac{\sin kl}{(kl)^2} \right)}}{\sqrt{1 - \frac{\sin kl}{kl}}} \quad (\text{micromodulus A}) \\
c_g &= \frac{ckle^{-\frac{(kl)^2}{4}}}{2\sqrt{1 - e^{-\frac{(kl)^2}{4}}}} \quad (\text{micromodulus B}) \\
c_g &= \frac{c\sqrt{6} \left(\frac{2(1 - \cos kl)}{(kl)^3} - \frac{\sin kl}{(kl)^2} \right)}{\sqrt{\frac{1}{2} - \frac{1 - \cos kl}{(kl)^2}}} \quad (\text{micromodulus C})
\end{aligned} \quad (77)$$

The non-dimensionalized dispersion curves and non-dimensionalized group velocities are plotted in Figs. 8 and 9, respectively. It is

interesting to observe from Figs. 8 and 9 that the micromodulus A and micromodulus C clearly show a negative group velocity for certain regions of the wavenumber k . This existence of the negative group velocities in the case of micromoduli A and C indicates that peridynamics can be used for modeling a material with anomalous dispersion.

7. Conclusions

Exact analytical solutions are obtained for peristatic and peridynamic problems for 1D infinite rod in the form of convergent integrals. Loadings considered are two concentrated point loads with or without time dependence. For each peristatic and peridynamic problem, two cases are treated with two corresponding micromoduli. The Case 1 of the peristatic problem and the special case of Case 2 of the peridynamic problem have been treated previously (Silling et al., 2003; Weckner and Abeyaratne, 2005; Emmrich and Weckner, 2007a,b), but other cases are believed to be new. Even for the Case 1 of the peristatic problem, which was treated before, the exact analytical solution is obtained for the first time in this paper. The key contribution of this paper is the development of a method to obtain exact analytical solutions for various peristatic and peridynamic problems. We have succeeded in obtaining the exact analytical solutions by transforming divergent integrals into singular solutions (generalized functions) plus convergent integrals. Also, by considering the classical limits of both the peristatic and peridynamic solutions as $l \rightarrow 0$, it is found that the corresponding elastostatic and elastodynamic solutions are recovered exactly. It is also found that, for the peridynamic problems considered in this paper, the peridynamic solutions can be expressed as a product of a peristatic solution and a certain time-dependent function plus other time-dependent integrals. Even though only two cases for each peristatic and peridynamic problem have been treated, the method developed in this paper can be applied to other peridynamic problems, provided that the form of the solution has certain similarities to the ones treated here. For example, a variety of other peridynamic problems for a 1D infinite rod can be solved by this method.

We have also examined the dispersion curves and the group velocities for the materials with three different micromoduli, two of which are used in the peridynamic problems considered in this paper. It is interesting to note that negative group velocities are produced for two of the three micromoduli investigated. The existence of the negative group velocities indicates that peridynamics can be used for modeling a material with anomalous dispersion. Lastly, during the course of investigation, minor errors were found in some of the mathematical formulas in Gradshteyn and Ryzhik (1980). Those formulas were corrected and improved. Independent proof of one of the key results of the formulas is also provided in Appendix B.

Acknowledgements

The author appreciates the helpful conversations he had with Dr. Stewart Silling on a number of occasions on the topic of peridynamics. The author also appreciates the anonymous reviewers for providing him a paper by Weckner et al. (2009), which the author was not aware of until the time of the review.

Appendix A. Proof of Convergence of Integrals

We want to prove the convergence of the following four integrals.

