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Higher order model for soft and hard elastic interfaces

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ABSTRACT

The present paper deals with the derivation of a higher order theory of interface models. In particular, it is studied the problem of two bodies joined by an adhesive interphase for which "soft" and "hard" linear elastic constitutive laws are considered. For the adhesive, interface models are determined by using two different methods. The first method is based on the matched asymptotic expansion technique, which adopts the strong formulation of classical continuum mechanics equations (compatibility, constitutive and equilibrium equations). The second method adopts a suitable variational (weak) formulation, based on the minimization of the potential energy. First and higher order interface models are derived for soft and hard adhesives. In particular, it is shown that the two approaches, strong and weak formulations, lead to the same asymptotic equations governing the limit behavior of the adhesive as its thickness vanishes. The governing equations derived at zero order are then put in comparison with the ones accounting for the first order of the asymptotic expansion, thus remarking the influence of the higher order terms and of the higher order derivatives on the interface response. Moreover, it is shown how the elastic properties of the adhesive enter the higher order terms. The effects taken into account by the latter ones could play an important role in the nonlinear response of the interface, herein not investigated. Finally, two simple applications are developed in order to illustrate the differences among the interface theories at the different orders.

1. Introduction

Interface models are widely used for structural analyses in several fields of engineering applications. They are adopted to simulate different structural situations as, for instance, to reproduce the crack evolution in a body according to the cohesive fracture mechanics (Barenblatt, 1962; Needleman, 1990), to study the delamination process for composite laminates (Corigliano, 1993; Point and Sacco, 1996, 1998), to simulate the presence of strain localization problems (Belytschko and Black, 1999; Moës and Belytschko, 2002; Ortiz et al., 1987) or to model the bond between two or more bodies (Frémond, 1987; Xu and Wei, 2012). Interfaces are mostly characterized by zero thickness even when the physical bond has a finite thickness, as in the case of glued bodies. This physical thickness of the adhesive can also be significant, as in the case of the mortar joining artificial bricks or natural blocks in the masonry material.

Interface models have the very attractive feature that the stress defined on the corresponding points of the two bonded surfaces, $\boldsymbol{\sigma}\mathbf{n}$ with $\mathbf{n}$ unit vector normal to the interface, assumes the same value, $[\boldsymbol{\sigma}\mathbf{n}] = 0$, and it is a function of the relative displacement, $[\mathbf{u}]$:

$$\boldsymbol{\sigma}\mathbf{n} \leftrightarrow [\mathbf{u}], \tag{1}$$

where the brackets $[\,]$ denote the jump in the enclosed quantity across the interface.

As a consequence, the interface constitutive law is assumed to relate the stress to the displacement jump. This constitutive relationship can be linear or it can take into account nonlinear effects, such as damage, plasticity, viscous phenomena, unilateral contact and friction (Alfano et al., 2006; Del Piero and Raous, 2010; Parrinello et al., 2009; Raous, 2011; Raous et al., 1999; Sacco and Lebon, 2012; Toti et al., 2013). As a consequence, different interface models have been proposed in the scientific literature. Moreover, interface models are implemented in many commercial and research codes as special finite elements.

Interface models can be categorized into two main groups. In the first group, the interface is characterized by a finite stiffness, so that relative displacements occur even for very low values of

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interface stresses; in such a case, the interface is often named in literature as "soft":

$$[\mathbf{u}] = f(\boldsymbol{\sigma}\mathbf{n}), \quad [\boldsymbol{\sigma}\mathbf{n}] = 0. \tag{2}$$

On the contrary, in the second group of models, interfaces are characterized by a rigid response, preceding the eventual damage or other inelastic phenomena; the interface is called "hard" and for the linear case it is governed by the equations:

$$[\mathbf{u}] = 0, \quad [\boldsymbol{\sigma}\mathbf{n}] = 0. \tag{3}$$

The interface models in the first group are widely treated in literature, as they are governed by smooth functions and, consequently, they can be more easily implemented in finite element codes; moreover, inelastic effects can be included as in a classical continuum material. In this instance, the numerical procedures and algorithms are derived and implemented as an extension of the ones typical of continuum mechanics.

The models in the second group are less studied in literature; they are governed by non-smooth functions when nonlinearities are considered and they require the use of quite powerful mathematical techniques; moreover, finite element implementations are more complicated (Dumont et al., 2014).

A rigorous and mathematically elegant way to recover the governing equations of both soft and hard interfaces is represented by the use of the concepts of the asymptotic expansion method. This method was developed by Sanchez-Palencia (1980) to derive the homogenized response of composites; it is based on the choice of a geometrically small parameter (e.g. the size of the microstructure) and on the expansion of the relevant fields (displacement, stress and strain) in a power series with respect to the chosen small parameter. This technique was successfully used to recover the plate and shell theories (Ciarlet, 1997; Ciarlet and Destuynder, 1987) or the governing equations of interface models (Geymonat and Krasucki, 1997; Klarbring and Movchan, 1998; Lebon et al., 1997; Licht and Michaille, 1996; Marigo et al., 1998).

When the thickness of the bonding material, ε, is not so small, higher order terms in the asymptotic expansions with respect to ε should be considered in the derivation of the interface governing equations. Previous studies have established that, if the stiffness of the adhesive material is comparable with the stiffness of the adherents, then various mathematical approaches (asymptotic expansions (Abdelmoula et al., 1998; Benveniste, 2006; Benveniste and Miloh, 2001; Geymonat et al., 1999; Hashin, 2002; Klarbring and Movchan, 1998; Lebon et al., 2004), Γ-convergence techniques (Caillerie, 1980; Lebon and Rizzoni, 2010; Licht, 1993; Licht and Michaille, 1997; Serpilli and Lenci, 2008), energy methods (Lebon and Rizzoni, 2011; Rizzoni and Lebon, 2012)) can be used to obtain the model of perfect interface at the first (zero) order in the asymptotic expansion. At the next (one) order, it is obtained a model of imperfect interface, which is non-local due to the presence of tangential derivatives entering the interface equations (Abdelmoula et al., 1998; Hashin, 2002; Lebon and Rizzoni, 2010, 2011; Rizzoni and Lebon, 2012, 2013).

The aim of this paper is the derivation of the governing equations for soft and hard anisotropic interfaces accounting for higher order terms in the asymptotic expansion, being the zero order terms classical and well-known in the literature. While the terms computed at the order one for hard interfaces (Eq. (64)) have been derived previously (Lebon and Rizzoni, 2010, 2011; Rizzoni and Lebon, 2013), the terms computed at the order one for soft interfaces (Eq. (56)) represent a new contribution. A novel asymptotic analysis is presented based on two different asymptotic methods: matching asymptotic expansions and an asymptotic method based on energy minimization. In the first method, the derivation of the governing equations is performed by adopting the strong formulation of the equilibrium problem, i.e. by writing the classical compatibility, constitutive and equilibrium equations. The second method relies on a weak formulation of the equilibrium problem and it is an original improvement of asymptotic methods proposed in Lebon and Rizzoni (2010), because the terms at the various orders in the energy expansion are minimized together and not successively starting from the term at the lowest order. The asymptotic analysis via the energy method is useful to ascertain the consistency and the equivalence with the method based on matched asymptotic expansions. Indeed, a main result of the paper consists in showing that the two approaches, one based on the strong and the other on the weak formulation, lead to the same governing equations. In addition, the derivation of the boundary conditions for an interface of finite length is straightforward via the energy method, while these conditions have to be specifically investigated using matched asymptotic expansions (Abdelmoula et al., 1998). Finally, the weak formulation is the basis of development of numerical procedures, such as finite element approaches, which can be used to perform numerical analyses in order to evaluate the influence and the importance of higher order effects in the response of the interface.

Another original result of the paper is a comparison of the equations governing the behavior of soft and hard interfaces obtained at order zero with the ones obtained at the first order in the asymptotic expansions. The influence of the higher order terms and of the higher order derivatives on the interface response is also highlighted and their dependence on the elastic properties of the adhesive is determined. Notably, the effects taken into account by the higher order terms in the asymptotic analysis could play an important role in the nonlinear response of the interface, herein not investigated.

The analysis of the regularity of the limit problems and of the singularities of the stress and displacement fields near the external boundary of the adhesive are not considered in this paper. For the model of soft interface computed at order zero, these questions are considered in Geymonat et al. (1999). Finally, it should be emphasized that the present analysis considers planar interphases of constant thickness. Thin layers of varying thickness have been considered in Ould Khaoua (1995) and higher order effects in curved interphases of constant thickness have been studied in Rizzoni and Lebon (2013) only for the case of a hard material.

The paper is organized as follows. In Section 2, the problem of two bodies in adhesion is posed, the rescaling technique is introduced and the governing equations of the adherents and of the adhesive are written, together with the matching conditions. In Section 3, the interface equations are derived for both the two cases of an adhesive constituted of a soft and a hard materials, and higher order terms in the asymptotic expansion are considered. In Section 4, the variational approach to the derivation of the governing equations of the interface is presented. Section 5 is devoted to the comparison between the lower and the higher order for both soft and hard interface models. Finally, two analytical examples are presented, the shear and the stretching of a two-dimensional composite block, and the main results are discussed.

2. Generalities of asymptotic expansions

A thin layer B^ε with cross-section S and uniform small thickness $\varepsilon \ll 1$ is considered, S being an open bounded set in R^2 with a smooth boundary. In the following B^ε and S will be called interphase and interface, respectively. The interphase lies between two bodies, named as adherents, occupying the reference configurations $\Omega^\varepsilon_\pm \subset R^3$. In such a way, the interphase represents the adhesive joining the two bodies Ω^ε_+ and Ω^ε_-. Let $S^\varepsilon_\pm$ be taken to denote the plane interfaces between the interphase and the adherents and let $\Omega^\varepsilon = \Omega^\varepsilon_\pm \cup S^\varepsilon_\pm \cup B^\varepsilon$ denote the composite system comprising the interphase and the adherents.

It is assumed that the adhesive and the adherents are perfectly bonded in order to ensure the continuity of the displacement and stress vector fields across $S^\varepsilon_\pm$.

2.1. Notations

An orthonormal Cartesian basis $(O, \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$ is introduced and let (x_1, x_2, x_3) be taken to denote the three coordinates of a particle. The origin lies at the center of the interphase midplane and the x_3-axis runs perpendicular to the open bounded set S, as illustrated in Fig. 1.

The materials of the composite system are assumed to be homogeneous and linearly elastic and let $\mathbf{a}_\pm, \mathbf{b}^\varepsilon$ be the elasticity tensors of the adherents and of the interphase, respectively. The tensors $\mathbf{a}_\pm, \mathbf{b}^\varepsilon$ are assumed to be symmetric, with the minor and major symmetries, and positive definite. The adherents are subjected to a body force density $\mathbf{f} : \Omega^\varepsilon_\pm \mapsto R^3$ and to a surface force density $\mathbf{g} : \Gamma^\varepsilon_g \mapsto R^3$ on $\Gamma^\varepsilon_g \subset (\partial\Omega^\varepsilon_+ \setminus S^\varepsilon_+) \cup (\partial\Omega^\varepsilon_- \setminus S^\varepsilon_-)$. Body forces are neglected in the adhesive.

On $\Gamma^\varepsilon_u = (\partial\Omega^\varepsilon_+ \setminus S^\varepsilon_+) \cup (\partial\Omega^\varepsilon_- \setminus S^\varepsilon_-) \setminus \Gamma^\varepsilon_g$, homogeneous boundary conditions are prescribed:

$$\mathbf{u}^\varepsilon = \mathbf{0} \quad \text{on} \quad \Gamma^\varepsilon_u, \tag{4}$$

where $\mathbf{u}^\varepsilon : \Omega^\varepsilon \mapsto R^3$ is the displacement field defined on Ω^ε. $\Gamma^\varepsilon_g, \Gamma^\varepsilon_u$ are assumed to be located far from the interphase, in particular the external boundary of the interphase B^ε , i.e. $\partial S \times (-\varepsilon/2, \varepsilon/2)$, is assumed to be stress-free. The fields of the external forces are endowed with sufficient regularity to ensure the existence of equilibrium configuration.

2.2. Rescaling

In the interphase, the change of variables $\hat{\mathbf{p}} : (x_1, x_2, x_3) \to (z_1, z_2, z_3)$ proposed by Ciarlet (1997) is operated, which is such that:

$$z_1 = x_1, \quad z_2 = x_2, \quad z_3 = \frac{x_3}{\varepsilon}, \tag{5}$$

resulting

$$\frac{\partial}{\partial z_1} = \frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial z_2} = \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial z_3} = \varepsilon\frac{\partial}{\partial x_3}. \tag{6}$$

Moreover, in the adherents the following change of variables $\bar{\mathbf{p}} : (x_1, x_2, x_3) \to (z_1, z_2, z_3)$ is also introduced:

$$z_1 = x_1, \quad z_2 = x_2, \quad z_3 = x_3 \pm \frac{1}{2}(1-\varepsilon), \tag{7}$$

where the plus (minus) sign applies whenever $x \in \Omega^\varepsilon_+$ ($x \in \Omega^\varepsilon_-$), with

$$\frac{\partial}{\partial z_1} = \frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial z_2} = \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial z_3} = \frac{\partial}{\partial x_3}. \tag{8}$$

After the change of variables (5), the interphase occupies the domain

$$B = \left\{ (z_1, z_2, z_3) \in R^3 : (z_1, z_2) \in S, |z_3| < \frac{1}{2} \right\} \tag{9}$$

and the adherents occupy the domains $\Omega_\pm = \Omega^\varepsilon_\pm \pm \frac{1}{2}(1-\varepsilon)i_3$, as shown in Fig. 1(b). The sets $S_\pm = \{(z_1, z_2, z_3) \in R^3 : (z_1, z_2) \in S, z_3 = \pm\frac{1}{2}\}$ are taken to denote the interfaces between B and $\Omega_\pm$ and $\Omega = \Omega_+ \cup \Omega_- \cup B \cup S_+ \cup S_-$ is the rescaled configuration of the composite body. Lastly, Γ_u and Γ_g indicates the images of Γ^ε_u and Γ^ε_g under the change of variables, and $\bar{\mathbf{f}} := \mathbf{f} \circ \bar{\mathbf{p}}^{-1}$ and $\bar{\mathbf{g}} := \mathbf{g} \circ \bar{\mathbf{p}}^{-1}$ the rescaled external forces.

