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# Generalized inner bending continua for linear fiber reinforced materials

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ABSTRACT

This paper deals with the effective behaviour of elastic materials periodically reinforced by linear slender elastic inclusions. Assuming a small scale ratio $\varepsilon$ between the cell size and the characteristic size of the macroscopic deformation, the macro-behaviour at the leading order is derived by the homogenization method of periodic media. Different orders of magnitude of the contrast between the shear modulus of the material $\mu_m$ and of the reinforcement $\mu_p$ are considered.

A contrast $\mu_m/\mu_p$ of the order of $\varepsilon^2$ leads to a full coupling between the beam behaviour of the inclusions and the elastic behaviour of the matrix. Under transverse motions, the medium behaves *at the leading order* as a generalized continuum that accounts for the inner bending introduced by the reinforcements and the shear of the matrix. Instead of the second degree balance equation of elastic Cauchy continua usually obtained for homogenized composites, the governing equation is of the fourth degree and the description differs from that of a Cosserat media.

This description degenerates into, (i) the usual continua behaviour of elastic composite materials when $O(\mu_m/\mu_p) \geqslant \varepsilon$, (ii) the usual Euler–Bernoulli beam behaviour when $O(\mu_m/\mu_p) \leqslant \varepsilon^3$.

The constitutive parameters are derived and can be computed or estimated from simplified geometries. Simple criteria are given to identify the appropriate model for real reinforcements under given loadings.


## 1. Introduction

Understanding the behaviour of reinforced materials is of interest in engineering mechanics of hand-made fiber oriented materials encountered in aeronautics (mat of carbon or glass fibers, etc.), in civil engineering (pile foundations massif, embankments of reinforced earth, etc.), or in the domain of natural materials studied in biomechanics (bones, vegetal tissues, etc.).

These materials belong to the wider class of composites media on which numerous studies aim at establishing the relation between the constituents, the local morphology and the global behaviour. This is justified by the fact that phenomena in heterogeneous media can be upscaled and formulated in terms of macroscopic behaviour, provided that the condition of scale separation is fulfilled. This latter condition requires a medium morphology sufficiently regular to be described by a representative elementary volume much smaller in size than the characteristic size of the phenomena (Auriault, 1991). In the literature, these conditions are systematically satisfied, implicitly or explicitly. Among the works on upscaling, let us mention the variational approaches, e.g. Hashin (1983), and the asymptotic methods of homogenization of periodic media (Sanchez-Palencia, 1980). For elastic constituents, the homogenization limited to the leading order proves that the macro-behaviour of composites is that of elastic equivalent Cauchy media where the elastic tensor can be determined as soon as the microstructure is known (Léné, 1978). Descriptions accounting for the higher terms introduce so-called "non-local" correctors (Gambin and Kröner, 1989; Boutin, 1996). The "non-local" denomination expresses that the stress state does not depend only on the strain state in an elementary representative volume but takes into account the strain gradient, or equivalently the strain in the neighbour representative volumes. In this sense the stress–strain relation is "non-local". In those previous works it was shown that the leading order description strictly applies for homogeneous macro-strain, whereas in other cases the effective behaviour involves higher gradients of strain (double gradient in most cases i.e., the curvature). Thus, in the range of loading where homogenization applies, composites appear as Cauchy media with small perturbations induced by the correctors, i.e., "slightly non-local" generalized media. As is the case for Cosserat's media or micromorphic media (Eringen, 1968) the fundamental difference with Cauchy media lies in the existence of an intrinsic finite length, related to the cell size of the composite. Numerous works are devoted to this topic and for a recent revue the reader may refer to Forest (2006).

In above mentioned results, the non-local effects in 3-D composites (made of constituents of properties of the same order) appear as correctors and not at the leading order. Conversely, in 1-D (homogeneous) beam theory, the curvature effect dominates. This leads to think that it should be possible to obtain non-local effects at the leading order in 3-D composite made of soft matrix and stiff parallel

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beams in finite concentration. This is the question investigated in this paper. Such a topic has been sparsely investigated in the literature. Through asymptotic method (Caillerie, 1980) studied the effect of a single stiff beam in a soft medium, and (Pideri and Seppecher, 1997a; Pideri and Seppecher, 1997b) then (Bellieud and Bouchitté, 2002) showed through a mathematical study that the energy of parallel beams in a soft matrix includes bending terms. For the same kind of material (Sudret and De Buhan, 1999; de Buhan and Hassen, 2008) developed a "multiphase model" based on a phenomenological approach that accounts for shear and bending effects. Auriault and Bonnet (1985) studied the dynamics of composites with non-connected soft inclusions. More recently, in the frame work of the homogenization of discrete media (Caillerie et al., 1989), different kinds of generalized media at the leading order were identified in reticulated media according to their morphology (Hans and Boutin, 2008). Among the earlier applications in the engineering field, let us mention the numerical works of Postel (1985) and the phenomenological attemps of Makris and Gazetas (1992) at describing the behaviour of massif of pile foundations.

The present contribution aims at deriving through asymptotic homogenization, the effective behaviour of elastic materials periodically reinforced by linear slender elastic inclusions. With a systematic use of the scaling and of the 1D geometry of the inclusions, the analysis is performed for finite concentration of fiber at different magnitudes of the contrast between the shear modulus of the material $\mu_m$ and of the reinforcement $\mu_p$. According to the asymptotic method, the contrast is weighted by the powers of the scale ratio $\varepsilon$ between the cell size and the characteristic size of the macroscopic deformation.

A contrast $\mu_m = \mu_p\varepsilon^2$ leads to a generalized continuum characterized at the leading order by a full coupling between the beam behaviour of the inclusions and the elastic matrix behaviour. Instead of the second degree differential equation of elastic Cauchy continua, the governing equation is of the fourth degree and differs from that of Cosserat media. This general situation degenerates either into the usual continua behaviour of elastic composite materials when $O(\mu_m/\mu_p) \geqslant \varepsilon$, or into the usual Euler–Bernoulli beam behaviour when $O(\mu_m/\mu_p) \leqslant \varepsilon^3$. Those results are established through formal expansions and the convergence is not handled here (on this point cf. Bellieud and Bouchitté, 2002).

The paper is divided into six sections. The elements necessary for the study concerning beam model and asymptotic approach are given in Section 2. Section 3 presents the macro-behaviour of the reinforced material for a stiffness contrast of $\varepsilon^2$ in the specific case of transverse shear. The general macroscopic constitutive law is established in Section 4 for the same contrast. Larger or weaker contrasts are investigated in Section 5. Section 6 is devoted to the practical applications of the results. The discussion emphasizes the domain of validity of the different descriptions and their possible extensions.

## 2. Derivation of beam model through asymptotic method

A beam is a slender cylindrical body (of section $S_p$ of any form) of axial dimension $L$ much larger than the typical dimension of section $h$ (Fig. 1(a)). This geometry naturally introduces:

- the small parameter $\varepsilon = h/L$, inverse of the slenderness, used in the expansions,
- the distinction between directions (i) in the axis (unitary vector $\underline{a}_1$) and (ii) in the plane ($\underline{a}_\alpha$; $\alpha = 2,3$) of the section. In the paper, greek indices run from 2 to 3; latin from 1 to 3.

Through asymptotic method (Trabucho and Viano, 1996) have shown that these geometrical features enable to move from the 3-D constitutive law of the material to the 1-D beam behaviour. This approach enables to handle statics and dynamics of homogeneous, heterogeneous or anisotropic beams. This section recall the developments strictly necessary for the sequel, and only homogeneous straight beams made of isotropic elastic material are considered.

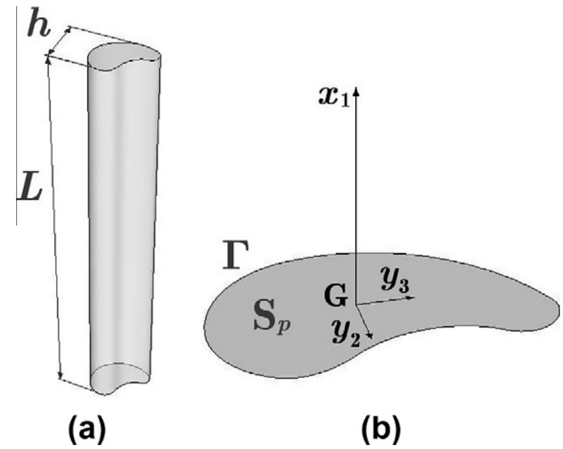


**Fig. 1.** Cylindrical beam of section of any form. (a) Dimensions. (b) Section notations.

### 2.1. Specificity of the beam's kinematic and appropriate space variables

The geometry of straight, homogeneous, unloaded beam suggests that the phenomena vary along the axis according to $L$ and within the section according to $h$. Moreover, generally, a beam is not loaded by external tangential forces on its contour $\Gamma$ (other cases will be studied in further sections). These two facts constrain the kinematic: denoting the motion by $\underline{u}$, the stress tensor by $\underline{\underline{\sigma}}$ and the normal of boundary $\Gamma$ of the straight beam by $\underline{n} = n_\alpha \cdot \underline{a}_\alpha$, one has:

$$(\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \underline{a}_1 = \sigma_{1\alpha} n_\alpha = 0 \quad \text{on } \Gamma \quad \text{where } \sigma_{1\alpha} = \mu(\mathrm{u}_{1,x_\alpha} + \mathrm{u}_{\alpha,x_1})$$

Here and in the following, the derivative according to a given variable, for instance $x_\alpha$, is denoted by $_{,x_\alpha}$, the second derivative according to $x_\alpha$ and $x_1$, by $_{,x_\alpha x_1}, \ldots$

Since $\mathrm{u}_{1,x_\alpha} = O(\mathrm{u}_1/h)$ and $\mathrm{u}_{\alpha,x_1} = O(\mathrm{u}_\alpha/L)$, the vanishing of $\sigma_{1\alpha}$ on $\Gamma$ requires:

$$O\left(\frac{\mathrm{u}_1}{h}\right) = O\left(\frac{\mathrm{u}_\alpha}{L}\right) \quad \text{i.e. } O(\mathrm{u}_1) = \varepsilon O(\mathrm{u}_\alpha)$$

This means that the transverse motions are associated with a normal motion of one order less. To respect this physical condition, the motions are rescaled in the following form:

$$\underline{\mathrm{u}} = \varepsilon u_1 \underline{a}_1 + u_\alpha \underline{a}_\alpha \quad \text{so that } O(u_1) = O(u_\alpha) \tag{1}$$

As the pertinent dimensionless space variables are $(x_1/L, x_2/h, x_3/h)$, the appropriate physical space variables reads $(x_1, y_2, y_3)$, where $y_\alpha = (L/h)x_\alpha = \varepsilon^{-1}x_\alpha$ (Fig. 1(b)). For quantity $\varphi$ expressed in function of $(x_1, y_\alpha)$ the gradient operator $\nabla = \partial_{x_i}\underline{a}_i$ – that applies on $\varphi(\underline{x})$ – becomes:

$$\nabla\varphi = (\partial_{x_1}\underline{a}_1 + \varepsilon^{-1}\partial_{y_\alpha}\underline{a}_\alpha)\varphi \quad \text{with } \varphi(x_1, \underline{y})$$

and the integrals are modified as ($ds = dy_2 dy_3$; $d\gamma = dy_\Gamma$):

$$\int_S \varphi(\underline{x}) dS = \varepsilon^2 \int_S \varphi(x_1, \underline{y}) ds; \quad \int_\Gamma \varphi(\underline{x}) d\Gamma = \varepsilon \int_\Gamma \varphi(x_1, \underline{y}) d\gamma$$

In this paper we use the following conventions:

- $|S| = \int_S dx_2 dx_3;\ |S'| = \int_S dy_2 dy_3 = \varepsilon^{-2}|S|$,
- $I_\alpha = \int_S (x_\alpha)^2 dx_2 dx_3;\ I'_\alpha = \int_S (y_\alpha)^2 dy_2 dy_3 = \varepsilon^{-4} I_\alpha$,
- the local problems are set on the "natural" $y$-frame originated at the center of "mass" of the beam section $S_p$ and orientated along its principal axis of inertia. Thus:

$$\int_{S_p} y_\alpha ds = 0; \quad \int_{S_p} y_\alpha y_\beta ds = 0 \quad \text{for } \alpha \neq \beta$$

*2.2. Formulation of the problem*

*2.2.1. Reduced strain and stress tensors*

The specificity of the axial direction leads to decompose any symmetric tensor $\underline{\underline{A}}$ into:

$$\underline{\underline{A}} = A_{ij}(\underline{a}_i \otimes \underline{a}_j + \underline{a}_j \otimes \underline{a}_i)/2 = A_n \underline{a}_1 \otimes \underline{a}_1 + (\underline{A}_t \otimes \underline{a}_1 + \underline{a}_1 \otimes \underline{A}_t) + \underline{\underline{A}}_s$$

where, for strain ($\underline{\underline{A}} = \underline{\underline{e}}$) or stress ($\underline{\underline{A}} = \underline{\underline{\sigma}}$), the three reduced tensors are respectively:

- $A_n = A_{11}$: the *scalar* axial strain or stress,
- $\underline{A}_t = A_{1\alpha}\underline{a}_\alpha$: the 2D *vector* of the strain or stress exerted *out of the plane* of the section,
- $\underline{\underline{A}}_s = A_{\alpha\beta}(\underline{a}_\alpha \otimes \underline{a}_\beta + \underline{a}_\beta \otimes \underline{a}_\alpha)/2$: the 2D second rank *tensor* of the strain or stress *in the plane* of the section.

Considering motions in the form (1), the reduced strain tensors are of different order:

$$e_n = \varepsilon u_{1,x_1}; \quad \underline{e}_t = [(u_{1,y_\alpha} + u_{\alpha,x_1})/2]\underline{a}_\alpha;$$
$$\underline{\underline{e}}_s = \varepsilon^{-1}[(u_{\alpha,y_\beta} + u_{\beta,y_\alpha})/2](\underline{a}_\alpha \otimes \underline{a}_\beta + \underline{a}_\beta \otimes \underline{a}_\alpha)/2 \tag{2}$$

The stress and strain tensors are related by the linear isotropic elasticity of the material:

$$\underline{\underline{\sigma}} = 2\mu\underline{\underline{e}} + \lambda \mathrm{tr}(\underline{\underline{e}})\underline{\underline{I}}; \quad \text{where } \lambda, \mu \text{ stands for Lame coefficients}$$

Thus, denoting $\underline{\underline{I}}_s = \underline{a}_2 \otimes \underline{a}_2 + \underline{a}_3 \otimes \underline{a}_3$:

$$\sigma_n = 2\mu e_n + \lambda(\mathrm{tr}(\underline{\underline{e}}_s) + e_n); \quad \underline{\sigma}_t = 2\mu\underline{e}_t;$$
$$\underline{\underline{\sigma}}_s = 2\mu\underline{\underline{e}}_s + \lambda(\mathrm{tr}(\underline{\underline{e}}_s) + e_n)\underline{\underline{I}}_s \tag{3}$$

Consequently, $\underline{\sigma}_t$ is of zero order while $\sigma_n$ and $\underline{\underline{\sigma}}_s$ contains terms of order $\varepsilon^{-1}$ and $\varepsilon$.

Using the Young modulus $E = 2\mu(1 + \nu)$ and the Poisson ratio $\nu = \lambda/2(\lambda + \mu)$, we also have:

$$\underline{\underline{\sigma}} = \frac{E}{1+\nu}\left[\underline{\underline{e}} + \frac{\nu}{1-2\nu}\mathrm{tr}(\underline{\underline{e}})\underline{\underline{I}}\right]; \quad \underline{\underline{e}} = \frac{1}{E}[(1+\nu)\underline{\underline{\sigma}} - \nu\mathrm{tr}(\underline{\underline{\sigma}})\underline{\underline{I}}]$$

so that $e_n$ and $\sigma_n$ are known as soon as $\mathrm{tr}(\underline{\underline{e}}_s)$ and $\mathrm{tr}(\underline{\underline{\sigma}}_s)$ are determined:

$$e_n = \frac{1}{2\nu}\left[\frac{(1-2\nu)(1+\nu)}{E}\mathrm{tr}(\underline{\underline{\sigma}}_s) - \mathrm{tr}(\underline{\underline{e}}_s)\right];$$
$$\sigma_n = \frac{1}{2\nu}[-E\mathrm{tr}(\underline{\underline{e}}_s) + (1-\nu)\mathrm{tr}(\underline{\underline{\sigma}}_s)] \tag{4}$$

For instance, in the case of in plane motion: $e_n = 0$ and $\sigma_n = \nu\mathrm{tr}(\underline{\underline{\sigma}}_s)$.

*2.2.2. Local balance equations and variational formulations*

Here, zero body and surface forces are assumed. The momentum balance $\underline{\mathrm{div}}(\underline{\underline{\sigma}}) = \underline{0}$, reads:

$$(\sigma_{i1,x_1} + \varepsilon^{-1}\sigma_{i\alpha,y_\alpha})\underline{a}_i = \underline{0}$$

The specificity of $\underline{a}_1$ direction leads to split the balance and boundary conditions ($\underline{\underline{\sigma}} \cdot \underline{n} = \underline{0}$) into:

- a scalar equation along $\underline{a}_1$ driving the vector stress $\underline{\sigma}_t$, the axial gradient of the scalar $\sigma_n$ being a forcing term, with homogeneous boundary conditions:
$\underline{a}_1$ direction:

$$\sigma_{n,x_1} + \varepsilon^{-1}\mathrm{div}_y(\underline{\sigma}_t) = 0 \quad \text{in } S_p \tag{5}$$
$$\underline{\sigma}_t \cdot \underline{n} = 0 \quad \text{on } \Gamma \tag{6}$$

The equivalent variational formulation is established classically as:

$$\forall w_1, \quad C^1 \text{ scalar defined on } S_p$$
$$\int_{S_p} \sigma_{n,x_1} w_1 ds = \varepsilon^{-1}\int_{S_p} \underline{\sigma}_t \cdot \underline{\mathrm{grad}}_y(w_1)ds$$

- a vectorial equation in the plane $(\underline{a}_2, \underline{a}_3)$ governing the in plane stress tensor $\underline{\underline{\sigma}}_s$, the axial gradient of vector $\underline{\sigma}_t$ being a forcing term, with homogeneous boundary conditions: $(\underline{a}_2, \underline{a}_3)$ directions:

$$\underline{\sigma}_{t,x_1} + \varepsilon^{-1}\underline{\mathrm{div}}_y(\underline{\underline{\sigma}}_s) = \underline{0} \quad \text{in } S_p \tag{7}$$
$$\underline{\underline{\sigma}}_s \cdot \underline{n} = \underline{0} \quad \text{on } \Gamma \tag{8}$$

whose equivalent variational formulation reads:

$$\forall \underline{w}_s, \text{ in plane } C^1 \text{ vector defined on } S_p$$
$$\int_{S_p} \underline{\sigma}_{t,x_1} \cdot \underline{w}_s ds = \varepsilon^{-1}\int_{S_p} \underline{\underline{\sigma}}_s : \underline{\underline{e}}_y(\underline{w}_s)ds$$

*2.2.3. Global balance equations of the section*

The balance equations of global forces, are derived by integrating (5) and (7) over $S_p$. Using the divergence theorem and the boundary conditions (6)–(8) give:

$$\int_{S_p} \mathrm{div}_y(\underline{\sigma}_t)ds = \int_\Gamma \underline{\sigma}_t \cdot \underline{n} d\gamma = 0 \quad \text{and} \quad \int_{S_p} \underline{\mathrm{div}}_y(\underline{\underline{\sigma}}_s)ds = \int_\Gamma \underline{\underline{\sigma}}_s \cdot \underline{n} d\gamma = \underline{0}$$

Thus, inverting $y_\alpha$-integration and $x_1$-derivate, provides the following balance equations over the section (valid when the beam is free of any surface or volume loading):

$$\text{along } \underline{a}_1: \left[\int_{S_p} \sigma_n ds\right]_{,x_1} = 0; \quad \text{along } \underline{a}_2, \underline{a}_3: \left[\int_{S_p} \underline{\sigma}_t ds\right]_{,x_1} = \underline{0} \tag{9}$$

Three global momentum equilibrium equations can also be established. Again, axial and in-plane directions must be distinguished. First, multiply (5) by $y_\alpha$ and integrate over $S_p$:

$$\int_{S_p} y_\alpha \sigma_{n,x_1} ds + \varepsilon^{-1}\int_{S_p} y_\alpha \mathrm{div}_y(\underline{\sigma}_t)ds = 0$$

Integrating the second integral by part and applying the divergence theorem yields:

$$\int_{S_p} \mathrm{div}_y(y_\alpha \underline{\sigma}_t)ds - \int_{S_p} \underline{\sigma}_t \cdot \underline{a}_\alpha ds = \int_\Gamma y_\alpha(\underline{\sigma}_t \cdot \underline{n})d\gamma - \int_{S_p} \underline{\sigma}_t \cdot \underline{a}_\alpha ds$$

and the integral over $\Gamma$ vanishes because of the free boundary condition (6). Finally, inverting $y_\alpha$-integration and $x_1$-derivate leads to the two momentum of momentum balance equations:

$$\text{along } \underline{a}_\alpha: \left[\int_{S_p} y_\alpha \sigma_n ds\right]_{,x_1} - \varepsilon^{-1}\int_{S_p} \underline{\sigma}_t \cdot \underline{a}_\alpha ds = 0$$

The global momentum of momentum balance in direction $\underline{a}_1$ is established by taking the vectorial product of (7) by the position vector $\underline{y} = y_\alpha \underline{a}_\alpha$ and integrating over the section:

$$\int_{S_p} \underline{y} \times \underline{\sigma}_{t,x_1} ds + \varepsilon^{-1}\int_{S_p} \underline{y} \times \underline{\mathrm{div}}_y(\underline{\underline{\sigma}}_s)ds$$
$$= \int_{S_p} \underline{y} \times \underline{\sigma}_{t,x_1} ds + \varepsilon^{-1}\underline{a}_1\left[\int_{S_p} \epsilon_{1\alpha\beta} y_\alpha \sigma_{\beta\gamma,y_\gamma} ds\right] = \underline{0}$$

where $\epsilon$ is the third rank tensor expressing the vectorial product. Integrating by part, then using the divergence theorem and the symmetry of $\underline{\underline{\sigma}}$, and finally the free boundary condition (8), gives:

$$\int_{S_p} \epsilon_{1\alpha\beta} y_\alpha \sigma_{\beta\gamma,y_\gamma} ds = -\int_{S_p} \epsilon_{1\alpha\beta}\sigma_{\beta\alpha} ds + \int_\Gamma \epsilon_{1\alpha\beta} y_\alpha \sigma_{\beta\gamma} n_\gamma d\gamma$$
$$= 0 + \underline{a}_1 \cdot \int_\Gamma \underline{y} \times (\underline{\underline{\sigma}}_s \cdot \underline{n}) d\gamma = 0$$

Consequently:

$$\left[\underline{a}_1 \cdot \int_{S_p} \underline{y} \times \underline{\sigma}_t ds\right]_{,x_1} = 0$$

To sum up, denoting by $N\underline{a}_1$ and $\underline{T} = T_\alpha \underline{a}_\alpha$ the normal and shear forces, and by $\underline{M} = M_\alpha \underline{a}_\alpha$ and $M_1\underline{a}_1$ the bending and torsion momentum, respectively, the balance equations of beams free of surface or volume loading are:

$$\text{along } \underline{a}_1 : N_{,x_1} = 0, \quad N = \int_{S_p} \sigma_n ds;$$

$$M_{1,x_1} = 0, \qquad M_1 = \underline{a}_1 \cdot \int_{S_p} \underline{y} \times \underline{\sigma}_t ds$$

$$\text{along } \underline{a}_\alpha : \underline{M}_{,x_1} - \varepsilon^{-1}\underline{T} = \underline{0}, \quad \underline{M} = \int_{S_p} \underline{y}\sigma_n ds;$$

$$\underline{T}_{,x_1} = \underline{0}, \quad \underline{T} = \int_{S_p} \underline{\sigma}_t ds$$

To go further it is necessary to relate the forces and the momentum to the motion. This is achieved by means of asymptotic expansions, the main steps of the process are presented in Appendix A.

