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A hybrid method for efficient solution of geometrically nonlinear structures

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ABSTRACT

We propose a novel hybrid method for calculating accurate responses of geometrically nonlinear structures exhibiting complex snap-through and snap-back behaviours. The proposed method employs a hybrid evolutionary-algebraic method to obtain each point of the equilibrium path, where the equilibrium is formulated generally as a minimisation problem. Genetic algorithm and Nelder–Mead simplex methods are used together as the hybrid optimiser. To obtain any arbitrary point of the equilibrium path, we only need a series of sparse matrix to vector multiplications and do not require information about previous equilibrium states, the assembly of tangent stiffness matrix, the solution of a set of system of linear equations or factorisation processes. Both primary and secondary paths can be followed. In addition, utilising the tangent stiffness matrix, this method can effectively find both the limit and bifurcation points directly. The state of non-proportional loading can also be considered successfully. Additionally, we show how to generate the solution of a structure whose geometrical and mechanical properties vary slightly, starting from the original solution. Finally, to demonstrate the efficiency and capabilities of the present approach, three examples that are well known for their complex snap-through snap-back load–deflection curves are comprehensively studied, and the results obtained are compared with those reported in the literature.

1. Introduction

The calculation of local and global collapse loads is critical in the reliable design of structures. For geometrically nonlinear structures, it is essential to trace the complete load–deflection curve as well as determine an accurate estimation of the limit and bifurcation points. The study of primary and secondary paths as well as the post-buckling behaviour of a structure enables us to achieve a deep insight into its real behaviour. The reader may refer to Crisfield (1991) to observe the detailed reasons why we require tracing the complete equilibrium path and its key points. Because of the importance of this issue, it has been extensively investigated by researchers. Different methods, including load controlled, displacement controlled and arc-length methods, have been developed for the analysis of geometrically nonlinear structures. Among the different methods developed for tracing nonlinear equilibrium path, the arc-length method is undoubtedly most appealing technique. The method has attracted extensive attention since it was first introduced by Riks (1972, 1979) and Wempner (1971). Bathe and Dvorkin (1983), Belleni and Chulya (1987), Crisfield (1981, 1983), Forde and Sttemer (1987), Ramm (1981), Schweizerhof and Wriggers (1986) and others have made different modifications and improvements to the original method. To identify singular points (limit or bifurcation points) on the equilibrium path, different approaches have been adopted. Among them, the formulation of the problem using an extended system of equations is most appealing; see, e.g. Ibrahimbegović and Al Mikdad (2000), Moore and Spence (1980), Weinitschke (1985), Wriggers et al. (1988) and Wriggers and Simo (1990). However, the extended system usually becomes ill conditioned near a singular point, making the solution procedure numerically unstable or very sensitive to small errors. Magnusson and Svensson (1998) developed a method for the direct computation of complete load–deflection curves, including primary and secondary paths as well as the exact calculation of singular points. The method employs an extended system of equations utilising the deflated block-elimination system (Chan and Resasco, 1986) to avoid the ill-conditioning problem.

The methods summarised so far are often classified as incremental-iterative methods. These methods are very popular and have been used extensively in commercial finite element software. Analytical representation of nonlinear equilibrium path (expanded in power series of a perturbation parameter) has also been adopted as an alternative approach even though it suffers from some limitations. The limitations of this method have been eliminated by combining a numerical approach with the perturbation technique, leading to a new approach known as asymptotic numerical method (ANM) (Damil and Potier-Ferry, 1990). ANM has been demonstrated to be very efficient as a solution strategy for nonlinear problems especially in structural mechanics. Cochelin (1994) proposed a path following method based on ANM (see also Baguet

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and Cochelin (2003) and Elhage-Hussein et al. (2000)). Najah et al. (1998) studied the performance of three types of ANM (the reduced basis, direct computation of series and Padé approximants) and showed that ANM with Padé approximants is more efficient than the others. For more applications of ANM in nonlinear structural mechanics the reader may also refer to Azrar et al. (1993), Cochelin et al. (1994a,b) and Vannucci et al. (1998).

Recently, Lee et al. (2011) developed an explicit arc-length method based on a dynamic relaxation method with kinetic damping for tracing the post-buckling equilibrium path of structures. To find the solution of large-scale structures with geometrical nonlinearity, Labeas and Belesis (2011) suggested a special methodology that exploits a series of incremental linear analyses and a set of fictitious forces in substitution of the typical Newton-type nonlinear analysis. The methods reviewed above briefly are fully algebraic. In this paper, a hybrid method is introduced for the efficient solution of geometrically nonlinear structures. This method is a hybrid of a genetic algorithm, as an evolutionary search approach, and the Nelder–Mead simplex method, as an algebraic heuristic search technique, to find points on the nonlinear equilibrium path as well as to directly calculate the limit and bifurcation points for both proportional and non-proportional loading. Our method also enables us to generate a solution of a slightly modified structure (modifications can be in both geometrical and mechanical properties) using the solution of the original structure without any need for reanalysis. Three examples are studied comprehensively, and the results obtained are compared with those reported in the literature. The comparisons clearly verify the accuracy and viability of our approach.

2. Path-following methods

The equilibrium path may be followed through several methods, including load-controlled, displacement-controlled and ANM techniques. Load-controlled methods cannot follow the post-buckling path for structures exhibiting snap-through and snap-back behaviour. Displacement-controlled methods can reach points beyond the limit points and can follow the post-buckling path even for a state of snap-through. However, the methods suffer from the same limitation as in the case of snap-back. To follow a complete equilibrium path, a general method, namely arc-length, has been introduced originally by Riks (1972, 1979) and Wempner (1971). ANM can also be used as a promising method for the path following problem. However, according to the popularity of the arc-length method, in the next section, we briefly describe this method.