2.3. Kinematics

After taking $\hat{\mathbf{u}}^\varepsilon = \mathbf{u}^\varepsilon \circ \hat{\mathbf{p}}^{-1}$ and $\bar{\mathbf{u}}^\varepsilon = \mathbf{u}^\varepsilon \circ \bar{\mathbf{p}}^{-1}$ to denote the displacement fields from the rescaled adhesive and adherents, respectively, the asymptotic expansions of the displacement fields with respect to the small parameter ε take the form:

$$\mathbf{u}^\varepsilon(x_1, x_2, x_3) = \mathbf{u}^0 + \varepsilon\mathbf{u}^1 + \varepsilon^2\mathbf{u}^2 + o(\varepsilon^2), \tag{10}$$

$$\hat{\mathbf{u}}^\varepsilon(z_1, z_2, z_3) = \hat{\mathbf{u}}^0 + \varepsilon\hat{\mathbf{u}}^1 + \varepsilon^2\hat{\mathbf{u}}^2 + o(\varepsilon^2), \tag{11}$$

$$\bar{\mathbf{u}}^\varepsilon(z_1, z_2, z_3) = \bar{\mathbf{u}}^0 + \varepsilon\bar{\mathbf{u}}^1 + \varepsilon^2\bar{\mathbf{u}}^2 + o(\varepsilon^2). \tag{12}$$

2.3.1. Interphase

The displacement gradient tensor of the field $\hat{\mathbf{u}}^\varepsilon$ in the rescaled interphase is computed as:

$$\hat{\mathbf{H}} = \varepsilon^{-1}\begin{bmatrix} 0 & \hat{u}^0_{\alpha,3} \\ 0 & \hat{u}^0_{3,3} \end{bmatrix} + \begin{bmatrix} \hat{u}^0_{\alpha,\beta} & \hat{u}^1_{\alpha,3} \\ \hat{u}^0_{3,\beta} & \hat{u}^1_{3,3} \end{bmatrix} + \varepsilon\begin{bmatrix} \hat{u}^1_{\alpha,\beta} & \hat{u}^2_{\alpha,3} \\ \hat{u}^1_{3,\beta} & \hat{u}^2_{3,3} \end{bmatrix} + O(\varepsilon^2), \tag{13}$$

where $\alpha = 1, 2$, so that the strain tensor can be obtained as:

$$\mathbf{e}(\hat{\mathbf{u}}^\varepsilon) = \varepsilon^{-1}\hat{\mathbf{e}}^{-1} + \hat{\mathbf{e}}^0 + \varepsilon\hat{\mathbf{e}}^1 + O(\varepsilon^2), \tag{14}$$

with:

$$\hat{\mathbf{e}}^{-1} = \begin{bmatrix} 0 & \frac{1}{2}\hat{u}^0_{\alpha,3} \\ \frac{1}{2}\hat{u}^0_{\alpha,3} & \hat{u}^0_{3,3} \end{bmatrix} = Sym(\hat{\mathbf{u}}^0_{,3} \otimes \mathbf{i}_3), \tag{15}$$

$$\begin{aligned} \hat{\mathbf{e}}^k &= \begin{bmatrix} Sym\left(\hat{u}^k_{\alpha,\beta}\right) & \frac{1}{2}\left(\hat{u}^k_{3,\alpha} + \hat{u}^{k+1}_{\alpha,3}\right) \\ \frac{1}{2}\left(\hat{u}^k_{3,\alpha} + \hat{u}^{k+1}_{\alpha,3}\right) & \hat{u}^{k+1}_{3,3} \end{bmatrix} \\ &= Sym\left(\hat{\mathbf{u}}^k_{,1} \otimes \mathbf{i}_1 + \hat{\mathbf{u}}^k_{,2} \otimes \mathbf{i}_2 + \hat{\mathbf{u}}^{k+1}_{,3} \otimes \mathbf{i}_3\right), \end{aligned} \tag{16}$$

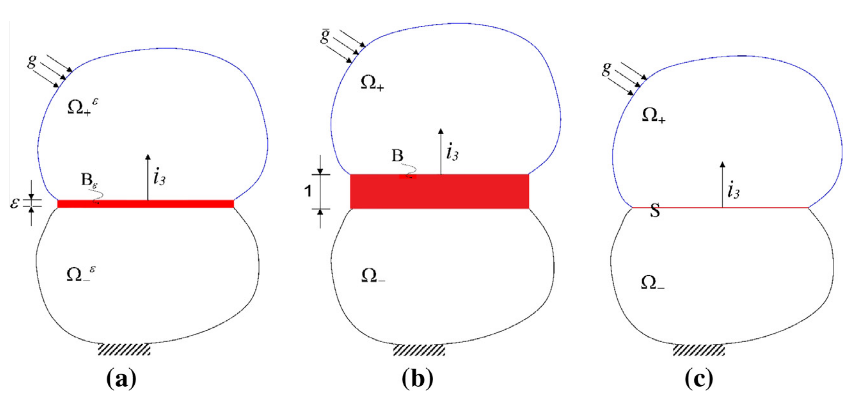

Fig. 1. Geometry of the assembled composite system.

where $Sym(\cdot)$ gives the symmetric part of the enclosed tensor and $k = 0, 1$.

2.3.2. Adherents

The displacement gradient tensor of the field $\bar{\mathbf{u}}^\varepsilon$ in the adherents is computed as:

$$\bar{\mathbf{H}} = \begin{bmatrix} \bar{u}^0_{\alpha,\beta} & \bar{u}^0_{\alpha,3} \\ \bar{u}^0_{3,\beta} & \bar{u}^0_{3,3} \end{bmatrix} + \varepsilon \begin{bmatrix} \bar{u}^1_{\alpha,\beta} & \bar{u}^1_{\alpha,3} \\ \bar{u}^1_{3,\beta} & \bar{u}^1_{3,3} \end{bmatrix} + O(\varepsilon^2), \tag{17}$$

so that the strain tensor can be obtained as:

$$\mathbf{e}(\bar{\mathbf{u}}^\varepsilon) = \varepsilon^{-1}\bar{\mathbf{e}}^{-1} + \bar{\mathbf{e}}^0 + \varepsilon\bar{\mathbf{e}}^1 + O(\varepsilon^2), \tag{18}$$

with:

$$\bar{\mathbf{e}}^{-1} = \mathbf{0}, \tag{19}$$

$$\bar{\mathbf{e}}^k = \begin{bmatrix} Sym\left(\bar{u}^k_{\alpha,\beta}\right) & \frac{1}{2}\left(\bar{u}^k_{3,\alpha} + \bar{u}^k_{\alpha,3}\right) \\ \frac{1}{2}\left(\bar{u}^k_{3,\alpha} + \bar{u}^k_{\alpha,3}\right) & \bar{u}^k_{3,3} \end{bmatrix} = Sym\left(\bar{\mathbf{u}}^k_{,1} \otimes \mathbf{i}_1 + \bar{\mathbf{u}}^k_{,2} \otimes \mathbf{i}_2 + \bar{\mathbf{u}}^k_{,3} \otimes \mathbf{i}_3\right), \tag{20}$$

$k = 0, 1$.

2.4. Stress fields

The stress fields in the rescaled adhesive and adherents, $\hat{\boldsymbol{\sigma}}^\varepsilon = \boldsymbol{\sigma} \circ \hat{\mathbf{p}}^{-1}$ and $\bar{\boldsymbol{\sigma}}^\varepsilon = \boldsymbol{\sigma} \circ \bar{\mathbf{p}}^{-1}$ respectively, are also represented as asymptotic expansions:

$$\boldsymbol{\sigma}^\varepsilon = \boldsymbol{\sigma}^0 + \varepsilon\boldsymbol{\sigma}^1 + O(\varepsilon^2), \tag{21}$$
$$\hat{\boldsymbol{\sigma}}^\varepsilon = \hat{\boldsymbol{\sigma}}^0 + \varepsilon\hat{\boldsymbol{\sigma}}^1 + O(\varepsilon^2), \tag{22}$$
$$\bar{\boldsymbol{\sigma}}^\varepsilon = \bar{\boldsymbol{\sigma}}^0 + \varepsilon\bar{\boldsymbol{\sigma}}^1 + O(\varepsilon^2). \tag{23}$$

2.4.1. Equilibrium equations in the interphase

As body forces are neglected in the adhesive, the equilibrium equation is:

$$div\hat{\boldsymbol{\sigma}}^\varepsilon = \mathbf{0}. \tag{24}$$

Substituting the representation form (22) into the equilibrium Eq. (24) and using (6), it becomes:

$$0 = \hat{\sigma}^\varepsilon_{i\alpha,\alpha} + \varepsilon^{-1}\hat{\sigma}^\varepsilon_{i3,3} = \varepsilon^{-1}\hat{\sigma}^0_{i3,3} + \hat{\sigma}^0_{i\alpha,\alpha} + \hat{\sigma}^1_{i3,3} + \varepsilon\hat{\sigma}^1_{i\alpha,\alpha} + O(\varepsilon), \tag{25}$$

where $\alpha = 1, 2$. Eq. (25) has to be satisfied for any value of ε, leading to:

$$\hat{\sigma}^0_{i3,3} = 0, \tag{26}$$
$$\hat{\sigma}^0_{i1,1} + \hat{\sigma}^0_{i2,2} + \hat{\sigma}^1_{i3,3} = 0, \tag{27}$$

where $i = 1, 2, 3$.

Eq. (26) shows that $\hat{\sigma}^0_{i3}$ is independent of z_3 in the adhesive, and thus it can be written:

$$\left[\hat{\sigma}^0_{i3}\right] = 0, \tag{28}$$

where $[.]$ denotes the jump between $z_3 = \frac{1}{2}$ and $z_3 = -\frac{1}{2}$.

In view of (28), Eq. (27) when $i = 3$ can be rewritten in the integrated form

$$\left[\hat{\sigma}^1_{33}\right] = -\hat{\sigma}^0_{13,1} - \hat{\sigma}^0_{23,2}. \tag{29}$$

2.4.2. Equilibrium equations in the adherents

The equilibrium equation in the adherents is:

$$\mathrm{div}\bar{\boldsymbol{\sigma}}^\varepsilon + \bar{\mathbf{f}} = \mathbf{0}. \tag{30}$$

Substituting the representation form (23) into the equilibrium Eq. (30) and taking into account that it has to be satisfied for any value of ε, it leads to:

$$\mathrm{div}\bar{\boldsymbol{\sigma}}^0 + \bar{\mathbf{f}} = \mathbf{0}, \tag{31}$$
$$\mathrm{div}\bar{\boldsymbol{\sigma}}^1 = \mathbf{0}. \tag{32}$$

2.5. Matching external and internal expansions

As a perfect contact law between the adhesive and the adherents is assumed, the continuity of the displacement and stress vector fields is enforced. In particular, the continuity of the displacements gives:

$$\mathbf{u}^\varepsilon\left(\bar{\mathbf{x}}, \pm\frac{\varepsilon}{2}\right) = \hat{\mathbf{u}}^\varepsilon\left(\bar{\mathbf{z}}, \pm\frac{1}{2}\right) = \bar{\mathbf{u}}^\varepsilon\left(\bar{\mathbf{z}}, \pm\frac{1}{2}\right), \tag{33}$$

where $\bar{\mathbf{x}} := (x_1, x_2)$, $\bar{\mathbf{z}} := (z_1, z_2) \in S$. Expanding the displacement in the adherent, $\mathbf{u}^\varepsilon$, in Taylor series along the x_3-direction and taking into account the asymptotic expansion (10), it results:

$$\mathbf{u}^\varepsilon\left(\bar{\mathbf{x}}, \pm\frac{\varepsilon}{2}\right) = \mathbf{u}^\varepsilon(\bar{\mathbf{x}}, 0^\pm) \pm \frac{\varepsilon}{2}\mathbf{u}^\varepsilon_{,3}(\bar{\mathbf{x}}, 0^\pm) + \cdots = \mathbf{u}^0(\bar{\mathbf{x}}, 0^\pm) + \varepsilon\mathbf{u}^1(\bar{\mathbf{x}}, 0^\pm) \pm \frac{\varepsilon}{2}\mathbf{u}^0_{,3}(\bar{\mathbf{x}}, 0^\pm) + \cdots \tag{34}$$

Substituting the expressions (11) and (12) together with formula (34) into the continuity condition (33), it holds true:

$$\mathbf{u}^0(\bar{\mathbf{x}}, 0^\pm) + \varepsilon\mathbf{u}^1(\bar{\mathbf{x}}, 0^\pm) \pm \frac{\varepsilon}{2}\mathbf{u}^0_{,3}(\bar{\mathbf{x}}, 0^\pm) + \cdots = \hat{\mathbf{u}}^0\left(\bar{\mathbf{z}}, \pm\frac{1}{2}\right) + \varepsilon\hat{\mathbf{u}}^1\left(\bar{\mathbf{z}}, \pm\frac{1}{2}\right) + \cdots = \bar{\mathbf{u}}^0\left(\bar{\mathbf{z}}, \pm\frac{1}{2}\right) + \varepsilon\bar{\mathbf{u}}^1\left(\bar{\mathbf{z}}, \pm\frac{1}{2}\right) + \cdots \tag{35}$$

After identifying the terms in the same powers of ε, Eq. (35) gives:

$$\mathbf{u}^0(\bar{\mathbf{x}}, 0^\pm) = \hat{\mathbf{u}}^0\left(\bar{\mathbf{z}}, \pm\frac{1}{2}\right) = \bar{\mathbf{u}}^0\left(\bar{\mathbf{z}}, \pm\frac{1}{2}\right), \tag{36}$$
$$\mathbf{u}^1(\bar{\mathbf{x}}, 0^\pm) \pm \frac{1}{2}\mathbf{u}^0_{,3}(\bar{\mathbf{x}}, 0^\pm) = \hat{\mathbf{u}}^1\left(\bar{\mathbf{z}}, \pm\frac{1}{2}\right) = \bar{\mathbf{u}}^1\left(\bar{\mathbf{z}}, \pm\frac{1}{2}\right). \tag{37}$$

Following a similar analysis for the stress vector, analogous results are obtained:

$$\sigma^0_{i3}(\bar{\mathbf{x}}, 0^\pm) = \hat{\sigma}^0_{i3}\left(\bar{\mathbf{z}}, \pm\frac{1}{2}\right) = \bar{\sigma}^0_{i3}B\left(\bar{\mathbf{z}}, \pm\frac{1}{2}\right), \tag{38}$$

$$\sigma^1_{i3}(\bar{\mathbf{x}}, 0^\pm) \pm \frac{1}{2}\sigma^0_{i3,3}(\bar{\mathbf{x}}, 0^\pm) = \hat{\sigma}^1_{i3}\left(\bar{\mathbf{z}}, \pm\frac{1}{2}\right) = \bar{\sigma}^1_{i3}\left(\bar{\mathbf{z}}, \pm\frac{1}{2}\right), \tag{39}$$

for $i = 1, 2, 3$.