### 2.3. Beam description in presence of body and contact forces

#### 2.3.1. Unloaded beam

The *unloaded* beam description at the leading order is split into sets of *uncoupled* equations (for bi-symmetric section) relating forces and momentum to motions. The results can be written with the unscaled variables $x_i$ by the inverse change of variable $x_\alpha = \varepsilon y_\alpha$. This leads to consider the *physically observable* quantities $\widetilde{Q}^i = \varepsilon^i Q^i$ instead of the scaled quantities $Q^i$, and to express the parameters in the system $x_i$ (i.e., practically, with the same units in the section and in the beam axis). Furthermore, there is no constraint on the relative order of magnitude of the uncoupled, hence independent, mechanisms. For this reason, the exponent specifying the order may be omitted (while keeping in mind that this is only the leading order description). Finally, one obtains the usual Euler–Bernoulli beam description in the absence of inner or external loading:

- Normal force $N$ and mean vertical motion $U_1$

  $N_{,x_1} = 0; \quad N = E|S_p|U_{1,x_1}$

- Transverse forces $T_\alpha$, momentum $M_\alpha$ and mean transverse motion $U_\alpha$

  $M_{\alpha,x_1} - T_\alpha = 0, \quad T_{\alpha,x_1} = 0; \quad M_\alpha = -EI_\alpha U_{\alpha,x_1x_1}$

- Torsion momentum $M_1$ and in-plane rotation of the section $\Omega$

  $M_{1,x_1} = 0; \quad M_1 = \mu I_t \Omega_{,x_1}$

#### 2.3.2. Loaded beam

Let us examine body forces $\underline{b} = b_i\underline{a}_i$ – such that $\underline{\mathrm{div}}(\underline{\underline{\sigma}}) = \underline{b}$ in $S_p$ – and contact forces $\underline{f} = f_i\underline{a}_i$ – such that $\underline{f} = \underline{\underline{\sigma}} \cdot \underline{n} = (\underline{\sigma}_t \cdot \underline{n})\underline{a}_1 + \underline{\underline{\sigma}}_s \cdot \underline{n}$ on $\Gamma$ – that can be applied while being compatible with a beam behaviour. First, they should not brake the axial/transerve scale separation so that they may be expressed as $\underline{b}(x_1, y_\alpha)$, $\underline{f}(x_1, y_\alpha)$. Second, they should be small enough not to disturb the leading kinematic of the section. This happens if $\underline{b}$ and $\underline{f}$ are of the orders:

$$b_1 = \varepsilon b_1^1, \quad f_1 = \varepsilon^2 f_1^2; \quad b_\alpha = \varepsilon^2 b_\alpha^2, \quad f_\alpha = \varepsilon^3 f_\alpha^3 \tag{10}$$

Indeed, in that case, the problems remain identical up to the fourth one (see Appendix A). Only the equilibrium is modified by $\underline{b}$ and $\underline{f}$ which averaged values on $S_p$ and $\Gamma$ act as sources. Denoting:

$$B_j^{i+2} = \int_{S_p} b_j^i ds; \quad F_j^{i+1} = \int_\Gamma f_j^i d\gamma$$

$$C_\alpha^4 = \int_{S_p} y_\alpha b_1^1 ds, \quad C_1^5 = \underline{a}_1 \cdot \int_{S_p} \underline{y} \times \underline{b}^2 ds;$$

$$G_\alpha^4 = \int_\Gamma y_\alpha f_1^2 d\gamma, \quad G_1^5 = \underline{a}_1 \cdot \int_\Gamma \underline{y} \times \underline{f}^3 d\gamma$$

the balance equations become (the uncoupling of bending and torsion requires that $b_1^1$ and $f_1^2$ respect the bi-symmetry of the section):

$$N^3_{,x_1} = B_1^3 + F_1^3$$
$$M^4_{\alpha,x_1} - T^4_\alpha = C^4_\alpha + G^4_\alpha; \quad T^4_{\alpha,x_1} = B^4_\alpha + F^4_\alpha$$
$$M^5_{1,x_1} = C^5_1 + G^5_1$$

With the unscaled variables, the loaded beam description reads (dropping the exponents since the order of magnitude of the compression, bending and torsion mechanisms are independent):

$$\underline{U} = (U_1 + x_\alpha U_{\alpha,x_1})\underline{a}_1 + U_\alpha\underline{a}_\alpha + \Omega\underline{a}_1 \times (x_\alpha\underline{a}_\alpha)$$

$$N_{,x_1} = \int_{S_p} b_1 dx_2 dx_3 + \int_\Gamma f_1 dx_\Gamma; \quad N = E|S_p|U_{1,x_1}$$

$$M_{\alpha,x_1} - T_\alpha = \int_{S_p} x_\alpha b_1 dx_2 dx_3 + \int_\Gamma x_\alpha f_1 dx_\Gamma; \quad M_\alpha = -EI_\alpha U_{\alpha,x_1x_1}$$

$$T_{\alpha,x_1} = \int_{S_p} b_\alpha dx_2 dx_3 + \int_\Gamma f_\alpha dx_\Gamma$$

$$M_{1,x_1} = \epsilon_{1\alpha\beta}\left(\int_{S_p} x_\alpha b_\beta dx_2 dx_3 + \int_\Gamma x_\alpha f_\beta dx_\Gamma\right); \quad M_1 = \mu I_t \Omega_{,x_1}$$

Smaller magnitudes of $\underline{b}$ and $\underline{f}$ leave the leading order unchanged and such a situation can be treated as unloaded beam; conversely, larger amplitudes are incompatible with a beam model.

## 3. Transverse behaviour of periodic parallel beams in a soft matrix

This section aims (i) to identify the conditions in which fiber reinforced materials behave as generalized continua and (ii) to derive the relevant modelling. The medium is made of a matrix (index$_m$) in which a periodic lattice of parallel identical homogeneous straight beams (index$_p$) is embedded with a perfect contact (Fig. 2(a)). The characteristic dimension $L$ along the beam axis is much larger than the lateral dimension $\ell$ of the period (Fig. 2(b)) of area $S = S_m \cup S_p$ and boundary $\partial S$; the typical size of the beam section $h$ is of the same order than $\ell$ so that the fibers are in finite concentration. This introduces the scale parameter $\varepsilon = \ell/L$. The materials $(_{m,p})$ are isotropic elastic. Obviously, the contrast between the elastic properties of the matrix and of the beam materials plays a crucial role:

- Without matrix, the beams clamped at their extremities are governed by bending.
- If $(_{m,p})$ materials are identical, one has a homogeneous medium governed by shear.

### 3.1. Contrast of beam-matrix stiffness

Section 2.3 suggests that, at the leading order of the upscaled description, the bending will survive when the contact forces

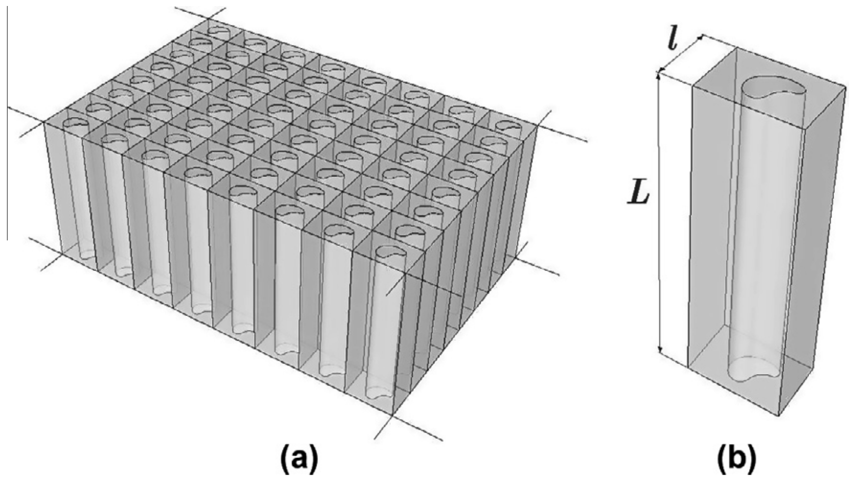


**Fig. 2.** Studied fiber reinforced material. (a) Periodic lattice of parallel identical homogeneous straight beams embedded in a matrix. (b) Period dimensions.

exerted by the matrix on the beam are of the order $f_1 = \varepsilon^2 f_1^2$ or smaller, and conversely, that the action of the matrix on the beam only remains when contact forces are $O(\varepsilon^2)$ or larger. Hence, denoting the variables in the matrix and in the beams by ante-exponent ${}^m$, ${}^p$, respectively, a description including both effects may be obtained when:

$$f_1 = {}^p\underline{\sigma}_t \cdot \underline{n} = -({}^m\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \underline{a}_1 = \varepsilon^2 f_1^2 \quad \text{on } \Gamma$$

Thus, under transverse motions the shear stress in the matrix $({}^m\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \underline{a}_1 = O(\mu_m {}^m u_{\alpha,x_1})$ is smaller by two order than the zero order reference stress on the beam $O(\mu_p {}^p u_{\alpha,x_1})$. The motion continuity on $\Gamma$, imposes ${}^m u_\alpha = O({}^p u_\alpha)$, and the estimate $\mu_m {}^m u_{\alpha,x_1} = \varepsilon^2 O(\mu_p {}^p u_{\alpha,x_1})$ imply:

$$\mu_m = \mu_p O(\varepsilon^2), \quad \text{and additionally } \lambda_m = \lambda_p O(\varepsilon^2)$$

The stiffness contrast has to be integrated in the asymptotic process. In this aim, we rescale the elastic coefficients of the matrix, taking those of the beam as reference:

$$\mu_m = \mu_p O(\varepsilon^2) = \mu'_m \varepsilon^2, \quad \lambda_m = \lambda_p O(\varepsilon^2) = \lambda'_m \varepsilon^2 \quad \text{so that}$$
$$\mu'_m = O(\mu_p), \quad \lambda'_m = O(\lambda_p)$$

and the stresses in both materials are written in the form below:

$${}^p\underline{\underline{\sigma}} = \lambda_p \mathrm{tr}({}^p\underline{\underline{\mathrm{e}}})\underline{\underline{I}} + 2\mu_p {}^p\underline{\underline{\mathrm{e}}}; \quad {}^m\underline{\underline{\sigma}} = \varepsilon^2[\lambda'_m \mathrm{tr}({}^m\underline{\underline{\mathrm{e}}})\underline{\underline{I}} + 2\mu'_m {}^m\underline{\underline{\mathrm{e}}}]$$

The general constitutive law under general motions will be derived in Section 4. To focus on the key point, this section deals with *dominating transverse* macroscopic motions varying macroscopically according to the *axial* direction only. Because of the plane geometry of the period, the quantities only depend locally on the variables $y_\alpha$ (Fig. 3). The motions of both constituents are rescaled in the form:

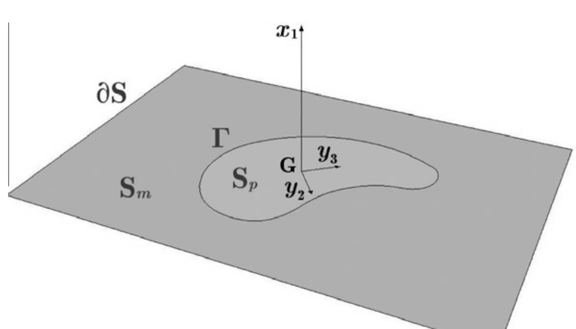


**Fig. 3.** Fiber reinforced material section. Notations.

$${}^q\underline{u}(x_1, y_\alpha) = \varepsilon({}^q u_1)\underline{a}_1 + {}^q u_\alpha \underline{a}_\alpha; \quad q = m, p \tag{11}$$

This enables to extend to the matrix the formulation developed for the beam and to express the balance equations and the perfect contact conditions at the beam/matrix interface as :
$\underline{a}_1$ direction:

$${}^q\sigma_{n,x_1} + \varepsilon^{-1}\mathrm{div}_y({}^q\underline{\sigma}_t) = 0 \quad \text{in } S_q \quad q = m,p \tag{12}$$

$${}^m\underline{\sigma}_t \cdot \underline{n} = {}^p\underline{\sigma}_t \cdot \underline{n} \quad \text{on } \Gamma \tag{13}$$

$${}^m u_1 = {}^p u_1 \tag{14}$$

$(\underline{a}_2, \underline{a}_3)$ directions:

$${}^q\underline{\sigma}_{t,x_1} + \varepsilon^{-1}\underline{\mathrm{div}}_y({}^q\underline{\underline{\sigma}}_s) = \underline{0} \quad \text{in } S_q \quad q = m,p \tag{15}$$

$${}^m\underline{\underline{\sigma}}_s \cdot \underline{n} = {}^p\underline{\underline{\sigma}}_s \cdot \underline{n} \quad \text{on } \Gamma \tag{16}$$

$${}^m u_\alpha \underline{a}_\alpha = {}^p u_\alpha \underline{a}_\alpha \tag{17}$$

### 3.2. Homogenized transverse behaviour

The motions are sought in the form of $S$-periodic expansions in power of $\varepsilon$. According to the scaling (11) the terms of the balance equation and the boundary condition in axial and in-plane directions jump from a factor $\varepsilon^2$, and as the contrast of elastic properties is also $\varepsilon^2$, it is sufficient to expand ${}^q u_\alpha$ and ${}^q u_1$ in the even powers of $\varepsilon$, i.e.:

$${}^q\underline{u} = \sum_{i=0} \varepsilon^{2i}\left[{}^q u_\alpha^{2i}\underline{a}_\alpha + \varepsilon({}^q u_1^{2i+1}\underline{a}_1)\right]$$
$$q = m,p; \quad {}^m u^i(x_1, y_\alpha) \quad \text{S-periodic in } y_\alpha \tag{18}$$

The expansions of the stresses are:

$${}^p\sigma_{ij} = \lambda_p \mathrm{e}_{kk}\delta ij + 2\mu_p \mathrm{e}_{ij} = \varepsilon^{-1}({}^p\sigma_{ij}^{-1}) + {}^p\sigma_{ij}^0 + \varepsilon({}^p\sigma_{ij}^1) + \varepsilon^2({}^p\sigma_{ij}^2) + \cdots$$

$${}^m\sigma_{ij} = \varepsilon^2(\lambda'_m \mathrm{e}_{kk}\delta ij + 2\mu'_m \mathrm{e}_{ij}) = \varepsilon({}^m\sigma_{ij}^1) + \varepsilon^2({}^m\sigma_{ij}^2) + \cdots$$

#### 3.2.1. Local problems in the beam and matrix

The analysis here below shows that because of the contrast of stiffness, the problems in the beam and those in the matrix are of different nature. As a matter of fact, the stiff beam imposes its motion to the soft matrix (then governed by problems of Dirichlet type) and in turn, the matrix imposes its stresses onto the beam (then governed by problems of Neumann type).

Since ${}^m\underline{\underline{\sigma}}$ is at the most of order $\varepsilon$, the presence of the soft matrix leaves unchanged the resolution of the first and second problems

(Appendix A.2) within the beam. Thus, disregarding the torsion at this order (which is consistent with a macro-rotation of the same order than the macro-distorsion) and the mean vertical motion $U_1^1(x_1)$ (because the compression kinematics is independent of the purely transverse kinematics on which we focus here):

$$^p\underline{u}^0 = \underline{U}^0 = U_\alpha^0(x_1)\underline{a}_\alpha, \quad ^p\underline{u}^1 = -\underline{y}\cdot\underline{U}^0_{,x_1}\underline{a}_1; \quad \text{and}$$
$$^p\underline{\underline{e}}^{-1} = {}^p\underline{\underline{e}}^0 = \underline{\underline{0}}, \quad {}^p\underline{\underline{\sigma}}^{-1} = {}^p\underline{\underline{\sigma}}^0 = \underline{\underline{0}}$$

In the matrix, the leading order problem ((15)-$\varepsilon^0$; (17)-$\varepsilon^0$) reads:

$$\begin{cases} \underline{\mathrm{div}}_y({}^m\underline{\underline{\sigma}}_s^1) = \underline{0} \quad \text{in } S_m \quad \text{with } {}^m\underline{\underline{\sigma}}_s^1 = \mu'_m\underline{\underline{e}}_y({}^m\underline{u}^0) + \lambda'_m\mathrm{div}_y({}^m\underline{u}^0)\underline{\underline{I}}_s \\ {}^m\underline{u}^0 = \underline{U}^0 \quad \text{on } \Gamma \\ {}^m\underline{u}^0 \quad y\text{-periodic on } \partial S \end{cases}$$

of which the solution ${}^m\underline{u}^0 = \underline{U}^0(x_1)$ means that the matrix follows the uniform translation of the beam section. It follows that ${}^m\underline{\underline{\sigma}}_s^1 = \underline{\underline{0}}$ then ${}^m\underline{\underline{\sigma}}^1 = \underline{\underline{0}}$. Thus, the resolution of the third problem in the beam (Appendix A.2) is unchanged giving the usual bending kinematic and stress state:

$$^p\underline{u}^2 = \nu U^0_{\alpha,x_1x_1}\underline{\xi}^\alpha + {}^p\underline{U}^2(x_1) + \Omega^1(x_1)\underline{a}_1\times\underline{y}$$
$$^p\underline{\underline{e}}^1 = -[\underline{a}_1\otimes\underline{a}_1 - \nu\underline{\underline{I}}_s](\underline{y}\cdot\underline{U}^0_{,x_1x_1}); \quad {}^p\underline{\underline{\sigma}}^1 = {}^p\sigma_n^1\underline{a}_1\otimes\underline{a}_1; \quad {}^p\sigma_n^1 = -E_p\underline{y}\cdot\underline{U}^0_{,x_1x_1}$$

At the next order, since ${}^m\underline{\underline{\sigma}}_n^1 = 0$, the problem in the matrix is ((12)-$\varepsilon$; (14)-$\varepsilon$):

$$\begin{cases} \mathrm{div}_y({}^m\underline{\sigma}_t^2) = 0 \quad \text{in } S_m \quad \text{with } {}^m\underline{\sigma}_t^2 = \mu'_m[{}^mu^1_{1,y_\alpha} + U^0_{\alpha,x_1}]\underline{a}_\alpha \\ {}^mu_1^1 = {}^pu_1^1 = -\underline{y}\cdot\underline{U}^0_{,x_1} \quad \text{on } \Gamma \\ {}^mu_1^1 \quad y\text{-periodic on } \partial S \end{cases}$$

From linearity the solution of this problem, where the forcing term results from the beam motion on $\Gamma$, is ($\psi_\alpha$ are the particular solutions for $U^0_{\alpha,x_1} = 1$):

$$^m\underline{u}^1 = {}^mu_1^1\underline{a}_1; \quad {}^mu_1^1 = (1/2)\psi_\alpha(\underline{y})U^0_{\alpha,x_1}$$
$$^m\underline{\sigma}_t^2 = \mu'_m[\underline{a}_\alpha + (1/2)\underline{\mathrm{grad}}_y(\psi_\alpha)]U^0_{\alpha,x_1}; \quad {}^m\sigma_n^2 = 0; \quad {}^m\underline{\underline{\sigma}}_s^2 = \underline{\underline{0}}$$

#### 3.2.2. Global beam-matrix balance

The global balance equation along $\underline{a}_1$ is derived by integrating ((12)-$\varepsilon$) over both $S_p$ and $S_m$ and summing. Using the divergence theorem one obtains (remind that ${}^m\sigma_n^1 = 0$):

$$\int_{S_p} {}^p\sigma^1_{n,x_1}ds + \int_\Gamma {}^p\underline{\sigma}_t^2\cdot\underline{n}d\gamma + \int_\Gamma {}^m\underline{\sigma}_t^2\cdot(-\underline{n})d\gamma + \int_{\partial S} {}^m\underline{\sigma}_t^2\cdot\underline{n}d\gamma = 0$$

With the stress continuity ((13)-$\varepsilon^1$) and the periodicity, the integrals on $\Gamma$ and $\partial S$ vanish. Introducing the expression of ${}^p\sigma_n^1$, consistently with the fact that the mean axial motion $U_1^0$ is taken to be null, we are left with the trivial equation:

$$0 = \int_{S_p} {}^p\sigma^1_{n,x_1}ds = -E_p\underline{U}^0_{,x_1x_1}\cdot\int_{S_p}\underline{y}ds = 0$$

Now, multiplying ((12)-$\varepsilon$) by $y_\alpha$, integrating over $S_p$ and $S_m$, and using the stress continuity ((13)-$\varepsilon^1$) give the momentum balance equations according to in-plane directions $\underline{a}_\alpha$:

$$\left[\int_{S_p} y_\alpha({}^p\sigma_n^1)ds\right]_{,x_1} - \int_{S_p} {}^p\underline{\sigma}_t^2\cdot\underline{a}_\alpha ds - \int_{S_m} {}^m\underline{\sigma}_t^2\cdot\underline{a}_\alpha ds$$
$$+ \int_{\partial S} y_\alpha({}^m\underline{\sigma}_t^2\cdot\underline{n})d\gamma = 0 \tag{19}$$

Then, introducing the previous expressions of ${}^p\sigma_n^1$ and ${}^m\underline{\sigma}_t^2$, dividing by $S'$ one obtains:

$$\langle\underline{\sigma}_t^2\cdot\underline{a}_\alpha\rangle = \mu'_m c^{*1\beta}_{1\alpha}U^0_{\beta,x_1} - \frac{E_pI'_{p\alpha}}{S'}U^0_{\alpha,x_1x_1x_1}$$

where, here and in the sequel, the mean value is denoted by $\langle\cdot\rangle$, for instance:

$$\langle\underline{\sigma}_t^2\rangle = \frac{1}{|S'|}\left(\int_{S_p}{}^p\underline{\sigma}_t^2ds + \int_{S_m}{}^m\underline{\sigma}_t^2ds\right);$$
$$\langle\underline{\underline{\sigma}}^k\rangle = \frac{1}{|S'|}\left(\int_{S_p}{}^p\underline{\underline{\sigma}}^kds + \int_{S_m}{}^m\underline{\underline{\sigma}}^kds\right)$$

and:

$$c^{*1\beta}_{1\alpha} = \left[\delta_{\alpha\beta} + \frac{1}{2}\frac{1}{S'}\int_{\partial S} y_\alpha\underline{\mathrm{grad}}_y(\psi_\beta)\cdot\underline{n}d\gamma\right]$$

Finally, the global equilibrium in directions $\underline{a}_\alpha$ is established by integrating ((15)-$\varepsilon^2$) over both $S_p$ and $S_m$, using the divergence theorem, the stress continuity ((16)-$\varepsilon^2$) and the periodicity:

$$\langle\underline{\sigma}_t^2\rangle_{,x_1} = 0$$

Recall that the physically observable stresses are $\tilde{\underline{\sigma}}_t^2 = \varepsilon^2\underline{\sigma}_t^2$ and not $\underline{\sigma}_t^2$. Coming back to the unscaled physical variables and parameters expressed in the system $x_i(x_\alpha = \varepsilon y_\alpha)$ since $\varepsilon^2\mu'_m = \mu_m$ and $\varepsilon^2I'_{p\alpha}/S' = I_{p\alpha}/S$ one obtains:

$$\langle\tilde{\underline{\sigma}}_t^2\rangle = \left(C^{1\beta}_{1\alpha}\frac{1}{2}U^0_{\beta,x_1} - \frac{E_pI_{p\alpha}}{S}U^0_{\alpha,x_1x_1x_1}\right)\underline{a}_\alpha; \quad C^{1\beta}_{1\alpha} = 2\mu_m c^{*1\beta}_{1\alpha} \tag{20}$$

$$\langle\tilde{\underline{\sigma}}_t^2\rangle_{,x_1} = \underline{0} \tag{21}$$

### 3.3. Discussion and physical interpretation

Eqs. (20) and (21) define the macroscopic behaviour of the reinforced material under macroscopic transverse motions $U_\alpha^0(x_1)\underline{a}_\alpha$. The balance equations (without body forces) includes:

- A classical shear contribution related to the distorsion $U^0_{\alpha,x_1}$. The elastic coefficients only depend on the local geometry and on the elastic parameters of the matrix. It is shown in Section 4.2.1 that they coincide with those given by the usual homogenization approach in the case of infinitely rigid reinforcements (Léné, 1978).
- A non-classical bending contribution related to the derivative of the curvature $U^0_{\alpha,x_1x_1x_1}$. The bending inertia parameter is exactly that of the beam (divided by the section). Conversely to usual composites, where higher gradient terms appear as correctors (Boutin, 1996), here the bending effect arises at the leading order.

