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journal homepage: www.elsevier.com/locate/ijsolstrNumerical analysis of a viscoelastic mixture problem[☆]M.I.M. Copetti^a, J.R. Fernández^{b,*}, M. Masid^b^a Laboratório de Análise Numérica e Astrofísica, Departamento de Matemática, Universidade Federal de Santa Maria, 97105-900, Santa Maria, RS, Brazil^b Departamento de Matemática Aplicada I, Universidade de Vigo, ETSI Telecomunicación, Campus As Lagoas Marcosende s/n, 36310 Vigo, Spain

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ABSTRACT

In this paper, we study, from the numerical point of view, a mixture problem involving a viscoelastic material and an elastic one. The mechanical problem is written as a linear system of two coupled hyperbolic partial differential equations. An existence and uniqueness result and an energy decay property are recalled. Then, fully discrete approximations are introduced by using the finite element method to approximate the spatial variable and the backward Euler scheme to discretize the time derivatives. A priori error estimates are proved from which, under suitable regularity conditions, the linear convergence of the algorithm is derived. Finally, some numerical simulations are presented to demonstrate the accuracy of the approximations and the behaviour of the solutions.

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1. Introduction

During the last decades, the number of papers dealing with problems of mixtures including viscoelastic and elastic materials has increased considerably. Since the first studies of Truesdell and Toupin (1960), Kelly (1964) or Green and Naghdi (1965, 1968), many papers have considered mathematical issues as the existence and uniqueness of solutions or the qualitative properties of the solutions as, for instance, the energy decay property (see, e.g., Alves et al. 2009a, 2009b, 2013; Ieşan 1992, 2006; Ieşan and Quintanilla 2007; Rivera et al. 2013). One of the main applications of these models is the theory of the well-known composites, with an increasing use in the automotive industry.

In this paper, we revisit a problem involving a mixture of a viscoelastic material and an elastic one which was considered by Quintanilla (2005) within the theory of viscoelastic mixtures. He continued the work of Ieşan (2006), for a linear theory, where the dissipation effects were determined by the viscosity of rate type of a constituent and the relative velocity. Thermal effects were also included. Existence and uniqueness of a weak solution were proved by using results of the semigroup of linear operators theory, and an energy

decay property was also shown. Moreover, limit cases, for instance corresponding to null viscosity coefficients, were analyzed.

Here, we assume the isothermal case and, for the sake of simplicity in the presentation, we restrict ourselves to the anti-plane shear deformations. We numerically study the problem, providing a priori error estimates, and we perform some numerical simulations to show the behaviour of the solutions.

The outline of this paper is as follows. In Section 2, we briefly describe the mathematical model and we introduce its variational formulation, for which an existence and uniqueness result, proved in Quintanilla (2005), is recalled. Then, fully discrete approximations are introduced in Section 3 by using a finite element method for the spatial approximation and the backward Euler scheme for the discretization of the time derivatives. An error estimate result is proved from which the linear convergence is deduced under suitable regularity assumptions. Finally, in Section 4 some numerical examples are shown to demonstrate the accuracy of the algorithm and the behaviour of the solution.

2. The model and its variational formulation

In this section, we present a brief description of the model (details can be found in Quintanilla (2005)).

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, be a domain with a Lipschitz boundary $\Gamma = \partial\Omega$ decomposed into two disjoint parts Γ_D and Γ_S such that $\text{meas}(\Gamma_D) > 0$, and denote by $[0, T]$, $T > 0$, the time interval of interest.

Let $\mathbf{x} \in \Omega$ and $t \in [0, T]$ be the spatial and time variables, respectively. In order to simplify the writing, we do not indicate

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the dependence of the functions on $\mathbf{x}$ and t . Moreover, a dot above a variable represents its derivative with respect to the time variable.

Let $\tilde{u}, \tilde{w} \in \mathbb{R}^d$ be the displacement of the viscoelastic and elastic materials, respectively. Assuming the case of the isothermal processes, we consider a particular kind of solutions corresponding to anti-plane shear deformations. Then, we are interested in solutions of the form $\tilde{u} = (u_1, u_2, u_3) = (0, 0, u(x_1, x_2))$ and $\tilde{w} = (w_1, w_2, w_3) = (0, 0, w(x_1, x_2))$. We note that this assumption is done for the sake of simplicity in the presentation and the general case can be analyzed in a straightforward way.

Therefore, following Quintanilla (2005) the mechanical problem of a mixture involving a viscoelastic material and an elastic one is written as follows.