Using the above results, it is possible to rewrite Eqs. (28) and (29) in the following form:

$$[[\sigma^0_{i3}]] = 0, \quad i = 1, 2, 3,$$
$$[[\sigma^1_{33}]] = -\sigma^0_{13,1} - \sigma^0_{23,2} - \langle\langle\sigma^0_{33,3}\rangle\rangle, \tag{40}$$

where $[[f]] := f(\bar{\mathbf{x}}, 0^+) - f(\bar{\mathbf{x}}, 0^-)$ is taken to denote the jump across the surface S of a generic function f defined on the limit configuration obtained as $\varepsilon \to 0$, as schematically illustrated in Fig. 1(c), while it is set $\langle\langle f\rangle\rangle := \frac{1}{2}(f(\bar{\mathbf{x}}, 0^+) + f(\bar{\mathbf{x}}, 0^-))$.

All equations written so far are general in the sense that they are independent of the constitutive behavior of the material.

2.6. Constitutive equations

The specific constitutive behavior of the materials is now introduced. In particular, the linearly elastic constitutive laws for the adherents and the interphase, relating the stress with the strain, are given by the equations:

$$\bar{\boldsymbol{\sigma}}^{\varepsilon} = \mathbf{a}^{\pm}(\mathbf{e}(\bar{\mathbf{u}}^{\varepsilon})), \tag{41}$$
$$\hat{\boldsymbol{\sigma}}^{\varepsilon} = \mathbf{b}^{\varepsilon}(\mathbf{e}(\hat{\mathbf{u}}^{\varepsilon})), \tag{42}$$

where $a^{\pm}_{ijkl}, b^{\varepsilon}_{ijkl}$ are the classical elastic constants of elasticity of the adherents and of the interphase, respectively.

The matrices $\mathbf{K}^{jl}_{\varepsilon}$ (with $j,l = 1,2,3$) are introduced, whose components are defined by the relation:

$$\left(K^{jl}_{\varepsilon}\right)_{ki} := b^{\varepsilon}_{ijkl}. \tag{43}$$

In view of the symmetry properties of the elasticity tensor $\mathbf{b}^{\varepsilon}$, it results that $\mathbf{K}^{jl}_{\varepsilon} = \left(\mathbf{K}^{lj}_{\varepsilon}\right)^{T}$, with $j,l = 1,2,3$.

3. Internal/interphase analysis

In the following, two specific cases of linearly elastic material are studied for the interphase. One, called "soft" material, is characterized by elastic moduli which are linearly rescaled with respect to the thickness ε; the second case, called "hard" material, is characterized by elastic moduli independent of the thickness ε. The two cases are relevant for the development of interface laws classically used in technical problems. Indeed, models of perfect and imperfect interfaces, which are currently used in finite element simulations, are known to arise from the hard and the soft cases, respectively, at the first (zero) order of the asymptotic expansion (Benveniste, 2006; Caillerie, 1980; Klarbring, 1991; Lebon and Rizzoni, 2010).

3.1. Soft interphase analysis

Assuming that the interphase is "soft", one defines:

$$\mathbf{b}^{\varepsilon} = \varepsilon\mathbf{b}, \tag{44}$$

where the tensor $\mathbf{b}$ does not depend on ε. Accordingly to position (43), it is set:

$$K^{jl}_{ki} := b_{ijkl}. \tag{45}$$

Taking into account relations (14) and (22), the stress–strain law takes the following form:

$$\hat{\boldsymbol{\sigma}}^{0} + \varepsilon\hat{\boldsymbol{\sigma}}^{1} = \mathbf{b}(\hat{\mathbf{e}}^{-1} + \varepsilon\hat{\mathbf{e}}^{0}) + o(\varepsilon). \tag{46}$$

As Eq. (46) is true for any value of ε, the following expressions are derived:

$$\begin{aligned} \hat{\boldsymbol{\sigma}}^{0} &= \mathbf{b}(\hat{\mathbf{e}}^{-1}), \\ \hat{\boldsymbol{\sigma}}^{1} &= \mathbf{b}(\hat{\mathbf{e}}^{0}). \end{aligned} \tag{47}$$

Substituting the expression (45) into $(47)_1$, it results:

$$\hat{\sigma}^{0}_{ij} = b_{ijkl}\hat{e}^{-1}_{kl} = K^{jl}_{ki}\hat{e}^{-1}_{kl}, \tag{48}$$

and using formula (15), it follows that:

$$\hat{\boldsymbol{\sigma}}^{0}\mathbf{i}_j = \mathbf{K}^{3j}\hat{\mathbf{u}}^{0}_{,3}, \tag{49}$$

for $j = 1,2,3$. Integrating Eq. (49) written for $j = 3$ with respect to z_3, it results:

$$\hat{\boldsymbol{\sigma}}^{0}\mathbf{i}_3 = \mathbf{K}^{33}\left[\hat{\mathbf{u}}^{0}\right], \tag{50}$$

which represents the classical law for a soft interface.

Analogously, substituting the expression (45) into $(47)_2$, and using formula (16) written for $k = 0$, one has:

$$\hat{\boldsymbol{\sigma}}^{1}\mathbf{i}_j = \mathbf{K}^{1j}\hat{\mathbf{u}}^{0}_{,1} + \mathbf{K}^{2j}\hat{\mathbf{u}}^{0}_{,2} + \mathbf{K}^{3j}\hat{\mathbf{u}}^{1}_{,3}, \tag{51}$$

for $j = 1,2,3$.

On the other hand, taking into account formula (49), written for $j = 1,2$, the equilibrium Eq. (27) explicitly becomes:

$$\left(\mathbf{K}^{31}\hat{\mathbf{u}}^{0}_{,3}\right)_{,1} + \left(\mathbf{K}^{32}\hat{\mathbf{u}}^{0}_{,3}\right)_{,2} + \left(\hat{\boldsymbol{\sigma}}^{1}\mathbf{i}_3\right)_{,3} = \mathbf{0}, \tag{52}$$

and thus, integrating with respect to z_3 between $-\frac{1}{2}$ and $\frac{1}{2}$, it gives:

$$\left[\hat{\boldsymbol{\sigma}}^{1}\mathbf{i}_3\right] = -\mathbf{K}^{31}\left[\hat{\mathbf{u}}^{0}\right]_{,1} - \mathbf{K}^{32}\left[\hat{\mathbf{u}}^{0}\right]_{,2}. \tag{53}$$

It can be remarked that, because of Eq. (26), the stress components $\hat{\sigma}^{0}_{i3}$, with $i = 1,2,3$, are independent of z_3. Consequently, taking into account Eq. (49) written for $j = 3$, the derivatives $\hat{u}^{0}_{i,3}$ are also independent of z_3; thus, the displacement components $\hat{u}^{0}_{i}$ are linear functions of z_3. Therefore, Eq. (53) reveals that the stress components $\hat{\sigma}^{1}_{i3}$, with $i = 1,2,3$, are linear functions of z_3, allowing to write the following representation form for the stress components:

$$\hat{\boldsymbol{\sigma}}^{1}\mathbf{i}_3 = \left[\hat{\boldsymbol{\sigma}}^{1}\mathbf{i}_3\right]z_3 + \langle\hat{\boldsymbol{\sigma}}^{1}\mathbf{i}_3\rangle, \tag{54}$$

where $\langle f\rangle(\bar{\mathbf{z}}) := \frac{1}{2}\left(f(\bar{\mathbf{z}},\frac{1}{2}) + f(\bar{\mathbf{z}},-\frac{1}{2})\right)$. Substituting Eq. (51) written for $j = 3$ into expression (54) and integrating with respect to z_3 it yields:

$$\langle\hat{\boldsymbol{\sigma}}^{1}\mathbf{i}_3\rangle = \mathbf{K}^{\alpha 3}\langle\hat{\mathbf{u}}^{0}\rangle_{,\alpha} + \mathbf{K}^{33}\left[\hat{\mathbf{u}}^{1}\right], \tag{55}$$

where the sum over $\alpha = 1,2$ is performed. Combining Eqs. (53)–(55), it results:

$$\begin{aligned} \hat{\boldsymbol{\sigma}}^{1}\left(\bar{\mathbf{z}},\pm\frac{1}{2}\right)\mathbf{i}_3 = {} & \mathbf{K}^{33}[\hat{\mathbf{u}}^{1}](\bar{\mathbf{z}}) + \frac{1}{2}(\mathbf{K}^{\alpha 3} \mp \mathbf{K}^{3\alpha})\hat{\mathbf{u}}^{0}_{,\alpha}\left(\bar{\mathbf{z}},\frac{1}{2}\right) + \frac{1}{2}(\mathbf{K}^{\alpha 3} \\ & \pm \mathbf{K}^{3\alpha})\hat{\mathbf{u}}^{0}_{,\alpha}\left(\bar{\mathbf{z}},-\frac{1}{2}\right). \end{aligned} \tag{56}$$

3.2. Hard interphase analysis

For a "hard" interphase, it is set:

$$\mathbf{b}^{\varepsilon} = \mathbf{b}, \tag{57}$$

where the tensor $\mathbf{b}$ does not depend on ε, and $\mathbf{K}^{jl}$ is still taken to denote the matrices such that $K^{jl}_{ki} := b_{ijkl}$.

Taking into account relations (14) and (22), the stress–strain equation takes the following form:

$$\hat{\boldsymbol{\sigma}}^{0} + \varepsilon\hat{\boldsymbol{\sigma}}^{1} = \mathbf{b}(\varepsilon^{-1}\hat{\mathbf{e}}^{-1} + \hat{\mathbf{e}}^{0} + \varepsilon\hat{\mathbf{e}}^{1}) + o(\varepsilon). \tag{58}$$

As Eq. (58) is true for any value of ε, the following conditions are derived:

$$\begin{aligned} \mathbf{0} &= \mathbf{b}(\hat{\mathbf{e}}^{-1}), \\ \hat{\boldsymbol{\sigma}}^{0} &= \mathbf{b}(\hat{\mathbf{e}}^{0}). \end{aligned} \tag{59}$$

Taking into account Eq. (15) and the positive definiteness of the tensor $\mathbf{b}$, relation $(59)_1$ gives:

$$\hat{\mathbf{u}}^{0}_{,3} = 0 \Rightarrow [\hat{\mathbf{u}}^{0}] = \mathbf{0}, \tag{60}$$

which corresponds to the kinematics of the perfect interface.

Substituting the expression (16) written for $k = 0$ into $(59)_2$, one has:

$$\hat{\boldsymbol{\sigma}}^{0}\mathbf{i}_j = \mathbf{K}^{1j}\hat{\mathbf{u}}^{0}_{,1} + \mathbf{K}^{2j}\hat{\mathbf{u}}^{0}_{,2} + \mathbf{K}^{3j}\hat{\mathbf{u}}^{1}_{,3}, \tag{61}$$

for $j = 1,2,3$. Integrating Eq. (61) written for $j = 3$ with respect to z_3, it results:

$$[\hat{\mathbf{u}}^{1}] = (\mathbf{K}^{33})^{-1}\left(\hat{\boldsymbol{\sigma}}^{0}\mathbf{i}_3 - \mathbf{K}^{\alpha 3}\hat{\mathbf{u}}^{0}_{,\alpha}\right). \tag{62}$$

Recalling the Eq. (61) written for $j = 1,2$, the equilibrium Eq. (27) explicitly becomes:

$$\begin{aligned} & \left(\mathbf{K}^{11}\hat{\mathbf{u}}^{0}_{,1} + \mathbf{K}^{21}\hat{\mathbf{u}}^{0}_{,2} + \mathbf{K}^{31}\hat{\mathbf{u}}^{1}_{,3}\right)_{,1} + \left(\mathbf{K}^{12}\hat{\mathbf{u}}^{0}_{,1} + \mathbf{K}^{22}\hat{\mathbf{u}}^{0}_{,2} + \mathbf{K}^{32}\hat{\mathbf{u}}^{1}_{,3}\right)_{,2} \\ & \quad + \left(\hat{\boldsymbol{\sigma}}^{1}\mathbf{i}_3\right)_{,3} = \mathbf{0}, \end{aligned} \tag{63}$$

and thus, integrating with respect to z_3 between $-1/2$ and $1/2$ and using (62), it gives:

$$[\hat{\boldsymbol{\sigma}}^1\mathbf{i}_3] = \left(-\mathbf{K}^{\alpha\beta}\hat{\mathbf{u}}^0_{,\beta} - \mathbf{K}^{3\alpha}[\hat{\mathbf{u}}^1]\right)_{,\alpha} = \left(-\mathbf{K}^{\alpha\beta}\hat{\mathbf{u}}^0_{,\beta} - \mathbf{K}^{3\alpha}(\mathbf{K}^{33})^{-1}\left(\hat{\boldsymbol{\sigma}}^0\mathbf{i}_3 - \mathbf{K}^{\beta 3}\hat{\mathbf{u}}^0_{,\beta}\right)\right)_{,\alpha}. \tag{64}$$

It can be noted that, in Eq. (64) higher order effects occur related to the appearance of in-plane derivatives, which are usually neglected in the classical first (zero) order theories of interfaces. These terms, which are related to second-order derivatives and thus, indirectly, to the curvature of the deformed interface, model a membrane effect in the adhesive.