#### 3.3.1. Generalized inner bending continua

The macroscopic behaviour differs from the description of composites usually derived by homogenization (e.g. Léné, 1978; Sanchez-Palencia, 1980; Postel, 1985). Here we obtain a generalized medium where the macroscopic variable is the translation $\underline{U}^0$, and the mean stress tensor $\langle\tilde{\underline{\underline{\sigma}}}^2\rangle = \langle\tilde{\underline{\sigma}}_t^2\rangle\otimes\underline{a}_1 + \underline{a}_1\otimes\langle\tilde{\underline{\sigma}}_t^2\rangle$ combines at the *same order* local and non-local terms related respectively to the strain tensor $\mathrm{e}_x(\underline{U}^0)$ and its second gradient. The behaviour can be seen either as the one of the *reinforced* matrix loaded by the body transverse forces induced by the bended beam (namely the gradient of the momentum, $T = M_{,x_1}$), i.e., with simplified obvious notations:

$$[CU_{,x_1}]_{,x_1} = \frac{1}{S}[M_{,x_1}]_{,x_1}, \quad M = EIU_{,x_1x_1}$$

or also as the one of the beam loaded by shear force induced by the reinforced matrix:

$$[T]_{,x_1} = S[CU_{,x_1}]_{,x_1}, \quad T - M_{,x_1} = 0, \quad M = EIU_{,x_1x_1}$$

The reinforced media differs essentially from Cosserat media since no out of plane rotation appears in the macro kinematic of the cell.

Note that, as the rigid beam "imposes" its motion to the soft matrix, the stiffnesses of the beam in bending and of the matrix in shear are combined "in parallel". Thus, the internal mechanism drastically differs from that of Timoshenko beams, where the bending and shear stiffnesses are somehow combined "in series".

Description (20) and (21) is similar to results derived by the phenomenological approach (Sudret and De Buhan, 1999), to the mathematical analysis of energy (Bellieud and Bouchitté, 2002) and to the behaviour of reticulated media made of beams regularly interconnected by tiny beams (Hans and Boutin, 2008). It provides a generalization of the work of (Pideri and Seppecher, 1997), which consider infinitesimal concentration of cylindrical fibers with extremely high modulus. Those latter assumptions lead to bending effects but neglect the stiffenning due to fibers in the effective shear behaviour of the reinforced matrix. Further, the extremely rigid fibers avoid any kinematics involving non-uniform vertical motion. Such a restriction to in plane motions is overcomed in the present approach, as presented in Section 4.

### 3.3.2. Macroscopic stress tensor and mean surface forces

By construction $\langle \tilde{\underline{\underline{\sigma}}}^2 \rangle$ is the mean of the symmetric stresses, and thus is a symmetric tensor. In the case of uniform strain, it reduces to the Cauchy stress classically obtained for composites. Conversely, for inhomogeneous strain, the macroscopic stress does not match the usual concept of Cauchy stress tensor in the sense that, on a surface oriented by $\underline{p}$ (different of the cell surface i.e., $\underline{p} \neq \underline{a}_1$), the average $\widetilde{\underline{T}}(\underline{p}) = \varepsilon^2 \underline{T}(\underline{p})$ of the stresses acting on this surface differs from $\langle \tilde{\underline{\underline{\sigma}}}^2 \rangle \cdot \underline{p} = \varepsilon^2 \langle \underline{\underline{\sigma}}^2 \rangle \cdot \underline{p}$ as proven here below. By definition of $\langle \underline{\sigma}_{\mathrm{t}}^2 \rangle$ as the mean value of $\underline{\sigma}_{\mathrm{t}}^2$ on $S$ we have:

$$\langle \underline{\underline{\sigma}}^2 \rangle \cdot \underline{a}_1 = \langle \underline{\sigma}_{\mathrm{t}}^2 \rangle = \underline{T}(\underline{a}_1)$$

However, for in plane orientation $\underline{p}$, expression (19) provides:

$$\langle \underline{\underline{\sigma}}^2 \rangle \cdot \underline{a}_\alpha = \{ \langle \underline{\sigma}_{\mathrm{t}}^2 \rangle \cdot \underline{a}_\alpha \} \underline{a}_1$$
$$= \left\{ \left[ \int_{S_p} y_\alpha ({}^p\sigma_{\mathrm{n}}^1) ds \right]_{,x_1} + \int_{\partial S} y_\alpha ({}^m\underline{\sigma}_{\mathrm{t}}^2 \cdot \underline{n}) d\gamma \right\} \underline{a}_1$$

Considering rectangular basic cell $S = |l_2 \underline{a}_2 \times l_3 \underline{a}_3|$ the last integral can be transformed into (taking for instance $\underline{a}_\alpha = \underline{a}_2$):

$$\frac{1}{S} \int_{\partial S} ({}^m\underline{\sigma}_{\mathrm{t}}^2 \cdot \underline{n}) y_2 d\gamma = \frac{1}{l_2 \cdot l_3} \left\{ \int_{-l_2/2}^{l_2/2} \left[ {}^m\underline{\sigma}_{\mathrm{t}}^2 (y_2, l_3/2) \right.\right.$$
$$\left. -{}^m\underline{\sigma}_{\mathrm{t}}^2 (y_2, -l_3/2) \right] \cdot \underline{a}_3 y_2 dy_2 + \int_{-l_3/2}^{l_3/2} \left[ (l_2/2)^m \underline{\sigma}_{\mathrm{t}}^2 (l_2/2, y_3) \right.$$
$$\left.\left. -(-l_2/2)^m \underline{\sigma}_{\mathrm{t}}^2 (-l_2/2, y_3) \right] \cdot \underline{a}_2 dy_3 \right\}$$

The first RHS integral vanishes because of the stress $y_3$-periodicity. The second integral simplifies to give the average of the stresses acting on the $\underline{a}_2$-oriented boundary of the cell:

$$\frac{1}{S} \int_{\partial S} ({}^m\underline{\sigma}_{\mathrm{t}}^2 \cdot \underline{n}) y_2 d\gamma = \frac{1}{l_3} \int_{-l_3/2}^{l_3/2} {}^m\underline{\sigma}_{\mathrm{t}}^2 (l_2/2, y_3) \cdot \underline{a}_2 dy_3 = \underline{T}(\underline{a}_2) \cdot \underline{a}_1$$

Consequently:

$$\langle \underline{\underline{\sigma}}^2 \rangle \cdot \underline{a}_\alpha = \underline{T}(\underline{a}_\alpha) + \left[ \int_{S_p} y_\alpha ({}^p\sigma_{\mathrm{n}}^1) ds \right]_{,x_1} \neq \underline{T}(\underline{a}_\alpha)$$

To sum up, the macroscopic balance equations apply to the symmetric stress tensor averaged on the cell section. However, the non-zero divergence of the stress in the cell makes that the classical Cauchy interpretation does not apply to this tensor.

### 3.3.3. Energy and boundary conditions

The higher order of differentiation in the equilibrium equations modifies the nature of the boundary conditions. These latter can be identified through the energy at the macroscale. Consider an infinite layer of reinforced material of height $H$ along $\underline{a}_1$, take the scalar product of (21) by a field test $\underline{U}^0$ and integrate over the height. One obtains, after two integrations by part:

$$0 = \int_0^H \langle \tilde{\underline{\sigma}}_{\mathrm{t}}^2 \rangle_{,x_1} \cdot \underline{U}^0 dx_1$$
$$= -\int_0^H \left[ \frac{1}{2} C_{1\alpha}^{1\beta} U_{\beta,x_1}^0 U_{\alpha,x_1}^0 - \frac{E_p I_{p\alpha}}{S} U_{\alpha,x_1x_1x_1}^0 U_\alpha^0 \right] dx_1 + \left[ \langle \tilde{\underline{\sigma}}_{\mathrm{t}}^2 \rangle \cdot \underline{U}^0 \right]_0^H$$

and

$$\int_0^H \left[ \frac{1}{2} C_{1\alpha}^{1\beta} U_{\beta,x_1}^0 U_{\alpha,x_1}^0 + \frac{E_p I_{p\alpha}}{S} U_{\alpha,x_1x_1}^0 U_{\alpha,x_1x_1}^0 \right] dx_1$$
$$= \left[ \langle \tilde{\underline{\sigma}}_{\mathrm{t}}^2 \rangle \cdot \underline{U}^0 + \frac{E_p I_{p\alpha}}{S} U_{\alpha,x_1x_1}^0 U_{\alpha,x_1}^0 \right]_0^H \qquad (22)$$

The elastic energy (LHS of (22)) accounts for both shear and bending deformations and balances the work (RHS of (22)) produced at the boundaries (normal $\pm\underline{a}_1$) by the mean surface stress $\langle \underline{\underline{\sigma}}^2 \rangle \cdot \underline{a}_1 = \langle \underline{\sigma}_{\mathrm{t}}^2 \rangle = \underline{T}(\underline{a}_1)$ submitted to the motion $\underline{U}^0$ and by the beam momenta $\frac{E_p I_{p\alpha}}{S} U_{\alpha,x_1x}^0$ submitted to the beam section rotations $\underline{U}_{\alpha,x_1}^0$. Hence, accordingly with the fourth degree differential equation, two boundary conditions must be specified at each extremities, in terms of displacement or stress as in usual media but also in rotation or momentum as for beams. By construction of the macroscopic modelling, the interpretation of these latter conditions is directly linked to the actual conditions imposed on the fibers. An illustration of the influence of the type of boundary conditions is given in Section 6.4.

## 4. Homogenized constitutive law of periodic parallel beams in a soft matrix

The study is here extended to macro kinematic $\underline{U} = U_i(x_1, x_\alpha)\underline{a}_i$. Since $U_1$ and $U_\alpha$ may be of the same order, the reduced strain tensors no more respect the orders given by (2). Hence the axial and in-plane problems differ at each order and it is necessary to consider full expansions:

$$\underline{u} = \sum_{i=0} \varepsilon^i \underline{u}^i = \sum_{i=0} \varepsilon^i u_j^i \underline{a}_j \quad \text{i.e.,} \; u_1 = \sum_{i=0} \varepsilon^i u_1^i; \quad u_\alpha = \sum_{i=0} \varepsilon^i u_\alpha^i \qquad (23)$$

Therefore, the set of balance equations becomes: $\underline{a}_1$ direction:

$${}^q\sigma_{\mathrm{n},x_1} + \mathrm{div}_x({}^q\underline{\sigma}_{\mathrm{t}}) + \varepsilon^{-1} \mathrm{div}_y({}^q\underline{\sigma}_{\mathrm{t}}) = 0 \quad \text{in } S_q; \quad q = m, p \qquad (24)$$

$(\underline{a}_2, \underline{a}_3)$ directions:

$${}^q\underline{\sigma}_{\mathrm{t},x_1} + \underline{\mathrm{div}}_x({}^q\underline{\underline{\sigma}}_s) + \varepsilon^{-1} \underline{\mathrm{div}}_y({}^q\underline{\underline{\sigma}}_s) = \underline{0} \quad \text{in } S_q; \quad q = m, p \qquad (25)$$

### 4.1. Leading order of motions

#### 4.1.1. In the beam

Two independent problems govern respectively ${}^p\underline{\underline{\sigma}}_s^{-1}$, ${}^pu_\alpha^0\underline{a}_\alpha$ and ${}^p\underline{\sigma}_{\mathrm{t}}^{-1}$, ${}^pu_1^0$. They are given by (25–16) and (24–13) both at order $(\varepsilon^{-2} - \varepsilon^{-1})$:

$$\begin{cases} \underline{\mathrm{div}}_y({}^p\underline{\underline{\sigma}}_s^{-1}) = \underline{0} \quad \text{in } S_p \\ {}^p\underline{\underline{\sigma}}_s^{-1} = 2\mu_p \underline{\underline{e}}_{sy}({}^p\underline{u}^0) + \lambda_p \mathrm{div}_y({}^p\underline{u}^0) \underline{\underline{I}}_s \\ {}^p\underline{\underline{\sigma}}_s^{-1} \cdot \underline{n} = \underline{0} \quad \text{on } \Gamma \end{cases}$$

$$\begin{cases} \mathrm{div}_y({}^p\underline{\sigma}_{\mathrm{t}}^{-1}) = 0 \quad \text{in } S_p \quad \text{with } {}^p\underline{\sigma}_{\mathrm{t}}^{-1} = \mu_p({}^pu_{1,y_\alpha}^0)\underline{a}_\alpha \\ {}^p\underline{\sigma}_{\mathrm{t}}^{-1} \cdot \underline{n} = 0 \quad \text{on } \Gamma \end{cases}$$

The solution is a rigid in-plane motion of the section and an out of plane translation:

$$ {}^{p}u_\alpha^0 = U_\alpha^0(\underline{x}) + \Xi^{-1}(\underline{x})[\underline{a}_1 \times \underline{y}]_\alpha, \quad {}^{p}u_1^0 = U_1^0(\underline{x}) \quad \text{and} $$

$$ {}^{p}\underline{\underline{e}}^{-1} = \underline{\underline{0}}, \quad {}^{p}\underline{\underline{\sigma}}^{-1} = \underline{\underline{0}} $$

We again assume that the macro-rotation is of the same order than the macro-distorsion and therefore we disregard the torsion at this order by taking $\Xi^{-1} = 0$.

#### 4.1.2. In the matrix

Similarly, two independent problems govern respectively ${}^{m}\underline{\underline{\sigma_s}}^1$, ${}^{m}u_\alpha^0\underline{a}_\alpha$ and ${}^{m}\underline{\sigma}_t^1$, ${}^{m}u_1^0$. They are given by (25–17) and (24–14) both at orders $(\varepsilon^0 - \varepsilon^0)$:

$$ \begin{cases} \underline{\mathrm{div}}_y({}^{m}\underline{\underline{\sigma}}_s^1) = \underline{0} \quad \text{in } S_m \\ {}^{m}\underline{\underline{\sigma}}_s^1 = 2\mu'_m \underline{\underline{e}}_{sy}({}^{m}\underline{u}^0) + \lambda'_m \mathrm{div}_y({}^{m}\underline{u}^0)\underline{\underline{I}}_s \\ {}^{m}u_\alpha^0 = U_\alpha^0 \quad \text{on } \Gamma \\ {}^{m}u_\alpha^0 \quad y\text{-periodic on } \partial S \end{cases} $$

$$ \begin{cases} \mathrm{div}_y({}^{m}\underline{\sigma}_t^1) = 0 \quad \text{in } S_m \quad \text{with } {}^{m}\underline{\sigma}_t^1 = \mu'_m({}^{m}u^0_{1,y_\alpha})\underline{a}_\alpha \\ {}^{m}u_1^0 = U_1^0 \quad \text{on } \Gamma \\ {}^{m}u_1^0 \quad y\text{-periodic on } \partial S \end{cases} $$

It is obvious that the matrix motion is homogeneous and identical to that of the beam:

$$ {}^{m}\underline{u}^0 = {}^{p}\underline{u}^0 = \underline{U}^0(\underline{x}) \quad \text{and} \quad {}^{m}\underline{\underline{e}}^{-1} = \underline{\underline{0}}, \quad {}^{m}\underline{\underline{\sigma}}^1 = \underline{\underline{0}} $$

### 4.2. Leading order of the beam and matrix stresses

#### 4.2.1. In the axial direction

The governing problem in the beam is defined by (24–13) at orders $(\varepsilon^{-1} - \varepsilon^0)$, i.e., using the results of the previous order:

$$ \begin{cases} \mathrm{div}_y({}^{p}\underline{\sigma}_t^0) = 0 \quad \text{in } S_p \quad \text{with } {}^{p}\underline{\sigma}_t^0 = \mu_p\left({}^{p}u^1_{1,y_\alpha} + \left[U^0_{1,x_\alpha} + U^0_{\alpha,x_1}\right]\right)\underline{a}_\alpha \\ {}^{p}\underline{\sigma}_t^0 \cdot \underline{n} = 0 \quad \text{on } \Gamma \end{cases} $$

and in the matrix by (24–14) at orders $(\varepsilon^1 - \varepsilon^1)$, giving with the previous results

$$ \begin{cases} \mathrm{div}_y({}^{m}\underline{\sigma}_t^2) = 0 \quad \text{in } S_p \quad \text{with} \quad {}^{m}\underline{\sigma}_t^2 = \mu'_m\left({}^{m}u^1_{1,y_\alpha} + \left[U^0_{1,x_\alpha} + U^0_{\alpha,x_1}\right]\right)\underline{a}_\alpha \\ {}^{m}u_1^1 = {}^{p}u_1^1 \quad \text{on } \Gamma \\ {}^{m}u_1^1 \quad y\text{-periodic on } \partial S \end{cases} $$

This is the sequence of problems already solved in Section 3.2.1 except that now the forcing term $\underline{e}_{tx}(\underline{U}^0) = (1/2)[U^0_{1,x_\alpha} + U^0_{\alpha,x_1}]\underline{a}_\alpha$ includes $U^0_{1,x_\alpha}$. Consequently:

$$ {}^{p}u_1^1 = -2\underline{y}\cdot\underline{e}_{tx}(\underline{U}^0) + U_1^1(\underline{x}); \quad {}^{p}\underline{e}_t^0 = \underline{0}; \quad {}^{p}\underline{\sigma}_t^0 = \underline{0} $$

$$ {}^{m}u_1^1 = \psi_\alpha(\underline{y})\mathrm{e}_{tx\alpha}(\underline{U}^0) + U_1^1(\underline{x}); \quad {}^{m}\underline{\sigma}_t^2 = 2\mu'_m[\underline{a}_\alpha + (1/2)\underline{\mathrm{grad}}_y(\psi_\alpha)]\mathrm{e}_{tx\alpha}(\underline{U}^0) $$

By construction, $\psi_\alpha$ only depends on the geometry of the period. Further, since the imposed displacements on $\Gamma$ correspond to a zero inner deformation of the beam (${}^{p}\underline{e}_t^0 = 0$), $\psi_\alpha$ are the solutions that would be obtained by usual homogenization of composites – with this geometry and under out of plane distorsion – in the case of an infinitely rigid body occupying $S_p$ (Léné, 1978). For a bi-symmetric cell, the solutions $\psi_\alpha$ respect the following properties:

$$ \psi_\alpha(y_\alpha, y_\beta) = -\psi_\alpha(-y_\alpha, y_\beta); \quad \psi_\alpha(y_\alpha, y_\beta) = \psi_\alpha(y_\alpha, -y_\beta) \tag{26} $$

#### 4.2.2. In-plane directions

The in-plane governing problem in the beam is defined by (25)–(16) at orders $(\varepsilon^{-1} - \varepsilon^0)$, i.e., using the previous results:

$$ \begin{cases} \underline{\mathrm{div}}_y({}^{p}\underline{\underline{\sigma}}_s^0) = \underline{0} \quad \text{in } S_p \\ {}^{p}\underline{\underline{\sigma}}_s^0 = 2\mu_p\left[\underline{\underline{e}}_{sy}({}^{p}\underline{u}^1) + \underline{\underline{e}}_{sx}(\underline{U}^0)\right] + \lambda_p\left[\mathrm{div}_y({}^{p}\underline{u}^1) + \mathrm{div}_x(\underline{U}^0)\right]\underline{\underline{I}}_s \\ {}^{p}\underline{\underline{\sigma}}_s^0 \cdot \underline{n} = \underline{0} \quad \text{on } \Gamma \end{cases} $$

The solution is derived by building an in-plane motion ${}^{p}\underline{v}^1$ where the local plane stress is identical to the plane stress induced by the 3-D motion $\underline{U}^0$, i.e.:

$$ 2\mu_p\underline{\underline{e}}_{sx}(\underline{U}^0) + \lambda_p\mathrm{div}_x(\underline{U}^0)\underline{\underline{I}}_s = 2\mu_p\underline{\underline{e}}_{sy}({}^{p}\underline{v}^1) + \lambda_p\mathrm{div}_y({}^{p}\underline{v}^1)\underline{\underline{I}}_s $$

Observing that:

$$ \underline{\underline{e}}_{sy}(\underline{\underline{e}}_{sx}(\underline{U}^0)\cdot\underline{y}) = \underline{\underline{e}}_{sx}(\underline{U}^0) \quad \text{and} \quad \mathrm{div}_y(\underline{\underline{e}}_{sx}(\underline{U}^0)\cdot\underline{y}) = U^0_{\alpha,x_\alpha} $$

we must have:

$$ \lambda_p U^0_{1,x_1}\underline{\underline{I}}_s = 2\mu_p\underline{\underline{e}}_{sy}({}^{p}\underline{v}^1 - \underline{\underline{e}}_{sx}(\underline{U}^0)\cdot\underline{y}) + \lambda_p\mathrm{div}_y({}^{p}\underline{v}^1 - \underline{\underline{e}}_{sx}(\underline{U}^0)\cdot\underline{y})\underline{\underline{I}}_s $$