Problem P. Find the displacement of the viscoelastic material $u : \Omega \times [0, T] \rightarrow \mathbb{R}$ and the displacement of the elastic material $w : \Omega \times [0, T] \rightarrow \mathbb{R}$ such that

$$\rho_1 \ddot{u} = \alpha \Delta u + \beta \Delta w - \xi(u - w) + \mu^* \Delta \dot{u} - \xi^*(\dot{u} - \dot{w}) + \rho_1 F_1 \quad \text{in } \Omega \times (0, T), \quad (1)$$

$$\rho_2 \ddot{w} = \beta \Delta u + \gamma \Delta w + \xi(u - w) + \xi^*(\dot{u} - \dot{w}) + \rho_2 F_2 \quad \text{in } \Omega \times (0, T), \quad (2)$$

$$u = 0, \quad w = 0 \quad \text{on } \Gamma_D \times (0, T), \quad (3)$$

$$u = 0, \quad w = 0 \quad \text{on } \Gamma_S \times (0, T), \quad (4)$$

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad w(\mathbf{x}, 0) = w_0(\mathbf{x}) \quad \text{for a.e. } \mathbf{x} \in \Omega, \quad (5)$$

$$\dot{u}(\mathbf{x}, 0) = v_0(\mathbf{x}), \quad \dot{w}(\mathbf{x}, 0) = e_0(\mathbf{x}) \quad \text{for a.e. } \mathbf{x} \in \Omega. \quad (6)$$

Here, ρ_1 and ρ_2 are the mass densities of each constituent, F_1 and F_2 represent the body forces on each constituent, α, β, ξ and γ are elastic coefficients, μ^* and ξ^* denote viscoelastic coefficients, and u_0, v_0, w_0, e_0 are given initial conditions. Moreover, homogeneous Dirichlet boundary conditions have been also assumed on $\Gamma_S \times (0, T)$, but we point out that Neumann conditions could be used with some minor modifications in the analysis presented below.

In order to obtain the variational formulation of Problem P, let $Y = L^2(\Omega)$ and $H = [L^2(\Omega)]^d$, and denote by $(\cdot, \cdot)_Y$ and $(\cdot, \cdot)_H$ the respective scalar products in these spaces, with corresponding norms $\|\cdot\|_Y$ and $\|\cdot\|_H$.

Moreover, let us define the variational space V as follows,

$$V = \{v \in H^1(\Omega); v = 0 \text{ on } \Gamma_D \cup \Gamma_S\},$$

with scalar product $(\cdot, \cdot)_V$ and norm $\|\cdot\|_V$.

By using the classical Green's formula and boundary conditions (3) and (4), we write the variational formulation of Problem P in terms of the velocities $v = \dot{u}$ and $e = \dot{w}$. Here, v denotes the velocity of the viscoelastic material and e is the velocity of the elastic material.

Problem VP. Find the velocity of the viscoelastic material $v : [0, T] \rightarrow V$ and the velocity of the elastic material $e : [0, T] \rightarrow V$ such that $v(0) = v_0$, $e(0) = e_0$ and, for a.e. $t \in (0, T)$,

$$\begin{aligned} \rho_1 (\dot{v}(t), z)_Y + \xi(u(t) - w(t), z)_Y + \xi^*(\dot{u}(t) - \dot{w}(t), z)_Y \\ + \beta(\nabla w(t), \nabla z)_H + \alpha(\nabla u(t), \nabla z)_H + \mu^*(\nabla \dot{u}(t), \nabla z)_H \\ = \rho_1 (F_1(t), z)_Y, \quad \forall z \in V, \end{aligned} \quad (7)$$

$$\begin{aligned} \rho_2 (\dot{e}(t), r)_Y + \xi(w(t) - u(t), r)_Y + \xi^*(\dot{w}(t) - \dot{u}(t), r)_Y \\ + \beta(\nabla u(t), \nabla r)_H + \gamma(\nabla w(t), \nabla r)_H \\ = \rho_2 (F_2(t), r)_Y, \quad \forall r \in V, \end{aligned} \quad (8)$$

where the displacement of the viscoelastic material $u(t)$ is obtained from the relation

$$u(t) = \int_0^t v(s) ds + u_0. \quad (9)$$

and the displacement of the elastic material $w(t)$ is calculated from

$$w(t) = \int_0^t e(s) ds + w_0. \quad (10)$$

The existence and uniqueness of weak solutions as well as an exponential energy decay property have been considered in Quintanilla (2005), where the thermal effects were also included. It is stated in the following.