4. Asymptotic analysis via an energy approach

In this section, an energy based approach is proposed to asymptotically analyze the limit behavior of the interphase as $\varepsilon \to 0$. This analysis is of great interest for at least three reasons. First, the aim is to prove the consistency and the equivalence of the two asymptotic approaches, one based on matched asymptotic expansions, considered in the previous section, and the other based on a variational formulation and illustrated in this section. Next, the energy approach allows to derive in a very straight way the boundary conditions when a finite length of the interface is considered. These conditions are not obvious and they have to be specifically studied using matched asymptotic expansions (Abdelmoula et al., 1998). Finally, variational formulations are the basis of development of numerical procedures, such as finite element approaches, which can be used to perform numerical analyses in order to evaluate the influence and the importance of higher order effects in the response of the interphase.

The energy approach is based on the fact that equilibrium configurations of the composite assemblage minimize the total energy:

$$E^\varepsilon(\mathbf{u}) = \int_{\Omega^\varepsilon_\pm}\left(\frac{1}{2}\mathbf{a}_\pm(\mathbf{e}(\mathbf{u}))\cdot\mathbf{e}(\mathbf{u}) - \mathbf{f}\cdot\mathbf{u}\right)dV_\mathbf{x} - \int_{\Gamma^\varepsilon_g}\mathbf{g}\cdot\mathbf{u}\,dA_\mathbf{x} + \int_{B^\varepsilon}\frac{1}{2}\mathbf{b}^\varepsilon(\mathbf{e}(\mathbf{u}))\cdot\mathbf{e}(\mathbf{u})\,dV_\mathbf{x}, \tag{65}$$

in the space of kinematically admissible displacements:

$$V^\varepsilon = \{\mathbf{u}\in H(\Omega^\varepsilon;R^3) : \mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma^\varepsilon_u\}, \tag{66}$$

where $H(\Omega^\varepsilon;R^3)$ is the space of the vector-valued functions on the set Ω^ε, which are continuous and differentiable as many times as necessary. Under suitable regularity assumptions, the existence of a unique minimizer $\mathbf{u}^\varepsilon$ in V^ε is ensured (Ciarlet, 1988, Theorem 6.3-2.).

Using the changes of variables (5)–(8), the rescaled energy takes the form:

$$\mathcal{E}^\varepsilon(\hat{\mathbf{u}}^\varepsilon,\bar{\mathbf{u}}^\varepsilon) := \int_{\Omega_\pm}\left(\frac{1}{2}\mathbf{a}_\pm(\mathbf{e}(\bar{\mathbf{u}}^\varepsilon))\cdot\mathbf{e}(\bar{\mathbf{u}}^\varepsilon) - \bar{\mathbf{f}}\cdot\bar{\mathbf{u}}^\varepsilon\right)dV_\mathbf{z} - \int_{\Gamma_g}\bar{\mathbf{g}}\cdot\bar{\mathbf{u}}^\varepsilon\,dA_\mathbf{z} + \int_B\frac{1}{2}\left(\varepsilon^{-1}\mathbf{K}^{33}_\varepsilon(\hat{\mathbf{u}}^\varepsilon_{,3})\cdot\hat{\mathbf{u}}^\varepsilon_{,3} + 2\mathbf{K}^{\alpha 3}_\varepsilon\left(\hat{\mathbf{u}}^\varepsilon_{,\alpha}\right)\cdot\hat{\mathbf{u}}^\varepsilon_{,3} + \varepsilon\mathbf{K}^{\alpha\beta}_\varepsilon\left(\hat{\mathbf{u}}^\varepsilon_{,\alpha}\right)\cdot\hat{\mathbf{u}}^\varepsilon_{,\beta}\right)dV_\mathbf{z}. \tag{67}$$

4.1. Soft interphase

Substituting position (44), (45) and the expansions (11), (12) into the rescaled energy (67), it is obtained:

$$\mathcal{E}^\varepsilon(\hat{\mathbf{u}}^\varepsilon,\bar{\mathbf{u}}^\varepsilon) = \int_{\Omega_\pm}\left(\frac{1}{2}\mathbf{a}_\pm(\mathbf{e}(\bar{\mathbf{u}}^\varepsilon))\cdot\mathbf{e}(\bar{\mathbf{u}}^\varepsilon) - \bar{\mathbf{f}}\cdot\bar{\mathbf{u}}^\varepsilon\right)dV_\mathbf{z} - \int_{\Gamma_g}\bar{\mathbf{g}}\cdot\bar{\mathbf{u}}^\varepsilon\,dA_\mathbf{z} + \int_B\frac{1}{2}\left(\mathbf{K}^{33}\left(\hat{\mathbf{u}}^\varepsilon_{,3}\right)\cdot\hat{\mathbf{u}}^\varepsilon_{,3} + 2\varepsilon\mathbf{K}^{\alpha 3}\left(\hat{\mathbf{u}}^\varepsilon_{,\alpha}\right)\cdot\hat{\mathbf{u}}^\varepsilon_{,3} + \varepsilon^2\mathbf{K}^{\alpha\beta}(\hat{\mathbf{u}}^\varepsilon_{,\alpha})\cdot\hat{\mathbf{u}}^\varepsilon_{,\beta}\right)dV_\mathbf{z} = \mathcal{E}^0(\hat{\mathbf{u}}^0,\bar{\mathbf{u}}^0) + \varepsilon\,\mathcal{E}^1(\hat{\mathbf{u}}^0,\hat{\mathbf{u}}^1,\bar{\mathbf{u}}^0,\bar{\mathbf{u}}^1) + o(\varepsilon), \tag{68}$$

where:

$$\mathcal{E}^0(\hat{\mathbf{u}}^0,\bar{\mathbf{u}}^0) := \int_{\Omega_\pm}\left(\frac{1}{2}\mathbf{a}_\pm(\mathbf{e}(\bar{\mathbf{u}}^0))\cdot\mathbf{e}(\bar{\mathbf{u}}^0) - \bar{\mathbf{f}}\cdot\bar{\mathbf{u}}^0\right)dV_\mathbf{z} - \int_{\Gamma_g}\bar{\mathbf{g}}\cdot\bar{\mathbf{u}}^0\,dA_\mathbf{z} + \int_B\frac{1}{2}\mathbf{K}^{33}(\hat{\mathbf{u}}^0_{,3})\cdot\hat{\mathbf{u}}^0_{,3}\,dV_\mathbf{z}, \tag{69}$$

$$\mathcal{E}^1(\hat{\mathbf{u}}^0,\hat{\mathbf{u}}^1,\bar{\mathbf{u}}^0,\bar{\mathbf{u}}^1) := \int_{\Omega_\pm}\left(\mathbf{a}_\pm(\mathbf{e}(\bar{\mathbf{u}}^0))\cdot\mathbf{e}(\bar{\mathbf{u}}^1) - \bar{\mathbf{f}}\cdot\bar{\mathbf{u}}^1\right)dV_\mathbf{z} - \int_{\Gamma_g}\bar{\mathbf{g}}\cdot\bar{\mathbf{u}}^1\,dA_\mathbf{z} + \int_B\left(\mathbf{K}^{33}(\hat{\mathbf{u}}^0_{,3})\cdot\hat{\mathbf{u}}^1_{,3} + \mathbf{K}^{3\alpha}(\hat{\mathbf{u}}^0_{,3})\cdot\hat{\mathbf{u}}^0_{,\alpha}\right)dV_\mathbf{z}. \tag{70}$$

The two energies $\mathcal{E}^0$ and $\mathcal{E}^1$, defined in Eqs. (132) and (148), respectively, are minimized with respect to couples $(\bar{\mathbf{u}}^i,\hat{\mathbf{u}}^i)$, $i = 0,1,2$ in the set:

$$V = \left\{(\hat{\mathbf{u}},\bar{\mathbf{u}})\in H\left(\Omega_\pm;R^3\right)\times H(B;R^3) : \bar{\mathbf{u}} = \mathbf{0} \text{ on } \Gamma_u,\ \hat{\mathbf{u}}\left(\bar{\mathbf{z}},\pm\left(\frac{1}{2}\right)^\mp\right) = \bar{\mathbf{u}}\left(\bar{\mathbf{z}},\pm\left(\frac{1}{2}\right)^\pm\right),\ \bar{\mathbf{z}}\in S\right\}. \tag{71}$$

The minimization of the energy functionals is performed using the classical rules of calculus of variations; specifically, the Euler–Lagrange differential equations for the two energies are determined:

- Euler–Lagrange equation for the energy $\mathcal{E}^0$:

$$\int_{\Omega_\pm}(\mathbf{a}_\pm(\mathbf{e}(\bar{\mathbf{u}}^0))\cdot\mathbf{e}(\bar{\boldsymbol{\eta}}|) - \bar{\mathbf{f}}\cdot\bar{\boldsymbol{\eta}}|)\,dV_\mathbf{z} - \int_{\Gamma_g}\bar{\mathbf{g}}\cdot\bar{\boldsymbol{\eta}}|\,dA_\mathbf{z} + \int_B\mathbf{K}^{33}(\hat{\mathbf{u}}^0_{,3})\cdot\hat{\boldsymbol{\eta}}_{,3}\,dV_\mathbf{z} = 0, \tag{72}$$

where $\bar{\boldsymbol{\eta}}|,\hat{\boldsymbol{\eta}}|\in V$ are perturbations of $\bar{\mathbf{u}}^0,\hat{\mathbf{u}}^0$, respectively;
- Euler–Lagrange equations for the energy $\mathcal{E}^1$:

$$\int_{\Omega_\pm}(\mathbf{a}_\pm(\mathbf{e}(\bar{\mathbf{u}}^0))\cdot\mathbf{e}(\bar{\boldsymbol{\eta}}|) - \bar{\mathbf{f}}\cdot\bar{\boldsymbol{\eta}}|)\,dV_\mathbf{z} - \int_{\Gamma_g}\bar{\mathbf{g}}\cdot\bar{\boldsymbol{\eta}}|\,dA_\mathbf{z} + \int_B\mathbf{K}^{33}\left(\hat{\mathbf{u}}^0_{,3}\right)\cdot\hat{\boldsymbol{\eta}}|_{,3}\,dV_\mathbf{z} = 0, \tag{73}$$

$$\int_{\Omega_\pm}\mathbf{a}_\pm(\mathbf{e}(\bar{\mathbf{u}}^1)\cdot\mathbf{e}(\bar{\boldsymbol{\eta}}|))\,dV_\mathbf{z} + \int_B\left(\mathbf{K}^{33}(\hat{\mathbf{u}}^1_{,3}) + \mathbf{K}^{\alpha 3}(\hat{\mathbf{u}}^0_{,\alpha})\right)\cdot\hat{\boldsymbol{\eta}}|_{,3}\,dV_\mathbf{z} + \int_B\mathbf{K}^{3\alpha}\left(\hat{\mathbf{u}}^0_{,3}\right)\cdot\hat{\boldsymbol{\eta}}|_{,\alpha}\,dV_\mathbf{z} = 0, \tag{74}$$

where $(\bar{\boldsymbol{\eta}}|,\hat{\boldsymbol{\eta}}|)\in V$ are perturbations of $(\bar{\mathbf{u}}^1,\hat{\mathbf{u}}^1)$ in (73) and of $(\bar{\mathbf{u}}^0,\hat{\mathbf{u}}^0)$ in (74), respectively.

In view of the arbitrariness of the perturbations, Eqs. (72) and (73) are identical, therefore it is sufficient to consider only the minimization of the highest order energy to derive all the Euler–Lagrange equations governing the problem.

Notably, the fact that the same equations are obtained (cfr. (72) and (73)) is at the base of the equivalence between the asymptotic method based on energy minimization and the asymptotic method based on the strong formulation and matching asymptotic expansions. A similar situation will occur in the case of a hard material, where the six Euler–Lagrange equations reduce to three equations (see Eqs. (93)–(95) below).

From (73), using standard arguments, the following equilibrium equations are obtained:

$$\mathrm{div}(\mathbf{a}_\pm(\mathbf{e}(\bar{\mathbf{u}}^0))) + \bar{\mathbf{f}} = \mathbf{0} \quad \text{in } \Omega_\pm, \tag{75}$$

$$\mathbf{a}_\pm(\mathbf{e}(\bar{\mathbf{u}}^0))\mathbf{n} = \bar{\mathbf{g}} \quad \text{on } \Gamma_g, \tag{76}$$

$$\mathbf{a}_\pm(\mathbf{e}(\bar{\mathbf{u}}^0))\mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega_\pm \setminus (\Gamma_g \cup \Gamma_u \cup S^\pm), \tag{77}$$

$$(\mathbf{K}^{33}\hat{\mathbf{u}}^0_{,3})_{,3} = \mathbf{0} \quad \text{in } B, \tag{78}$$

$$\mathbf{a}_\pm(\mathbf{e}(\bar{\mathbf{u}}^0))\mathbf{i}_3 = \mathbf{K}^{33}\hat{\mathbf{u}}^0_{,3} \quad \text{on } S^\pm, \tag{79}$$

where $\mathbf{n}$ is taken to denote the outward normal. Eqs. (75)–(77) are the equilibrium equations of the adherents, with the suitable boundary conditions. Eq. (78) shows that $\mathbf{K}^{33}\hat{\mathbf{u}}^0_{,3}$ does not depend on z_3 in B. This result together with condition (79) imply the continuity of the traction vector, and thus Eq. (28) is reobtained. Integration of Eq. (78) with respect to z_3 and use of (79) give again relationship (49) and then (50), up to substituting $\bar{\mathbf{u}}^0$ with $\hat{\mathbf{u}}^0$ which are equal at the surfaces $S^\pm$.

Conversely, it can be remarked that Eq. (78) can be also obtained by the combination of Eq. (49), written for $j = 3$, and (28). Note also that Eqs. (36) and (38) together with (41) and (49), written for $j = 3$, imply (79).