$$\begin{aligned} J_1 &= \int_0^\infty \sin ka \sin kx \left(\frac{1 - \cos \left[\frac{\sqrt{6}ct}{l} \sqrt{1 - \frac{\sin kl}{kl}} \right]}{1 - \frac{\sin kl}{kl}} - 1 + \cos \left[\frac{\sqrt{6}ct}{l} \right] \right) dk \\ J_2 &= \int_0^\infty \sin ka \sin kx \left(\frac{1 - \cos \left[\frac{2ct}{l} \sqrt{1 - e^{-\left(\frac{k}{2}\right)^2}} \right]}{1 - e^{-\left(\frac{k}{2}\right)^2}} - 1 + \cos \left[\frac{2ct}{l} \right] \right) dk \\ J_3 &= \int_0^\infty \sin ka \sin kx \frac{\cos \left[\frac{\sqrt{6}ct}{l} \right] - \cos \left[\frac{\sqrt{6}ct}{l} \sqrt{1 - \frac{\sin kl}{kl}} \right]}{1 - \frac{\sin kl}{kl}} dk \\ J_4 &= \int_0^\infty \sin ka \sin kx \frac{\cos \left[\frac{2ct}{l} \right] - \cos \left[\frac{2ct}{l} \sqrt{1 - e^{-\frac{k^2 l^2}{4}}} \right]}{1 - e^{-\frac{k^2 l^2}{4}}} dk \end{aligned} \quad (A-1)$$

The above integrals can be rewritten as

$$\begin{aligned} J_1 &= \int_0^\infty \frac{\sin \frac{a}{l} y \sin \frac{x}{l} y}{\sqrt{y}} \sqrt{y} \left(\frac{1 - \cos \left[pt \sqrt{1 - \frac{\sin y}{y}} \right]}{1 - \frac{\sin y}{y}} - 1 + \cos pt \right) dy \\ J_2 &= \int_0^\infty \frac{\sin \frac{a}{l} y \sin \frac{x}{l} y}{\sqrt{y}} \sqrt{y} \left(\frac{1 - \cos \left[qt \sqrt{1 - e^{-\left(\frac{y}{2}\right)^2}} \right]}{1 - e^{-\left(\frac{y}{2}\right)^2}} - 1 + \cos qt \right) dy \\ J_3 &= \int_0^\infty \frac{\sin \frac{a}{l} y \sin \frac{x}{l} y}{\sqrt{y}} \frac{\sqrt{y} (\cos pt - \cos \left[pt \sqrt{1 - \frac{\sin y}{y}} \right])}{1 - \frac{\sin y}{y}} dy \\ J_4 &= \int_0^\infty \frac{\sin \frac{a}{l} y \sin \frac{x}{l} y}{\sqrt{y}} \frac{\sqrt{y} (\cos qt - \cos \left[qt \sqrt{1 - e^{-\left(\frac{y}{2}\right)^2}} \right])}{1 - e^{-\left(\frac{y}{2}\right)^2}} dy \end{aligned} \quad (A-2)$$

Let us note that, by using 3.762 of Gradshteyn and Ryzhik (1980), we obtain

$$\int_0^\infty \frac{\sin ax \sin bx}{\sqrt{x}} dx = \frac{1}{2} \sqrt{\frac{\pi}{2}} \left(\frac{1}{\sqrt{|a-b|}} - \frac{1}{\sqrt{a+b}} \right) \quad (A-3)$$

Using (A-3), we have

$$\int_0^\infty \frac{\sin \frac{a}{l} y \sin \frac{x}{l} y}{\sqrt{y}} dy = \frac{1}{2} \sqrt{\frac{\pi}{2}} \left(\frac{1}{\sqrt{\left| \frac{a}{l} - \frac{x}{l} \right|}} - \frac{1}{\sqrt{\frac{a}{l} + \frac{x}{l}}} \right) \quad (A-4)$$

Now, the above four integrals in (A-2) can be rewritten as

$$\begin{aligned} J_1 &= \int_0^\infty \frac{\sin \frac{a}{l} y \sin \frac{x}{l} y}{\sqrt{y}} g_1(y) dy \\ J_2 &= \int_0^\infty \frac{\sin \frac{a}{l} y \sin \frac{x}{l} y}{\sqrt{y}} g_2(y) dy \\ J_3 &= \int_0^\infty \frac{\sin \frac{a}{l} y \sin \frac{x}{l} y}{\sqrt{y}} g_3(y) dy \\ J_4 &= \int_0^\infty \frac{\sin \frac{a}{l} y \sin \frac{x}{l} y}{\sqrt{y}} g_4(y) dy \end{aligned} \quad (A-5)$$