2.1. Spherical and cylindrical arc-length methods

In general, the equilibrium can be formulated using an out-of-balance force vector as follows.

$$\mathbf{g}(\mathbf{u},\lambda) = \mathbf{q}_i(\mathbf{u}) - \lambda\mathbf{q}_e = \mathbf{0} \tag{1}$$

In Eq. (1), $\mathbf{g}(\mathbf{u},\lambda)$ is the out-of-balance force vector, which is a function of the internal forces $\mathbf{q}_i(\mathbf{u})$ obtained from the true displacements vector $\mathbf{u}$ and the external loading vector $\mathbf{q}_e$ (independent of displacement vector) multiplied by the scalar loading parameter λ (load factor). Note that Eq. (1) defines the equilibrium in the state of proportional loading.

In the arc-length method, a single constraint equation $a(\mathbf{u},\lambda)$ is coupled with Eq. (1). The incremental form of this constraint (using incremental form of displacements $\Delta\mathbf{u}$ and loading parameter $\Delta\lambda$) is given in Eq. (2)

$$a(\Delta\mathbf{u},\Delta\lambda) = (\Delta\mathbf{u}^t\Delta\mathbf{u} + \Delta\lambda^2\psi^2\mathbf{q}_e^t\mathbf{q}_e) - \Delta l^2 = 0 \tag{2}$$

where Δl is the fixed given radius of the desired intersection that is an approximation to the incremental arc-length and ψ is a scaling parameter.

By applying the Newton–Raphson method to Eqs. (1) and (2) and using the Batoz and Dhatt (1979) approach, iterative displacements $\delta\mathbf{u}$ can be obtained as follows.

$$\delta\mathbf{u} = \delta\bar{\mathbf{u}} + \delta\lambda\,\delta\mathbf{u}_e \tag{3}$$

where

$$\delta\bar{\mathbf{u}} = -\mathbf{K}_t^{-1}\mathbf{g}_i, \qquad \delta\mathbf{u}_e = \mathbf{K}_t^{-1}\mathbf{q}_e \tag{4}$$

In Eq. (4), $\mathbf{K}_t$ is the tangential stiffness matrix. In addition, the incremental and iterative changes in displacements and load parameter are defined as given in Eq. (5)

$$\Delta\lambda_{i+1} = \Delta\lambda_i + \delta\lambda_i, \qquad \Delta\mathbf{u}_{i+1} = \Delta\mathbf{u}_i + \delta\mathbf{u}_i \qquad i = 0, 1, \ldots \tag{5}$$

where, $\Delta\lambda_0 = 0$ and $\Delta\mathbf{u}_0 = \mathbf{0}$. Fig. 1 shows the variables defined for a one degree of freedom system with $\psi = 1$.

At this stage, using Eqs. (2–5), a scalar quadratic equation is obtained as given in Eq. (6), which can be solved for $\delta\lambda_i$, leading to a new incremental load.

$$a_1\delta\lambda_i^2 + a_2\delta\lambda_i + a_3 = 0 \tag{6}$$

in which

$$a_1 = \delta\mathbf{u}_e^t\delta\mathbf{u}_e + \psi^2\mathbf{q}_e^t\mathbf{q}_e \tag{6a}$$

$$a_2 = 2\delta\mathbf{u}_e^t(\Delta\mathbf{u}_i + \delta\bar{\mathbf{u}}) + 2\Delta\lambda_i^2\psi^2\mathbf{q}_e^t\mathbf{q}_e \tag{6b}$$

$$a_3 = (\Delta\mathbf{u}_i + \delta\bar{\mathbf{u}})^t(\Delta\mathbf{u}_i + \delta\bar{\mathbf{u}}) - \Delta l^2 + \Delta\lambda_i^2\psi^2\mathbf{q}_e^t\mathbf{q}_e \tag{6c}$$

Eq. (6) is a quadratic equation and may have different roots, including complex roots. When we have two real roots, a suitable one could be selected by minimising the angle between the two successive incremental displacements (see, e.g. Crisfield, 1991, for more detail). The case of complex roots may be avoided by decreasing the step length Δl. However, a more advanced method to address complex roots was proposed by Kouhia and Mikkola

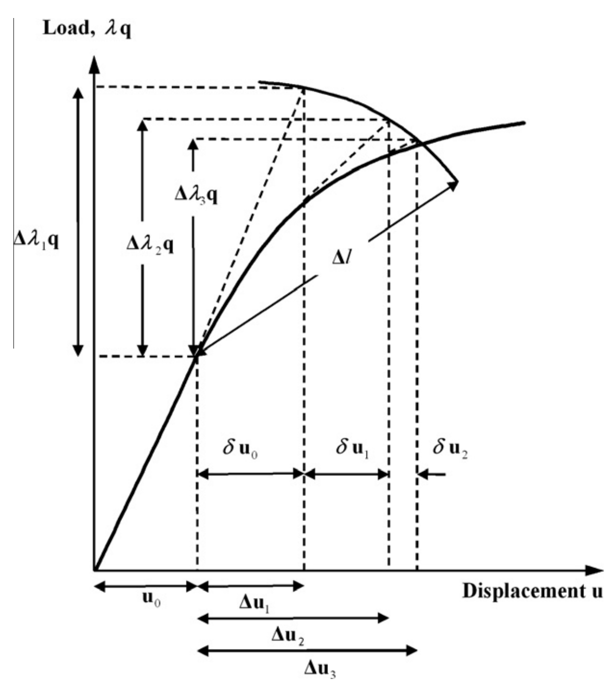

Fig. 1. Schematic view of the arc-length procedure and the defined variables with $\psi = 1$ (Crisfield, 1991).