On use of the divergence and Gauss Green theorems, Eq. (74) yields the equilibrium equations:

$$\mathrm{div}(\mathbf{a}_\pm(\mathbf{e}(\bar{\mathbf{u}}^1))) = \mathbf{0} \quad \text{in } \Omega_\pm, \tag{80}$$

$$\mathbf{a}_\pm(\mathbf{e}(\bar{\mathbf{u}}^1))\mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_g, \tag{81}$$

$$\mathbf{a}_\pm(\mathbf{e}(\bar{\mathbf{u}}^1))\mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega_\pm \setminus (\Gamma_g \cup \Gamma_u \cup S^\pm), \tag{82}$$

$$\mathbf{K}^{33}\hat{\mathbf{u}}^1_{,33} + (\mathbf{K}^{\alpha 3} + \mathbf{K}^{3\alpha})\hat{\mathbf{u}}^0_{,3\alpha} = \mathbf{0} \quad \text{in } B, \tag{83}$$

$$\mathbf{a}_\pm(\mathbf{e}(\bar{\mathbf{u}}^1))\mathbf{i}_3 = \mathbf{K}^{33}\hat{\mathbf{u}}^1_{,3} + \mathbf{K}^{\alpha 3}\hat{\mathbf{u}}^0_{,\alpha} \quad \text{on } S^\pm, \tag{84}$$

up to a term on the lateral boundary of B, which will be discussed in Section 5.4. Eqs. (80)–(82) are the equilibrium equations of the adherents at the higher (one) order, with the suitable boundary conditions.

Eqs. (83), (84) are equivalent to Eqs. (51) and (56), up to the continuity conditions (37), (39) and the constitutive equation of the adherent (41), thus providing the interface laws (56). To show it, note that in view of (78), $\hat{\mathbf{u}}^0$ can be written in the useful form:

$$\hat{\mathbf{u}}^0(\bar{\mathbf{z}}, z_3) = [\bar{\mathbf{u}}^0](\bar{\mathbf{z}})z_3 + \langle\bar{\mathbf{u}}^0\rangle(\bar{\mathbf{z}}), \tag{85}$$

where the condition $\hat{\mathbf{u}}^0 = \bar{\mathbf{u}}^0$ on $S^\pm$ has been also taken into account. Integrating Eq. (83) with respect to z_3 gives:

$$\mathbf{K}^{33}\hat{\mathbf{u}}^1_{,3} + (\mathbf{K}^{\alpha 3} + \mathbf{K}^{3\alpha})\hat{\mathbf{u}}^0_{,\alpha} = \boldsymbol{\phi}(\bar{\mathbf{z}}), \quad \text{in } B, \tag{86}$$

with $\boldsymbol{\phi}$ independent of z_3 and to be determined. Substituting (85) into (86) and integrating with respect to z_3 between $-1/2$ and $1/2$ allow to determine $\boldsymbol{\phi}(\bar{z})$:

$$\mathbf{K}^{33}[\hat{\mathbf{u}}^1] + (\mathbf{K}^{\alpha 3} + \mathbf{K}^{3\alpha})\langle\hat{\mathbf{u}}^0\rangle_{,\alpha} = \boldsymbol{\phi}(\bar{\mathbf{z}}), \quad \bar{\mathbf{z}} \text{ in } S. \tag{87}$$

Eliminating $\boldsymbol{\phi}$ from (86) and (87) and rearranging the terms give:

$$\mathbf{K}^{33}\hat{\mathbf{u}}^1_{,3} = \mathbf{K}^{33}[\hat{\mathbf{u}}^1] - (\mathbf{K}^{\alpha 3} + \mathbf{K}^{3\alpha})[\hat{\mathbf{u}}^0]_{,\alpha} z_3, \quad (\bar{\mathbf{z}}, z_3) \text{ in } B. \tag{88}$$

Substituting the latter result into Eq. (84), using the definition of $\langle\bar{\mathbf{u}}^0\rangle$, simplifying and introducing the notation $\bar{\boldsymbol{\sigma}}^1_\pm := \mathbf{a}_\pm(\mathbf{e}(\bar{\mathbf{u}}^1))$, relation (84) leads to Eq. (56).

The converse equivalence, which ensures that the equations determined via strong formulation can lead to the one recovered via variational approach, can also be proved.

On the basis of the above results, the two approaches, the matching expansions method and the energy based formulation, are equivalent, leading to the same governing equations.

4.2. Hard interphase

For "hard" interphase, one can proceed as done for the case of a soft interphase, substituting position (57) and the expansions (11), (12) into the rescaled energy (67); then, it is obtained:

$$\begin{aligned} \mathcal{E}^\varepsilon(\hat{\mathbf{u}}^\varepsilon, \bar{\mathbf{u}}^\varepsilon) &= \int_{\Omega_\pm} \left(\frac{1}{2}\mathbf{a}_\pm(\mathbf{e}(\bar{\mathbf{u}}^\varepsilon)) \cdot \mathbf{e}(\bar{\mathbf{u}}^\varepsilon) - \bar{\mathbf{f}} \cdot \bar{\mathbf{u}}^\varepsilon\right) dV_{\mathbf{z}} - \int_{\Gamma_g} \bar{\mathbf{g}} \cdot \bar{\mathbf{u}}^\varepsilon \, dA_{\mathbf{z}} \\ &\quad + \int_B \frac{1}{2}\Big(\varepsilon^{-1}K^{33}(\hat{\mathbf{u}}^\varepsilon_{,3}) \cdot \hat{\mathbf{u}}^\varepsilon_{,3} + 2\mathbf{K}^{\alpha 3}(\hat{\mathbf{u}}^\varepsilon_{,\alpha}) \cdot \hat{\mathbf{u}}^\varepsilon_{,3} \\ &\quad + \varepsilon\mathbf{K}^{\alpha\beta}(\hat{\mathbf{u}}^\varepsilon_{,\alpha}) \cdot \hat{\mathbf{u}}^\varepsilon_{,\beta}\Big) \, dV_{\mathbf{z}} \\ &= \mathcal{E}^{-1}(\hat{\mathbf{u}}^0) + \mathcal{E}^0(\bar{\mathbf{u}}^0, \hat{\mathbf{u}}^0, \hat{\mathbf{u}}^1) + \varepsilon\, \mathcal{E}^1(\bar{\mathbf{u}}^0, \hat{\mathbf{u}}^0, \bar{\mathbf{u}}^1, \hat{\mathbf{u}}^1, \hat{\mathbf{u}}^2) + o(\varepsilon), \end{aligned} \tag{89}$$

where:

$$\mathcal{E}^{-1}(\hat{\mathbf{u}}^0) := \int_B \frac{1}{2}\mathbf{K}^{33}(\hat{\mathbf{u}}^0_{,3}) \cdot \hat{\mathbf{u}}^0_{,3} \, dV_{\mathbf{z}}, \tag{90}$$

$$\begin{aligned} \mathcal{E}^0(\bar{\mathbf{u}}^0, \hat{\mathbf{u}}^0, \hat{\mathbf{u}}^1) &:= \int_{\Omega_\pm} \left(\frac{1}{2}\mathbf{a}_\pm(\mathbf{e}(\bar{\mathbf{u}}^0)) \cdot \mathbf{e}(\bar{\mathbf{u}}^0) - \bar{\mathbf{f}} \cdot \bar{\mathbf{u}}^0\right) dV_{\mathbf{z}} \\ &\quad - \int_{\Gamma_g} \bar{\mathbf{g}} \cdot \bar{\mathbf{u}}^0 \, dA_{\mathbf{z}} + \int_B (\mathbf{K}^{33}(\hat{\mathbf{u}}^0_{,3}) \cdot \hat{\mathbf{u}}^1_{,3} \\ &\quad + \mathbf{K}^{3\alpha}(\hat{\mathbf{u}}^0_{,3}) \cdot \hat{\mathbf{u}}^0_{,\alpha}) \, dV_{\mathbf{z}}, \end{aligned} \tag{91}$$

$$\begin{aligned} &\mathcal{E}^1(\bar{\mathbf{u}}^0, \hat{\mathbf{u}}^0, \bar{\mathbf{u}}^1, \hat{\mathbf{u}}^1, \hat{\mathbf{u}}^2) : \\ &= \int_{\Omega_\pm} (\mathbf{a}_\pm(\mathbf{e}(\bar{\mathbf{u}}^0)) \cdot \mathbf{e}(\bar{\mathbf{u}}^1) - \bar{\mathbf{f}} \cdot \bar{\mathbf{u}}^1) \, dV_{\mathbf{z}} - \int_{\Gamma_g} \bar{\mathbf{g}} \cdot \bar{\mathbf{u}}^1 \, dA_{\mathbf{u}} \\ &\quad + \int_B \left(\mathbf{K}^{33}(\hat{\mathbf{u}}^0_{,3}) \cdot \hat{\mathbf{u}}^2_{,3} + \frac{1}{2}\mathbf{K}^{33}(\hat{\mathbf{u}}^1_{,3}) \cdot \hat{\mathbf{u}}^1_{,3}\right) dV_{\mathbf{z}} \\ &\quad + \int_B \left(\mathbf{K}^{\alpha 3}(\hat{\mathbf{u}}^0_{,\alpha}) \cdot \hat{\mathbf{u}}^1_{,3} + \mathbf{K}^{3\alpha}(\hat{\mathbf{u}}^0_{,3}) \cdot \hat{\mathbf{u}}^1_{,\alpha} + \frac{1}{2}\mathbf{u}^{\alpha\beta}(\hat{\mathbf{u}}^0_{,\alpha}) \cdot \hat{\mathbf{u}}^0_{,\beta}\right) dV_{\mathbf{u}}. \end{aligned} \tag{92}$$

Minimizing these three energies in the set V defined as in (71) gives six Euler–Lagrange equations, which can be reduced to the following three independent equations:

$$\int_B \mathbf{K}^{33}(\hat{\mathbf{u}}^0_{,3}) \cdot \hat{\boldsymbol{\eta}}|_{,3} \, dV_{\mathbf{z}} = 0, \tag{93}$$

$$\begin{aligned} &\int_{\Omega_\pm} (\mathbf{a}_\pm(\mathbf{e}(\bar{\mathbf{u}}^0)) \cdot e(\bar{\boldsymbol{\eta}}|) - \bar{\mathbf{f}} \cdot \bar{\boldsymbol{\eta}}|) \, dV_{\mathbf{z}} - \int_{\Gamma_g} \bar{\mathbf{g}} \cdot \bar{\boldsymbol{\eta}}| \, dA_{\mathbf{z}} + \int_B \Big(\mathbf{K}^{33}(\hat{\mathbf{u}}^1_{,3}) \cdot \hat{\mathbf{u}}_{,3} \\ &\quad + K^{\alpha 3}(\hat{\mathbf{u}}^0_{,\alpha}) \cdot \hat{\boldsymbol{\eta}}|_{,3} + \mathbf{K}^{3\alpha}(\hat{\mathbf{u}}^0_{,3}) \cdot \hat{\boldsymbol{\eta}}|_{,\alpha}\Big) \, dV_{\mathbf{z}} = 0, \end{aligned} \tag{94}$$

$$\begin{aligned} &\int_{\Omega_\pm} \mathbf{a}_\pm(\mathbf{e}(\bar{\mathbf{u}}^1) \cdot e(\bar{\boldsymbol{\eta}}|)) \, dV_{\mathbf{z}} + \int_B \left(\mathbf{K}^{33}(\hat{\mathbf{u}}^2_{,3}) + \mathbf{K}^{\alpha 3}(\hat{\mathbf{u}}^1_{,\alpha})\right) \cdot \hat{\boldsymbol{\eta}}|_{,3} \, dV_{\mathbf{z}} \\ &\quad + \int_B \left(\mathbf{K}^{3\beta}(\hat{\mathbf{u}}^1_{,3}) + \mathbf{K}^{\alpha\beta}(\hat{\mathbf{u}}^0_{,\alpha})\right) \cdot \hat{\boldsymbol{\eta}}|_{,\beta} \, dV_{\mathbf{z}} = 0, \end{aligned} \tag{95}$$

with $(\bar{\boldsymbol{\eta}}|, \hat{\boldsymbol{\eta}}|)$ perturbations in V. From Eq. (93) and the arbitrariness of $\hat{\boldsymbol{\eta}}|$, it results:

$$\mathbf{K}^{33}(\hat{\mathbf{u}}^0_{,33}) = \mathbf{0} \quad \text{in } B, \tag{96}$$

$$\mathbf{K}^{33}(\hat{\mathbf{u}}^0_{,3}) = \mathbf{0} \quad \text{on } S^\pm, \tag{97}$$

implying $\hat{\mathbf{u}}^0_{,3} = \mathbf{0}$ in B, which is exactly (60).