Theorem 2.1. Assume that

$$\mu^* > 0, \quad \xi > 0, \quad \xi^* > 0, \quad \alpha > 0, \quad \gamma > 0, \quad \rho_2 > 0, \quad \rho_1 > 0, \quad (11)$$

$$\gamma\alpha - \beta^2 > 0, \quad (12)$$

and also

$$F_1, F_2 \in C^1([0, T]; Y). \quad (13)$$

Therefore, Problem VP has a unique solution with the regularity:

$$u \in C^1([0, T]; H^1(\Omega)) \cap C^2([0, T]; Y),$$

$$w \in C([0, T]; H^1(\Omega)) \cap C^2([0, T]; Y).$$

Moreover, if we define the energy function

$$\begin{aligned} E(t) = \frac{1}{2} \int_{\Omega} (\rho_1 \dot{u}^2 + \rho_2 \dot{w}^2 + \xi(u - w)^2 + \alpha \nabla u \cdot \nabla u \\ + \gamma \nabla w \cdot \nabla w + 2\beta \nabla w \cdot \nabla u) d\mathbf{x}, \end{aligned}$$

and we assume that $F_1 = F_2 = 0$, then there exist two positive constants $M, \lambda > 0$ such that

$$E(t) \leq ME(0)e^{-\lambda t} \quad \text{for } t \geq 0.$$

3. Fully discrete approximations: a priori error estimates

In this section, we now consider a fully discrete approximation of Problem VP. This is done in two steps. First, we assume that $\bar{\Omega}$ is a polyhedral domain and we consider a finite dimensional space $V^h \subset V$, approximating the variational space V , given by

$$V^h = \{v^h \in C(\bar{\Omega}); v^h|_K \in P_1(K) \quad K \in \mathcal{T}^h, \quad v^h = 0 \text{ on } \Gamma_D \cup \Gamma_S\}, \quad (14)$$

where $P_1(K)$ represents the space of polynomials of global degree less or equal to one in K and we denote by $(\mathcal{T}^h)_{h>0}$ a regular family of triangulations of $\bar{\Omega}$ (in the sense of Ciarlet (1991)), compatible with the partition of the boundary $\Gamma = \partial\Omega$ into Γ_D and Γ_S ; i.e. the finite element space V^h is composed of continuous and piecewise affine functions. Let h_K be the diameter of an element $K \in \mathcal{T}^h$ and let $h = \max_{K \in \mathcal{T}^h} h_K$ denote the spatial discretization parameter. Moreover, we assume that the discrete initial conditions, denoted by u_0^h, v_0^h, w_0^h and e_0^h , are given by

$$u_0^h = \mathcal{P}^h u_0, \quad v_0^h = \mathcal{P}^h v_0, \quad w_0^h = \mathcal{P}^h w_0, \quad e_0^h = \mathcal{P}^h e_0. \quad (15)$$

where $\mathcal{P}^h$ is the $L^2(\Omega)$ -projection operator over V^h .

Secondly, we consider a partition of the time interval $[0, T]$, denoted by $0 = t_0 < t_1 < \dots < t_N = T$. In this case, we use a uniform partition of the time interval $[0, T]$ with step size $k = T/N$ and nodes $t_n = nk$ for $n = 0, 1, \dots, N$. For a continuous function $z(t)$, we use the notation $z_n = z(t_n)$ and, for the sequence $\{z_n\}_{n=0}^N$, we denote by $\delta z_n = (z_n - z_{n-1})/k$ its corresponding divided differences.

Therefore, using the backward Euler scheme, the fully discrete approximations are considered as follows.

Problem VP^{hk} Find the discrete velocity of the viscoelastic material $v^{hk} = \{v_n^{hk}\}_{n=0}^N \subset V^h$ and the discrete velocity of the elastic material $e^{hk} = \{e_n^{hk}\}_{n=0}^N \subset V^h$ such that $v_0^{hk} = v_0^h$, $e_0^{hk} = e_0^h$ and, for

$n = 1, \dots, N$,

$$\begin{aligned} & \rho_1(\delta v_n^{hk}, z^h)_Y + \xi(u_n^{hk} - w_n^{hk}, z^h)_Y + \xi^*(v_n^{hk} - e_n^{hk}, z^h)_Y \\ & + \beta(\nabla w_n^{hk}, \nabla z^h)_H + \alpha(\nabla u_n^{hk}, \nabla z^h)_H + \mu^*(\nabla v_n^{hk}, \nabla z^h)_H \\ & = \rho_1(F_{1n}, z^h)_Y, \quad \forall z^h \in V^h, \end{aligned} \quad (16)$$

$$\begin{aligned} & \rho_2(\delta e_n^{hk}, r^h)_Y + \xi(w_n^{hk} - u_n^{hk}, r^h)_Y + \xi^*(e_n^{hk} - v_n^{hk}, r^h)_Y \\ & + \beta(\nabla u_n^{hk}, \nabla r^h)_H + \gamma(\nabla w_n^{hk}, \nabla r^h)_H = \rho_2(F_{2n}, r^h)_Y, \quad \forall r^h \in V^h, \end{aligned} \quad (17)$$

where the two discrete displacement fields $u^{hk} = \{u_n^{hk}\}_{n=0}^N \subset V^h$ and $w^{hk} = \{w_n^{hk}\}_{n=0}^N \subset V^h$ are then given by

$$u_n^{hk} = \sum_{j=1}^n k v_j^{hk} + u_0^h, \quad (18)$$

$$w_n^{hk} = \sum_{j=1}^n k e_j^{hk} + w_0^h. \quad (19)$$

The existence and uniqueness of discrete solutions to Problem VP^{hk} are obtained in a straightforward way using the classical Lax-Milgram lemma and taking into account conditions (11)–(12).