(1989). The method briefly mentioned above is usually referred to the spherical arc-length method. Crisfield (1981) and Ramm (1981) studied the effect of the scaling parameter on the efficiency of the arc-length method. According to their studies, it is possible to set this parameter to zero ($\psi = 0$) without considering significant changes in the convergence properties. This case is referred to the cylindrical arc-length method.

3. Proposed method

3.1. General formulation

In this section, we propose a direct formulation of the nonlinear equilibrium path (material nonlinearity is not considered) based on a known displacement field.

Let $\mathbf{x}$ be a vector of nodal coordinates for the initial model. In addition, let $\mathbf{u}$ be a known displacement field (nodal displacements). The new geometry of structure including updated nodal coordinates, denoted by $\mathbf{x}_n$, can be obtained using Eq. (7) as follows.

$$\mathbf{x}_n = \mathbf{x} + \mathbf{u} \tag{7}$$

Now, for each element, the deformation gradient $\mathbf{F}$ is formed using original and new geometries as given in Eq. (8).

$$\mathbf{F} = \begin{bmatrix} 1 + \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & 1 + \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & 1 + \frac{\partial w}{\partial z} \end{bmatrix} \tag{8}$$

Additionally, u, v and w are displacement functions in the x-, y- and z-directions, respectively, based on the element's shape functions and nodal displacements.

For large deflections, different strain measures, including Green–Lagrange, Almansi and Hencky (log-strain), may be used. All of these strain measures can be derived from the deformation gradient (see, e.g. Bathe, 1996 or Crisfield, 1991). However, in this study, we use the Green–Lagrange strain measure that is given in Eq. (9) in tensor form as:

$$\mathbf{E}_g = \frac{1}{2}(\mathbf{F}^t\mathbf{F} - \mathbf{I}) \tag{9}$$

where $\mathbf{I}$ is a second order identity tensor. The Green–Lagrange strain measure can be represented for truss elements in a very simple form using initial and new lengths as follows.

$$\varepsilon_g = \frac{l_n^2 - l^2}{2l^2} \tag{9a}$$

In Eq. (9a), l_n is the length of element after applying the nodal displacements, while l is the original length.

Using Eq. (9) and the constitutive law considered, the second Piola–Kirchhoff stresses (conjugated with Green–Lagrange strain measure) are calculated. These stresses provide a vector of element forces, denoted by $\mathbf{p}$, in the local coordinate system of each element. Finally, a rectangular equilibrium matrix, which is widely used in the context of force method, is used to transform the vector of element forces to the nodal internal forces in the global coordinate system as given in Eq. (10)

$$\mathbf{q}_i(\mathbf{u}) = \mathbf{H}\mathbf{p} \tag{10}$$

where $\mathbf{q}_i(\mathbf{u})$ is the internal nodal force in the global coordinate system created from the known nodal displacement field $\mathbf{u}$ and $\mathbf{H}$ is the rectangular equilibrium matrix assembled from the equilibrium matrix of all elements. The reader may refer to Prezemieniecki (1968) for more details about this matrix and the generation method for different finite elements in the context of the force method.

At this stage, the state of equilibrium can be written by setting the out-of-balance force vector $\mathbf{g}$ to zero as previously given in Eq. (1).

3.2. Treatment of the equilibrium path

According to the procedure described above, it is possible to find the state of equilibrium by minimising the objective function given in Eq. (11)

$$\text{Minimise} \quad \|\mathbf{q}_i(\mathbf{u}) - \lambda\mathbf{q}_e\|_2 \tag{11}$$

$$\text{Subject to} \qquad u_{il} \leqslant u_i \leqslant u_{iu}, \qquad \lambda_l \leqslant \lambda \leqslant \lambda_u$$

where u_{il} and u_{iu} are the lower and upper bounds of displacements, respectively. In addition, λ_l and λ_u are the lower and upper bounds of the load factor, respectively. For some points, especially when there is a complex load–deflection curve, it is possible that the objective function defined in Eq. (11) converges to a local minimum that is not as small as required (according to the tolerance defined). In such cases, a modified objective function given in Eq. (12) should be examined to reach a better point. However, two points obtained as solutions of Eq. (11) and Eq. (12) may be used as new start points for the unconstrained minimisation technique that we will use.

$$\frac{\|\mathbf{q}_i(\mathbf{u}) - \lambda\mathbf{q}_e\|_2}{\|\mathbf{q}_i(\mathbf{u})\|_2} \tag{12}$$

3.3. Direct evaluation of limit or bifurcation points

The formulation given in Eq. (11) could simply be modified to enable us to directly find limit or bifurcation points. In a limit or bifurcation point, the tangent stiffness matrix is no longer positive definite, and full rank, i.e., the determinant of tangent stiffness matrix, will be zero or a very small magnitude. However, when we are working with real numbers, the determinant is not a proper criterion for checking the rank of matrices. As an example, it can be simply shown that a perfectly conditioned matrix may have a very small determinant (with respect to the computer *eps* that is 2.22e−16 for 64 bit real numbers). As a result, we use eigenvalues of the tangent stiffness matrix as a suitable indicator of its rank. Eq. (13) shows the new optimisation problem defined for the direct evaluation of limit or bifurcation points.