where

$$\begin{aligned}
g_1(y) &= \sqrt{y} \left(\frac{1 - \cos \left[\frac{pt \sqrt{1 - \frac{\sin y}{y}}}{1 - \frac{\sin y}{y}} \right] - 1 + \cos pt}{1 - \frac{\sin y}{y}} \right) \\
g_2(y) &= \sqrt{y} \left(\frac{1 - \cos \left[\frac{qt \sqrt{1 - e^{-\left(\frac{y}{2}\right)^2}}}{1 - e^{-\left(\frac{y}{2}\right)^2}} \right] - 1 + \cos qt}{1 - e^{-\left(\frac{y}{2}\right)^2}} \right) \\
g_3(y) &= \frac{\sqrt{y} \left(\cos pt - \cos \left[\frac{pt \sqrt{1 - \frac{\sin y}{y}}}{1 - \frac{\sin y}{y}} \right] \right)}{1 - \frac{\sin y}{y}} \\
g_4(y) &= \frac{\sqrt{y} \left(\cos qt - \cos \left[\frac{qt \sqrt{1 - e^{-\left(\frac{y}{2}\right)^2}}}{1 - e^{-\left(\frac{y}{2}\right)^2}} \right] \right)}{1 - e^{-\left(\frac{y}{2}\right)^2}}
\end{aligned} \quad (\text{A-6})$$

By using the Dirichlet (or Chartier's) test (see [Greenberg, 1978](#)), the above four integrals in (A-5) converge if we can prove the following equalities (limits), respectively.

$$\begin{aligned}
\lim_{y \rightarrow \infty} g_1(y) &= 0, \quad \lim_{y \rightarrow \infty} g_2(y) = 0, \\
\lim_{y \rightarrow \infty} g_3(y) &= 0, \quad \lim_{y \rightarrow \infty} g_4(y) = 0
\end{aligned} \quad (\text{A-7})$$

By using the Taylor expansion, we obtain

$$\begin{aligned}
g_1(y) &= \sqrt{y} \left[\left(1 - \cos pt - \frac{pt}{2} \sin pt \right) \frac{\sin y}{y} \right. \\
&\quad \left. + \left(1 - \cos pt + \frac{(pt)^2}{8} \cos pt - \frac{5pt}{8} \sin pt \right) \frac{\sin^2 y}{y^2} + O\left(\frac{1}{y^3}\right) \right] \\
g_2(y) &= \sqrt{y} \left[\left(1 - \cos qt - \frac{qt}{2} \sin qt \right) e^{-\frac{y^2}{4}} \right. \\
&\quad \left. + \left(1 - \cos qt + \frac{(qt)^2}{8} \cos qt - \frac{5qt}{8} \sin qt \right) e^{-\frac{y^2}{2}} + O\left(e^{-\frac{3y^2}{4}}\right) \right] \\
g_3(y) &= \frac{\sqrt{y}}{1 - \frac{\sin y}{y}} \left[-\frac{pt}{2} \sin pt \frac{\sin y}{y} + \frac{pt}{8} (pt \cos pt - \sin pt) \frac{\sin^2 y}{y^2} + O\left(\frac{1}{y^3}\right) \right] \\
g_4(y) &= \frac{\sqrt{y}}{1 - e^{-\frac{y^2}{4}}} \left[-\frac{qt}{2} e^{-\frac{y^2}{4}} \sin qt + \frac{qt}{8} (qt \cos qt - \sin qt) e^{-\frac{y^2}{2}} + O\left(e^{-\frac{3y^2}{4}}\right) \right]
\end{aligned} \quad (\text{A-8})$$

By using (A-8), Eq. (A-7) can be easily proven. Therefore, all of the integrals J_1, J_2, J_3, J_4 are convergent. \square

Appendix B. Independent Proof of A Key Formula

We want to prove

$$\int_0^\infty \cos(by) \left(\frac{\sin y}{y} \right)^n dy = 0 \quad \text{for } b \geq n \geq 1 \text{ except for } b = n = 1 \quad (\text{B-1})$$

from

$$\int_0^\infty \cos(by) \left(\frac{\sin y}{y} \right)^n dy = \begin{cases} \frac{n!}{2^n} \sum_{k=0}^{\lfloor \frac{n+b}{2} \rfloor} \frac{(-1)^k (n+b-2k)^{n-1}}{k!(n-k)!}, & n \geq 1 \text{ (integer)}, b \geq 0 \\ \frac{\pi}{4}, & \text{if } n = b = 1 \end{cases} \quad (\text{B-2})$$