Remark 3.1. We note that we have chosen continuous and piecewise affine functions because of the form of the weak formulation, involving H^1 functions, and for the sake of simplicity in the calculations presented in this section. Anyway, we notice that the a priori error estimates could be easily extended to higher order schemes assuming additional regularity on the continuous solution.

Now, we have a stability result.

Theorem 3.2. Under the assumptions of Theorem 2.1, it follows that the sequences $\{u^{hk}, v^{hk}, w^{hk}, e^{hk}\}$ generated by Problem VP^{hk} satisfy the stability estimate:

$$\begin{aligned} & \rho_1 \|v_n^{hk}\|_Y^2 + \xi \|u_n^{hk}\|_Y^2 + (\alpha - \zeta \beta) \|\nabla u_n^{hk}\|_H^2 + \left(\gamma - \frac{\beta}{\zeta}\right) \|\nabla w_n^{hk}\|_H^2 \\ & + \xi \|w_n^{hk}\|_Y^2 + \rho_2 \|e_n^{hk}\|_Y^2 \leq C, \end{aligned}$$

where C is a positive constant and $\zeta > 0$ is assumed to be such that

$$\frac{\beta}{\gamma} < \zeta < \frac{\alpha}{\beta}.$$

Proof. In order to simplify the writing, we remove in this proof the superscripts h and k in all the variables.

Taking $z^h = v_n$ as a test function in the discrete variational equation (16) it follows that

$$\begin{aligned} & \rho_1(\delta v_n, v_n)_Y + \xi(u_n - w_n, v_n)_Y + \xi^*(v_n - e_n, v_n)_Y + \beta(\nabla w_n, \nabla v_n)_H \\ & + \alpha(\nabla u_n, \nabla v_n)_H + \mu^*(\nabla v_n, \nabla v_n)_H = \rho_1(F_{1n}, v_n)_Y. \end{aligned}$$

Taking $r^h = e_n$ as a test function in the discrete variational equation (17) we have

$$\begin{aligned} & \rho_2(\delta e_n, e_n)_Y + \xi(w_n - u_n, e_n)_Y + \xi^*(e_n - v_n, e_n)_Y + \beta(\nabla u_n, \nabla e_n)_H \\ & + \gamma(\nabla w_n, \nabla e_n)_H = \rho_2(F_{2n}, e_n)_Y. \end{aligned}$$

Keeping in mind that

$$\begin{aligned} & (\delta v_n, v_n)_Y = \frac{1}{2k} [\|v_n\|_Y^2 - \|v_{n-1}\|_Y^2 + \|v_n - v_{n-1}\|_Y^2], \\ & (u_n, v_n)_Y = \frac{1}{2k} [\|u_n\|_Y^2 - \|u_{n-1}\|_Y^2 + \|u_n - u_{n-1}\|_Y^2], \\ & (\nabla u_n, \nabla v_n)_H = \frac{1}{2k} [\|\nabla u_n\|_H^2 - \|\nabla u_{n-1}\|_H^2 + \|\nabla(u_n - u_{n-1})\|_H^2], \\ & (\delta e_n, e_n)_Y = \frac{1}{2k} [\|e_n\|_Y^2 - \|e_{n-1}\|_Y^2 + \|e_n - e_{n-1}\|_Y^2], \\ & (w_n, e_n)_Y = \frac{1}{2k} [\|w_n\|_Y^2 - \|w_{n-1}\|_Y^2 + \|w_n - w_{n-1}\|_Y^2], \end{aligned}$$

$$\begin{aligned} & (\nabla w_n, \nabla e_n)_H = \frac{1}{2k} [\|\nabla w_n\|_H^2 - \|\nabla w_{n-1}\|_H^2 + \|\nabla(w_n - w_{n-1})\|_H^2], \\ & \text{adding the previous two equations, we find that} \\ & \frac{\rho_1}{2k} [\|v_n\|_Y^2 - \|v_{n-1}\|_Y^2 + \|v_n - v_{n-1}\|_Y^2] \end{aligned}$$