The left hand side isotropic term due to the axial gradient of the vertical component can be written as $\nu_p U^0_{1,x_1}[2\mu_p\underline{\underline{e}}_{sy}(\underline{y}) + \lambda_p\mathrm{div}_y(\underline{y})\underline{\underline{I}}_s]$ and finally:

$$ {}^{p}\underline{v}^1 = \underline{\underline{e}}_{sx}(\underline{U}^0)\cdot\underline{y} + \nu_p U^0_{1,x_1}\underline{y} $$

From the same reasoning than in Appendix A.2 (third problem), it follows that:

$$ \underline{\underline{e}}_{sy}({}^{p}u_\alpha^1\underline{a}_\alpha + {}^{p}\underline{v}^1) = 0 \quad \text{and} \quad {}^{p}\underline{\underline{\sigma}}_s^0 = 0 $$

Consequently, ${}^{p}u_\alpha^1\underline{a}_\alpha + {}^{p}\underline{v}^1$ is a rigid in-plane motion of the section, and:

$$ \underline{\underline{e}}_{sy}({}^{p}\underline{u}^1) = -\underline{\underline{e}}_{sy}({}^{p}\underline{v}^1) = -\underline{\underline{e}}_{sx}(\underline{U}^0) - \nu_p U^0_{1,x_1}\underline{\underline{I}}_s \tag{27} $$

The in-plane strain field in the beam is:

$$ \begin{aligned} {}^{p}\underline{\underline{e}}_s^0 &= \underline{\underline{e}}_{sy}({}^{p}\underline{u}^1) + \underline{\underline{e}}_{sx}(\underline{U}^0) = -\underline{\underline{e}}_{sy}({}^{p}\underline{v}^1) + \underline{\underline{e}}_{sx}(\underline{U}^0) = -\underline{\underline{e}}_{sy}(\nu_p\underline{y}U^0_{1,x_1}) \\ &= -\nu_p U^0_{1,x_1}\underline{\underline{I}}_s \end{aligned} $$

and the isotropic elasticity relations (4) lead to the normal stress and strain:

$$ {}^{p}\mathrm{e}_n^0 = U^0_{1,x_1}; \quad {}^{p}\sigma_n^0 = E_p U^0_{1,x_1} $$

#### 4.2.3. In the whole section

Accounting for the above results, (25–14) at orders $(\varepsilon^1 - \varepsilon^1)$ and the constraint (27), give:

$$ \begin{cases} \underline{\mathrm{div}}_y({}^{m}\underline{\underline{\sigma}}_s^2) = \underline{0} \quad \text{in } S_m \\ {}^{m}\underline{\underline{\sigma}}_s^2 = 2\mu'_m\left[\underline{\underline{e}}_{sy}({}^{m}\underline{u}^1) + \underline{\underline{e}}_{sx}(\underline{U}^0)\right] + \lambda'_m[\mathrm{div}_y({}^{m}\underline{u}^1) + \mathrm{div}_x(\underline{U}^0)]\underline{\underline{I}}_s \\ \underline{\underline{e}}_{sy}({}^{p}\underline{u}^1) = -\underline{\underline{e}}_{sx}(\underline{U}^0) - \nu_p U^0_{1,x_1}\underline{\underline{I}}_s \quad \text{in } S_p \\ {}^{m}u_\alpha^1 = {}^{p}u_\alpha^1 \quad \text{on } \Gamma \\ {}^{m}u_\alpha^1 \quad y\text{-periodic on } \partial S \end{cases} $$

This is a 2D elastic problem in which the forcing terms are $\underline{\underline{e}}_{sx}(\underline{U}^0)$ and $U^0_{1,x_1}$. The existence and uniqueness of a zero mean value solution is established from Stampaccia theorem in a similar way as Levy and Sanchez-Palencia (1983). Consider the convex $K$ of the in-plane motions $\underline{u} = u_\alpha\underline{a}_\alpha$ within $S({}^{m}\underline{u}$ in $S_m$, ${}^{p}\underline{u}$ in $S_p)$ such that:

$$ K = \left\{\underline{u}, C^1, S\text{-periodic}; \quad \underline{\underline{e}}_{sy}(\underline{u}) = -\underline{\underline{e}}_{sx}(\underline{U}^0) - \nu_p U^0_{1,x_1}\underline{\underline{I}}_s \quad \text{in } S_p; \quad \int_S \underline{u}ds = \underline{0}\right\} $$

Noticing that the balance of ${}^{m}\underline{\underline{\sigma}}_s^2$ simplifies into:

$$ \underline{\mathrm{div}}_y({}^{m}\underline{\underline{\sigma}}_s'^2) = \underline{0} \quad \text{in } S_m; \quad {}^{m}\underline{\underline{\sigma}}_s'^2 = 2\mu'_m\underline{\underline{e}}_{sy}({}^{m}\underline{u}^1) + \lambda'_m\mathrm{div}_y({}^{m}\underline{u}^1)\underline{\underline{I}}_s $$

we have for any field $\underline{v}$ of $K$:

$$0=\int_{S_m}\underline{\mathrm{div}}_y({}^m\underline{\underline{\sigma}}{}'^2_s)\cdot(\underline{v}-{}^m\underline{u}^1)ds$$
$$=-\int_{S_m}{}^m\underline{\underline{\sigma}}{}'^2_s\cdot\underline{\underline{e}}_y(\underline{v}-{}^m\underline{u}^1)ds-\int_\Gamma({}^m\underline{\underline{\sigma}}{}'^2_s\cdot\underline{n})\cdot(\underline{v}-{}^p\underline{u}^1)d\gamma$$

By construction $\underline{\underline{e}}_{sy}(\underline{v}-{}^p\underline{u}^1)=\underline{\underline{0}}$ in $S_p$ (including $\Gamma=\partial S_p$), so that $\underline{v}-{}^p\underline{u}^1$ is an in-plane rigid body motion of the form $\underline{A}+B\underline{a}_1\times\underline{y}$. Thus, the last integral representing the virtual energy $\mathcal{E}^p$ in the section $S_p$ becomes:

$$\underline{A}\cdot\int_\Gamma{}^m\underline{\underline{\sigma}}{}'^2_s\cdot\underline{n}d\gamma-B\underline{a}_1\cdot\int_\Gamma({}^m\underline{\underline{\sigma}}{}'^2_s\cdot\underline{n})\times\underline{y}d\gamma=\mathcal{E}^p$$

By considering the limit case of a composite where the inclusion (index $i$) becomes infinitely rigid compared to the matrix (Léné, 1978) has shown that:

$$\mathcal{E}^p=0$$

The physical reason lies in the fact that, as the stresses in both materials are of the same order ($\sigma=O(\mu_m{}^mu/l)=O(\mu_i{}^iu/l)$), the elastic energy in the inclusion, $\mathcal{E}^i=O(\mu_i({}^iu/l)^2)$, becomes negligible compared to the energy of the matrix, $\mathcal{E}^m=O(\mu_m({}^mu/l)^2)$, when the inclusion tends to be rigid: $\mathcal{E}^i/\mathcal{E}^m=O(\mu_m/\mu_i)\to0$. Besides, the divergence theorem applied to $\underline{\mathrm{div}}_y({}^m\underline{\underline{\sigma}}{}'^2_s)$ (with the periodicity) and to $\underline{\mathrm{div}}_y({}^m\underline{\underline{\sigma}}{}'^2_s)\times\underline{y}$ provide (see Section 2.2.3):

$$\int_{S_m}\underline{\mathrm{div}}_y({}^m\underline{\underline{\sigma}}{}'^2_s)ds=\int_\Gamma{}^m\underline{\underline{\sigma}}{}'^2_s\cdot\underline{n}d\gamma=\underline{0};$$
$$\int_{S_m}\underline{\mathrm{div}}_y({}^m\underline{\underline{\sigma}}{}'^2_s)\times\underline{y}ds=\int_{\Gamma\cup\partial S}({}^m\underline{\underline{\sigma}}{}'^2_s\cdot\underline{n})\times\underline{y}d\gamma=\underline{0}$$

so that the vanishing of $\mathcal{E}^p$ implies:

$$\int_{\partial S}({}^m\underline{\underline{\sigma}}{}'^2_s\cdot\underline{n})\times\underline{y}d\gamma=\underline{0}\tag{28}$$

Finally, we are left with the problem:

$$\forall\underline{v}\in K\quad\mathcal{A}({}^m\underline{u}^1,\underline{v}-{}^m\underline{u}^1)=0;$$
$$\mathcal{A}(\underline{u},\underline{v})=\int_{S_m}\left[2\mu'_m\underline{\underline{e}}_{sy}(\underline{u}):\underline{\underline{e}}_{sy}(\underline{v})+\lambda'_m\mathrm{div}_y(\underline{u})\mathrm{div}_y(\underline{v})\right]ds$$

where the solution $\underline{u}^1$ in $S$ is unique since $\mathcal{A}$ is a bilinear symmetric coercive form. By linearity, the solution (of non zero mean value) reads:

$$u^1_\delta\underline{a}_\delta=\underline{\Phi}^{\alpha\beta}(\underline{y})\mathrm{e}_{x\alpha\beta}(\underline{U}^0)+\nu_p\underline{\Upsilon}(\underline{y})U^0_{1,x_1}+U^1_\delta\underline{a}_\delta$$

where the 2D in-plane fields $\underline{\Phi}^{\alpha\beta}$ and $\underline{\Upsilon}=\underline{\Phi}^{\alpha\alpha}=\underline{\Phi}^{22}+\underline{\Phi}^{33}$, are respectively the particular zero mean value solutions for $\underline{\underline{e}}_{sx}(\underline{U}^0)=(\underline{a}_\alpha\otimes\underline{a}_\beta+\underline{a}_\beta\otimes\underline{a}_\alpha)/2$ and $U^0_{1,x_1}=1$. By construction, $\underline{\Phi}^{\alpha\beta}$ depends on the period's geometry of and on $\nu_m$. As the imposed displacements on $\Gamma$ correspond to a zero inner deformation of the beam section (${}^p\underline{\underline{e}}{}^0_s=\underline{\underline{0}}$ when $U^0_{1,x_1}=0$), $\underline{\Phi}^{\alpha\beta}$ are the same solutions that would be obtained by standart homogenization of composites – with this geometry and under $\underline{\underline{e}}_{sx}(\underline{U}^0)$ – in the case of an infinitely rigid body occupying $S_p$ (Léné, 1978). In accordance with the constraint (27), $\underline{\Phi}^{\alpha\beta}$ consist, within $S_p$, into a rigid body motion (of translation $\underline{\phi}^{\alpha\beta}$ and rotation $\theta^{\alpha\beta}$) and the deformation field associated to the imposed unit strains, i.e.:

$$\underline{\Phi}^{\alpha\beta}=\underline{\phi}^{\alpha\beta}+\theta^{\alpha\beta}\underline{a}_1\times\underline{y}-(y_\alpha\underline{a}_\beta+y_\beta\underline{a}_\alpha)/2\quad\text{in }S_p$$

In the case of bi-symmetric cells, the solutions $\underline{\Phi}^{\alpha\beta}$ respect the following properties of symmetry (no summations on repeated indices and $\alpha\neq\beta$):

$$\Phi^{23}_\alpha(y_\alpha,y_\beta)=\Phi^{23}_\alpha(-y_\alpha,y_\beta);\quad\Phi^{23}_\alpha(y_\alpha,y_\beta)=-\Phi^{23}_\alpha(y_\alpha,-y_\beta)$$
$$\Phi^{\alpha\alpha}_\alpha(y_\alpha,y_\beta)=-\Phi^{\alpha\alpha}_\alpha(-y_\alpha,y_\beta);\quad\Phi^{\alpha\alpha}_\alpha(y_\alpha,y_\beta)=\Phi^{\alpha\alpha}_\alpha(y_\alpha,-y_\beta)$$
$$\Phi^{\alpha\alpha}_\beta(y_\alpha,y_\beta)=\Phi^{\alpha\alpha}_\beta(-y_\alpha,y_\beta);\quad\Phi^{\alpha\alpha}_\beta(y_\alpha,y_\beta)=-\Phi^{\alpha\alpha}_\beta(y_\alpha,-y_\beta)\tag{29}$$

and consequently $\theta^{\alpha\alpha}=0$, and $\underline{\phi}^{\alpha\beta}=\underline{\phi}^{\alpha\alpha}=\underline{0}$.

The in-plane stress state in the matrix is given by:

$${}^m\sigma^2_{s\alpha\beta}=[2\mu'_m(\mathrm{e}_{\alpha\beta}(\underline{\Phi}^{\zeta\eta})+\delta_{\alpha\zeta}\delta_{\beta\eta})+\lambda'_m(\mathrm{div}_y(\underline{\Phi}^{\zeta\eta})+\delta_{\zeta\eta})\delta_{\alpha\beta}]\mathrm{e}_{x\zeta\eta}(\underline{U}^0)$$
$$+[2\mu'_m\nu_p\mathrm{e}_{\alpha\beta}(\underline{\Phi}^{\eta\eta})+\lambda'_m(\nu_p\mathrm{div}_y(\underline{\Phi}^{\eta\eta})+1)\delta_{\alpha\beta}]U^0_{1,x_1}\tag{30}$$

Finally, the normal stress is:

$${}^m\sigma^2_n=2\mu'_mU^0_{1,x_1}+\lambda'_m[\mathrm{div}_y({}^m\underline{u}^1)+\mathrm{div}_x(\underline{U}^0)]$$
$$=[2\mu'_mU^0_{1,x_1}+\lambda'_m\mathrm{div}_x(\underline{U}^0)]+\lambda'_m[\nu_p\mathrm{div}_y(\underline{\Phi}^{\eta\eta})U^0_{1,x_1}$$
$$+\mathrm{div}_y(\underline{\Phi}^{\alpha\beta})\mathrm{e}_{x\alpha\beta}(\underline{U}^0)]$$

To sum up, the leading order of the stress in the matrix reads:

$${}^m\underline{\underline{\sigma}}^2=\underline{\underline{\underline{\underline{c'}}}}(\underline{y}):\underline{\underline{e}}_x(\underline{U}^0)$$

and can be formulated with the usual $6\times6$ matricial notation:

$$\begin{pmatrix}{}^m\sigma^2_{11}\\{}^m\sigma^2_{22}\\{}^m\sigma^2_{33}\\{}^m\sigma^2_{23}\\{}^m\sigma^2_{13}\\{}^m\sigma^2_{12}\end{pmatrix}=\begin{pmatrix}c'^{11}_{11}&c'^{22}_{11}&c'^{33}_{11}&c'^{23}_{11}&0&0\\c'^{11}_{22}&c'^{22}_{22}&c'^{33}_{22}&c'^{23}_{22}&0&0\\c'^{11}_{33}&c'^{22}_{33}&c'^{33}_{33}&c'^{23}_{33}&0&0\\c'^{11}_{23}&c'^{22}_{23}&c'^{33}_{23}&c'^{23}_{23}&0&0\\0&0&0&0&c'^{13}_{13}&c'^{12}_{13}\\0&0&0&0&c'^{13}_{12}&c'^{12}_{12}\end{pmatrix}\begin{pmatrix}\mathrm{e}_{x11}(\underline{U}^0)\\\mathrm{e}_{x22}(\underline{U}^0)\\\mathrm{e}_{x33}(\underline{U}^0)\\\mathrm{e}_{x23}(\underline{U}^0)\\\mathrm{e}_{x13}(\underline{U}^0)\\\mathrm{e}_{x12}(\underline{U}^0)\end{pmatrix}$$

where, without summation on repeated indices:

$$c'^{11}_{11}=\lambda'_m[1+\nu_p\mathrm{div}_y(\underline{\Upsilon})]+2\mu'_m;\quad c'^{\alpha\alpha}_{11}=\lambda'_m[1+\mathrm{div}_y(\underline{\Phi}^{\alpha\alpha})]$$
$$c'^{\beta\beta}_{\alpha\alpha}=\lambda'_m[1+\mathrm{div}_y(\underline{\Phi}^{\beta\beta})]+2\mu'_m[\delta_{\alpha\beta}+\mathrm{e}_{\alpha\alpha}(\underline{\Phi}^{\beta\beta})];$$
$$c'^{11}_{\alpha\alpha}=\lambda'_m[1+\nu_p\mathrm{div}_y(\underline{\Upsilon})]+2\mu'_m\nu_p\mathrm{e}_{\alpha\alpha}(\underline{\Upsilon})$$
$$c'^{23}_{23}=2\mu'_m[1+2\mathrm{e}_{23}(\underline{\Phi}^{23})];\quad c'^{1\alpha}_{1\alpha}=2\mu'_m[1+(1/2)\psi_{\alpha,y_\alpha}]\tag{31}$$
$$c'^{23}_{11}=2\lambda'_m\mathrm{div}_y(\underline{\Phi}^{23});\quad c'^{11}_{23}=2\mu'_m\nu_p\mathrm{e}_{23}(\underline{\Upsilon})$$
$$c'^{23}_{\alpha\alpha}=2[\lambda'_m\mathrm{div}_y(\underline{\Phi}^{23})+2\mu'_m\mathrm{e}_{\alpha\alpha}(\underline{\Phi}^{23})];\quad c'^{\alpha\alpha}_{23}=2\mu'_m\mathrm{e}_{23}(\underline{\Phi}^{\alpha\alpha})$$
$$c'^{12}_{13}=\mu'_m\psi_{2,y_3};\quad c'^{13}_{12}=\mu'_m\psi_{3,y_2}$$

This ends the resolution within the matrix. In the beam, the leading stress and the first order displacement are given here below. Conveniently, in-plane rotation of the section is decomposed into the in-plane rotation $\Psi^0$ inherent to $\underline{U}^0$ and the additional torsion term ${}^p\Omega^0(\underline{x})$:

$${}^p\underline{\underline{\sigma}}^0=E_pU^0_{1,x_1}\underline{a}_1\otimes\underline{a}_1$$
$${}^p\underline{u}^1=\underline{U}^1(\underline{x})-2\underline{y}\cdot\underline{\underline{e}}_{tx}(\underline{U}^0)\underline{a}_1-\underline{y}\cdot\underline{\underline{e}}_{sx}(\underline{U}^0)-\nu_p\underline{y}U^0_{1,x_1}$$
$$+{}^p\underline{U}^1_s(\underline{x})+[\Psi^0(\underline{x})+{}^p\Omega^0(\underline{x})]\underline{a}_1\times\underline{y}$$
$${}^p\underline{U}^1_s(\underline{x})=\underline{\phi}^{\alpha\beta}\mathrm{e}_{x\alpha\beta}(\underline{U}^0)+\nu_p\underline{\phi}^{\alpha\alpha}U^0_{1,x_1}$$
$$\Psi^0(\underline{x})=-(1/2)\underline{\mathrm{curl}}_x(\underline{U}^0)\cdot\underline{a}_1=(U^0_{2,x_3}-U^0_{3,x_2})/2$$
$${}^p\Omega^0(\underline{x})=\theta^{\alpha\beta}\mathrm{e}_{x\alpha\beta}(\underline{U}^0)+\nu_p\theta^{\alpha\alpha}U^0_{1,x_1}-(U^0_{2,x_3}-U^0_{3,x_2})/2$$

### 4.3. Global axial balance at the leading order – First order of the beam stresses

#### 4.3.1. In the axial direction

The next order field in the beam is governed by (24)–(13) at orders $(\varepsilon^0-\varepsilon^1)$. As ${}^p\underline{\sigma}^0_t=\underline{0}$ and ${}^m\underline{\sigma}^1_t=\underline{0}$, one obtains :

$$\begin{cases}({}^p\sigma^0_n)_{,x_1}+\mathrm{div}_y({}^p\underline{\sigma}^1_t)=0\quad\text{in }S_p\\{}^p\underline{\sigma}^1_t=\mu_p({}^pu^2_{1,y_\alpha}+[{}^pu^1_{1,x_\alpha}+{}^pu^1_{\alpha,x_1}])\underline{a}_\alpha\\{}^p\underline{\sigma}^1_t\cdot\underline{n}=0\quad\text{on }\Gamma\end{cases}$$

Here $({}^p\sigma^0_n)_{,x_1}$ acts as a source term and $[{}^pu^1_{1,x_\alpha}+{}^pu^1_{\alpha,x_1}]$ as forcing term. The condition of compatibility of the source is established by integrating the balance equation on $S_p$. This provides the axial balance equation at the leading order:

$$\left[\int_{S_p} {}^p\sigma_n^0 ds\right]_{,x_1} = |S'_p|({}^p\sigma_n^0)_{,x_1} = 0 \quad \text{i.e.}: \ [E_p U^0_{1,x_1}]_{,x_1} = 0 \tag{32}$$

Thus, the source term vanishes. Now, the determination of ${}^p u_1^2$ is performed in two steps.

First, the forcing term related to ${}^p\underline{u}^1 - {}^p\Omega^0(\underline{x})\underline{a}_1 \times \underline{y}$ can be directly integrated and provides a first contribution ${}^p\mathrm{v}_1^2$ (disregarding the integration constant) to ${}^p u_1^2$:

$$\begin{aligned}{}^p\mathrm{v}_1^2 &= -2\underline{y}\cdot\underline{\mathrm{e}}_{\mathrm{tx}}(\underline{U}^1 + {}^p\underline{U}_s^1)\\ &\quad + \left\{2\underline{y}\cdot\underline{\mathrm{grad}}_x[\underline{\mathrm{e}}_{\mathrm{tx}}(\underline{U}^0)\cdot\underline{y}] + \underline{y}\cdot[\underline{\underline{\mathrm{e}}}_{sx}(\underline{U}^0)]_{,x_1}\cdot\underline{y} + \nu_p|y|^2 U^0_{1,x_1x_1}\right\}\Big/2\end{aligned}$$

Consequently ${}^p\underline{u}^1 - {}^p\Omega^0(\underline{x})\underline{a}_1 \times \underline{y}$ does not create any out of plane stress in the section.