$$\text{Minimise} \quad t_1 + t_2 \tag{13}$$

$$\text{Subject to} \qquad u_{il} \leqslant u_i \leqslant u_{iu}, \qquad \lambda_l \leqslant \lambda \leqslant \lambda_u$$

where

$$t_1 = \|\mathbf{q}_i(\mathbf{u}) - \lambda\mathbf{q}_e\|_2, \tag{14a}$$

$$t_2 = \min|diag(\mathbf{D})| \tag{14b}$$

In Eq. (14b), $\mathbf{D}$ is a diagonal matrix consisting of all eigenvalues of the tangent stiffness matrix that could be calculated using Eq. (15) as follows

$$\mathbf{K}_t\Phi = \Phi\mathbf{D} \tag{15}$$

$\mathbf{K}_t$ is the tangent stiffness matrix formed using the current displacement field, Φ is the matrix of eigenvectors and $\mathbf{D}$ is the diagonal matrix of eigenvalues. In fact, t_2 is the smallest absolute eigenvalue of the tangent stiffness matrix. It is clear that when the objective function defined in Eq. (13) is zero or has a very small value, the calculated point is on the equilibrium path and the tangent stiffness matrix is rank deficient. In relation to Eq. (15), it is useful to note that the computational cost to calculate all eigenvalues and eigenvectors of $\mathbf{K}_t$ (within the minimisation process) is very significant, especially for large scale structural models. Fortunately, we only need to calculate the smallest absolute eigenvalue of $\mathbf{K}_t$, hence the *inverse iteration* method, known as the natural generalisation of the *power* method, can be employed effectively. Inverse iteration method enables us to find smallest absolute eigenvalue and corresponding eigenvector of a matrix by solving a set of linear equations. The rate of convergence of inverse iteration method can significantly be accelerated by setting a suitable shift (Wilkinson, 1965). For special cases (if $\mathbf{K}_t$ is semi-positive definite at limit or bifurcation points) original power method with a shift can also be employed that is very faster than the inverse iteration method. However, this step is still computationally expensive for large scale finite element models.

3.4. Direct evaluation of bifurcation points

The solution obtained from the previous section may be a limit or bifurcation point. It is straightforward to determine which point is reached. For each point as a solution of the minimisation problem defined in Eq. (13), a quantity can be calculated as follows

$$t_3 = |\mathbf{q}_e^t \boldsymbol{\varphi}| \tag{16}$$

where $\boldsymbol{\varphi}$ is the eigenvector corresponding to the zero eigenvalue of the tangent stiffness matrix. If t_3 is zero, then the point obtained will be a bifurcation point; otherwise, it will be a limit point. By checking this value, it is possible to determine the type of point obtained. However, another minimisation problem may be defined as given in Eq. (17), where a bifurcation point may be detected directly as follows.

$$\text{Minimise} \quad t_1 + t_2 + t_3 \tag{17}$$

$$\text{Subject to} \qquad u_{il} \leqslant u_i \leqslant u_{iu}, \qquad \lambda_l \leqslant \lambda \leqslant \lambda_u$$

The solution process described thus far can directly detect the limit and bifurcation points without any need to trace whole equilibrium path and without any need to find a solution of a system of linear equations, which is usually ill conditioned near the limit and bifurcation points.

3.5. Non-proportional loading

The equilibrium relationship given in Eq. (1) is valid for a state of proportional loading. In other words, there is only a single loading parameter, λ, and all loads are scaled proportionally via this parameter. For many practical structural problems, this loading regime is too restrictive (Crisfield, 1991) and a structure may be subjected to a non-proportional loading where loads are scaled by different loading parameters. In this case, there are different scenarios for which a structure can be unstable according to different combinations of loading parameters. This state is very expected especially for the space structures that their live loads (e.g., snow load) may be distributed non-uniformly.

The nonlinear equilibrium equations for a state of non-proportional loading can be represented using matrix of loading parameters as given in Eq. (18).

$$\mathbf{g}(\mathbf{u}, \boldsymbol{\Lambda}) = \mathbf{q}_i(\mathbf{u}) - \boldsymbol{\Lambda}\mathbf{q}_e = \mathbf{0} \tag{18}$$

where, $\boldsymbol{\Lambda}$ is a diagonal matrix of loading parameters as follows.

$$\Lambda = diag([\lambda_1, \lambda_2, \ldots, \lambda_n]) \tag{19}$$

In Eq. (19), n is the total number of degrees of freedom (for all free nodes) and λ_i is the loading parameter associated with ith degree of freedom. For a state of proportional loading, we have a single loading parameter, i.e., $\lambda_i = \lambda, i = 1, 2, \ldots, n$ and $\boldsymbol{\Lambda} = \lambda\mathbf{I}$ ($\mathbf{I}$ is an identity matrix). Substituting $\boldsymbol{\Lambda}$ into Eq. (18) gives the equilibrium equations in the state of proportional loading, Eq. (1). The formulation proposed in Section 3.2 can be modified easily to solve Eq. (18) as given in Eq. (20).

$$\text{Minimise} \quad \|\mathbf{q}_i(\mathbf{u}) - \boldsymbol{\Lambda}\mathbf{q}_e\|_2 \tag{20}$$

$$\text{Subject to} \qquad u_{il} \leqslant u_i \leqslant u_{iu}, \qquad \lambda_{il} \leqslant \lambda_i \leqslant \lambda_{iu}$$

Unfortunately, this subject has not been addressed adequately in the literature. As a result, it may still be a challenge to obtain the nonlinear response of a structure to a non-proportional loading via available solution strategies. However, as given in Eq. (20), our method deals with this problem in the same way as the proportional loading.