Using the divergence theorem and the arbitrariness of $\bar{\eta}, \hat{\eta} \in V$ in (94), Eqs. (75)–(77) are reobtained. Moreover, the following additional conditions are recovered:

$$\mathbf{K}^{33}\hat{\mathbf{u}}^1_{,33} + (\mathbf{K}^{\alpha 3} + \mathbf{K}^{3\alpha})\hat{\mathbf{u}}^0_{,3\alpha} = \mathbf{0} \quad \text{in } B, \tag{98}$$

$$\mathbf{a}_\pm(\mathbf{e}(\bar{\mathbf{u}}^0))\mathbf{i}_3 = \mathbf{K}^{33}\hat{\mathbf{u}}^1_{,3} + \mathbf{K}^{\alpha 3}\hat{\mathbf{u}}^0_{,\alpha} \quad \text{on } S^\pm, \tag{99}$$

and a term on the lateral boundary of B, which vanishes because $\hat{\mathbf{u}}^0_{,3} = \mathbf{0}$ in B. For the same reason, Eq. (98) implies that $\hat{\mathbf{u}}^1_{,33} = \mathbf{0}$ in B, i.e. $\hat{\mathbf{u}}^1$ admits a representation of the form:

$$\hat{\mathbf{u}}^1(\bar{\mathbf{z}}, z_3) = [\bar{\mathbf{u}}^1](\bar{\mathbf{z}})z_3 + \langle\bar{\mathbf{u}}^1\rangle(\bar{\mathbf{z}}), \tag{100}$$

where the continuity condition $\hat{\mathbf{u}}^1 = \bar{\mathbf{u}}^1$ on $S^{\pm}$ has been taken into account. Recalling that the terms on the right-hand side of Eq. (99) do not depend on z_3, the jump of stress vector across B vanishes, i.e.

$$\mathbf{a}_+\left(\mathbf{e}\left(\bar{\mathbf{u}}^0\left(\bar{\mathbf{z}},+\frac{1}{2}\right)\right)\right)\mathbf{i}_3 = \mathbf{a}_-\left(\mathbf{e}\left(\bar{\mathbf{u}}^0\left(\bar{\mathbf{z}},-\frac{1}{2}\right)\right)\right)\mathbf{i}_3, \tag{101}$$

which is just the continuity of the stress vector at the order zero (cf. Eq. (28)). Deriving expression (100) with respect to z_3 and substituting into (99), the following condition is obtained:

$$[\bar{\mathbf{u}}^1] = (\mathbf{K}^{33})^{-1}\left(\mathbf{a}_{\pm}(\mathbf{e}(\bar{\mathbf{u}}^0))\mathbf{i}_3 - \mathbf{K}^{\alpha3}\hat{\mathbf{u}}^0_{,\alpha}\right), \tag{102}$$

which allows to determine the displacement jump at the order 1, i.e. the Eq. (62) is recovered.

Using the divergence theorem and the arbitrariness of $\bar{\boldsymbol{\eta}}|, \hat{\boldsymbol{\eta}}| \in V$ in the stationary condition (95), Eqs. (80)–(82) are reobtained. In addition, the following equations are recovered:

$$\mathbf{K}^{33}\hat{\mathbf{u}}^2_{,33} + (\mathbf{K}^{\alpha3} + \mathbf{K}^{3\alpha})\hat{\mathbf{u}}^1_{,3\alpha} + \mathbf{K}^{\alpha\beta}\hat{\mathbf{u}}^0_{,\alpha\beta} = \mathbf{0} \quad \text{in } B, \tag{103}$$

$$\mathbf{a}_{\pm}(\mathbf{e}(\bar{\mathbf{u}}^1))\mathbf{i}_3 = \mathbf{K}^{33}\hat{\mathbf{u}}^2_{,3} + \mathbf{K}^{\alpha3}\hat{\mathbf{u}}^1_{,\alpha} \quad \text{on } S^{\pm}, \tag{104}$$

and a term on the lateral boundary of B, which will be discussed in Section 5.4. Using a procedure similar to the one adopted in Eqs. (86)–(88), Eq. (103) gives:

$$\mathbf{K}^{33}\hat{\mathbf{u}}^2_{,3} = \mathbf{K}^{33}[\hat{\mathbf{u}}^2] - \left((\mathbf{K}^{\alpha3}+\mathbf{K}^{3\alpha})[\hat{\mathbf{u}}^1]_{,\alpha} + \mathbf{K}^{\alpha\beta}\hat{\mathbf{u}}^0_{,\alpha\beta}\right)z_3, \quad (\bar{\mathbf{z}},z_3) \text{ in } B. \tag{105}$$

Substituting the latter result into Eq. (104), evaluated at $z_3 = \pm\frac{1}{2}$, gives the two relations:

$$\mathbf{a}_{\pm}(\mathbf{e}(\bar{\mathbf{u}}^1))\mathbf{i}_3 = \mathbf{K}^{33}[\hat{\mathbf{u}}^2] \mp \frac{1}{2}\left(\mathbf{K}^{3\alpha}[\hat{\mathbf{u}}^1]_{,\alpha} + \mathbf{K}^{\alpha\beta}\hat{\mathbf{u}}^0_{,\alpha\beta}\right) + \mathbf{K}^{\alpha3}\langle\bar{\mathbf{u}}^1\rangle_{,\alpha} \text{ on } S^{\pm}. \tag{106}$$

which, subtracted each other, give in turn:

$$[\bar{\boldsymbol{\sigma}}^1\mathbf{i}_3] = -\left(\mathbf{K}^{3\alpha}[\hat{\mathbf{u}}^1]_{,\alpha} + \mathbf{K}^{\alpha\beta}\hat{\mathbf{u}}^0_{,\alpha\beta}\right) \quad \text{on } S^{\pm}. \tag{107}$$

with $\bar{\boldsymbol{\sigma}}^1_{\pm} := \mathbf{a}_{\pm}(\mathbf{e}(\bar{\mathbf{u}}^1))$. In view of expression (102), the latter formula leads exactly to Eq. (64).

The converse equivalence, which ensures that the equations determined via strong formulation can lead to the one recovered via variational approach, can also be proved for the zero and one order.

5. Overall response of the interface and concluding remarks

5.1. Comparison of interphase laws

In Table 1 the comparison of the equations governing the soft and the hard interphase models at the different levels is reported. In the upper part of the table, the references to the equations arising from the two different approaches (i.e., the matching and the energy approaches) at the various orders are summarized. Then, the equations governing the interphase laws at the orders zero and one are indicated.

The aim of the table is to illustrate the similarities and the differences between the soft and the hard interphase laws.

As for the similarities, the equilibrium equations for the adherents do not change in the two cases, being (75)–(77) at order zero and (80)–(82) at order one. Another common fact is the presence of jumps in the displacement and stress vector fields at the higher (one) order.

On the other hand, the interphase laws for soft and hard cases are quite different. In the soft case for the higher (one) order, the jumps in the stress and displacement vector fields (see Eq. (56)) depend on the first derivatives of the in-plane displacement at the lowest (zero) order, while in the hard case the same jumps are functions of the first and second derivatives of the in-plane displacement. Furthermore, in the case of the soft interphase, the relation between the stress vector and the displacement jump (see Eq. (64)) involves terms of the same order, both at the first (zero) and higher (one) order. On the contrary, this does not occur for the hard interphase, where the stress vector at higher (one) order depends only on terms at the lowest (zero) order.

It can be remarked that at the level minus one, no constitutive equations are written for the soft interphase; analogously, at level one, no constitutive equations are written for the hard interphase. Indeed in this second case, constitutive equations could be written if further terms were considered in the asymptotic development.

5.2. Soft interface

Using the matching relations (38) and (39), the interface laws calculated at order zero and order one can be rewritten in the final configuration represented in Fig. 2(c) as follows:

$$\boldsymbol{\sigma}^0(\cdot,0)\mathbf{i}_3 = \mathbf{K}^{33}[[\mathbf{u}^0]], \tag{108}$$

$$\begin{aligned}\boldsymbol{\sigma}^1(\cdot,0^{\pm})\mathbf{i}_3 = {}& \mathbf{K}^{33}\left([[\mathbf{u}^1]] + \langle\langle\mathbf{u}^0_{,3}\rangle\rangle\right) + \frac{1}{2}(\mathbf{K}^{\alpha3} \mp \mathbf{K}^{3\alpha})\mathbf{u}^0_{,\alpha}(\cdot,0^+) \\ &+ \frac{1}{2}(\mathbf{K}^{\alpha3} \pm \mathbf{K}^{3\alpha})\mathbf{u}^0_{,\alpha}(\cdot,0^-) \mp \frac{1}{2}\boldsymbol{\sigma}^0_{,3}(\cdot,0^{\pm})\mathbf{i}_3.\end{aligned} \tag{109}$$

Eq. (108) is the classical imperfect (spring-type) interface law characterized by a finite stiffness of the interphase. Eq. (109) allows to evaluate the stress vector at the higher (one) order which depends not only on displacement jump at the higher (one) order but also on the displacement and stress fields evaluated at the first (zero) order and their derivatives.

The stress field in the interface can be obtained from Eq. (22), by taking into account the response at the orders zero and one given by (108) and (109), respectively. Finally, it results:

$$\begin{aligned}\boldsymbol{\sigma}^{\varepsilon}(\cdot,0^{\pm})\mathbf{i}_3 \approx {}& \mathbf{K}^{33}[[\mathbf{u}^0]] + \varepsilon\Big(\mathbf{K}^{33}\left([[\mathbf{u}^1]] + \langle\langle\mathbf{u}^0_{3}\rangle\rangle\right) \\ &+ \frac{1}{2}(\mathbf{K}^{\alpha3} \mp \mathbf{K}^{3\alpha})\mathbf{u}^0_{,\alpha}(\cdot,0^+) + \frac{1}{2}(\mathbf{K}^{\alpha3} \pm \mathbf{K}^{3\alpha})\mathbf{u}^0_{,\alpha}(\cdot,0^-) \mp \frac{1}{2}\boldsymbol{\sigma}^0_{,3}(\cdot,0^{\pm})\mathbf{i}_3\Big).\end{aligned} \tag{110}$$

It can be remarked that the latter relation improves the classic interface law at order zero by linearly linking the stress vector and the relative displacement via a higher order term, involving the in-plane first derivatives of the displacement. Moreover, the stress vector is no longer continuous, as it occurs at the lowest (zero) order; indeed, inspection of (110) clearly shows that the

Table 1
Synthesis of the asymptotic analysis.

Kind of interface	Soft	Hard
Constitutive equation level −1	–	Eq. $(59)_1$
Constitutive equation level 0	Eq. $(47)_1$	Eq. $(59)_2$
Constitutive equation level 1	Eq. $(47)_2$	–
Interphase laws Order 0	Eq. (50) [Eq. (78) and (79)]	Eq. (60) [Eqs. (96) and (97)]
Interphase laws Order 1	Eq. (56) [Eq. (83) and (84)]	Eq. (64) [Eqs. (103) and (104)]

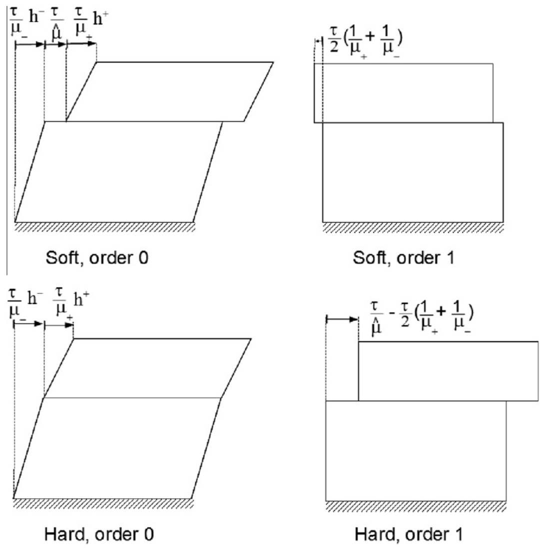

Fig. 2. Deformed configurations of composite blocks with soft and hard adhesive undergoing shear.

stress vector takes different values on the top and the bottom of the interface.

5.3. Hard interface

Using the matching relations (36)–(39) the interface laws calculated at order one can be rewritten in the final configuration represented in Fig. 1(c) as follows:

$$[[\mathbf{u}^0]] = \mathbf{0}, \tag{111}$$

$$[[\mathbf{u}^1]] = -(\mathbf{K}^{33})^{-1}\left(\boldsymbol{\sigma}^0\mathbf{i}_3 - \mathbf{K}^{\alpha 3}\mathbf{u}^0_{,\alpha}\right) - \langle\langle \mathbf{u}^0_{,3}\rangle\rangle, \tag{112}$$

$$[[\boldsymbol{\sigma}^0\ \mathbf{i}_3]] = \mathbf{0}, \tag{113}$$

$$[[\boldsymbol{\sigma}^1\ \mathbf{i}_3]] = \left(-\mathbf{K}^{\alpha\beta}\mathbf{u}^0_{,\beta} + \mathbf{K}^{3\alpha}(\mathbf{K}^{33})^{-1}\left(\boldsymbol{\sigma}^0\mathbf{i}_3 - \mathbf{K}^{\beta 3}\mathbf{u}^0_{,\beta}\right)\right)_{,\alpha} - \langle\langle \boldsymbol{\sigma}^0_{,3}\ \mathbf{i}_3\rangle\rangle. \tag{114}$$

Eqs. (111) and (113) represent the classical perfect interface law characterized by the continuity of the displacement and stress vector fields. Eqs. (112) and (114) are imperfect interface conditions, allowing jumps in the displacement and in the stress vector fields at the higher (one) order across S. In fact, these jumps depend on the displacement and the stress fields at the first (zero) order and on their first and second derivatives.

The constitutive law for the hard interface written in terms of jumps in the displacement and in the stress from the final configuration (Fig. 2(c)) can be obtained from Eq. (10) with (111) and (112), and from Eqs. (21), (113) and (114), respectively, leading to:

$$[[\mathbf{u}^\varepsilon]] = -\varepsilon\left((\mathbf{K}^{33})^{-1}\left(\boldsymbol{\sigma}^0\mathbf{i}_3 + \mathbf{K}^{\alpha 3}\mathbf{u}^0_{,\alpha}\right) - \langle\langle \mathbf{u}^0_{,3}\rangle\rangle\right), \tag{115}$$

$$[[\boldsymbol{\sigma}^\varepsilon\ \mathbf{i}_3]] = \varepsilon\left(\left(-\mathbf{K}^{\alpha\beta}\mathbf{u}^0_{,\beta} + \mathbf{K}^{3\alpha}(\mathbf{K}^{33})^{-1}\left(\boldsymbol{\sigma}^0\mathbf{i}_3 - \mathbf{K}^{\beta 3}\mathbf{u}^0_{,\beta}\right)\right)_{,\alpha} - \langle\langle \boldsymbol{\sigma}^0_{,3}\ \mathbf{i}_3\rangle\rangle\right). \tag{116}$$

5.4. Emerging forces at the adhesive-adherent interface boundary

Integrating by part the last term of the weak form of the equilibrium Eq. (74), stresses arise on the boundary $\partial S \times \{-\frac{1}{2}, \frac{1}{2}\}$. The resultant of these stresses can be considered as a force applied at boundary of interface S, and it can be evaluated as:

$$\mathbf{F}_{soft} = \varepsilon\mathbf{K}^{3\alpha}(\hat{\mathbf{u}}^0_{,3})n_\alpha = \varepsilon\mathbf{K}^{3\alpha}(\mathbf{K}^{33})^{-1}(\hat{\sigma}^0\mathbf{i}_3)n_\alpha, \tag{117}$$

where Eq. (49) is used.