$$\begin{aligned} & + \frac{\xi}{2k} [\|u_n\|_Y^2 - \|u_{n-1}\|_Y^2 + \|u_n - u_{n-1}\|_Y^2] \\ & + \frac{\alpha}{2k} [\|\nabla u_n\|_H^2 - \|\nabla u_{n-1}\|_H^2 + \|\nabla(u_n - u_{n-1})\|_H^2] \\ & + \frac{\rho_2}{2k} [\|e_n\|_Y^2 - \|e_{n-1}\|_Y^2 + \|e_n - e_{n-1}\|_Y^2] \\ & + \frac{\gamma}{2k} [\|\nabla w_n\|_H^2 - \|\nabla w_{n-1}\|_H^2 + \|\nabla(w_n - w_{n-1})\|_H^2] \\ & + \frac{\xi}{2k} [\|w_n\|_Y^2 - \|w_{n-1}\|_Y^2 + \|w_n - w_{n-1}\|_Y^2] \\ & + \beta(\nabla w_n, \nabla v_n)_H + \beta(\nabla u_n, \nabla e_n)_H \\ & \leq C(\|v_n\|_Y^2 + \|e_n\|_Y^2 + \|u_n\|_Y^2 + \|w_n\|_Y^2), \end{aligned} \quad (20)$$

where $C > 0$ is a positive constant assumed to be independent of the discretization parameters h and k .

Observing that

$$\begin{aligned} & (\nabla w_n, \nabla(u_n - u_{n-1}))_H + (\nabla u_n, \nabla(w_n - w_{n-1}))_H = (\nabla w_n, \nabla u_n)_H \\ & - (\nabla w_{n-1}, \nabla u_{n-1})_H + (\nabla(u_n - u_{n-1}), \nabla(w_n - w_{n-1}))_H, \end{aligned}$$

and, due to inequality (12), taking into account that

$$\begin{aligned} & \alpha \|\nabla(u_n - u_{n-1})\|_H^2 + \gamma \|\nabla(w_n - w_{n-1})\|_H^2 \\ & + 2\beta(\nabla(u_n - u_{n-1}), \nabla(w_n - w_{n-1}))_H \geq 0, \end{aligned}$$

and

$$\begin{aligned} & \alpha \|\nabla u_n\|_H^2 + \gamma \|\nabla w_n\|_H^2 + 2\beta(\nabla u_n, \nabla w_n)_H \\ & \geq (\alpha - \beta\zeta) \|\nabla u_n\|_H^2 + (\gamma - \frac{\beta}{\zeta}) \|\nabla w_n\|_H^2, \end{aligned}$$

where $\zeta > 0$ is assumed to be such that

$$\frac{\beta}{\gamma} < \zeta < \frac{\alpha}{\beta},$$

summing up to n estimates (20) and using a discrete version of Gronwall's inequality we obtain the desired stability property. $\square$

As a consequence of Theorem 3.2 we can prove a discrete version of the energy decay property.

Corollary 3.3. If we define the discrete energy at time $t = t_n$, E_n^{hk} , as follows

$$\begin{aligned} E_n^{hk} &= \rho_1 \|v_n^{hk}\|_Y^2 + \rho_2 \|e_n^{hk}\|_Y^2 + \xi \|u_n^{hk} - w_n^{hk}\|_Y^2 + \alpha \|\nabla u_n^{hk}\|_H^2 \\ & + \gamma \|\nabla w_n^{hk}\|_H^2 + 2\beta(\nabla u_n^{hk}, \nabla w_n^{hk})_H, \end{aligned} \quad (21)$$

and we assume that the volume forces $F_1 = F_2 = 0$, then we have

$$\frac{E_n^{hk} - E_{n-1}^{hk}}{k} \leq 0.$$

Remark 3.4. From the previous corollary we can only conclude that the discrete energy decreases when time increases. An investigation of its decay rate is beyond the scope of this paper. We refer to Ervedoza and Zuazua (2009); Münch and Pazoto (2007); Tébou and Zuazua (2003) for works that discuss long time decay for numerical approximations of locally damped wave equations using observability inequalities and multipliers.

In the rest of this section, our aim is to obtain a priori error estimates for the numerical errors $u_n - u_n^{hk}$, $v_n - v_n^{hk}$, $w_n - w_n^{hk}$ and $e_n - e_n^{hk}$.