Second, the problem related to ${}^p\Omega^0(\underline{x})\underline{a}_1 \times \underline{y}$ is identical to the wrapping problem in the beam, see Appendix A.3. It results that the second contribution to ${}^p u_1^2$ reads $w(\underline{y})^p\Omega^0_{,x_1}$. Finally, summing both contributions and including the integration constant, give:

$${}^p u_1^2 = {}^p U_1^2(\underline{x}) + {}^p\mathrm{v}_1^2 + w(\underline{y})^p\Omega^0_{,x_1} \tag{33}$$

and the stress ${}^p\underline{\sigma}_t^1$ is exclusively due to the vertical gradient of torsion ${}^p\Omega^0_{,x_1}$:

$${}^p\underline{\mathrm{e}}_t^1 = (1/2)[\underline{\mathrm{grad}}_y(w) + \underline{a}_1 \times \underline{y}]^p\Omega^0_{,x_1}; \quad {}^p\underline{\sigma}_t^1 = 2\mu_p{}^p\underline{\mathrm{e}}_t^1 \tag{34}$$

The properties (61) (Appendix A.3) of the wrapping stresses vector $\underline{\zeta} = \mu(\underline{\mathrm{grad}}_y(w) + \underline{a}_1 \times \underline{y})$ yield to the vanishing of the mean value of ${}^p\underline{\sigma}_t^1$ and the skew symmetry of $\int_{S_p}\underline{y}\otimes{}^p\underline{\sigma}_t^1 ds$:

$$\int_{S_p} {}^p\underline{\sigma}_t^1 ds = \underline{0};$$

$$\int_{S_p}[\underline{y}\otimes{}^p\underline{\sigma}_t^1 + {}^p\underline{\sigma}_t^1\otimes\underline{y}]ds = \underline{\underline{0}} \quad \text{hence} \quad \int_{S_p}\underline{y}\cdot{}^p\underline{\sigma}_t^1 ds = 0$$

Denoting by $\mathcal{J}'_p$ the "wrapping inertia" $\mathcal{J}'_p$ ($= I'_p$ for circular beam section, cf. Section A.3):

$$\mathcal{J}'_p = \int_{S_p} y_2(w_{,y_3} + y_2)ds = \int_{S_p} y_3(-w_{,y_2} + y_3)ds$$

one has:

$$\int_{S_p}\underline{y}\otimes{}^p\underline{\sigma}_t^1 ds = \mu_p\mathcal{J}'_p{}^p\Omega^0_{,x_1}[\underline{a}_2\otimes\underline{a}_3 - \underline{a}_3\otimes\underline{a}_2] \tag{36}$$

*4.3.2. In-plane directions*

The problem in the beam defined by (25)–(16) at orders $(\varepsilon^0 - \varepsilon^1)$, simplified by the results ${}^p\underline{\sigma}_t^0 = \underline{0}$, ${}^p\underline{\underline{\sigma}}_s^0 = \underline{\underline{0}}$ and ${}^m\underline{\underline{\sigma}}^1 = \underline{\underline{0}}$, takes the form:

$$\begin{cases}\underline{\mathrm{div}}_y({}^p\underline{\underline{\sigma}}_s^1) = \underline{0} \quad \text{in } S_p\\ {}^p\underline{\underline{\sigma}}_s^1 = 2\mu_p[\underline{\underline{\mathrm{e}}}_{sy}({}^p\underline{u}^2) + \underline{\underline{\mathrm{e}}}_{sx}({}^p\underline{u}^1)] + \lambda_p[\mathrm{div}_y({}^p\underline{u}^2) + \mathrm{div}_x({}^p\underline{u}^1)]\underline{\underline{I}}_s\\ {}^p\underline{\underline{\sigma}}_s^1\cdot\underline{n} = \underline{0} \quad \text{on } \Gamma\end{cases}$$

The resolution is close to the one developed for ${}^p\underline{\underline{\sigma}}_s^0$. Recalling that $\underline{\underline{\mathrm{e}}}_{sy}(\underline{\xi}^\alpha) = y_\alpha\underline{\underline{I}}_s$ and noticing that the in-plane field $\underline{\mathrm{grad}}_{sx}(\underline{y}\cdot\underline{\underline{\mathrm{e}}}_{sx}(\underline{U}^0)\cdot\underline{y})$ has the following plane strain:

$$\underline{\underline{\mathrm{e}}}_{sy}(\underline{\mathrm{grad}}_{sx}(\underline{y}\cdot\underline{\underline{\mathrm{e}}}_{sx}(\underline{U}^0)\cdot\underline{y})) = 2\underline{\underline{\mathrm{e}}}_{sx}(\underline{\underline{\mathrm{e}}}_{sx}(\underline{U}^0)\cdot\underline{y})$$

the plane stress induced by ${}^p\underline{u}^1$ is re-expressed as the plane stress of the in-plane motion ${}^p\underline{\mathrm{v}}_s^2$, i.e.:

$$2\mu_p\underline{\underline{\mathrm{e}}}_{sx}({}^p\underline{u}^1) + \lambda_p\mathrm{div}_x({}^p\underline{u}^1)]\underline{\underline{I}}_s = 2\mu_p\underline{\underline{\mathrm{e}}}_{sy}({}^p\underline{\mathrm{v}}_s^2) + \lambda_p\mathrm{div}_y({}^p\underline{\mathrm{v}}_s^2)\underline{\underline{I}}_s$$

$$\begin{aligned}{}^p\underline{\mathrm{v}}_s^2 &= \underline{\underline{\mathrm{e}}}_{sx}(\underline{U}^1)\cdot\underline{y} + \nu_p U^1_{1,x_1}\underline{y} - 2\nu_p[\mathrm{e}_{\mathrm{tx}\alpha}(\underline{U}^0)]_{,x_1}\underline{\xi}^\alpha\\ &\quad - (1/2)[\underline{\mathrm{grad}}_{sx}(\underline{y}\cdot\underline{\underline{\mathrm{e}}}_{sx}(\underline{U}^0)\cdot\underline{y}) + \nu_p|y|^2\underline{\mathrm{grad}}_{sx}(U^0_{1,x_1})]\end{aligned}$$

It follows that:

$$\underline{\underline{\mathrm{e}}}_{sy}({}^p u_\alpha^2\underline{a}_\alpha + {}^p\underline{\mathrm{v}}_s^2) = \underline{\underline{0}} \quad \text{and} \quad {}^p\underline{\underline{\sigma}}_s^1 = \underline{\underline{0}}$$

Then ${}^p u_\alpha^2\underline{a}_\alpha + {}^p\underline{\mathrm{v}}_s^2$ is a rigid in-plane motion of the section, and:

$${}^p u_\alpha^2\underline{a}_\alpha = -{}^p\underline{\mathrm{v}}_s^2 + U_\alpha^2(\underline{x})\underline{a}_\alpha + \Omega^1(\underline{x})\underline{a}_1 \times \underline{y} \tag{37}$$

The in-plane strain field in the beam is:

$$\begin{aligned}{}^p\underline{\underline{\mathrm{e}}}_s^1 &= \underline{\underline{\mathrm{e}}}_{sy}({}^p\underline{u}^2) + \underline{\underline{\mathrm{e}}}_{sx}(\underline{u}^1) = -\underline{\underline{\mathrm{e}}}_{sy}({}^p\underline{\mathrm{v}}_s^2) + \underline{\underline{\mathrm{e}}}_{sx}(\underline{u}^1)\\ &= -\underline{\underline{\mathrm{e}}}_{sy}(\nu_p\underline{y}U^1_{1,x_1} - 2\nu_p\underline{\xi}^\alpha[\mathrm{e}_{\mathrm{tx}\alpha}(\underline{U}^0)]_{,x_1})\\ &= -\nu_p(U^1_{1,x_1} - 2y_\alpha[\mathrm{e}_{\mathrm{tx}\alpha}(\underline{U}^0)]_{,x_1})\underline{\underline{I}}_s = -\nu_p{}^p u^1_{1,x_1}\underline{\underline{I}}_s\end{aligned}$$

The isotropic elasticity leads to the normal component that ends the determination of both stress and strain states (${}^p\underline{\mathrm{e}}_t^1$ and ${}^p\underline{\underline{\sigma}}^1$ are given by (34)):

$${}^p\mathrm{e}_n^1 = {}^p u^1_{1,x_1}; \quad {}^p\underline{\underline{\mathrm{e}}}^1 = [\underline{a}_1\otimes\underline{a}_1 - \nu_p\underline{\underline{I}}_s]^p u^1_{1,x_1} + {}^p\underline{\mathrm{e}}_t^1\otimes\underline{a}_1 + \underline{a}_1\otimes{}^p\underline{\mathrm{e}}_t^1$$

$${}^p\sigma_n^1 = E_p{}^p u^1_{1,x_1}; \quad {}^p\underline{\underline{\sigma}}^1 = {}^p\sigma_n^1\underline{a}_1\otimes\underline{a}_1 + {}^p\underline{\sigma}_t^1\otimes\underline{a}_1 + \underline{a}_1\otimes{}^p\underline{\sigma}_t^1$$

### 4.4. Global momentum equilibrium at the leading order

*4.4.1. Axial direction*

The global balance results from (24–13) at orders $(\varepsilon - \varepsilon^2)$. As ${}^m\underline{\sigma}_t^1 = \underline{0}$, one obtains:

$$\begin{cases}({}^p\sigma_n^1)_{,x_1} + \mathrm{div}_x({}^p\underline{\sigma}_t^1) + \mathrm{div}_y({}^p\underline{\sigma}_t^2) = 0 \quad \text{in } S_p\\ \mathrm{div}_y({}^m\underline{\sigma}_t^2) = 0 \quad \text{in } S_m\\ {}^p\underline{\sigma}_t^2\cdot\underline{n} = {}^m\underline{\sigma}_t^2\cdot\underline{n} \quad \text{on } \Gamma\end{cases}$$

In a similar way than in Section 3.2.2, the normal (scalar) global balance equation is established:

$$\left[\int_{S_p}{}^p\sigma_n^1 ds\right]_{,x_1} + \mathrm{div}_x\left[\int_{S_p}{}^p\underline{\sigma}_t^1 ds\right] = 0 \quad \text{i.e., from(35-a)}:$$

$$\left[E_p\frac{|S'_p|}{|S'|}U^1_{1,x_1}\right]_{,x_1} = 0 \tag{38}$$

This equality is identical to the axial balance at the previous order. It leads to identical result shifted of one order and hence can be disregarded. Following again Section 3.2.2, the momentum (vectorial) global balance equation reads:

$$\left[\int_{S_p}{}^p\sigma_n^1\underline{y}ds\right]_{,x_1} + \underline{\mathrm{div}}_x\left[\int_{S_p}\underline{y}\otimes{}^p\underline{\sigma}_t^1 ds\right] + \int_{\partial S}({}^m\underline{\sigma}_t^2\cdot\underline{n})\underline{y}d\gamma = |S'|\langle\underline{\sigma}_t^2\rangle \tag{39}$$

$$\langle\underline{\sigma}_t^2\rangle = \frac{1}{|S'|}\left(\int_{S_p}{}^p\underline{\sigma}_t^2 ds + \int_{S_m}{}^m\underline{\sigma}_t^2 ds\right) \tag{40}$$

As ${}^m\underline{\sigma}_t^2$, ${}^p\underline{\sigma}_t^1$ and ${}^p\sigma_n^1$ are already known, the relation (39) provides the mean stress $\langle\underline{\sigma}_t^2\rangle$. Notice that according to expression (36):

$$\underline{\mathrm{div}}_x\left[\int_{S_p}\underline{y}\otimes{}^p\underline{\sigma}_t^1 ds\right] = \mu_p\mathcal{J}'_p\underline{\mathrm{curl}}_x({}^p\Omega^0_{,x_1}\underline{a}_1)$$

*4.4.2. In-plane directions*

The global balance is established from (25–16) at orders $(\varepsilon^1 - \varepsilon^2)$. Using the results ${}^p\underline{\underline{\sigma}}_s^1 = \underline{\underline{0}}$ and ${}^m\underline{\underline{\sigma}}^1 = \underline{\underline{0}}$, the problem takes the form:

$$\begin{cases} ({}^p\underline{\sigma}_t^1)_{,x_1} + \underline{\mathrm{div}}_y({}^p\underline{\underline{\sigma}}_s^2) = \underline{0} & \text{in } S_p \\ \underline{\mathrm{div}}_y({}^m\underline{\underline{\sigma}}_s^2) = \underline{0} & \text{in } S_m \\ {}^p\underline{\underline{\sigma}}_s^2 \cdot \underline{n} = {}^m\underline{\underline{\sigma}}_s^2 \cdot \underline{n} & \text{on } \Gamma \end{cases}$$

Consider first $\underline{y} \times \underline{\mathrm{div}}_y({}^q\underline{\underline{\sigma}}_s^2)$ and integrate on $S_q$. Making the usual integral transformations (Section 2.2.3) with the divergence theorem, the stress tensor symmetry and the stress continuity on $\Gamma$, one obtains with the help of (28):

$$\left[\int_{S_p} \underline{y} \times {}^p\underline{\sigma}_t^1 ds\right]_{,x_1} = -\int_{\partial S} \underline{y} \times ({}^m\underline{\underline{\sigma}}_s^2 \cdot \underline{n}) d\gamma = \underline{0}$$

This result associated to the skew symmetry of the tensor $\underline{y} \otimes {}^p\underline{\sigma}_t^1$ (35-b) proves that:

$$\left[\int_{S_p} \underline{y} \otimes {}^p\underline{\sigma}_t ds\right]_{,x_1} = \underline{\underline{0}} \quad \text{and from (36)} \quad {}^p\Omega^0_{,x_1x_1} = 0 \qquad (41)$$

Considering now $\underline{y} \otimes \underline{\mathrm{div}}_y({}^q\underline{\underline{\sigma}}_s^2)$ and integrating on $S_q$ leads, with the divergence theorem and the stress continuity on $\Gamma$, to:

$$\int_{S_p} {}^p\underline{\underline{\sigma}}_s^2 ds + \int_{S_m} {}^m\underline{\underline{\sigma}}_s^2 ds = \int_{\partial S} \underline{y} \otimes ({}^m\underline{\underline{\sigma}}_s^2 \cdot \underline{n}) d\gamma + \left[\int_{S_p} \underline{y} \otimes {}^p\underline{\sigma}_t^1 ds\right]_{,x_1}$$

and, accounting from (41) the tensorial equality that defines the mean stress $\langle \underline{\underline{\sigma}}_s^2 \rangle$ reads:

$$\langle \underline{\underline{\sigma}}_s^2 \rangle = \frac{1}{|S'|} \int_{\partial S} \underline{y} \otimes ({}^m\underline{\underline{\sigma}}_s^2 \cdot \underline{n}) d\gamma \qquad (42)$$

#### 4.4.3. Mean normal stress

The mean normal stress $\langle \sigma_n^2 \rangle$ remains to determine. For this, we have to come back to the linear problem governing ${}^p\underline{\underline{\sigma}}_s^2$ :

$$\begin{cases} ({}^p\underline{\sigma}_t^1)_{,x_1} + \underline{\mathrm{div}}_y({}^p\underline{\underline{\sigma}}_s^2) = \underline{0} & \text{in } S_p; \\ {}^p\underline{\underline{\sigma}}_s^2 = 2\mu_p[\underline{\underline{e}}_{sy}({}^p\underline{u}^3) + \underline{\underline{e}}_{sx}({}^p\underline{u}^2)] + \lambda_p[\mathrm{div}_y({}^p\underline{u}^3) + \mathrm{div}_x({}^p\underline{u}^2)]\underline{\underline{I}}_s \\ {}^p\underline{\underline{\sigma}}_s^2 \cdot \underline{n} = {}^m\underline{\underline{\sigma}}_s^2 \cdot \underline{n} & \text{on } \Gamma \end{cases}$$

The solution is decomposed in three parts associated to the forcing terms (i) of "bending" induced by ${}^p\underline{u}^2$, (ii) related to the confining exerted by ${}^m\underline{\underline{\sigma}}_s^2$, and (iii) of torsion associated to ${}^p\underline{\sigma}_t^1$.

For the "bending" contribution (denoted with ante index $^b$) associated to the ${}^p\underline{u}^2$-forcing terms (taking ${}^m\underline{\underline{\sigma}}_s^2 = \underline{\underline{0}}$ and $[{}^p\underline{\sigma}_t^1]_{,x_1} = \underline{0}$), we have:

$$\int_{S_p} {}^{pb}\underline{\underline{\sigma}}_s^2 ds = \int_{\Gamma} \underline{y} \otimes ({}^{pb}\underline{\underline{\sigma}}_s^2 \cdot \underline{n}) d\gamma = \underline{\underline{0}}, \quad \text{thus} \quad \int_{S_p} \mathrm{tr}({}^{pb}\underline{\underline{\sigma}}_s^2) ds = 0$$

With this result, the integration of relation (4) on $S_p$, provides:

$$\int_{S_p} {}^{pb}\sigma_n^2 ds = E_p \int_{S_p} {}^{pb}e_n^2 ds$$

Further ${}^{pb}e_n^2 = {}^pu^2_{1,x_1}$, and according to (33):

$$\begin{aligned} {}^pu^2_{1,x_1} = {}& {}^pU^2_{1,x_1} - 2\underline{y} \cdot [\underline{e}_{tx}(\underline{U}^1 + {}^p\underline{U}_s^1)]_{,x_1} + \{2\underline{y} \cdot \underline{\mathrm{grad}}_x([\underline{e}_{tx}(\underline{U}^0) \cdot \underline{y}]_{,x_1}) \\ & + \underline{y} \cdot [\underline{\underline{e}}_{sx}(\underline{U}^0)]_{,x_1x_1} \cdot \underline{y} + \nu_p |y|^2 U^0_{1,x_1x_1}\}/2 + w(\underline{y})^p\Omega^0_{,x_1x_1} \end{aligned}$$

After integration, the ${}^p\underline{u}^2$-forcing contribution is (recall that $\int_{S_p} w ds = 0$, see Appendix A):

$$\begin{aligned} \int_{S_p} {}^{pb}\sigma_n^2 ds = {}& E_p |S'_p| {}^pU^2_{1,x_1} + E_p I'_{p\alpha}[U^0_{1,x_1x_\alpha x_\alpha} + 2U^0_{\alpha,x_\alpha x_1 x_1}]/2 \\ & + \nu_p E_p (I'_{p2} + I'_{p3})/2 U^0_{1,x_1x_1x_1} \end{aligned}$$

For the "confining" contribution (ante index $^c$), taking $[{}^p\underline{\sigma}_t^1]_{,x_1} = \underline{0}$ and ${}^p\underline{u}^2 = \underline{0}$ (then ${}^{pc}e_n^2 = 0$), we have:

$$\int_{S_p} {}^{pc}\underline{\underline{\sigma}}_s^2 ds = \int_{\Gamma} \underline{y} \otimes ({}^m\underline{\underline{\sigma}}_s^2 \cdot \underline{n}) d\gamma; \quad \text{and} \quad {}^{pc}\sigma_n^2 = \nu_p \mathrm{tr}({}^{pc}\underline{\underline{\sigma}}_s^2)$$

Combining both results with the expression of ${}^m\underline{\underline{\sigma}}_s^2$ yields (dimensionless moduli $D_{11}'^{ij}$ are $O(1)$):

$$\int_{S_p} {}^{pc}\sigma_n^2 ds = \nu_p \int_{\Gamma} \underline{y} \cdot {}^m\underline{\underline{\sigma}}_s^2 \cdot \underline{n} d\gamma = \nu_p \mu'_m [D_{11}'^{11} U^0_{1,x_1} + D_{11}'^{\alpha\beta} e_{x\alpha\beta}(\underline{U}^0)]$$

Finally, for the torsion contribution, (ante index $^t$), taking ${}^m\underline{\underline{\sigma}}_s^2 = \underline{\underline{0}}$ and ${}^p\underline{u}^2 = \underline{0}$, then ${}^{pt}e_n^2 = 0$ and ${}^{pt}\sigma_n^2 = \nu_p \mathrm{tr}({}^{pt}\underline{\underline{\sigma}}_s^2)$, we have:

$$\int_{S_p} {}^{pt}\underline{\underline{\sigma}}_s^2 ds = \int_{S_p} \underline{y} \otimes [{}^p\underline{\sigma}_t^1]_{,x_1} ds$$

then from (35-c)

$$\int_{S_p} \mathrm{tr}({}^{pt}\underline{\underline{\sigma}}_s^2) ds = \left[\int_{S_p} \underline{y} \cdot {}^p\underline{\sigma}_t^1 ds\right]_{,x_1} = 0$$

so that the torsion contribution vanishes: $\frac{1}{|S'|}\int_{S_p} {}^{pt}\sigma_n^2 ds = 0$. To sum up:

$$\begin{aligned} \frac{1}{|S'|} \int_{S_p} {}^p\sigma_n^2 ds = {}& E_p \left[\frac{I'_{p\alpha}}{2|S'|}(U^0_{1,x_1x_\alpha x_\alpha} + 2U^0_{\alpha,x_\alpha x_1 x_1}) + \nu_p \frac{I'_{p2} + I'_{p3}}{2|S'|} U^0_{1,x_1x_1x_1} \right. \\ & \left. + \frac{|S'_p|}{|S|} U^2_{1,x_1}\right] + D_{11}'^{11} U^0_{1,x_1} + D_{11}'^{\alpha\beta} e_{x\alpha\beta}(\underline{U}^0) \end{aligned}$$

### 4.5. Global in-plane equilibrium at the leading order

#### 4.5.1. Axial direction

The global balance results from (24)–(13) at orders ($\varepsilon^1 - \varepsilon^3$):

$$\begin{cases} ({}^p\sigma_n^2)_{,x_1} + \mathrm{div}_x({}^p\underline{\sigma}_t^2) + \mathrm{div}_y({}^p\underline{\sigma}_t^3) = 0 & \text{in } S_p \\ ({}^m\sigma_n^2)_{,x_1} + \mathrm{div}_x({}^m\underline{\sigma}_t^2) + \mathrm{div}_y({}^m\underline{\sigma}_t^3) = 0 & \text{in } S_m \\ {}^p\underline{\sigma}_t^3 \cdot \underline{n} = {}^m\underline{\sigma}_t^3 \cdot \underline{n} & \text{on } \Gamma \end{cases}$$

The integration and the usual transformations give the normal balance at the second order:

$$\left[\int_{S_p} {}^p\sigma_n^2 ds + \int_{S_m} {}^m\sigma_n^2 ds\right]_{,x_1} + \mathrm{div}_x\left[\int_{S_p} {}^p\underline{\sigma}_t^2 ds + \int_{S_m} {}^m\underline{\sigma}_t^2 ds\right] = 0$$

$$\langle \sigma_n^2 \rangle_{,x_1} + \mathrm{div}_x(\langle \underline{\sigma}_t^2 \rangle) = 0$$

This equation generally provides a corrector of the leading and first order Eqs. (32) and (38). Nevertheless, when $U_1$ is independent of $x_1$, the two previous equations become trivial and the normal balance is governed by the present equation. Two momentum balance equations could also be established. However, in any case, they constitute the first corrector of (39) and are not necessary for the leading order description.