3.6. Solution for slight variation in initial data

The formulation of the problem with our method has an interesting advantage over conventional methods. Suppose we have obtained the complete solution of a geometrically nonlinear problem. This solution may be achieved using any solution strategy, including the method introduced here. Now, if one requires the solution of a problem with slight changes in both the geometrical and mechanical properties of the initial problem, conventional solution procedures should be started with new initialisation. However, the proposed method can simply be adapted to form the solution of the new problem using the solution of old problem. In fact, it is sufficient to find the solution of the new structure by minimising the objective function defined in Eq. (11) (without constraints) using an unconstrained minimisation technique where the previous solution is considered as a good starting point. This notable feature is useful especially for nonlinear analyses of structures where a considerable computational effort is usually required. The efficiency of this approach will be shown in Example 5.1.

4. Optimisation methods

4.1. Genetic algorithm

Genetic algorithms are used as a global search technique based on Darwin's evolution theory of 'survival of the fittest'. The method was first inspired by Holland (1975) and used by many others as one of the most popular and practical meta-heuristic approaches. In this method, an initial random population of feasible solutions

Table 1
Main parameters of the genetic algorithm used.

Parameter name	Type or value
Bounds of variables	According to the type of problem
Population (type, size)	Real valued, 20 and 50 for examples 1 and (2, 3) respectively
Selection	Stochastic uniform
Elitism count	2
Crossover (type, rate)	Heuristic (ratio = 1.7), 0.9
Mutation (type, rate)	Adaptive feasible, 0.1
Stopping criteria (fitness limit)	1e−2, 1e−3
(generation number)	2000

Table 2
Main parameters of the Nelder–Mead simplex method.

Parameter name	Type or value
Start point	Solution generated by genetic algorithm or any available solution
Stopping criteria	
(Max number of iterations)	200 × number of variables
(Max number of function evaluations)	200 × number of variables
Function tolerance	1e−10
Variables tolerance (Euclidian distance)	1e−10

Table 3
Representation of an individual for the sate of proportional and non-proportional loading.

Chromosome representation (proportional loading)					Chromosome representation (non-proportional loading)							
Genes corresponding to displacements				Gene corresponding to single loading parameter	Genes corresponding to displacements				Genes corresponding to loading parameters			
u_1	u_2	...	u_n	λ	u_1	u_2	...	u_n	λ_1	λ_2	...	λ_n

(individuals) are evolved to generate better solutions based on genetic operators (selection, crossover, mutation, etc.). We also use a genetic algorithm as a global optimiser to solve the minimisation problems defined in Section 3. The objective functions defined earlier are selected as our desired fitness functions. The main parameters, including the selection method and genetic operations used in our genetic algorithm, are presented in Table 1.

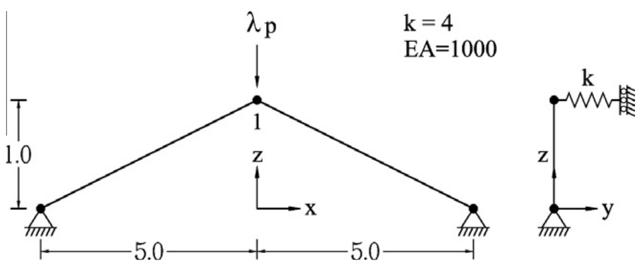

Fig. 2. A two-member shallow truss.

4.2. Nelder–Mead simplex method

The Nelder–Mead simplex method (Nelder and Mead, 1965) is a well-known heuristic optimisation method that finds the minimum (or maximum) of an unconstrained multivariable function. This method does not use any numerical or analytical gradients. A simplex in n-dimensional space is characterised by the $n+1$ distinct vectors that are its vertices. In two-dimensional space, a simplex is a triangle, and in three-dimensional space, it is a pyramid. At each step of the search, a new point in or near the current simplex is generated. The function value (objective function) at the new point is compared with the function's values at the vertices of the simplex, and usually, one of the vertices is replaced by the new point using operations including reflection and expansion, giving a new simplex. This step is repeated until the diameter of

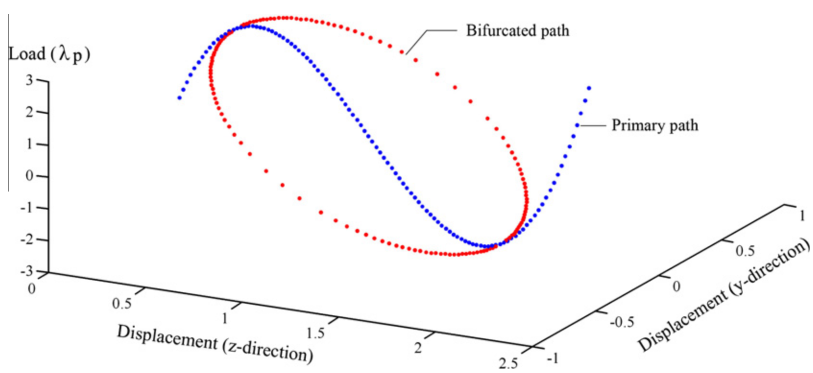

Fig. 3. Primary (blue dots) and bifurcated paths (red dots) for Example 5.1. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Table 4
Detailed results for Example 1 (bifurcation point).