Analogously, after integrating by part the last term of Eq. (95), the following emerging force arises:

$$\begin{aligned}\mathbf{F}_{hard} &= \varepsilon\left(\mathbf{K}^{3\beta}(\hat{\mathbf{u}}^1_{,3}) + \mathbf{K}^{\alpha\beta}(\hat{\mathbf{u}}^0_{,\alpha})\right)_{,\beta} n_\beta \\ &= \varepsilon\left(\mathbf{K}^{3\alpha}(\mathbf{K}^{33})^{-1}(\hat{\sigma}^0\mathbf{i}_3) + (\mathbf{K}^{\alpha\beta} - \mathbf{K}^{3\alpha}(\mathbf{K}^{33})^{-1}\mathbf{K}^{\beta 3})\hat{\mathbf{u}}^0_{,\beta}\right)n_\alpha. \end{aligned}\tag{118}$$

The presence of these forces is not directly taken into account by the interface laws. Therefore, in order to satisfy the equilibrium Eqs. (74) for a soft interface and (95) for a hard interface, additional terms have to be introduced in the expansions of the stress or of the displacement fields. In particular, in Le Dret and Raoult (1995), the authors have inserted in the expansion of the displacement at order one in the adherent a term denoted as $\bar{w}^1$. The introduction of this term yields the further condition

$$\bar{\sigma}^1\mathbf{n} = \bar{\mathbf{F}} \quad \text{on } \partial S \times \{-1/2, 1/2\}, \tag{119}$$

with $\mathbf{n} = n_\alpha\mathbf{i}_\alpha$, and $\mathbf{F} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{\mathbf{F}}\, dz_3$ given by (117) and (118) in the soft and the hard case, respectively.

5.5. Formal equivalence between soft and hard theories

It is well known that, at order zero, it is possible to recover from the soft interface model the hard interface model by infinitely increasing the stiffness of the interphase material. The question is if it is possible to obtain the same result at order one.

To this end, let us denote $\tilde{\mathbf{u}} := \bar{\mathbf{u}}^0 + \varepsilon\bar{\mathbf{u}}^1$ and $\tilde{\sigma} := \bar{\sigma}^0 + \varepsilon\bar{\sigma}^1$. These fields are approximations of $\bar{u}^\varepsilon, \bar{\sigma}^\varepsilon$ at the first order.

Using the interface laws (50) and (55), for the soft interphase, it results:

$$\begin{aligned}[\tilde{\mathbf{u}}] &= (\mathbf{K}^{33})^{-1}\left(\bar{\sigma}^0\mathbf{i}_3 + \varepsilon\langle\bar{\sigma}^1\mathbf{i}_3\rangle + \varepsilon\mathbf{K}^{\alpha 3}\langle\bar{\mathbf{u}}^0_{,\alpha}\rangle\right) \\ &= (\mathbf{K}^{33})^{-1}\left(\langle\tilde{\sigma}\mathbf{i}_3\rangle + \varepsilon\mathbf{K}^{\alpha 3}\langle\tilde{\mathbf{u}}_{,\alpha}\rangle\right) + o(\varepsilon^2),\end{aligned}\tag{120}$$

It is now shown that this interface law is general enough to describe the interface laws prescribing the displacement jump in the case of a hard interface, after a suitable rescaling of the matrices $\mathbf{K}^{ij}$ and up to neglecting higher order terms in ε. Indeed, to simulate the case of a hard interface, the matrices $\mathbf{K}^{ij}$ with $\varepsilon^{-1}\mathbf{K}^{ij}$ and the relations:

$$\varepsilon\langle\tilde{\sigma}\mathbf{i}_3\rangle = \varepsilon(\bar{\sigma}^0 + \varepsilon\langle\bar{\sigma}^1\rangle)i_3 = \varepsilon\,\bar{\sigma}^0\mathbf{i}_3 + o(\varepsilon^2) = \varepsilon\,\tilde{\sigma}\mathbf{i}_3 + o(\varepsilon^2), \tag{121}$$

$$\varepsilon\langle\tilde{\mathbf{u}}_{,\alpha}\rangle = \varepsilon(\bar{\mathbf{u}}^0_{,\alpha} + \varepsilon\langle\bar{\mathbf{u}}^1_{,\alpha}\rangle) = \varepsilon\,\bar{\mathbf{u}}^0_{,\alpha} + o(\varepsilon^2) = \varepsilon\,\tilde{\mathbf{u}}_{,\alpha} + o(\varepsilon^2), \tag{122}$$

are substituted into (120) to obtain:

$$[\tilde{\mathbf{u}}] = (\mathbf{K}^{33})^{-1}\left(\varepsilon\tilde{\sigma}\mathbf{i}_3 + \varepsilon\,\mathbf{K}^{\alpha 3}\tilde{\mathbf{u}}_{,\alpha}\right) + o(\varepsilon^2), \tag{123}$$

which, formally, is the interface law governing the displacement jump for the hard case.

Note that replacing $\mathbf{K}^{ij}$ with $\varepsilon\mathbf{K}^{ij}$ in (123) and taking into account the discontinuity of the traction vector yield back (120).

Analogously, in view of (67) and (72) the following general relation holds true for the hard case:

$$[\tilde{\sigma}\mathbf{i}_3] = -\varepsilon\mathbf{K}^{3\alpha}(\mathbf{K}^{33})^{-1}(\tilde{\sigma}\mathbf{i}_3 - \mathbf{K}^{\beta 3}\mathbf{u}_{,\alpha})_{,\alpha} - \mathbf{K}^{\alpha\beta}\mathbf{u}_{,\alpha\beta}. \tag{124}$$

To simulate the soft case, we substitute $\mathbf{K}^{ij}$ with $\varepsilon\mathbf{K}^{ij}$ in (124) to get:

$$\begin{aligned}[\tilde{\sigma}\mathbf{i}_3] &= -\varepsilon\mathbf{K}^{3\alpha}(\mathbf{K}^{33})^{-1}(\tilde{\sigma}\mathbf{i}_3 - \varepsilon\mathbf{K}^{\beta 3}\mathbf{u}_{,\alpha})_{,\alpha} - \varepsilon\mathbf{K}^{\alpha\beta}\mathbf{u}_{,\alpha\beta} \\ &= \varepsilon\mathbf{K}^{3\alpha}(\mathbf{K}^{33})^{-1}(\tilde{\sigma}\mathbf{i}_3)_{,\alpha} + o(\varepsilon^2),\end{aligned}\tag{125}$$

which is formally equivalent to the relation governing the traction jump in the soft case.

5.6. Other form of the soft interface law

The hard interface law is often written in terms of jumps in the displacement and in the stress, whereas the soft interface law is written as a relation between stress vector and jump in the displacement. In this section, the soft interface law is rewritten in terms of jumps, as done for the law of a hard interface. For this purpose, the conditions (56) written on S^+ and on S^- are added together to obtain:

$$[\bar{\mathbf{u}}^1] = (\mathbf{K}^{33})^{-1}\left(\langle\bar{\sigma}^1\mathbf{i}_3\rangle + \mathbf{K}^{\alpha 3}\langle\bar{\mathbf{u}}^0_{,\alpha}\rangle\right). \tag{126}$$

On the other hand, subtracting the two conditions (56) gives:

$$[\bar{\sigma}^1\mathbf{i}_3] = -\mathbf{K}^{3\alpha}\langle\bar{\mathbf{u}}^0\rangle_{,\alpha}. \tag{127}$$

The two conditions (126) and (127) taken together are equivalent to conditions (56) and they show that the soft interface laws at order one prescribe the jumps in the displacement and in the stress vector fields.

5.7. Condensed form of the hard interface law

In this section, a condensed form of the hard interface law is proposed, i.e. a form which summarizes the interface laws at orders zero and one in only one couple of equations. To this end, after taking into account Eqs. (11), (60) and (62), the jump of displacement in the rescaled adhesive results:

$$[\hat{\mathbf{u}}^\varepsilon] \approx -\varepsilon(\mathbf{K}^{33}_\varepsilon)^{-1}\left(\hat{\boldsymbol{\sigma}}^\varepsilon\mathbf{i}_3 - \mathbf{K}^{\alpha 3}_\varepsilon\hat{\mathbf{u}}^\varepsilon_{,\alpha}\right). \tag{128}$$

Analogously, after taking into account Eqs. (22), (61) and (64), the jump of the interface stress in the rescaled adhesive results:

$$\begin{aligned}[\hat{\boldsymbol{\sigma}}^\varepsilon\mathbf{i}_3] &\approx \varepsilon\left(-\mathbf{K}^{\alpha\beta}\hat{\mathbf{u}}^\varepsilon_{,\beta} - \mathbf{K}^{3\alpha}[\hat{\mathbf{u}}^\varepsilon]\right)_{,\alpha}\\ &\approx \varepsilon\left(-\mathbf{K}^{\alpha\beta}_\varepsilon\hat{\mathbf{u}}^\varepsilon_{,\beta} + \mathbf{K}^{3\alpha}_\varepsilon(\mathbf{K}^{33}_\varepsilon)^{-1}\left(\hat{\boldsymbol{\sigma}}^\varepsilon\mathbf{i}_3 - \mathbf{K}^{\beta 3}_\varepsilon\hat{\mathbf{u}}^\varepsilon_{,\beta}\right)\right)_{,\alpha}.\end{aligned} \tag{129}$$

This implicit formulation could be more useful for a numerical implementation.

5.8. Examples

To conclude this Section and to remark the differences among the zero and one order interface models and among the soft and hard constitutive laws, two examples are reported.

The first one is the shear of a composite block. Due to its simplicity, the closed form solution for a block with an interphase is available and directly comparable to the approximated solution obtained with the interface laws calculated in this paper. An even simplified version of this example was given in Lebon and Rizzoni (2010) only for the case of a hard interface, and it is reproposed here because it allows a direct and interesting comparison of the two cases of soft and hard interfaces.

The second example is the stretching of a two dimensional solid composed of two identical adherents separated by a soft or a hard interface. By using the interface laws proposed in this paper, the (average) elastic modulus of the solid is calculated by taking into account the presence of the adhesive up to the first order.

5.8.1. Shear of a composite block

The shear test of a composite body is considered in the plane (x_2, x_3). Two elastic isotropic rectangular blocks Ω^ε_- and Ω^ε_+, with the same length and heights h_- and h_+ respectively, are joined by a thin elastic isotropic glue and subjected to a pure shear stress τ on the boundary, so that the resulting stress tensor is $\sigma^\varepsilon = \tau(\mathbf{i}_2\otimes\mathbf{i}_3 + \mathbf{i}_3\otimes\mathbf{i}_2)$. The displacement is assumed to be equal to zero on the lower edge of Ω^ε_-. The Lamé constants of the three different materials are λ_-, μ_- and λ_+, μ_+ for the two adherents, and λ, μ for the glue.

The solution for a block with an interphase of finite thickness ε in terms of displacements is given by

$$\mathbf{u}^\varepsilon = u^\varepsilon_2\,\mathbf{i}_2, \quad u^\varepsilon_2 = \begin{cases} \frac{\tau}{\mu_-}\left(x_3 + h_- + \frac{\varepsilon}{2}\right) & \text{in } \Omega^\varepsilon_-,\\ \frac{\tau}{\mu}\left(x_3 + \frac{\varepsilon}{2}\right) + \frac{\tau}{\mu_-}h_- & \text{in } B^\varepsilon,\\ \frac{\tau}{\mu_+}\left(x_3 - \frac{\varepsilon}{2}\right) + \frac{\tau}{\mu}\varepsilon + \frac{\tau}{\mu_-}h_- & \text{in } \Omega^\varepsilon_+. \end{cases} \tag{130}$$

For the soft case, i.e. considering $\lambda = \varepsilon\bar{\lambda}, \mu = \varepsilon\bar{\mu}$, the problem at the order zero is given by Eqs. (75)–(77) and Eq. (108) which it is written as

$$[[u^0_2]] = \frac{\tau}{\bar{\mu}}. \tag{131}$$

A straightforward calculation gives the following solution at the order zero in terms of displacements

$$\mathbf{u}^0 = u^0_2\,\mathbf{i}_2, \quad u^0_2 = \begin{cases} \frac{\tau}{\mu_-}(x_3 + h_-) & \text{in } \Omega^0_-,\\ \frac{\tau}{\mu_+}x_3 + \frac{\tau}{\bar{\mu}}\varepsilon + \frac{\tau}{\mu_-}h_- & \text{in } \Omega^0_+, \end{cases} \tag{132}$$

and the solution $\sigma^0 = \tau(\mathbf{i}_2\otimes\mathbf{i}_3 + \mathbf{i}_3\otimes\mathbf{i}_2)$ in terms of stress. This solution corresponds to the shearing of the two adherents given by the amounts $\tau/\mu_\pm h^\pm$, and a sliding of the upper adherent $(+)$ of the amount $\tau/\bar{\mu}$ in the direction of the applied load (cf. Fig. 2). The sliding is clearly due to the spring-type response of the adhesive interface, mimicking the shear deformability of the interphase.