#### 4.5.2. In-plane directions

The global balance is driven by (25)–(16) at orders ($\varepsilon^2 - \varepsilon^3$) with ${}^m\underline{\sigma}^1 = \underline{0}$:

$$\begin{cases} {}^p\underline{\sigma}^2_{t,x_1} + \underline{\mathrm{div}}_x({}^p\underline{\underline{\sigma}}_s^2) + \underline{\mathrm{div}}_y({}^p\underline{\underline{\sigma}}_s^3) = \underline{0} & \text{in } S_p \\ {}^m\underline{\sigma}^2_{t,x_1} + \underline{\mathrm{div}}_x({}^m\underline{\underline{\sigma}}_s^2) + \underline{\mathrm{div}}_y({}^m\underline{\underline{\sigma}}_s^3) = \underline{0} & \text{in } S_m \\ {}^p\underline{\underline{\sigma}}_s^3 \cdot \underline{n} = {}^m\underline{\underline{\sigma}}_s^3 \cdot \underline{n} & \text{on } \Gamma \end{cases}$$

The compatibility of the source term gives the leading order of the in-plane macroscopic balance:

$$\left[\int_{S_p} {}^p\underline{\sigma}_t^2 ds + \int_{S_m} {}^m\underline{\sigma}_t^2 ds\right]_{,x_1} + \underline{\mathrm{div}}_x\left[\int_{S_p} {}^p\underline{\underline{\sigma}}_s^2 ds + \int_{S_m} {}^m\underline{\underline{\sigma}}_s^2 ds\right] = \underline{0}$$

$$\langle \underline{\sigma}_t^2 \rangle_{,x_1} + \underline{\mathrm{div}}_x(\langle \underline{\underline{\sigma}}_s^2 \rangle) = \underline{0}$$

#### 4.5.3. Equilibrium with body forces

If the medium is submitted to body forces $\underline{\tilde{b}}^2 = \varepsilon^2 \underline{b}^2$, the only modification occurs in the balance equations of $\underline{\underline{\sigma}}^2$ that become:

$$\langle\sigma_{\mathrm{n}}^{2}\rangle_{,x_1} + \mathrm{div}_x(\langle\underline{\sigma_{\mathrm{t}}^{2}}\rangle) = \langle b_1^2\rangle$$
$$\langle\underline{\sigma_{\mathrm{t}}^{2}}\rangle_{,x_1} + \underline{\mathrm{div}}_x(\langle\underline{\underline{\sigma_{s}^{2}}}\rangle) = \langle b_\alpha^2\rangle\underline{a}_\alpha$$

That is, in the usual compact form:

$$\underline{\mathrm{div}}_x(\langle\underline{\underline{\sigma}}^2\rangle) = \langle\underline{b}^2\rangle$$

### 4.6. Synthesis: macroscopic description

The above results extend the description (20) and (21) limited to purely transverse kinematic. As the first order axial balance does not introduce any information compared to the leading order it is disregarded. After coming back to the physical unscaled quantities ($\underline{\widetilde{Q}}^2 = \varepsilon^2 Q^2$) and parameters ($\lambda_m, \mu_m, I_{p\alpha}, |S_p|, |S|$), the complete description is summarized as follows:

$$\underline{U}(\underline{x}) = \underline{U}^0(\underline{x}) + \varepsilon^2\underline{U}^2(\underline{x}) + \cdots = \underline{U}^0(\underline{x}) + \underline{\widetilde{U}}^2(\underline{x}) + \cdots$$

$$\langle\underline{\underline{\sigma}}\rangle(\underline{x}) = \langle\underline{\underline{\sigma}}^0\rangle(\underline{x}) + \varepsilon^2\langle\underline{\underline{\sigma}}^2\rangle(\underline{x}) + \cdots = \langle\underline{\underline{\sigma}}^0\rangle(\underline{x}) + \langle\underline{\underline{\tilde{\sigma}}}^2\rangle(\underline{x}) + \cdots$$

$$\underline{\mathrm{div}}_x(\langle\underline{\underline{\sigma}}^0\rangle) = \underline{0}$$

$$\langle\underline{\underline{\sigma}}^0\rangle = E_p\frac{|S_p|}{|S|}U^0_{1,x_1}\underline{a}_1\otimes\underline{a}_1$$

$$\underline{\mathrm{div}}_x(\langle\underline{\underline{\tilde{\sigma}}}^2\rangle) = \langle\underline{\tilde{b}}^2\rangle$$

$$\langle\underline{\underline{\tilde{\sigma}}}^2\rangle = \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\mathrm{e}}}_x(\underline{U}^0) + E_p\frac{|S_p|}{|S|}\widetilde{U}^2_{1,x_1}\underline{a}_1\otimes\underline{a}_1 - \underline{\underline{\mathbb{S}}} - \underline{\underline{\mathbb{S}}}'$$

$$\underline{\underline{\mathbb{S}}} = -E_p\left\{\frac{I_{p\alpha}}{2|S|}([U^0_{1,x_\alpha} + U^0_{\alpha,x_1}]_{,x_1x_\alpha} + [U^0_{\alpha,x_\alpha}]_{,x_1x_1}) + \nu_p\frac{I_{p2}+I_{p3}}{2|S|}[U^0_{1,x_1}]_{,x_1x_1}\right\}\underline{a}_1\otimes\underline{a}_1$$
$$+ E_p\frac{I_{p\alpha}}{|S|}[U^0_{1,x_\alpha} + U^0_{\alpha,x_1}]_{,x_1x_1}(\underline{a}_1\otimes\underline{a}_\alpha + \underline{a}_\alpha\otimes\underline{a}_1)$$

$$\underline{\underline{\mathbb{S}}}' = \mu_p\mathcal{J}_p[\underline{a}_1\otimes\underline{\mathrm{curl}}_x({}^p\Omega^0_{,x_1}\underline{a}_1) + \underline{\mathrm{curl}}_x({}^p\Omega^0_{,x_1}\underline{a}_1)\otimes\underline{a}_1]$$

The non-zero components of the macroscopic elastic tensor ($\{ij\} = \{11,22,33,23\}$) and the wrapping inertia coefficient are given below where $c^{ij}_{kl}$ have the same expression than $c'^{ij}_{kl}$ except that $\lambda'_m$, $\mu'_m$ are replaced by $\lambda_m$, $\mu_m$:

$$\begin{aligned}
C^{ij}_{11} &= \frac{1}{|S'|}\left[\int_{S_m} c^{ij}_{11}ds + \nu_p\int_\Gamma y_\alpha c^{ij}_{\alpha\beta}n_\beta d\gamma\right]\\
C^{ij}_{\alpha\beta} &= \frac{1}{|S'|}\int_{\partial S} y_\alpha c^{ij}_{\beta\delta}n_\delta d\gamma\\
C^{1\beta}_{1\alpha} &= \frac{1}{|S'|}\int_{\partial S} y_\alpha c^{1\beta}_{1\delta}n_\delta ds\\
\mathcal{J}_p &= \frac{|S'|}{|S|}\int_{S_p} y_2(w_{,y_3} - y_2)ds = -\int_{S_p} y_3(w_{,y_2} - y_3)ds
\end{aligned}\qquad (43)$$

The macroscopic behaviour is that of a generalized medium and comments made in Section 3.3 also apply here. The kinematic variable is the translation $\underline{U}$ and the mean stress $\langle\underline{\underline{\tilde{\sigma}}}^2\rangle$ combines at the *same order*, (i) local terms related to the strain tensor $\mathrm{e}_x(\underline{U}^0)$ and $\underline{U}^1_{,x_1}$, and (ii) non-local terms related to the second gradient of the strain tensor. This leads to decompose the mean stress tensor $\langle\underline{\underline{\tilde{\sigma}}}^2\rangle$ into the "Cauchy" tensor related to the reinforced matrix and the "non-Cauchy" tensors $\underline{\underline{\mathbb{S}}}$ and $\underline{\underline{\mathbb{S}}}'$ related to the beam. Following the reasoning of Section 3.3.2 we have: $\langle\underline{\underline{\tilde{\sigma}}}^2\rangle\cdot\underline{a}_1 = \underline{T}(\underline{a}_1)$ while $\langle\underline{\underline{\sigma}}^2\rangle\cdot\underline{a}_\alpha = \underline{T}(\underline{a}_\alpha) - (\underline{\underline{\mathbb{S}}}\cdot\underline{a}_\alpha + \underline{\underline{\mathbb{S}}}'\cdot\underline{a}_\alpha)$. Tensor $\underline{\underline{\mathbb{S}}}$ arises from the bending, the Poisson effect under inhomogeneous compression, the effect of inhomogeneous confining, $\underline{\underline{\mathbb{S}}}'$ is due to the wrapping under torsion. Note that, because of (41):

$$\underline{\mathrm{div}}_x(\underline{\underline{\mathbb{S}}}') = \underline{0}$$

The formulation can be simplified in two ways:

- From the leading order balance equation $U^0_{1,x_1x_1} = 0$. Thus, the second and higher derivatives of $U^0_1$ vanish in $\langle\underline{\underline{\sigma}}^2\rangle$ (in presence of constant body force along $x_1$, the third and higher derivatives vanish).
- for bi-symmetric cell, the symmetry (26) of $\psi_\alpha$, (29) of $\Phi^{\alpha\beta}$, implies from the expression of the local stresses (31), that $C^{23}_{11} = C^{23}_{ii} = C^{13}_{12} = C^{12}_{13} = 0$.

Because of the large contrast in stiffness and of the parallel orientation of fibers, either the strain tensor components are all of the same order and the normal stress is of two orders higher than the other stress components, or conversely, the stress tensor components are all of the same order and the normal strain is of two orders smaller than the other components.

In the first case, the leading order description is:

$$\underline{U}(\underline{x}) = U^0_i(\underline{x})\underline{a}_i + \cdots;\quad \langle\underline{\underline{\sigma}}\rangle(\underline{x}) = \langle\underline{\underline{\sigma}}^0\rangle(\underline{x}) + \langle\underline{\underline{\tilde{\sigma}}}^2\rangle(\underline{x}) + \cdots$$
$$\left[\langle\sigma^0\rangle_{11}\right]_{,x_1} = 0;\quad \langle\sigma^0\rangle_{11} = E_p\frac{S_p}{S}U^0_{1,x_1}$$
$$\left[\langle\tilde{\sigma}^2\rangle_{\alpha i} + \mathbb{S}'_{\alpha i}\right]_{,x_i} = \langle\underline{\tilde{b}}^2_\alpha\rangle;\quad \langle\tilde{\sigma}^2\rangle_{\alpha i} + \mathbb{S}'_{\alpha i} = C^{ij}_{\alpha i}\mathrm{e}_{xij}(\underline{U}^0) - \delta_{i1}E_p\frac{I_{p\alpha}}{|S|}U^0_{\alpha,x_1x_1x_1}$$

The corresponding energy formulation for a infinite layer of reinforced material of height $H$ along $\underline{a}_1$ combines terms of different order of magnitude and reads:

$$\begin{aligned}
&\int_0^H\left\{E_p\frac{S_p}{S}(U^0_{1,x_1})^2 + \frac{1}{2}\underline{\underline{\mathrm{e}}}_x(\underline{U}^0) : \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\mathrm{e}}}_x(\underline{U}^0) + \frac{E_pI_{p\alpha}}{S}(U^0_{\alpha,x_1x_1})^2\right\}dx_1\\
&= \left[E_p\frac{S_p}{S}(U^0_{1,x_1})U^0_1 + ((\langle\underline{\underline{\tilde{\sigma}}}^2\rangle + \underline{\underline{\mathbb{S}}}')\cdot\underline{a}_1)\cdot\underline{U}^0 + \frac{E_pI_{p\alpha}}{S}U^0_{\alpha,x_1x_1}U^0_{\alpha,x_1}\right]_0^H\\
&\quad + \int_0^H\langle\underline{\tilde{b}}^2\rangle\cdot\underline{U}^0dx_1
\end{aligned}\qquad (44)$$

Hence, the boundary conditions are expressed in terms of displacement and stress for the components in the axial direction $\underline{a}_1$, and, for the in-plane components, in terms of mean stress and transverse motion and rotation and momentum (these latter conditions being related to the actual conditions imposed on the fibers).

In the second case, the leading order description is:

$$\underline{U}(\underline{x}) = U^0_\alpha(\underline{x})\underline{a}_\alpha + \widetilde{U}^2_1(\underline{x})\underline{a}_1 + \cdots;\quad \langle\underline{\underline{\sigma}}\rangle(\underline{x}) = \langle\underline{\underline{\tilde{\sigma}}}^2\rangle(\underline{x}) + \cdots$$

$$\mathrm{div}_x(\langle\underline{\underline{\sigma}}^2\rangle + \underline{\underline{\mathbb{S}}}') = \langle\underline{\tilde{b}}^2\rangle$$

$$\begin{aligned}
\langle\underline{\underline{\sigma}}^2\rangle + \underline{\underline{\mathbb{S}}}' &= \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\mathrm{e}}}_x(U^0_\alpha\underline{a}_\alpha) + \underline{a}_1\otimes\underline{a}_1E_p\frac{|S_p|}{|S|}\widetilde{U}^2_{1,x_1}\\
&\quad - E_p\frac{I_{p\alpha}}{|S|}\left\{U^0_{\alpha,x_1x_1x_1}(\underline{a}_1\otimes\underline{a}_\alpha + \underline{a}_\alpha\otimes\underline{a}_1) - U^0_{\alpha,x_\alpha x_1x_1}\underline{a}_1\otimes\underline{a}_1\right\}
\end{aligned}$$

The corresponding energy formulation reads:

$$\begin{aligned}
&\int_0^H\left\{E_p\frac{S_p}{S}(\widetilde{U}^2_{1,x_1})^2 + \frac{1}{2}\underline{\underline{\mathrm{e}}}_x(\underline{U}^0) : \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\mathrm{e}}}_x(\underline{U}^0) + \frac{E_pI_{p\alpha}}{S}(U^0_{\alpha,x_1x_1})^2\right.\\
&\left.+ \frac{E_pI_{p\alpha}}{S}(U^0_{\alpha,x_\alpha x_1})\widetilde{U}^2_{1,x_1x_1}\right\}dx_1 = \left[E_p\frac{S_p}{S}(\widetilde{U}^2_{1,x_1})\widetilde{U}^2_1 + ((\langle\underline{\underline{\tilde{\sigma}}}^2\rangle + \underline{\underline{\mathbb{S}}}')\cdot\underline{a}_1)\cdot\underline{U}^0\right.\\
&\left.+ \frac{E_pI_{p\alpha}}{S}U^0_{\alpha,x_1x_1}U^0_{\alpha,x_1} + \frac{E_pI_{p\alpha}}{S}(U^0_{\alpha,x_\alpha x_1})\widetilde{U}^2_{1,x_1}\right]_0^H + \int_0^H\langle\underline{\tilde{b}}^2\rangle\cdot\underline{U}^0dx_1
\end{aligned}$$

Thus, the boundary conditions for any direction of the components are expressed in terms of mean stress and transverse motion and rotation and momentum.

## 5. Models for other beam/matrix stiffness contrasts

### 5.1. Very soft matrix or matrix as stiff as the beam

A change in the order of magnitude of the stiffness of the matrix is directly reflected in the order of magnitude of the stress in the matrix.

Consequently, if the contrast is increased to have extremely soft matrix with $\mu_m/\mu_p \leqslant O(\varepsilon^3)$, following the arguments of Section 2.3, the beam at the leading order works as in absence of matrix. The medium will behaves as an assembly of parallel beams and the constitutive law will be given by set (43) disregarding the matrix contribution $\underline{\underline{C}} : \underline{\underline{e}}_x(\underline{U}^0)$.

Conversely, if the contrast is decreased so that $\mu_m/\mu_p = O(1)$ i.e., a matrix as stiff as the beam material, the classical homogenization results apply and the model is a Cauchy elastic media. The elastic tensor (orthotropic because of the cell geometry) differs from $C$ since the stresses and strains in the matrix and in the beam are of the same order. The local deformation in the beam does not respect the usual beam kinematics. The non-local terms are masked at the leading order and would only appear in the correctors as studied in Boutin (1996).

It remains to investigate the intermediate contrast $\mu_m = \mu_p\varepsilon$.

### 5.2. ε-Soft matrix

A contrast $\mu_m = \mu_p O(\varepsilon)$ imposes to consider full expansions in power of $\varepsilon$ as (23), and the balance equations are (24) and (25) while the expansion of the stresses in the matrix is now (using the rescaled coefficients $\mu'_m = \mu_m/\varepsilon,\ \lambda'_m = \lambda_m/\varepsilon$):

$$^m\sigma_{ij} = \varepsilon(\lambda'_m e_{kk}\delta ij + 2\mu'_m e_{ij}) = {}^m\sigma^0_{ij} + \varepsilon{}^m\sigma^1_{ij} + \varepsilon^{2m}\sigma^2_{ij} + \cdots$$

#### 5.2.1. Leading order of motions and stresses in the beam and the matrix

The problems driving the leading order of motions are unchanged (although the stress in the matrix is one order higher). As for the leading order of stresses, one obtains the same sequence of problems and the solution only differs by the order of ${}^m\underline{\underline{\sigma}}$. Thus, with the same notations:

$$\begin{aligned}
&{}^m\underline{u}^0 = {}^p\underline{u}^0 = \underline{U}^0(\underline{x})\\
&\underline{u}^1 = \underline{U}^1(\underline{x}) + [\psi_\alpha(\underline{y}) e_{tx\alpha}(\underline{U}^0)]\underline{a}_1 + \underline{\Phi}^{\alpha\beta}(\underline{y})\underline{\underline{e}}_{x\alpha\beta}(\underline{U}^0) + \underline{\Upsilon}(\underline{y})U^0_{1,x_1}\\
&{}^p\underline{\underline{\sigma}}^{-1} = \underline{\underline{0}}\\
&{}^p\underline{\underline{\sigma}}^0 = E_p U^0_{1,x_1}\underline{a}_1 \otimes \underline{a}_1\\
&{}^m\underline{\underline{\sigma}}^0 = \underline{\underline{0}}\\
&{}^m\underline{\sigma}^1_t = 2\mu'_m[\underline{a}_\alpha + \underline{\mathrm{grad}}_y(\psi_\alpha)] e_{tx\alpha}(\underline{U}^0)\\
&{}^m\underline{\underline{\sigma}}^1_{s\alpha\beta} = [2\mu'_m(\underline{\underline{e}}_{\alpha\beta}(\underline{\Phi}^{\zeta\eta}) + \delta_{\alpha\zeta}\delta_{\beta\eta}) + \lambda'_m(\mathrm{div}_y(\underline{\Phi}^{\zeta\eta}) + \delta_{\zeta\eta})\delta_{\alpha\beta}]\underline{\underline{e}}_{x\zeta\eta}(\underline{U}^0)\\
&\qquad + [2\mu'_m\underline{\underline{e}}_{\alpha\beta}(\underline{\Upsilon}) + \lambda'_m(\mathrm{div}_y(\underline{\Upsilon}) + 1)\delta_{\alpha\beta}]U^0_{1,x_1}\\
&{}^m\sigma^1_n = [2\mu'_m + \lambda'_m \mathrm{div}_y(\underline{\Upsilon})]U^0_{1,x_1} + \lambda'_m[\mathrm{div}_y(\underline{\Phi}^{\alpha\beta})\underline{\underline{e}}_{x\alpha\beta}(\underline{U}^0) + \mathrm{div}_x(\underline{U}^0)]
\end{aligned} \tag{45}$$

#### 5.2.2. Global axial balance at the leading order and first order of stress in the beam

*In the axial direction* the fields are governed by (24–13) at orders $(\varepsilon^0 - \varepsilon^1)$, with ${}^p\underline{\sigma}^0_t = {}^m\underline{\sigma}^0_t = \underline{0}$. Here ${}^m\underline{\sigma}^1_t \neq \underline{0}$ and one obtains:

$$\begin{cases}
({}^p\sigma^0_n)_{,x_1} + \mathrm{div}_y({}^p\underline{\sigma}^1_t) = 0 \quad \text{in } S_p\\
\mathrm{div}_y({}^m\underline{\sigma}^1_t) = 0 \quad \text{in } S_m\\
{}^p\underline{\sigma}^1_t \cdot \underline{n} = {}^m\underline{\sigma}^1_t \cdot \underline{n} \quad \text{on } \Gamma
\end{cases}$$

The condition of compatibility of the source term $({}^p\sigma^0_n)_{,x_1}$ is established by integrating the balance equation on $S_p$ and $S_m$ and by using the continuity and periodicity condition on $\Gamma$ and $\partial S$. This leads to the same leading order axial balance equation:

$$\left[\int_{S_p} {}^p\sigma^0_n ds\right]_{,x_1} = |S'_p|\left[{}^p\sigma^0_n\right]_{,x_1} = 0 \quad \text{i.e.}: [E_p U^0_{1,x_1}]_{,x_1} = 0$$

Thus, the source term vanishes and the problem is reformulated in a more compact form:

$$\begin{cases}
\mathrm{div}_y({}^q\underline{\sigma}^1_t) = 0 \quad \text{in } S_q, \quad \mathrm{q = m,\ p}\\
{}^q\underline{\sigma}^1_t \cdot \underline{n} \quad \text{continuous on } \Gamma\\
{}^q\underline{\sigma}^1_t \cdot \underline{n} \quad y\text{-periodic on } \partial S
\end{cases}$$

The leading order description does not need the fields within the beam. Taking the scalar product of the balance equation by $\underline{y}$, integrating and making the usual transformations give:

$$\int_{S_p} {}^p\underline{\sigma}^1_t ds + \int_{S_m} {}^m\underline{\sigma}^1_t ds = \int_{\partial S} ({}^m\underline{\sigma}^1_t \cdot \underline{n})\underline{y} d\gamma$$

*In the in-plane directions* the problem defined by (25–16) at orders $(\varepsilon^0 - \varepsilon^1)$, simplified by the results ${}^p\underline{\sigma}^0_t = \underline{0}$, ${}^p\underline{\underline{\sigma}}^0_s = \underline{\underline{0}}$, and ${}^m\underline{\underline{\sigma}}^0_s = \underline{\underline{0}}$, takes the form:

$$\begin{cases}
\underline{\mathrm{div}}_y({}^q\underline{\underline{\sigma}}^1_s) = \underline{0} \quad \text{in } S_q, \quad \mathrm{q = m,\ p}\\
{}^q\underline{\underline{\sigma}}^1_s \cdot \underline{n} \qquad \text{continuous on } \Gamma\\
{}^q\underline{\underline{\sigma}}^1_s \cdot \underline{n} \qquad y - \text{periodic on } \partial S
\end{cases}$$

As above, the resolution in the beam is unnecessary. Taking the tensorial product of the balance equation by $\underline{y}$, integrating and making the usual transformations gives:

$$\int_{S_p} {}^p\underline{\underline{\sigma}}^1_s ds + \int_{S_m} {}^m\underline{\underline{\sigma}}^1_s ds = \int_{\partial S} \underline{y} \otimes ({}^m\underline{\underline{\sigma}}^1_s \cdot \underline{n}) d\gamma$$

i.e., the same expression of the mean stress (shifted of one order see (42)) as for a $\varepsilon^2$-contrast.