	Generation/ iteration	$\|\mathbf{q}_i(\mathbf{u}) - \lambda\mathbf{q}_e\|_2$	Displacement (u_{1z})	Load (λp)	Computational times (second)
Hybrid method	1	1.800950	0.213974	1.582241	
	2	0.917897	0.263471	2.172845	
	3	0.590701	0.349074	2.585803	
	5	0.198088	0.301845	2.634877	
	14	0.061466	0.319827	2.748782	
	15	0.046919	0.311922	2.751487	0.2571
	23	0.024285	0.313895	2.722590	
	34	0.004967	0.314182	2.739831	
	39	0.002045	0.314500	2.741392	
	41	0.001159	0.314567	2.741415	
	42*	0.000904	0.314587	2.741422	
	43	0.000904	0.314587	2.741422	
	69	6.89e−05	0.314652	2.741326	0.0354
	111	8.06e−08	0.314652	2.741392	
	147	7.54e−11	0.314652	2.741392	
Analytical, Krenk (2009)			0.314652	2.741392	
Numerical, Magnusson and Svensson (1998)			0.3147	2.741	

* After this generation the hybrid method switches from genetic algorithm to the Nelder–Mead simplex method.

the simplex is less than the specified tolerance. However, to find a minimum, the simplex method requires a starting point that should be sufficiently close to the final desired point. In addition, note that we use Nelder–Mead version of the simplex method as described in Lagarias et al. (1998).

4.3. Proposed hybrid optimisation method

The optimisation methods described above have advantages and disadvantages depending on what we are looking for. A genetic algorithm as a global search technique employs different starting points (a population of random individuals). This exceptional feature increases the chance of finding the global minimum rather than a local one. However, the convergence rate of the method is not as fast as a local optimiser. On the other hand, the simplex method is very efficient as a local optimiser and works properly and rapidly when a good (sufficiently close to the local minimum) starting point is available. For complex problems, setting up a good starting point is almost impossible. To overcome this weakness of the simplex method, our hybrid solution procedure uses the solution of a genetic algorithm as a good starting point for the simplex method.

First, we use the genetic algorithm to find a minimum according to the tolerance adopted. We do not consider a hard tolerance for this step. In other words, we aim to find a solution with a maximum of one to three significant digits. The solution obtained from this step is close to the final solution. However, if the solution does not converge to the defined tolerance, it may still be used as a starting point for the next step and be able to converge to a valid point according to the final convergence criterion. Second, the solution obtained from the previous step is used as a starting point for the simplex algorithm. This step is the final step to find a solution with high accuracy. Hence, a hard stopping criterion should be considered. In this study, the stopping criterion is set to 1e−10 (see Table 2).

The above process can be used similarly for non-proportional loading and the only modification required is to add sufficient number of genes (corresponding to the number of loading parameters) to each chromosome (Table 3).

Obviously, the best solution obtained from the genetic algorithm is used as a good starting point for Nelder–Mead simplex method, allowing the new variables (loading parameters) to be included in the local search as well (see example 5.3).

5. Numerical examples

5.1. A two-member truss

In Fig. 2, a shallow truss consisting of two members is shown. This example has been studied by several authors e.g., Krenk (2009), Magnusson and Svensson (1998) and Pecknold et al. (1985). Here, we consider the specific case associated with the perfect structure where the geometrical imperfection is ignored in the y-direction. We aim to find the primary and bifurcated equilibrium paths using the proposed method. Fig. 3 shows the primary and bifurcated paths obtained using our method where the step size for displacement in the z-direction is set to 0.02.

Tables 4 and 5 provide the detailed results obtained for the direct calculation of the bifurcation and limit points. The results obtained are in complete agreement with the analytical results derived by Krenk (2009) and the numerical results reported by Magnusson and Svensson (1998) (up to 10 significant digits are the same according to the stopping criteria adopted). The comparisons clearly demonstrate the accuracy and efficiency of the proposed method for the direct calculation of the limit and bifurcation points without any need for tracing the complete equilibrium path. In addition, to demonstrate the efficiency of the hybrid method (to emphasise the role of the Nelder–Mead method), the bifurcation point is also obtained using a simple genetic

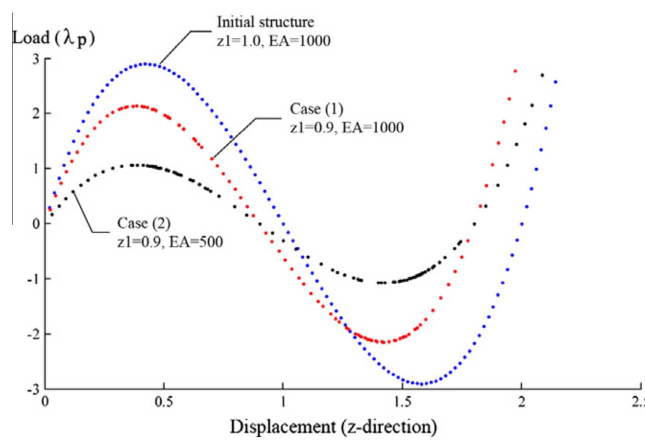

Fig. 4. Generated equilibrium paths for the modified structures using procedure described.

Table 5
Detailed results for Example 1 (limit point).