First, we obtain some estimates on the displacement field of the viscoelastic material. Then, we subtract variational equation (7) at

time $t = t_n$ for a test function $z = z^h \in V^h \subset V$ and discrete variational equation (16) to obtain

$$\begin{aligned} & \rho_1(\dot{v}_n - \delta v_n^{hk}, z^h)_Y + \xi(u_n - u_n^{hk} - (w_n - w_n^{hk}), z^h)_Y \\ & + \beta(\nabla(w_n - w_n^{hk}), \nabla z^h)_H + \xi^*(v_n - v_n^{hk} - (e_n - e_n^{hk}), z^h)_Y \\ & + \alpha(\nabla(u_n - u_n^{hk}), \nabla z^h)_H + \mu^*(\nabla(v_n - v_n^{hk}), \nabla z^h)_H = 0, \\ & \forall z^h \in V^h, \end{aligned}$$

and so, for all $z^h \in V^h$,

$$\begin{aligned} & \rho_1(\dot{v}_n - \delta v_n^{hk}, v_n - v_n^{hk})_Y + \xi(u_n - u_n^{hk} - (w_n - w_n^{hk}), v_n - v_n^{hk})_Y \\ & + \xi^*(v_n - v_n^{hk} - (e_n - e_n^{hk}), v_n - v_n^{hk})_Y \\ & + \beta(\nabla(w_n - w_n^{hk}), \nabla(v_n - v_n^{hk}))_H \\ & + \alpha(\nabla(u_n - u_n^{hk}), \nabla(v_n - v_n^{hk}))_H \\ & + \mu^*(\nabla(v_n - v_n^{hk}), \nabla(v_n - v_n^{hk}))_H \\ & = \rho_1(\dot{v}_n - \delta v_n^{hk}, v_n - z^h)_Y \\ & + \xi(u_n - u_n^{hk} - (w_n - w_n^{hk}), v_n - z^h)_Y \\ & + \xi^*(v_n - v_n^{hk} - (e_n - e_n^{hk}), v_n - z^h)_Y \\ & + \beta(\nabla(w_n - w_n^{hk}), \nabla(v_n - z^h))_H \\ & + \alpha(\nabla(u_n - u_n^{hk}), \nabla(v_n - z^h))_H \\ & + \mu^*(\nabla(v_n - v_n^{hk}), \nabla(v_n - z^h))_H. \end{aligned}$$

Now, taking into account that

$$\begin{aligned} \dot{v}_n - \delta v_n^{hk} &= \dot{v}_n - \delta v_n + \delta v_n - \delta v_n^{hk}, \\ (\delta v_n - \delta v_n^{hk}, v_n - v_n^{hk})_Y &\geq \frac{1}{2k} \{ \|v_n - v_n^{hk}\|_Y^2 - \|v_{n-1} - v_{n-1}^{hk}\|_Y^2 \}, \\ v_n - v_n^{hk} &= \dot{u}_n - \delta u_n + \delta u_n - \delta u_n^{hk}, \\ (u_n - u_n^{hk}, \delta u_n - \delta u_n^{hk})_Y &\geq \frac{1}{2k} \{ \|u_n - u_n^{hk}\|_Y^2 - \|u_{n-1} - u_{n-1}^{hk}\|_Y^2 \}, \\ \alpha \left(\nabla(u_n - u_n^{hk}), \frac{\nabla(u_n - u_{n-1} - (u_n^{hk} - u_{n-1}^{hk}))}{k} \right)_H \\ &= \frac{\alpha}{2k} \{ \|\nabla(u_n - u_n^{hk})\|_H^2 - \|\nabla(u_{n-1} - u_{n-1}^{hk})\|_H^2 \\ &+ \|\nabla(u_n - u_{n-1} - (u_n^{hk} - u_{n-1}^{hk}))\|_H^2 \}, \end{aligned}$$

where $\delta v_n = (v_n - v_{n-1})/k$ and $\delta u_n = (u_n - u_{n-1})/k$, using several times Cauchy-Schwarz inequality and the inequality $ab \leq \epsilon a^2 + \frac{1}{2\epsilon} b^2$, for $a, b, \epsilon \in \mathbb{R}$ with $\epsilon > 0$, we find that