Finally, proceeding as for a $\varepsilon^2$-contrast, the mean normal stress in the beam is deduced from the linear problem governing ${}^p\underline{\underline{\sigma}}^1_s$:

$$\begin{cases}
\underline{\mathrm{div}}_y({}^p\underline{\underline{\sigma}}^1_s) = \underline{0} \quad \text{in } S_p\\
{}^p\underline{\underline{\sigma}}^1_s = 2\mu_p[\underline{\underline{e}}_{sy}({}^p\underline{u}^2) + \underline{\underline{e}}_{sx}({}^p\underline{u}^1)] + \lambda_p[\mathrm{div}_y({}^p\underline{u}^2) + \mathrm{div}_x({}^p\underline{u}^1)]\underline{\underline{I}}_s\\
{}^p\underline{\underline{\sigma}}^1_s \cdot \underline{n} = {}^m\underline{\underline{\sigma}}^1_s \cdot \underline{n} \quad \text{on } \Gamma
\end{cases}$$

The solution is decomposed in two parts associated with the forcing induced by ${}^p\underline{u}^1$ and ${}^m\underline{\underline{\sigma}}^1_s$. Both problems have been solved in Sections 4.3.2 and 4.4.2. Combining the contributions yields:

$$\frac{1}{|S'|}\int_{S_p} {}^p\sigma^1_n ds = \nu_p\mu_m(D'^{11}_{11}U^0_{1,x_1} + D'^{\alpha\beta}_{11} e_{x\alpha\beta}(\underline{U}^0)) + E_p\frac{|S'_p|}{|S'|}U^1_{1,x_1}$$

and, at the first order, the unscaled expression of mean normal stress in the section reads:

$$\begin{aligned}
\langle\tilde{\sigma}^1_n\rangle &= \frac{\varepsilon}{|S'|}\left[\int_{S_p} {}^p\sigma^1_n ds + \int_{S_m} {}^m\sigma^1_n ds\right]\\
&= C^{11}_{11}U^0_{1,x_1} + C^{\alpha\beta}_{11} e_{x\alpha\beta}(\underline{U}^0) + E_p\frac{|S_p|}{|S|}U^1_{1,x_1}
\end{aligned}$$

#### 5.2.3. Global balance and constitutive law at the leading order

In the axial and in-plane directions, the global balance results from (24)–(13) and (25)–(16) both at orders $(\varepsilon - \varepsilon^2)$. These two sets can be expressed in the compact form:

$$\begin{cases}
\underline{\mathrm{div}}_x({}^q\underline{\underline{\sigma}}^1) + \underline{\mathrm{div}}_y({}^q\underline{\underline{\sigma}}^2) = \underline{0} \quad \text{in } S_q, \quad \mathrm{q} = m, p\\
{}^q\underline{\underline{\sigma}}^2 \cdot \underline{n} \quad \text{continuous on } \Gamma\\
{}^q\underline{\underline{\sigma}}^2 \cdot \underline{n} \quad y\text{-periodic on } \partial S
\end{cases}$$

Integrating on $S_q$ leads – with the help of the divergence theorem and the stress continuity on $\Gamma$ – to the following macroscopic description (including body force $\underline{\tilde{b}}^1 = \varepsilon \underline{b}^1$):

$$\underline{\mathrm{div}}_x(\langle \underline{\underline{\sigma}}^0 \rangle) = \underline{0} \tag{46}$$

$$\langle \underline{\underline{\sigma}}^0 \rangle = \underline{a}_1 \otimes \underline{a}_1 E_p \frac{|S_p|}{|S|} U^0_{1,x_1} \tag{47}$$

$$\underline{\mathrm{div}}_x(\langle \underline{\underline{\tilde{\sigma}}}^1 \rangle) = \langle \underline{\tilde{b}}^1 \rangle \tag{48}$$

$$\langle \underline{\underline{\tilde{\sigma}}}^1 \rangle = \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{e}}_x(\underline{U}^0) + \underline{a}_1 \otimes \underline{a}_1 E_p \frac{|S_p|}{|S|} \widetilde{U}^1_{1,x_1}$$

For this intermediate contrast, the leading order behaviour is that of an elastic Cauchy media, where the elastic tensor is the same than for a $\varepsilon^2$-contrast. The non-local (bending, ...) effects are of one smaller order, then masked. However the local deformation in the beam still respects the Euler–Bernoulli beam kinematics.

## 6. Application to real media

This section deals with the practical applications of the theoretical results. This question is treated with the example of a periodic reinforced layer made of by-symmetric squared cells of area $S = \ell^2$, with fiber of section $S_p = h^2$, i.e., a surface concentration of fiber $c = S_p/S = (h/\ell)^2$. The medium is assumed of infinite lateral extension, of finite height $H > \ell$ along the fibers and submitted to transverse shear $\underline{U}^0(x_1) = U^0_\alpha(x_1)\underline{a}_\alpha$ for which the non-local effects are necessarily present in the case of $\varepsilon^2$-contrast (conveniently, the motion is supposed along $\underline{a}_2$ and $U^0_2$ is denoted $U$ to save notations). In addition to the geometric lengths $\ell$ and $H$, we are able to define independently two physical lengths, namely the macroscopic length related to the phenomena and the intrinsic length related to the microstructure.

### 6.1. Macroscopic length and relevant description

Real media are of finite geometrical dimensions and constituted by cells of finite size. This mismatch between the reality and the ideal conditions of homogenization stipulating that the scale ratio should tend to zero implies that:

- The homogenized descriptions are only approximations of the actual behaviour.
- An argument has to be proposed to identify the relevant description for a real media, i.e., the appropriate scaling of a finite stiffness contrast (indeed if $\varepsilon \to 0$, the generalized media would only exist for infinitely soft matrix!).

The answer to this question lies in the assessment of the macroscopic length $L$. This latter is evaluated by a dimensional analysis at the macroscopic scale:

$$L = O\left(\frac{|U|}{|U_{,x_1}|}\right) \tag{49}$$

This estimate is consistent with the asymptotic expansion since the increment of the macroscopic variable on one cell, $\ell\partial_x U$, has to be of order $\varepsilon$ compared to its current value, $U$. This implies the equality: $\ell\partial_x U = O(\varepsilon U) = O(\ell U/L)$, leading to (49) (Boutin and Auriault, 1990).

In a given material, the assessment of the physical macro-length $L$ enables to quantify the *actual finite* scale ratio $\tilde{\varepsilon} = \ell/L$ for the considered phenomena. Then, the known finite stiffness contrast of the real media can be equalized to the physical scale ratio $\tilde{\varepsilon}$ at a particular power. This latter power – replaced by a close integer – supplies unambiguously the physical *scaling consistent with the real problem* in consideration.

Performing homogenization with this particular scaling consists in replacing the actual finite value $\tilde{\varepsilon}$ by a mathematical $\varepsilon$ that one makes tend to zero. Doing so, by construction, the relative orders of magnitude of the physical terms are kept identical whatever the cell size is, and consequently for both the real cell (finite $\ell$) and the continuous model (infinitesimal $\ell$) obtained at the limit. Finally, the real structure can be considered as an imperfect realization (for the small mathematical value $\varepsilon = \tilde{\varepsilon}$) of the homogenized continuous model built with the proper scaling. The smaller $\tilde{\varepsilon}$ is, the better would be the continuous approximation.

Thus, in real cases it is possible to define physically the appropriate continuous macro-description, provided that the macro-length $L$ is reliably estimated.

### 6.2. Intrinsic length versus stiffness contrast

The generalized media description (20) and (21) includes the other descriptions as degenerated cases. The leading order governing equation, valid in the absence of body forces or for perturbations from equilibrium in the presence of body forces, reads:

$$C^{12}_{12}\frac{1}{2}U_{,x_1x_1} - \frac{E_p I_p}{|S|}U_{,x_1x_1x_1x_1} = 0 \tag{50}$$

Rewritten in a dimensionless form and accounting for the neglected term $O(\varepsilon)$ (relatively to the leading terms) one has:

$$\frac{|U|}{L^2} - \mathcal{L}^2\frac{|U|}{L^4} = O(\varepsilon); \quad \mathcal{L} = \sqrt{\frac{2E_p I_p}{C^{12}_{12}|S|}} \tag{51}$$

where $\mathcal{L}$ is the *intrinsic* length of the generalized media. Assuming a small amount of reinforcements $C^{12}_{12} \approx 2\mu_m$ and $I_p/|S| = h^4/(12\ell^2) = c^2\ell^2/12$. Consequently:

$$\mathcal{L} \approx \ell c\sqrt{\frac{E_p}{12\mu_m}}, \quad \text{thus } \mathcal{L} = O(\ell) \quad \text{when } E_p = O(\mu_m);$$

$$\mathcal{L} \gg O(\ell) \quad \text{when } E_p \gg O(\mu_m)$$

The formulation (51) enables to relate the nature of the behaviour to $(\mathcal{L}/L)^2$: if $(\mathcal{L}/L)^2 = O(\varepsilon)$ then the bending term is negligible at the leading order and the effective behaviour is governed by shear, if $(L/\mathcal{L})^2 = O(\varepsilon)$ the shear is negligible and the behaviour is governed by bending, if $(\mathcal{L}/L)^2 = O(1)$ both terms are of the same order and the behaviour is that of a generalized media. This classification is consistent with that based on the stiffness contrast since:

$$\left(\frac{\mathcal{L}}{L}\right)^2 \approx \frac{\ell^2c^2}{L^2}\frac{E_p}{12\mu_m} = O(\varepsilon^2)c^2\frac{E_p}{\mu_m}$$

### 6.3. Mapping of the relevant macroscopic modeling

It is of interest to identify the relevant modelling from the known geometrical lengths $\ell$ and $H$ and the intrinsic length $\mathcal{L}$. This is performed by mapping the domain of validity of the different behaviours according to the two dimensionless parameters $h^* = \mathrm{Log}(H/\ell)$ and $k^* = \mathrm{Log}(\mathcal{L}/\ell)$. Following the assumptions of this study, $H > \ell$ then $h^* > 0$ and $E_p \geqslant \mu_m$ then $k^* \geqslant 0$.

The general solution of (50) reads:

$$U = ax_1 + b + d_+ \exp(x_1/\mathcal{L}) + d_- \exp(-x_1/\mathcal{L})$$

then, without specifying the boundary conditions (that determine the coefficients $a$, $b$, $d_+$, $d_-$ see next section), three main situations arise:

- if $H \leqslant \sqrt{\tilde{\varepsilon}}\mathcal{L}$, the terms $\exp(\pm x_1/\mathcal{L})$ are different of zero but negligible, then

$$L = O\left(\frac{|U|}{|U_{,x_1}|}\right) \approx \frac{|U(H) - U(0)|}{|U_{,x_1}|} \approx H \quad \text{and} \quad \tilde{\varepsilon} = \ell/H$$

Consequently, this case is reached when:

$$H \leqslant \sqrt{\frac{\ell}{H}}\mathcal{L} \quad \text{i.e. when } 3h^* \leqslant 2k^*$$

Now, as $H \leqslant \sqrt{\tilde{\varepsilon}}\mathcal{L}$ is equivalent to $\ell/\tilde{\varepsilon} \leqslant \sqrt{\tilde{\varepsilon}}\mathcal{L}$, one derives the following estimate of the stiffness contrast:

$$\left(\frac{\ell}{\mathcal{L}}\right)^2 = O(\mu_m/E_p) \leqslant \tilde{\varepsilon}^3$$

This scaling shows that in the domain $3h^* \leqslant 2k^*$ the macroscopic behaviour is governed by bending at the leading order.

- if $H = O(\mathcal{L})$, these lengths are the macroscopic size of the three terms of $U(ax_1$ and $\exp(\pm x_1/\mathcal{L}))$ so that:

  $$L = O(H) = O(\mathcal{L}) \quad \text{and} \quad \tilde{\varepsilon} = \ell/H = \ell/\mathcal{L} \quad \text{when } h^* = O(k^*)$$

  Since $H = O(\mathcal{L})$ is equivalent to $\ell/\tilde{\varepsilon} = O(\mathcal{L})$ the stiffness contrast reads:

  $$\left(\frac{\ell}{\mathcal{L}}\right)^2 = O(\mu_m/E_p) = O(\tilde{\varepsilon}^2)$$

  Consequently, when $h^* = O(k^*)$ the material behaves as a generalized medium.
- if $\mathcal{L} \leqslant \sqrt{\tilde{\varepsilon}}H$, the terms $d_+ \exp(x_1/\mathcal{L})$ and $d_- \exp(-x_1/\mathcal{L})$ introduce boundary layers of thickness $\mathcal{L}$ at both extremities (decreasing as $\exp(-x_1/\mathcal{L})$ close to $x_1 = 0$ and as $\exp([x_1 - H]/\mathcal{L})$ close to $x_1 = H$). Outside of these boundary layers, the terms $\exp(\pm x_1/\mathcal{L})$ are negligible. Consequently, two regions behave differently in the medium.
  - Within the boundary layers $L = O(\mathcal{L})$, $\tilde{\varepsilon} = \ell/\mathcal{L}$ and the material respond as a generalized medium. Thus, this situation occurs when $\mathcal{L} \leqslant \sqrt{\ell/\mathcal{L}}H$, i.e., $3k^* \leqslant 2h^*$.
  - Between the two boundary layers $L = O(H)$, $\tilde{\varepsilon} = \ell/H$. Thus, in the inner region, the inequality $\mathcal{L} \leqslant \sqrt{\tilde{\varepsilon}}H$ is equivalent to $\mathcal{L} \leqslant \sqrt{\tilde{\varepsilon}}\ell/\tilde{\varepsilon}$, that is:

  $$\left(\frac{\ell}{\mathcal{L}}\right)^2 = O(\mu_m/E_p) \geqslant \tilde{\varepsilon}$$

  meaning that the material responds as a classical Cauchy medium. As mentioned above, $\mu_m/E_p = O(1)$, leads to $\mathcal{L} = O(\ell)$. In that case, $\mathcal{L}$ is a microscopic size and the mechanisms within this thin layer are not correctly described by the generalized media.

The Fig. 4 presents the different situations in the plane $(h^*; k^*)$. The effective domain of validity of the generalized media lies in between the two lines $3h^* \leqslant 2k^*$ and $3k^* \leqslant 2h^*$. For a given matrix and a reinforcement modulus, $k^*$ = *cste* corresponds to an identical amount of fiber concentration $c$. Consequently, the reinforced materials on a vertical line of the plane $(h^*; k^*)$ have identical *axial* modulus, but different transverse behaviours, varying from dominating bending to generalized media and Cauchy media when $\ell/H$ decreases. In other words, the apparent transverse deformability varies with the cell size.

As an example, consider a pile foundation massif constituted by a weak modulus soil ($\mu_m = 4 \times 10^6$ Pa) reinforced by cylindrical concrete piles ($E_p = 4 \times 10^{10}$ Pa) of 1.2 m of diameter ($I_p \approx 1 \times 10^{-1}$ m$^4$), regularly spaced of 4 m ($S = 16$ m$^2$) in two orthogonal directions. The intrinsic length is $\mathcal{L} \approx 8$ m. If the thickness of the layer is $H = 10$ m, the reinforced soil behaves as a generalized medium (with $\tilde{\varepsilon} \approx 0.5$), if $H$ is about 30 m the reinforced soil behaves as an elastic Cauchy medium (with $\tilde{\varepsilon} \approx 0.1$) except on both extremities over a length of about 8 m. The same vertical modulus would be obtained for piles of 0.30 m of diameter ($I_p \approx 4 \times 10^{-4}$ m$^4$), regularly spaced of 1 m ($S = 1$ m$^2$). However the intrinsic length will be $\mathcal{L}' \approx 2$ m and for $H = 10$ m the reinforced soil will behave as an elastic Cauchy medium (with $\tilde{\varepsilon} \approx 0.1$).

### *6.4. Influence of the boundary conditions*

As seen in Section 3.3.3 the boundary conditions cannot be formulated in term of the macroscopic motions and stress tensor only, but require also the conditions imposed to the fibers. They can be clamped into a rigid basement so that the section rotation $U'$ vanishes, or they can be free of momentum, then curvature $U''$ vanishes. As an example, when the layer is fixed at $x_1 = 0$ and submitted to an imposed transverse displacement on $x_1 = H$, one obtains by integrating (50):

- fibers clamped at both extremities:

$$U(x_1) = a\left\{x_1 - \frac{\mathcal{L}}{\sinh(H/\mathcal{L})}[\cosh(x_1/\mathcal{L}) - \cosh((x_1 - H)/\mathcal{L}) + \cosh(H/\mathcal{L}) - 1]\right\}$$

- fibers clamped on $x_1 = H$, free of momentum on $x_1 = 0$:

$$U(x_1) = a\left\{x_1 - \frac{\mathcal{L}}{\cosh(H/\mathcal{L})}\sinh(x_1/\mathcal{L})\right\}$$

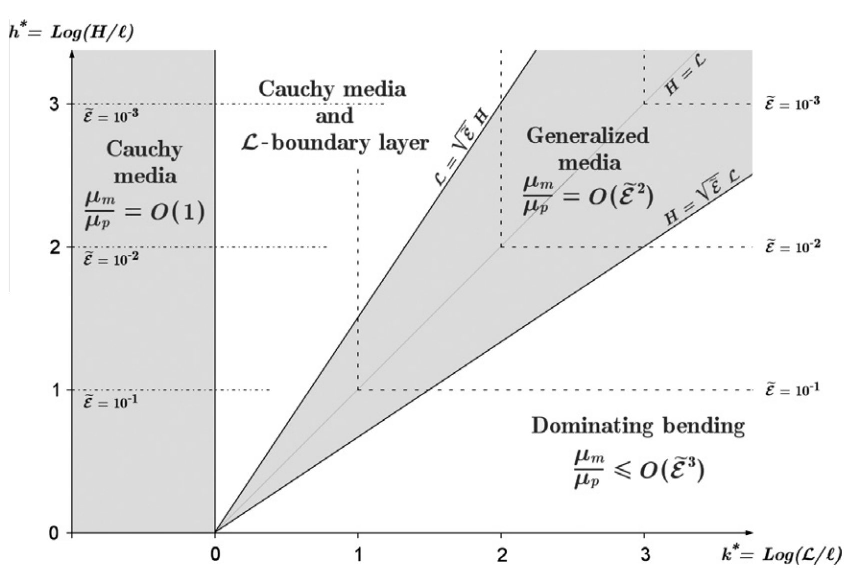


**Fig. 4.** Mapping of the macroscopic behaviours according to the two parameters $h^* = \mathrm{Log}(H/\ell)$ and $k^* = \mathrm{Log}(\mathcal{L}/\ell)$.

- fibers free of momentum at both extremities:

$$U(x_1) = ax_1$$

In the first two cases, the previous analysis applies. Conversely, in the last case, the reinforced material behaves as a classical medium even if $H = \mathcal{L}$, because the particular boundary conditions avoid the inner bending.

### 6.5. Remark on inner torsion

Note finally that the geometry of fiber reinforced materials offer the possibility of unusual loading of the material. For instance, in case of large fibers (like pile foundations) a torsion $\Omega(\underline{x})$ could be applied to each pile without acting on the matrix. Conversely to the above analysis (where the inner kinematic results from $\underline{U}^0$), such a situation would introduce an additional degree of freedom implying the macroscopic field of inner torsion, *independent and complementary* to $\underline{U}^0$. Although, in other composites, similar loading would involve a layer of the size of the elementary representative volume, the fiber facilitates the penetration of the imposed torsion. The study of this mechanism is not detailed here.

## 7. Conclusion

Through asymptotic homogenization, we have derived the effective behaviour of elastic materials periodically reinforced by fiber for different orders of magnitude of the contrast between the shear modulus of the matrix and of the fiber.

A contrast $\mu_m/\mu_p = O(\varepsilon^2)$ leads to a full coupling between the beam behaviour of the fibers and the elastic behaviour of the matrix. Under macroscopic transverse motions, the medium behaves *at the leading order* as a generalized continuum that accounts for the inner bending within the fibers and the shear of the matrix. Non-local terms also appear on axial stresses along fibers, under inhomogeneous axial and lateral confining, and, in the case of non-symmetric cell, because of wrapping under torsion and Poisson effect. The description degenerates into the classical behaviour of elastic composites for stiffer matrix and the usual Euler–Bernoulli beam behaviour for softer matrix.

The constitutive parameters can be computed rigorously, or also be simply estimated from the self consistent approach (Christensen and Lo, 1979; Hashin, 1983), with an excellent approximation for weak concentrations of fibers. For instance, considering bi-symmetric cells (e.g. circle in a square):

$$C^{23}_{23} \approx 2\mu_m\left[1 + c\left(1 + \frac{1}{3-4\nu_m}\right)\left(1 - \frac{\mu_m}{\mu_p}\left(1 + \frac{1}{3-4\nu_m}\right)\right)\right];$$

$$C^{12}_{12} = C^{13}_{13} \approx 2\mu_m\left[1 + \frac{2c}{1-c}\left(1 - \frac{\mu_m}{\mu_p}\frac{2c}{1-c}\right)\right]$$

Simple criteria based on the comparison between the geometric data – size of the cell $\ell$, overall dimension of the media $H$ – and the physical lengths – intrinsic length $\mathcal{L} \approx \sqrt{E_p I_p/(|S|\mu_m)}$ and macroscopic length $L$ – enable to identify the appropriate model for real reinforced media. Provided that the intrinsic length of the material is larger than the size of the cell, the analysis shows that in general the actual response is influenced by the non-local effect at least on a boundary layer of the size of the intrinsic length.

The unidirectional morphology treated in this paper is not an absolute requirement. 3-D cells could also be considered for fibers periodically heterogeneous along their axis, or for fibers oriented in the three orthogonal directions. However, in this latter case, to keep the inner bending effect at the macroscale (in the three orthogonal directions of fibers) the fibers must necessarily be fully embedded in the matrix without fiber–fiber connexion. Interconnecions between orthogonal fibers would drastically increase the interaction forces at the contact, preventing the bending mechanism to occur as described in Section 2.3. This situation would lead to classical composite description (cf. also Bellieud and Bouchitté (2002)).

In quasi statics, using complex modulus in the Fourier domain, the results can be extended to viscoelastic constituents or to elastic fibers embedded in a viscous matrix. Extension to weakly compressible matrix, dynamic loadings and wave propagation may also be considered.