	Generation/ iteration	$\lVert \mathbf{q}_i(\mathbf{u}) - \lambda \mathbf{q}_e \rVert_2$	Displacement (u_{1z})	Load (λp)	Computational times (second)
Hybrid method	1	1.628271	0.340253	1.441836	
	2	1.118989	0.425915	1.869237	
	3	0.16154	0.312999	2.880681	
	5	0.152083	0.419197	2.964717	
	10	0.076628	0.425477	2.906092	0.2103
	14	0.068693	0.422400	2.965443	
	22	0.007848	0.422738	2.908817	
	30	0.004419	0.422738	2.90116	
	34	0.003443	0.422541	2.902666	
	35	0.001662	0.422638	2.901928	
	36*	0.000686	0.422638	2.902905	
	61	0.000384	0.422648	2.903611	
	70	0.000102	0.422648	2.903317	0.0451
	118	1.06e−07	0.422650	2.903275	
	157	8.94e−11	0.422650	2.903274	
Analytical, Krenk (2009)			0.422650	2.903274	
Numerical, Magnusson and Svensson (1998)			0.4226	2.903	

* After this generation the hybrid method switches from genetic algorithm to the Nelder–Mead simplex method.

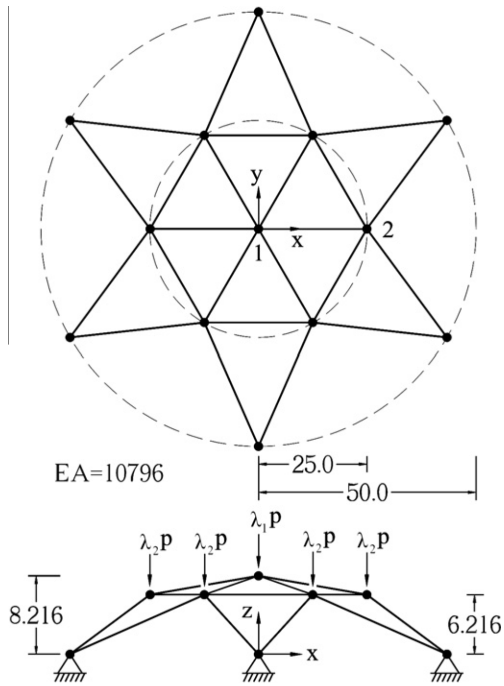

Fig. 5. A star-shaped truss dome.

algorithm. In this case, the method converges to the desired point after 1520 generations where the fitness value becomes 9.04e−11. The computational time for this process is 6.834 s, which shows a dramatic increase in comparison with the 0.2925 s of the hybrid method, verifying the significant role of local search. Note that all computations are performed on an Intel Pentium 4, 1.5 GHz processor.

Furthermore, to show the viability of the proposed method in relation with the procedure described in Section 3.6, two different changes in the initial data were made, and the solution of the new structures is provided. In the first case, the height of the truss was set to 0.9, and in the second case, the height and the *EA* were set to 0.9 and 500, respectively. The equilibrium paths related to these cases were generated using the suggested procedure and are illustrated in Fig. 4. The results obtained are very accurate, and the maximum residual for these cases in comparison with the solutions obtained by reanalysis is 1e−10, according to the tolerance adopted (Table 2).

5.2. A star-shaped 3D truss dome

Fig. 5 shows a star-shaped truss dome. Several researchers have addressed this example extensively considering different load cases. Here, we study the case that there is only a point load at the top of the dome, i.e. $\lambda_1 = \lambda$ and $\lambda_2 = 0$. The complete load–deflection curve, including several limit and bifurcation points, has been provided, e.g., by Magnusson and Svensson (1998) and Wriggers et al. (1988) (see also Wriggers, 2010). This dome has also been investigated in Crisfield (1997). We find the entire equilibrium path using the developed method in 1700 points. The load–deflection curves for the displacement of nodes 1 and 2 in the *z*-direction and the radial displacement of node 2 are illustrated in Fig. 6. Obviously, the equilibrium path exhibits different snap-through and snap-back events, which lead to a complex load–deflection curve. Each point on the equilibrium path is obtained using the hybrid algorithm developed in this paper. In addition, the first limit point is calculated using the proposed method directly (Table 6 provides the detailed results) and compared with that obtained by the authors mentioned above, thus verifying the accuracy of the results.

5.3. 3D dome with non-proportional loading

In this example the 3D dome studied in the previous example is considered again in a state of non-proportional loading, i.e.