The problem at order one is given by Eqs. (80)–(82) and Eq. (109) which, in view of (136), it is written as

$$\boldsymbol{\sigma}^1(\cdot, 0^\pm)\mathbf{i}_2 = \mathbf{K}^{33}\left([[\mathbf{u}^1]] + \frac{1}{2}\left(\frac{\tau}{\mu_+} + \frac{\tau}{\mu_-}\right)\mathbf{i}_2\right), \tag{133}$$

with

$$K^{33} = \begin{pmatrix} \bar{\mu} & 0 & 0\\ 0 & \bar{\mu} & 0\\ 0 & 0 & 2\bar{\mu} + \bar{\lambda} \end{pmatrix}. \tag{134}$$

A solution in terms of displacements to the problem at order one is

$$\mathbf{u}^1 = u^1_2\,\mathbf{i}_2, \quad u^1_2 = \begin{cases} 0 & \text{in } \Omega^0_-,\\ -\frac{1}{2}\left(\frac{\tau}{\mu_+} + \frac{\tau}{\mu_-}\right) & \text{in } \Omega^0_+, \end{cases} \tag{135}$$

and the corresponding solution in terms of stress is $\sigma^1 = 0$. This solution corresponds to the rigid body motion obtained by sliding the upper adherent in the direction opposite to the shear load of the amount $\tau/2(1/\mu_+ + 1/\mu_-)$. The superposition of the two solutions at the orders zero and one, i.e. $\mathbf{u}^0 + \varepsilon\mathbf{u}^1$, gives back the exact solution (130) in the adherents up to the substitution $\bar{\mu} = \mu\varepsilon^{-1}$. For the hard case, i.e. considering $\lambda = \bar{\lambda}, \mu = \bar{\mu}$, the problem at the order zero is given by Eqs. (75)–(77) and Eqs. (111), (113) which prescribe a vanishing jump of the displacement and stress vectors at the interface. The corresponding solution in terms of displacements is

$$\mathbf{u}^0 = u^0_2\,\mathbf{i}_2, \quad u^0_2 = \begin{cases} \frac{\tau}{\mu_-}(x_3 + h_-) & \text{in } \Omega^0_-,\\ \frac{\tau}{\mu_+}x_3 + \frac{\tau}{\mu_-}h_- & \text{in } \Omega^0_+, \end{cases} \tag{136}$$

and the solution in terms of stress is $\sigma^0 = \tau(\mathbf{i}_2\otimes\mathbf{i}_3 + \mathbf{i}_3\otimes\mathbf{i}_2)$. This solution corresponds to the shearing of the two adherents and to a perfect interface behavior of the adhesive (cf. Fig. 2).

The problem at order one for the hard interface is given by Eqs. (80)–(82) and Eqs. (112), (114) which take the form

$$[[\mathbf{u}^1]] = \left(\frac{\tau}{\bar{\mu}} - \frac{1}{2}\left(\frac{\tau}{\mu_+} + \frac{\tau}{\mu_-}\right)\right)\mathbf{i}_2, \tag{137}$$

$$[[\boldsymbol{\sigma}^1\ \mathbf{i}_3]] = 0. \tag{138}$$

A solution in terms of displacements is

$$\mathbf{u}^1 = u_2^1\,\mathbf{i}_2, \quad u_2^1 = \begin{cases} 0 & \text{in } \Omega_-^0, \\ \frac{\tau}{\bar{\mu}} - \frac{1}{2}\left(\frac{\tau}{\mu_+} + \frac{\tau}{\mu_-}\right) & \text{in } \Omega_+^0, \end{cases} \tag{139}$$

and the corresponding solution in terms of stress is $\sigma^1 = 0$. This solution corresponds to the rigid body motion obtained by sliding the upper adherent in the direction of the load of the amount $\tau/\bar{\mu} - \tau/2(1/\mu_+ + 1/\mu_-)$. The superposition of the two solutions at the orders zero and one, i.e. $\mathbf{u}^0 + \varepsilon\mathbf{u}^1$, gives back the exact solution (130) in the adherents up to the substitution $\bar{\mu} = \mu$. The deformed configurations of the composite blocks at the different orders and for the two cases of a soft and a hard adhesive are compared in Fig. 2. It can be noted that the two cases differ for the sliding of the upper adherent of the amount $\tau/\bar{\mu}$. This rigid body motion reproduces the shear deformation of the interphase, which is captured at the order zero in the case of a soft adhesive and at the order one in the case of a hard adhesive.

5.8.2. Stretching of a composite block with identical adherents

The two-dimensional block considered in the previous example is now subjected to a tensile load q on the upper and lower boundary, so that the resulting stress tensor is $\sigma^\varepsilon = q(\mathbf{i}_3 \otimes \mathbf{i}_3)$. The origin is fixed in order to prevent rigid body motions of the block. The two adherents are assumed to be composed of the same elastic isotropic material, i.e. $\lambda_- = \lambda_+ =: \tilde{\lambda}, \mu_- = \mu_+ =: \tilde{\mu}$.

First, a soft isotropic adhesive is considered, with Lamé constants $\lambda = \varepsilon\bar{\lambda}, \mu = \varepsilon\bar{\mu}$.

It is convenient to introduce the Young's modulus and the Poisson's ratio of the adherents, $\widetilde{E} = \tilde{\mu}(3\tilde{\lambda} + 2\tilde{\mu})/(\tilde{\lambda} + \tilde{\mu})$ and $\tilde{\nu} = \tilde{\lambda}/(2\tilde{\lambda} + \tilde{\mu})$, respectively, and the rescaled Young's modulus and the Poisson's ratio of the adhesive, $\overline{E} = \bar{\mu}(3\bar{\lambda} + 2\bar{\mu})/(\bar{\lambda} + \bar{\mu})$ and $\bar{\nu} = \bar{\lambda}/(2\bar{\lambda} + \bar{\mu})$, respectively.

The problem at the order zero is given again by Eqs. (75)–(77) and by Eq. (108). The solution in terms of stress is $\boldsymbol{\sigma}^0 = q(\mathbf{i}_3 \otimes \mathbf{i}_3)$. The corresponding solution in terms of displacements is

$$\mathbf{u}_s^0(x_2, x_3) = -q\frac{\tilde{\nu}}{\widetilde{E}}x_2\mathbf{i}_2 + \left(\frac{q}{\widetilde{E}}x_3 \pm \frac{q}{2\overline{E}}\,\frac{(1+\bar{\nu})(1-2\bar{\nu})}{(1-\bar{\nu})}\right)\mathbf{i}_3, \quad (x_2, x_3) \in \Omega_\pm^0. \tag{140}$$

This solution corresponds to a mode I-type (opening) deformation of the adhesive, described by the jump

$$[[\mathbf{u}_s^0]] \cdot \mathbf{i}_3 = \frac{q}{\overline{E}}\,\frac{(1+\bar{\nu})(1-2\bar{\nu})}{(1-\bar{\nu})} \tag{141}$$

superimposed to a uniform stretching of the adherents.

The "macroscopic" response of the block composed of the two identical adherents and the soft interface at the order zero is

$$q = E_s^0\,\frac{(\mathbf{u}_s^0(x_2, h^+) - \mathbf{u}_s^0(x_2, -h^-)) \cdot \mathbf{i}_3}{(h^+ + h^-)} \tag{142}$$

with

$$E_s^0 = \frac{\widetilde{E}}{\left(1 + \frac{\widetilde{E}}{\overline{E}}\,\frac{(1+\bar{\nu})(1-2\bar{\nu})}{(1-\bar{\nu})(h^+ + h^-)}\right)} \tag{143}$$

the homogenized elastic modulus of the block at the order zero.

The problem at order one is given by Eqs. (80)–(82) and Eq. (109) which, in view of (134), (140), and of the relations

$$K^{13} = \begin{pmatrix} 0 & 0 & \hat{\mu} \\ 0 & 0 & 0 \\ \hat{\lambda} & 0 & 0 \end{pmatrix}, \quad K^{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \hat{\mu} \\ 0 & \hat{\lambda} & 0 \end{pmatrix}. \tag{144}$$

it is written as

$$\boldsymbol{\sigma}^1(x_2, 0^\pm)\mathbf{i}_2 = \frac{\overline{E}}{2(1+\bar{\nu})}([[\mathbf{u}_s^1]] \cdot \mathbf{i}_2)\mathbf{i}_2 + \left(\frac{\overline{E}(1-\bar{\nu})}{(1+\bar{\nu})(1-2\bar{\nu})}([[\mathbf{u}_s^1]] \cdot \mathbf{i}_3) - q\frac{\overline{E}}{\widetilde{E}}\,\frac{2\tilde{\nu}\bar{\nu}}{(1+\bar{\nu})(1-2\bar{\nu})}\right)\mathbf{i}_3. \tag{145}$$

The solution in terms of stress is $\sigma^1 = 0$, and the corresponding solution in terms of displacements is

$$\mathbf{u}_s^1(x_2, x_3) = (\mp\frac{q}{2\widetilde{E}} \pm \frac{q\tilde{\nu}}{\widetilde{E}}\,\frac{\bar{\nu}}{(1-\bar{\nu})})\mathbf{i}_3, \quad (x_2, x_3) \in \Omega_\pm^0. \tag{146}$$

i.e., a mode I-type (opening) deformation of the adhesive described by the jump

$$[[\mathbf{u}_s^1]] \cdot \mathbf{i}_3 = -\frac{q}{\widetilde{E}} + \frac{q}{\widetilde{E}}\,\frac{2\tilde{\nu}\bar{\nu}}{(1-\bar{\nu})}. \tag{147}$$

The field $\tilde{\mathbf{u}}_s^\varepsilon := \mathbf{u}_s^0 + \varepsilon\mathbf{u}_s^1$ is an approximated solution in the adherents to the original equilibrium problem of the block composed by the adherents and the elastic interphase, and it gives the approximated macroscopic response formally analogous to (142) but with E_s^0 substituted by the approximated "homogenized" elastic modulus

$$\widetilde{E}_s^\varepsilon = \widetilde{E}\frac{1}{\left(1 + \frac{\varepsilon}{(h^+ + h^-)}\frac{1}{(1-\bar{\nu})}\left(\frac{\widetilde{E}}{\varepsilon\overline{E}}(1+\bar{\nu})(1-2\bar{\nu}) + 2\bar{\nu}\tilde{\nu} - 1 + \bar{\nu}\right)\right)}. \tag{148}$$

A hard isotropic adhesive is considered next, with Lamé constants $\lambda = \bar{\lambda}, \mu = \bar{\mu}$ and Young's modulus and Poisson's coefficient, $\widetilde{E}, \tilde{\nu}$, defined as for the soft case. The problem at the order zero for the hard case is given by Eqs. (75)–(77) and Eqs. (111), (113) which prescribe perfect interface conditions. The corresponding solution in terms of stress is again $\sigma^0 = q(\mathbf{i}_3 \otimes \mathbf{i}_3)$, and the solution in terms of displacements is just a uniform stretching of the adherents

$$\mathbf{u}_h^0(x_2, x_3) = -q\frac{\tilde{\nu}}{\widetilde{E}}x_2\mathbf{i}_2 + \frac{q}{\widetilde{E}}x_3\mathbf{i}_3, \quad (x_2, x_3) \in \Omega_\pm^0. \tag{149}$$

The response of the block at the order zero is just described by the Young's modulus of the adherents, $E_h^0 := \widetilde{E}$.

The problem at order one for the hard interface is given by Eqs. (80)–(82) and Eqs. (112), (114) which now take the form

$$[[\mathbf{u}_h^1]] = \left(\frac{q}{\overline{E}}\,\frac{(1+\bar{\nu})(1-2\bar{\nu})}{(1-\bar{\nu})} - \frac{q}{\widetilde{E}} + \frac{q}{\widetilde{E}}\,\frac{2\tilde{\nu}\bar{\nu}}{(1-\bar{\nu})}\right)\mathbf{i}_3. \tag{150}$$

A solution in terms of stress is $\sigma^1 = 0$, and the corresponding solution in terms of displacements is just a relative displacement along the x_3-axis of the two adherents of the amount given by the jump (150)

$$\mathbf{u}_h^1(x_2, x_3) = \pm\frac{1}{2}\left(\frac{q}{\overline{E}}\,\frac{(1+\bar{\nu})(1-2\bar{\nu})}{(1-\bar{\nu})} - \frac{q}{\widetilde{E}} + \frac{q}{\widetilde{E}}\,\frac{2\tilde{\nu}\bar{\nu}}{(1-\bar{\nu})}\right)\mathbf{i}_3, \quad (x_2, x_3) \in \Omega_\pm^0. \tag{151}$$

By considering the approximated solution $\tilde{\mathbf{u}}_h^\varepsilon := \mathbf{u}_h^0 + \varepsilon\mathbf{u}_h^1$, the macroscopic response is described by the following homogenized elastic modulus

$$\widetilde{E}_h^\varepsilon = \widetilde{E}\frac{1}{\left(1 + \frac{\varepsilon}{(h^+ + h^-)}\frac{1}{(1-\bar{\nu})}\left(\frac{\widetilde{E}}{\overline{E}}(1+\bar{\nu})(1-2\bar{\nu}) + 2\bar{\nu}\tilde{\nu} - 1 + \bar{\nu}\right)\right)}. \tag{152}$$

The two moduli (148) and (152) formally coincide up to the substitution of $\varepsilon\overline{E}$ with $\overline{E}$.

6. Conclusion

Higher order theories of interface models have been obtained for two bodies joined by an adhesive interphase for which "soft" and "hard" linear elastic constitutive laws have been considered. In order to obtain the interface model, two different methods were applied, one based on the matched asymptotic expansion technique and the strong formulation of the equilibrium problem, and the other based on the minimization of the potential energy, i.e. on the weak formulation of the equilibrium problem. First and higher order interface models have been derived for soft and hard adhesives and it is shown that the two approaches, strong and weak formulations, lead to the same asymptotic equations governing the behavior of the interface, geometrical limit of the adhesive as its thickness vanishes.

The governing equations derived at zero order have been compared with the ones accounting for the first order of the asymptotic expansion. Approximated constitutive law for soft and hard interfaces have been proposed, obtained by superimposing to the law calculated at the order zero the law calculated at the order one, rescaled with ε. The result is the constitutive law for a soft interface given by Eq. (110), and the constitutive law for a hard interface given by Eq. (115) and (116).

The asymptotic method based on energy minimization allows to calculate the expressions of emerging forces at the adhesive-adherent interface boundary, whose presence is not directly taken into account by the interface laws.

Finally, two simple applications have been developed in order to illustrate the differences among the interface theories at the different orders. Both applications, based on a homogenous deformation on the adherents, show that the two cases of soft and hard adhesive differ for the fact that the deformation of the adhesive, described by a relative sliding in the example with shear and by a relative opening in the example with stretching, is captured at the order zero and order one in the case of a soft adhesive and at the order one in the case of a hard adhesive.

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