$$\begin{aligned} & \frac{\rho_1}{2k} \|v_n - v_n^{hk}\|_Y^2 + \mu^* \|\nabla(v_n - v_n^{hk})\|_H^2 + \frac{\xi}{2k} \|u_n - u_n^{hk}\|_Y^2 \\ & + \frac{\alpha}{2k} \|\nabla(u_n - u_n^{hk})\|_H^2 + \xi^* \|v_n - v_n^{hk}\|_Y^2 \\ & + \frac{\beta}{k} (\nabla(u_n - u_n^{hk} - (u_{n-1} - u_{n-1}^{hk})), \nabla(w_n - w_n^{hk}))_H \\ & + \frac{\alpha}{2k} \|\nabla(u_n - u_{n-1} - (u_n^{hk} - u_{n-1}^{hk}))\|_H^2 \\ & \leq C(\|\nabla(u_n - u_n^{hk})\|_H^2 + \|\nabla(\dot{u}_n - \delta u_n)\|_H^2 \\ & + \|u_n - u_n^{hk}\|_Y^2 + \|v_n - v_n^{hk}\|_Y^2 \\ & + \|v_n - z^h\|_Y^2 + \|\dot{v}_n - \delta v_n\|_Y^2 + \|w_n - w_n^{hk}\|_Y^2 + \|e_n - e_n^{hk}\|_Y^2 \\ & + \|\nabla(v_n - z^h)\|_H^2 + \|\dot{u}_n - \delta u_n\|_Y^2 + (\delta v_n - \delta v_n^{hk}, v_n - z^h)_Y \\ & + \epsilon \|\nabla(v_n - v_n^{hk})\|_H^2 + \epsilon \|\nabla(w_n - w_n^{hk})\|_H^2) \\ & + \frac{\rho_1}{2k} \|v_{n-1} - v_{n-1}^{hk}\|_Y^2 + \frac{\xi}{2k} \|u_{n-1} - u_{n-1}^{hk}\|_Y^2 \\ & + \frac{\alpha}{2k} \|\nabla(u_{n-1} - u_{n-1}^{hk})\|_H^2, \end{aligned} \quad (22)$$

where $\epsilon > 0$ is assumed small enough.

Secondly, we obtain the estimates on the displacement field of the elastic material. Thus, we subtract now variational equation (8)

at time $t = t_n$ for a test function $r = r^h \in V^h \subset V$ and discrete variational equation (17), and we have

$$\begin{aligned} & \rho_2(\dot{e}_n - \delta e_n^{hk}, r^h)_Y + \xi(w_n - w_n^{hk} - (u_n - u_n^{hk}), r^h)_Y \\ & + \xi^*(e_n - e_n^{hk} - (v_n - v_n^{hk}), r^h)_Y + \beta(\nabla(u_n - u_n^{hk}), \nabla r^h)_H \\ & + \gamma(\nabla(w_n - w_n^{hk}), \nabla r^h)_H = 0, \quad \forall r^h \in V^h. \end{aligned}$$

Therefore, we find that

$$\begin{aligned} & \rho_2(\dot{e}_n - \delta e_n^{hk}, e_n - e_n^{hk})_Y + \xi(w_n - w_n^{hk} - (u_n - u_n^{hk}), e_n - e_n^{hk})_Y \\ & + \xi^*(e_n - e_n^{hk} - (v_n - v_n^{hk}), e_n - e_n^{hk})_Y \\ & + \beta(\nabla(u_n - u_n^{hk}), \nabla(e_n - e_n^{hk}))_H \\ & + \gamma(\nabla(w_n - w_n^{hk}), \nabla(e_n - e_n^{hk}))_H \\ & = \rho_2(\dot{e}_n - \delta e_n^{hk}, e_n - r^h)_Y + \xi(w_n - w_n^{hk} - (u_n - u_n^{hk}), e_n - r^h)_Y \\ & + \xi^*(e_n - e_n^{hk} - (v_n - v_n^{hk}), e_n - r^h)_Y \\ & + \beta(\nabla(u_n - u_n^{hk}), \nabla(e_n - r^h))_H \\ & + \gamma(\nabla(w_n - w_n^{hk}), \nabla(e_n - r^h))_H, \quad \forall r^h \in V^h. \end{aligned}$$

Taking into account that

$$\begin{aligned} \dot{e}_n - \delta e_n^{hk} &= \dot{e}_n - \delta e_n + \delta e_n - \delta e_n^{hk}, \\ (\delta e_n - \delta e_n^{hk}, e_n - e_n^{hk})_Y &\geq \frac{1}{2k} \{ \|e_n - e_n^{hk}\|_Y^2 - \|e_{n-1} - e_{n-1}^{hk}\|_Y^2 \}, \\ e_n - e_n^{hk} &= \dot{w}_n - \delta w_n + \delta w_n - \delta w_n^{hk}, \\ (w_n - w_n^{hk}, \delta w_n - \delta w_n^{hk})_Y &\geq \frac{1}{2k} \{ \|w_n - w_n^{hk}\|_Y^2 - \|w_{n-1} - w_{n-1}^{hk}\|_Y^2 \}, \\ \gamma \left(\nabla(w_n - w_n^{hk}), \frac{\nabla(w_n - w_{n-1} - (w_n^{hk} - w_{n-1}^{hk}))}{k} \right)_H \\ &= \frac{\gamma}{2k} \{ \|\nabla(w_n - w_n^{hk})\|_H^2 - \|\nabla(w_{n-1} - w_{n-1}^{hk})\|_H^2 \\ &+ \|\nabla(w_n - w_{n-1} - (w_n^{hk} - w_{n-1}^{hk}))\|_H^2 \}, \end{aligned}$$