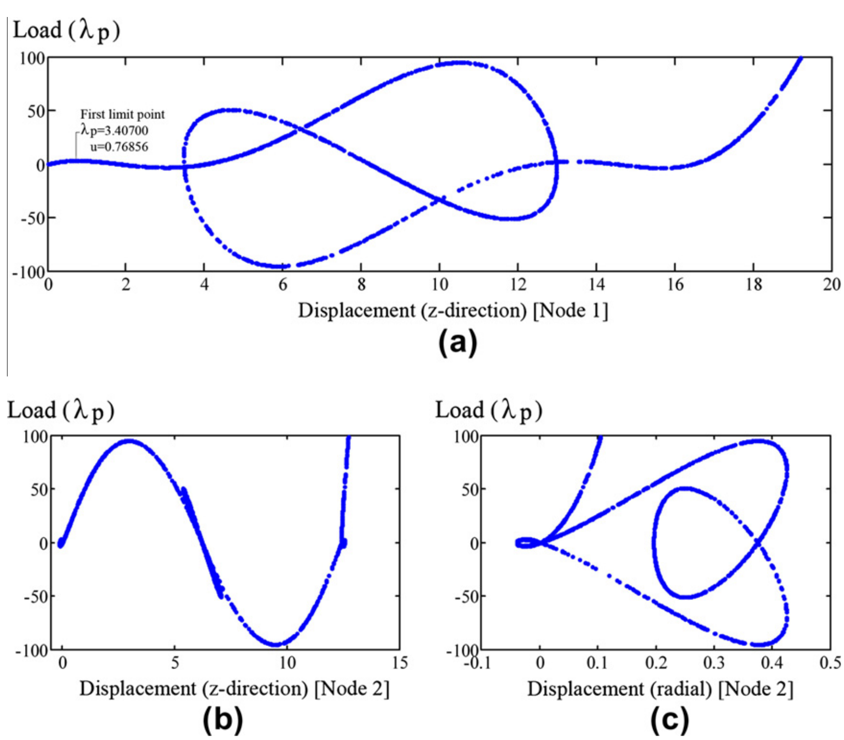

Fig. 6. Load–deflection curves for Example 2 (a) load and displacement in the *z*-direction (node 1); (b) load and displacement in the *z*-direction (node 2); (c) load and displacement in the radial direction (node 2).

Table 6
Detailed results for Example 2 (first limit point).

	Generation / iteration	$\|\mathbf{q}_i(\mathbf{u}) - \lambda\mathbf{q}_e\|_2$	Displacement (u_{1z})	Load (λp)	Computational times (second)
Hybrid method	1	42.66941	2.93063	−28.1198	
	4	27.06217	0.85330	5.900099	
	12	13.11603	1.25552	0.805619	
	22	5.428611	1.19697	3.026898	
	37	3.365671	1.24153	2.301372	6.5722
	54	1.968832	0.84776	2.213790	
	68	0.506961	0.76518	3.772206	
	81	0.172269	0.77222	3.396856	
	100	0.025596	0.76814	3.411469	
	115	0.010087	0.76815	3.401420	
	125*	0.009452	0.76815	3.409965	
	130	0.007123	0.76835	3.409186	
	185	0.000242	0.76857	3.406993	0.2618
	264	7.17E−08	0.76856	3.407001	
	306	9.83E−11	0.76856	3.407001	
Numerical, Magnusson and Svensson (1998)				3.407	
Numerical, Wriggers et al. (1988)				3.407	

* After this generation the hybrid method switches from genetic algorithm to the Nelder–Mead simplex method.

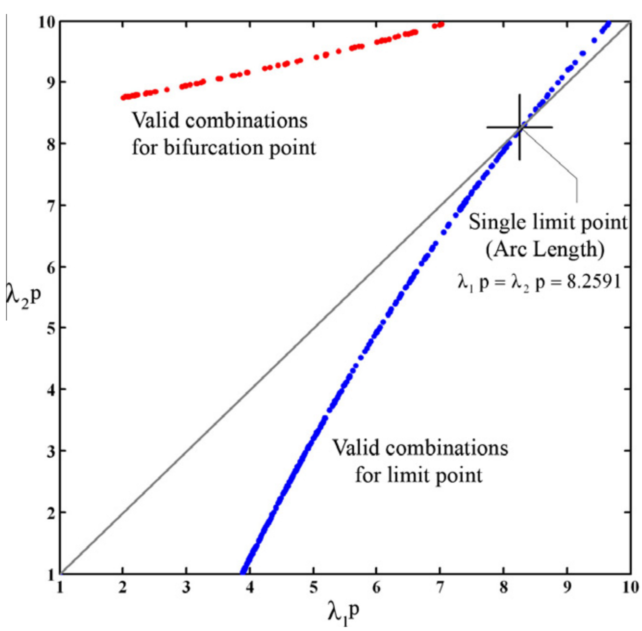

Fig. 7. Different combinations of loads to make the structure unstable (state of non-proportional loading).

$\lambda_1 \neq \lambda_2 \neq 0$. We assume that the lower and upper bounds for the load parameters are set to 1 and 10, respectively and we aim to find the possible limit and bifurcation points within these bounds. The arc-length method can only find one limit point within the bounds where loads are obtained as $\lambda_1 p = \lambda_2 p = 8.2591$, while our method finds different combinations of loads to make the structure unstable. These combinations of loads are depicted in Fig. 7, where blue and red points are for limit and bifurcation points, respectively. All these points are obtained using our method via 400 runs. A simple investigation of this graph shows that without considering non-proportional loading, a designer never finds a real estimation of buckling loads. For example the structure will reach to a limit point when $\lambda_1 p = 3.888$ and $\lambda_2 p = 1.0$, however; these load levels are very lower than the one obtained in the case of proportional loading.

Furthermore, arc-length method can not detect any bifurcation point within bounds while our method finds a set of valid combinations for bifurcation (see Fig. 7). This simple example is provided to show the importance of the non-proportional loading for the reliable design of space structures and the viability of our method to deal with this problem.

6. Conclusions

In this paper, a novel hybrid method is introduced for finding the solution of geometrically nonlinear structures. The major advantages of the methods and formulation suggested are as follows:

1. Each point on the equilibrium path can be obtained directly without any need to solve a set of linear equations or form a tangent stiffness matrix, instead using a series of sparse matrix to vector multiplications through a special optimisation process.
2. Limit and bifurcation points in any section of the load–deflection curves for both proportional and non-proportional loading can be found directly without tracing the entire equilibrium path (formation of the tangent stiffness matrix is required for this feature).
3. The nonlinear response of a structure to a slight variation in the geometrical and mechanical properties can be obtained directly.

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