## Appendix A. Derivation of beam behaviour through asymptotic expansions

The beam behaviour attained at large slender ratio, i.e., small $\varepsilon$, is determined by seeking for the variables in the form of expansions in power of $\varepsilon$ (Trabucho and Viano, 1996). The kinematic condition (1) imposes to formulate the expansions as:

$$\underline{u} = \sum_{i=0}^{\infty}\varepsilon^i\underline{u}^i = \sum_{i=1}^{\infty}\varepsilon^i u^i_1\underline{a}_1 + \sum_{i=0}^{\infty}\varepsilon^i u^i_\alpha\underline{a}_\alpha \quad \text{i.e.,}$$

$$u_1 = \sum_{i=0}^{\infty}\varepsilon^{i+1}u^{i+1}_1; \quad \underline{u}_\alpha = \sum_{i=0}^{\infty}\varepsilon^i\underline{u}^i_\alpha$$

### A.1. Appropriate asymptotic expansions

Inserting the reduced strain and stress tensors (2) and (3) into the balance and boundary Eqs. (5)–(8) yield to scaled problems expressed in function of $u_1$ and $u_\alpha$. The axial balance contains terms in $\varepsilon^{-1}$ and $\varepsilon$ (in $\varepsilon^0$ as for the boundary condition), and the in-plane balance contains terms in $\varepsilon^{-2}$ and $\varepsilon^0$ (in $\varepsilon^{-1}$ and $\varepsilon$ as for the boundary condition). Since the terms of the equations "jump" from a factor $\varepsilon^2$, it is sufficient to expand the components $u_i$ according to the even powers of $\varepsilon$. Thus, the appropriate expansion reads:

$$\underline{u} = \sum_{i=0}^{\infty}\varepsilon^{2i}[u^{2i}_\alpha\underline{a}_\alpha + \varepsilon(u^{2i+1}_1\underline{a}_1)] \quad \text{i.e.,}$$

$$u_1 = \sum_{i=0}^{\infty}\varepsilon^{2i+1}u^{2i+1}_1; \quad u_\alpha = \sum_{i=0}^{\infty}\varepsilon^{2i}u^{2i}_\alpha \tag{53}$$

Consequently, the axial ($_\mathrm{n}$) and in-plane ($_\mathrm{s}$) – respectively out of plane ($_\mathrm{t}$) – reduced strain and stress tensors (2) and (3) are expanded in odd – respectively even – powers of $\varepsilon$:

$$\underline{\underline{e}}_\mathrm{s} = \varepsilon\sum_{i=-1}\varepsilon^{2i}\underline{\underline{e}}^{2i+1}_\mathrm{s}; \quad \underline{e}_\mathrm{t} = \sum_{i=0}\varepsilon^{2i}\underline{e}^{2i}_\mathrm{t}; \quad e_\mathrm{n} = \varepsilon\sum_{i=0}\varepsilon^{2i}e^{2i+1}_\mathrm{n} \tag{54}$$

$$\underline{\underline{\sigma}}_\mathrm{s} = \varepsilon\sum_{i=-1}\varepsilon^{2i}\underline{\underline{\sigma}}^{2i+1}_\mathrm{s}; \quad \underline{\sigma}_\mathrm{t} = \sum_{i=0}\varepsilon^{2i}\underline{\sigma}^{2i}_\mathrm{t}; \quad \sigma_\mathrm{n} = \varepsilon\sum_{i=-1}\varepsilon^{2i}\sigma^{2i+1}_\mathrm{n} \tag{55}$$

Expansions (53)–(55) indicate that the reference value (i.e., of zero order) of the displacements, strains and stress are respectively, $u^0_\alpha,\ u^0_{\alpha,x_1},\ \mu u^0_{\alpha,x_1}$.

### A.2. Asymptotic solution

The solution is derived by introducing expansions (55) in (5)–(8). Separating the terms of different orders leads to a series of problems. The first non-trivial equations expressed with $x_1$ provides the leading order of the beam description. It results from the even or odd expansions of the reduced tensors (54) and (55), that the axial and the in-plane problems (5)–(8) are separated by one order. Thus, the resolution is achieved by treating alternatively the equilibrium of the section, then the axial equilibrium, and so on.

**The first problem** (Eqs. (7) and (8), order $\varepsilon^{-2}-\varepsilon^{-1}$) deals with $\underline{\underline{\sigma}}_s^{-1}$ and $\underline{u}^0 = u^0_\alpha \underline{a}_\alpha$.

$$\begin{cases} \underline{\mathrm{div}}_y(\underline{\underline{\sigma}}_s^{-1}) = \underline{0} \quad \text{in } S_p \quad \text{with} \quad \underline{\underline{\sigma}}_s^{-1} = 2\mu \underline{\underline{e}}_{sy}(\underline{u}^0) + \lambda \mathrm{div}_y(\underline{u}^0)\underline{\underline{I}}_s \\ \underline{\underline{\sigma}}_s^{-1} \cdot \underline{n} = \underline{0} \quad \text{on } \Gamma \end{cases}$$

The equivalent variational formulation is:

$$\forall \underline{w}_s \quad \text{vector } C^1 \text{defined on } S_p, \quad \int_{S_p} \underline{\underline{\sigma}}_s^{-1} : \underline{\underline{e}}_{sy}(\underline{w}_s) ds = 0$$

Taking for the virtual field $\underline{w}_s = \underline{u}^0$it turns out that:

$$\int_{S_p} [\lambda(\mathrm{div}_y(\underline{u}^0))^2 + 2\mu \underline{\underline{e}}_{sy}(\underline{u}^0) : \underline{\underline{e}}_{sy}(\underline{u}^0)] ds = 0$$

The positiveness of the Lame constants implies that $\underline{\underline{e}}_{sy}(\underline{u}^0) = \underline{\underline{0}}$ (and $\mathrm{div}_y(\underline{u}^0) = 0$). Therefore $\underline{u}^0$ is a rigid motion of the section in its plane, i.e., a translation $\underline{U}^0$ and a rotation $\Omega^{-1}\underline{a}_1$:

$$\underline{u}^0 = u^0_\alpha \underline{a}_\alpha; \quad u^0_\alpha = U^0_\alpha + \Omega^{-1}[\underline{a}_1 \times \underline{y}]_\alpha$$

Moreover since $\underline{\underline{e}}_{sy}(\underline{u}^0) = \underline{\underline{0}}$ then $\underline{\underline{\sigma}}_s^{-1} = \underline{\underline{0}}$ and $\sigma_n^{-1} = \lambda \mathrm{div}_y(\underline{u}^0) = 0$. Therefore:

$$\underline{\underline{e}}^{-1} = \underline{\underline{0}}; \quad \underline{\underline{\sigma}}^{-1} = \underline{\underline{0}}$$

The translation $\underline{U}^0$ and the rotation $\Omega^{-1}$ (of order $-1$ to respect the scaling of the zero order motion $\Omega^{-1}h = O(1)$) are two independent kinematics. They appear at the same order because the assumption of zero order transverse motion does not distinguish translation and rotation. Nevertheless, physically, their relative order of magnitude may differ. Without restricting the generality of the further developments, we will consider that the rotation is of lesser order than the translation i.e., $\Omega^{-1} = 0$, and leave the treatment of the section rotation for higher orders.

**The second problem** (Eqs. (5) and (6), order $\varepsilon^{-1}-\varepsilon^{0}$) deals with $u_1^1$ and $\underline{\sigma}_t^0$. As $\sigma_n^{-1} = 0$, we have:

$$\begin{cases} \mathrm{div}_y(\underline{\sigma}_t^0) = 0 \quad \text{in } S_p \quad \text{with} \quad \underline{\sigma}_t^0 = \mu(u^1_{1,y_\alpha} + U^0_{\alpha,x_1})\underline{a}_\alpha \\ \underline{\sigma}_t^0 \cdot \underline{n} = 0 \quad \text{on } \Gamma \end{cases}$$

This problem admits the following equivalent variational formulation:

$$\forall w_1 \text{scalar } C^1 \text{defined on} S_p, \quad \int_{S_p} \underline{\sigma}_t^0 \cdot \underline{\mathrm{grad}}_y(w_1) ds = 0$$

Choosing $w_1 = u_1^1 + \underline{y} \cdot \underline{U}^0_{,x_1}$ yields:

$$\int_{S_p} (\mu \|\underline{\mathrm{grad}}_y(u_1^1) + \underline{U}^0_{,x_1}\|^2) ds = 0, \quad \text{then,} \quad \underline{\mathrm{grad}}_y(u_1^1) + \underline{U}^0_{,x_1} = \underline{0}$$

and by integration the solution is:

$$\underline{u}^1 = u_1^1 \underline{a}_1, \quad u_1^1 = -\underline{y} \cdot \underline{U}^0_{,x_1} + U_1^1(x_1)$$
$$\underline{e}_t^0 = \underline{0}, \quad \underline{\sigma}_t^0 = \underline{0}, \quad \underline{\underline{e}}^0 = \underline{\underline{0}}, \quad \underline{\underline{\sigma}}^0 = \underline{\underline{0}}$$

At the leading order, the out of plane motion of the section consists into (i) a rigid out of plane rotation (of vector $\underline{U}^0_{,x_1} \times \underline{a}_1$) and (ii) a uniform vertical translation $U_1^1 \underline{a}_1$; i.e., the usual kinematics of the Euler–Bernoulli beam. Although their relative magnitude may physically differ, we will treat them conjointly, considering them to be of identical order for convenience.

**The third problem** (Eqs. (7) and (8), order $\varepsilon^0-\varepsilon$) concerns $\underline{\underline{\sigma}}_s^1$ and $\underline{u}^2$. As $\underline{\sigma}_t^0 = \underline{0}$, it takes the form:

$$\begin{cases} \underline{\mathrm{div}}_y(\underline{\underline{\sigma}}_s^1) = \underline{0} \quad \text{in } S_p \\ \underline{\underline{\sigma}}_s^1 = 2\mu \underline{\underline{e}}_{sy}(\underline{u}^2) + \lambda[\mathrm{div}_y(\underline{u}^2) + u^1_{1,x_1}]\underline{\underline{I}}_s \\ \underline{\underline{\sigma}}_s^1 \cdot \underline{n} = \underline{0} \quad \text{on } \Gamma \end{cases}$$

Noticing that the plane strains of the particular fields $\underline{y}$ and $\underline{\xi}^\alpha = y_\alpha \underline{y} - \frac{1}{2}\|y\|^2 \underline{a}_\alpha$ are:

$$\underline{\underline{e}}_{sy}(\underline{y}) = \underline{\underline{I}}_s \quad \text{and} \quad \underline{\underline{e}}_{sy}(\underline{\xi}^\alpha) = y_\alpha \underline{\underline{I}}_s$$

the plane isotropic stress induced by $u^1_{1,x_1} = -\underline{y} \cdot \underline{U}^0_{,x_1x_1} + U^1_{1,x_1}$ can be re-expressed as the plane stress resulting from the particular in-plane motion $\underline{v}^2$:

$$\lambda u^1_{1,x_1} \underline{\underline{I}}_s = \lambda(U^1_{1,x_1} - \underline{y} \cdot \underline{U}^0_{1,x_1x_1})\underline{\underline{I}}_s$$
$$= 2\mu \underline{\underline{e}}_{sy}(\underline{v}^2) + \lambda \mathrm{div}_y(\underline{v}^2)\underline{\underline{I}}_s = (\mu + \lambda)\mathrm{div}_y(\underline{v}^2)\underline{\underline{I}}_s$$
$$\underline{v}^2 = \nu(\underline{y} U^1_{1,x_1} - \underline{\xi}^\alpha U^0_{\alpha,x_1x_1})$$

Consequently, $\underline{\underline{\sigma}}_s^1$ may be rewritten as:

$$\underline{\underline{\sigma}}_s^1 = 2\mu \underline{\underline{e}}_{sy}(\underline{u}^2 + \underline{v}^2) + \lambda \mathrm{div}_y(\underline{u}^2 + \underline{v}^2)\underline{\underline{I}}_s$$

Setting $\underline{w}_s = \underline{u}^2 + \underline{v}^2$ in the variational formulation associated to $\underline{\underline{\sigma}}_s^1$ (cf. Section 2.2.2):

$$\forall \underline{w}_s \text{vector } C^1 \text{defined on } S_p, \quad \int_S \underline{\underline{\sigma}}_s^1 : \underline{\underline{e}}_{sy}(\underline{w}_s) ds = 0$$

provides:

$$\int_{S_p} \lambda[\mathrm{div}_y(\underline{u}^2 + \underline{v}^2)]^2 + 2\mu \underline{\underline{e}}_{sy}(\underline{u}^2 + \underline{v}^2) : \underline{\underline{e}}_{sy}(\underline{u}^2 + \underline{v}^2) ds = 0$$

Consequently, $\underline{\underline{e}}_{sy}(\underline{u}^2 + \underline{v}^2) = \underline{\underline{0}}$, then $\underline{u}^2 + \underline{v}^2$ is a rigid in-plane motion of the section, and:

$$\underline{u}^2 = -\nu(\underline{y} U^1_{1,x_1} - \underline{\xi}^\alpha U^0_{\alpha}, x_1x_1) + \underline{U}^2(x_1) + \Omega^1(x_1)\underline{a}_1 \times \underline{y} \tag{56}$$

Moreover, by construction:

$$\underline{\underline{e}}_s^1 = \underline{\underline{e}}_{sy}(\underline{u}^2) = -\underline{\underline{e}}_{sy}(\underline{v}^2) = -\nu \underline{\underline{I}}_s u^1_{1,x_1}, \quad e_n^1 = u^1_{1,x_1};$$
$$\text{thus} \quad \underline{\underline{e}}^1 = [\underline{a}_1 \otimes \underline{a}_1 - \nu \underline{\underline{I}}_s]u^1_{1,x_1} \tag{57}$$

Now, since $\underline{\underline{e}}_{sy}(\underline{u}^2 + \underline{v}^2) = \underline{\underline{0}}$, then $\underline{\underline{\sigma}}_s^1 = \underline{\underline{0}}$. Hence $\sigma_n^1$ is deduced from the constitutive elastic law:

$$\sigma_n^1 = Eu^1_{1,x_1} = E(-\underline{y} \cdot \underline{U}^0_{,x_1x_1} + U^1_{1,x_1}); \quad \underline{\underline{\sigma}}^1 = \sigma_n^1 \underline{a}_1 \otimes \underline{a}_1 \tag{58}$$

**The fourth problem** (Eqs. (5) and (6), order $\varepsilon-\varepsilon^2$) concerns the axial balance of $\underline{\sigma}_t^2$:

$$\begin{cases} \sigma^1_{n,x_1} + \mathrm{div}_y(\underline{\sigma}_t^2) = 0 \quad \text{in } S_p \quad \text{with} \quad \underline{\sigma}_t^2 = \mu[u^3_{1,y_\alpha} + u^2_{\alpha,x_1}]a_\alpha \\ \underline{\sigma}_t^2 \cdot \underline{n} = 0 \quad \text{on } \Gamma \end{cases}$$

Here $\sigma^1_{n,x_1}$ acts as a source term. Following Section 2.2.3, three non-trivial balance equations of the section are established and complemented by the relations between (i) normal force and longitudinal strain, and (ii) transverse momentum and curvature. The uncoupling of compression and bending mechanisms is obvious when the beam behavioural laws are expressed in the "natural" $y$-frame of the section. Reminding that, as the scaling makes the $y$-derivative of quantity of order $i$ to be of order $i-1$, in a similar way, the $y$-integral over the section of a quantity of order $i$ is of order $i+2$- and of order $i+\ (2+j)$ if multiplied by $y^j_\alpha$ – one obtains:

$$N^3_{,x_1} = 0; \quad N^3 = \int_{S_p} \sigma_n^1 ds = E|S'_p|U^1_{1,x_1} \tag{59}$$

$$\underline{M}^4_{,x_1} - \underline{T}^4 = \underline{0}; \quad M^4_\alpha = \int_{S_p} \sigma_n^1 y_\alpha ds = -EI'_\alpha U^0_{\alpha,x_1x_1} \tag{60}$$

The derivation of $u_1^3$, not necessary for the leading order behaviour, is reported in Section A.3.

**The fifth problem** (Eqs. (7) and (8), order $\varepsilon^2-\varepsilon^3$) expresses the balance of $\underline{\underline{\sigma}}_s^3$ under the forcing term $\underline{\sigma}^2_{t,x_1}$.

$$\begin{cases} \underline{\sigma}^2_{t,x_1} + \underline{\mathrm{div}}_y(\underline{\underline{\sigma}}_s^3) = \underline{0} \quad \text{in } S_p \\ \underline{\underline{\sigma}}_s^3 \cdot \underline{n} = \underline{0} \quad \text{on } \Gamma \end{cases}$$

As above, according to Section 2.2.3 two non-trivial balance equations are deduced:

$$\underline{T}^4_{,x_1} = \underline{0}; \quad T^4_\alpha = \int_{S_p} \underline{\sigma}^2_t \cdot \underline{a}_\alpha ds = -EI'_\alpha U^0_{\alpha,x_1x_1x_1}$$

$$M^5_{1,x_1} = 0; \quad M^5_1 = \int_{S_p} [\underline{y} \times \underline{\sigma}^2_t] \cdot \underline{a}_1 ds = \mu I'_t \Omega^1_{,x_1}$$

The torsion law relating $M^5_1$ to $\Omega^1_{,x_1}$ is valid for sections having two orthogonal axis of symmetry. Non bi-symmetric sections introduce a torsion-bending coupling as shown in the next section.

*A.3. Wrapping and torsion*

We determine here $u^3_1$ and the expression of $M^5_1$. The global axial balance of the section (59) implies $U^1_{1,x_1x_1} = 0$ which simplifies the expressions (58) of $\sigma^1_n$, (56) of $\underline{u}^2$, and (57) of $\mathrm{div}_y(\underline{u}^2)$. Thus, the fourth problem (Eqs. (5) and (6), order $\varepsilon - \varepsilon^2$), after simplifying by $\mu$, is rewritten as:

$$\begin{cases} 2\underline{y} \cdot \underline{U}^0_{,x_1x_1x_1} + \mathrm{div}_y(\underline{\mathrm{grad}}_y(u^3_1)) = 0 & \text{in } S_p \\ (u^3_{1,y_\alpha} + u^2_{\alpha,x_1}) \cdot n_\alpha = 0 & \text{on } \Gamma \end{cases}$$

with $\quad u^2_{\alpha,x_1} = \nu \xi^\beta_\alpha U^0_{\beta,x_1x_1x_1} + U^2_{\alpha,x_1} + \Omega^1_{,x_1}(\underline{a}_1 \times \underline{y})_\alpha$

The solution $u^3_1$ of this linear problem is the sum of the contributions of each forcing term.

The problem related to $U^2_{\alpha,x_1}$ is identical to that treated for $U^1_1$ and the solution is $-\underline{U}^2_{,x_1} \cdot \underline{y}$.

The problem related to $\Omega^1_{,x_1}$ is new. The solution reads $w(\underline{y})\Omega^1(x_1)_{,x_1}$ where the wrapping function $w(\underline{y})$ is solution of (the zero mean value condition provides the unicity):

$$\begin{cases} \mathrm{div}_y(\underline{\mathrm{grad}}_y(w) + \underline{a}_1 \times \underline{y}) = \Delta_y(w) = 0 & \text{in } S_p \\ (\underline{\mathrm{grad}}_y(w) + \underline{a}_1 \times \underline{y}) \cdot \underline{n} = 0 & \text{on } \Gamma \\ \int_{S_p} w ds = 0 \text{ (unicity)} \end{cases}$$

Note that for circular section $w = 0$ since $(\underline{a}_1 \times \underline{y}) \cdot \underline{n} = 0$ on $\Gamma$. Besides for any section, as the wrapping stresses vector $\underline{\zeta} = \mu(\underline{\mathrm{grad}}_y(w) + \underline{a}_1 \times \underline{y})$ is of free divergence with free boundary condition, one has for any $C^1$ function $f$:

$$0 = \int_{S_p} f \mathrm{div}_y(\underline{\zeta}) ds = -\int_{S_p} \underline{\mathrm{grad}}_y(f) \cdot \underline{\zeta} ds + \int_\Gamma f \underline{\zeta} \cdot \underline{n} ds$$
$$= -\int_{S_p} \underline{\mathrm{grad}}_y(f) \cdot \underline{\zeta} ds$$

Taking $f = y_\alpha$, and $f = y_\alpha y_\beta$, one deduces the following properties:

$$\int_{S_p} \underline{\zeta} ds = \underline{0}; \quad \int_{S_p} [\underline{y} \otimes \underline{\zeta} + \underline{\zeta} \otimes \underline{y}] ds = \underline{\underline{0}}; \quad \int_{S_p} \underline{y} \cdot \underline{\zeta} ds = 0 \tag{61}$$

The problem related to $\underline{U}^0_{,x_1x_1x_1}$ via the Poisson effect, introduces a vector of particular solutions, whose components $\chi_\alpha(\underline{y}, \nu)$ are associated to $\underline{U}^0_{\alpha,x_1x_1x_1}$ so that the solution reads $\underline{\chi} \cdot \underline{U}^0_{,x_1x_1x_1}$. Moreover $\underline{\chi}(\underline{y}, \nu)$ can be decomposed into two vectors $\underline{\eta}(\underline{y})$, $\underline{\theta}(\underline{y})$ independent of the Poisson ratio:

$$\underline{\chi}(\underline{y}, \nu) = (1 + \nu)\underline{\eta}(\underline{y}) + \nu \underline{\theta}(\underline{y})$$

where the functions $\eta_\alpha$ and $\theta_\alpha$ are solutions of (the zero mean value provides the unicity):

$$\begin{cases} -2y_\alpha + \Delta_y(\eta_\alpha) = 0 & \text{in } S_p \\ \underline{\mathrm{grad}}_y(\eta_\alpha) \cdot \underline{n} = 0 & \text{on } \Gamma \\ \int_{S_p} \eta_\alpha ds = 0 \text{ (unicity)} \end{cases}$$

$$\begin{cases} 2y_\alpha + \Delta_y(\theta_\alpha) = 0 & \text{in } S_p \\ (\underline{\mathrm{grad}}_y(\theta_\alpha) + \underline{\xi}^\alpha) \cdot \underline{n} = 0 & \text{on } \Gamma \\ \int_{S_p} \theta_\alpha ds = 0 \text{ (unicity)} \end{cases}$$

The solutions $w$, $\eta$, $\theta$, only depend on the section's geometry. In case of two orthogonal axis of symmetry (then the principal axis of inertia $y_2$, $y_3$) the following properties are satisfied:

$$\begin{aligned} &w(-y_\alpha, y_\beta) = -w(y_\alpha, y_\beta); \quad w(y_\alpha, -y_\beta) = -w(y_\alpha, y_\beta) \\ &\chi_\alpha(-y_\alpha, y_\beta) = -\chi_\alpha(y_\alpha, y_\beta); \quad \chi_\alpha(y_\alpha, -y_\beta) = \chi_\alpha(y_\alpha, y_\beta) \end{aligned} \tag{62}$$

$M^5_1$ is derived by replacing $\underline{\sigma}^2_t$ by its expression:

$$M^5_1 = \int_{S_p} \epsilon_{1\alpha\beta} y_\alpha \sigma^2_{t\beta} ds$$
$$= \int_{S_p} \epsilon_{1\alpha\beta} y_\alpha \mu [\underline{\chi}_{,y_\beta} \cdot \underline{U}^0_{,x_1x_1x_1} + w_{,y_\beta} \Omega^1_{,x_1} + \nu \xi^\gamma_\beta U^0_{\gamma,x_1x_1x_1} + \Omega^1_{,x_1}(\underline{a}_1 \times \underline{y})_\beta] ds$$

In general, their is a coupling between the torsion and bending mechanisms via the Poisson effect (and, in case of inhomogeneous body forces, the contribution of $U^1_{1,x_1x_1}$ would also lead to a coupling with compression). Now, if the section presents two orthogonal axis of symmetry:

$$\int_S y_\alpha y_\beta y_\gamma ds = \int_S y_\alpha \|y\|^2 ds = 0,$$

and from (62) : $\displaystyle\int_{S_p} y_\alpha \xi_{,y_\beta} ds = \int_S y_\alpha \chi_{\gamma,y_\beta} ds = 0$

In that bi-symmetric case, the bending term vanishes and it only remains the uncoupled law:

$$M^5_1 = \mu \left[ \int_{S_p} (\underline{a}_1 \cdot (\underline{y} \times \underline{\mathrm{grad}}_y(w)) + y_2^2 + y_3^2) ds \right] \Omega^1_{,x_1} = \mu I_t \Omega^1_{,x_1}$$

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