In order to prove (B-1) from (B-2), we need to prove that

$$\sum_{k=0}^{\lfloor \frac{n+b}{2} \rfloor} \frac{(-1)^k (n+b-2k)^{n-1}}{k!(n-k)!} = 0 \quad \text{for } b \geq n \geq 1 \text{ except for } b = n = 1 \quad (\text{B-3})$$

It is easily seen that (B-3) is true even for $b = n = 1$, even though this fact is not needed or related to our goal. Thus, it is easier to deal with this extended equality including the case of $b = n = 1$.

$$\sum_{k=0}^{\lfloor \frac{n+b}{2} \rfloor} \frac{(-1)^k (n+b-2k)^{n-1}}{k!(n-k)!} = 0 \quad \text{for } b \geq n \geq 1 \quad (\text{B-4})$$

Now it can be easily shown that (B-4) is equivalent to

$$\sum_{k=0}^{\lfloor r \rfloor} {}_n C_k (-1)^k (r-k)^{n-1} = 0 \quad \text{for } r \geq n \geq 1 \quad (\text{B-5})$$

Let us note that (B-5) is equivalent to

$$\sum_{k=0}^n {}_n C_k (-1)^k (r-k)^{n-1} = 0 \quad \text{for } r \geq n \geq 1 \quad (\text{B-6})$$

since

$${}_n C_k = 0 \quad \text{for } k = n+1, \dots, [r] \quad (\text{B-7})$$

if $[r] \geq n+1$. We will now prove the family of equalities

$$\sum_{k=0}^n {}_n C_k (-1)^k (r-k)^m = 0 \quad \text{for } r \geq n \geq 1 \quad \text{and } m = 1, 2, 3, \dots, n-1 \quad (\text{B-8})$$

which includes (B-6). For $m = 1$, we proceed as follows. We have

$$(x+1)^n = \sum_{k=0}^n {}_n C_k x^{n-k} \quad (\text{B-9})$$

Multiplying (B-9) by x^{r-n} , we have

$$x^{r-n} (x+1)^n = \sum_{k=0}^n {}_n C_k x^{r-k} \quad (\text{B-10})$$

Define a differential operator L as

$$L = x \frac{d}{dx} \quad (\text{B-11})$$

Applying L on the both sides of (B-10), and setting the value of x as -1 , we have

$$L \sum_{k=0}^n {}_n C_k x^{r-k} \Big|_{x=-1} = L [x^{r-n} (x+1)^n] \Big|_{x=-1} = 0 \quad (\text{B-12})$$

From (B-12), we obtain

$$\sum_{k=0}^n {}_n C_k (r-k) (-1)^{r-k} = 0 \quad (\text{B-13})$$

Multiplying (B-13) by $(-1)^{2n-r}$, and noting that $(-1)^{2k} = 1$ for any integer k , we finally obtain

$$\sum_{k=0}^n {}_n C_k (-1)^k (r-k) = 0 \quad (\text{B-14})$$

which proves (B-8) for $m = 1$. In a similar manner, (B-8) can be proven for all the other values of m up to and including $m = n-1$. The last case is shown below. Applying L^{n-1} on the both sides of (B-10), and setting the value of x as -1 , we have

$$L^{n-1} \sum_{k=0}^n {}_n C_k x^{r-k} \Big|_{x=-1} = L^{n-1} [x^{r-n} (x+1)^n] \Big|_{x=-1} = 0 \quad (\text{B-15})$$

where it should be pointed out that the last equality would not hold if the differential operator is L^n instead of L^{n-1} . From (B-15), we obtain

$$\sum_{k=0}^n {}_n C_k (r-k)^{n-1} (-1)^{r-k} = 0 \quad (\text{B-16})$$

Multiplying (B-16) by $(-1)^{2n-r}$, and noting that $(-1)^{2k} = 1$ for any integer k , we finally obtain

$$\sum_{k=0}^n C_k (-1)^k (r-k)^{n-1} = 0 \quad (\text{B-17})$$

□

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