where $\delta e_n = (e_n - e_{n-1})/k$ and $\delta w_n = (w_n - w_{n-1})/k$, proceeding as in the estimates of the velocity field of the viscoelastic material it follows that

$$\begin{aligned} & \frac{\rho_2}{2k} \|e_n - e_n^{hk}\|_Y^2 + \frac{\xi}{2k} \|w_n - w_n^{hk}\|_Y^2 + \frac{\gamma}{2k} \|\nabla(w_n - w_n^{hk})\|_H^2 \\ & + \xi^* \|e_n - e_n^{hk}\|_Y^2 \\ & + \frac{\beta}{k} (\nabla(u_n - u_n^{hk}), \nabla(w_n - w_n^{hk} - (w_{n-1} - w_{n-1}^{hk})))_H \\ & + \frac{\gamma}{2k} \|\nabla(w_n - w_{n-1} - (w_n^{hk} - w_{n-1}^{hk}))\|_H^2 \\ & \leq C(\|\dot{e}_n - \delta e_n\|_Y^2 + \|e_n - r^h\|_Y^2 + \|w_n - w_n^{hk}\|_Y^2 \\ & + \|u_n - u_n^{hk}\|_Y^2 + \|\dot{w}_n - \delta w_n\|_Y^2 + \|e_n - e_n^{hk}\|_Y^2 \\ & + \|v_n - v_n^{hk}\|_Y^2 + \|\nabla(u_n - u_n^{hk})\|_H^2 + \|\nabla(e_n - r^h)\|_H^2 \\ & + \|\nabla(w_n - w_n^{hk})\|_H^2 + \|\nabla(\dot{w}_n - \delta w_n)\|_H^2 \\ & + (\delta e_n - \delta e_n^{hk}, e_n - r^h)_Y) \\ & + \frac{\rho_2}{2k} \|e_{n-1} - e_{n-1}^{hk}\|_Y^2 + \frac{\xi}{2k} \|w_{n-1} - w_{n-1}^{hk}\|_Y^2 \\ & + \frac{\gamma}{2k} \|\nabla(w_{n-1} - w_{n-1}^{hk})\|_H^2. \end{aligned} \quad (23)$$

Using the fact that $\alpha\gamma - \beta^2 > 0$ (see (12)), we have

$$\begin{aligned} & \alpha \|\nabla(u_n - u_{n-1} - (u_n^{hk} - u_{n-1}^{hk}))\|_H^2 \\ & + \gamma \|\nabla(w_n - w_{n-1} - (w_n^{hk} - w_{n-1}^{hk}))\|_H^2 \\ & + 2\beta (\nabla(u_n - u_{n-1} - (u_n^{hk} - u_{n-1}^{hk})), \\ & \nabla(w_n - w_{n-1} - (w_n^{hk} - w_{n-1}^{hk})))_H \geq 0. \end{aligned}$$

Keeping in mind that

$$\begin{aligned} & \frac{\beta}{k} (\nabla(w_n - w_n^{hk} - (w_{n-1} - w_{n-1}^{hk})), \nabla(u_n - u_n^{hk}))_H \\ & + \frac{\beta}{k} (\nabla(u_n - u_n^{hk} - (u_{n-1} - u_{n-1}^{hk})), \nabla(w_n - w_n^{hk}))_H \\ & = \frac{\beta}{k} (\nabla(w_n - w_n^{hk}), \nabla(u_n - u_n^{hk}))_H \\ & - \frac{\beta}{k} (\nabla(w_{n-1} - w_{n-1}^{hk}), \nabla(u_{n-1} - u_{n-1}^{hk}))_H \\ & + \frac{\beta}{k} (\nabla(w_n - w_{n-1} - (w_n^{hk} - w_{n-1}^{hk})), \\ & \nabla(u_n - u_{n-1} - (u_n^{hk} - u_{n-1}^{hk})))_H, \end{aligned}$$

combining estimates (22) and (23) we find that

$$\begin{aligned} & \frac{\rho_1}{2k} \|v_n - v_n^{hk}\|_Y^2 + \mu^* \|\nabla(v_n - v_n^{hk})\|_H^2 \\ & + \frac{\xi}{2k} \|u_n - u_n^{hk}\|_Y^2 + \frac{\alpha}{2k} \|\nabla(u_n - u_n^{hk})\|_H^2 \\ & + \frac{\rho_2}{2k} \|e_n - e_n^{hk}\|_Y^2 + \frac{\xi}{2k} \|w_n - w_n^{hk}\|_Y^2 + \frac{\gamma}{2k} \|\nabla(w_n - w_n^{hk})\|_H^2 \\ & + \frac{\beta}{k} (\nabla(w_n - w_n^{hk}), \nabla(u_n - u_n^{hk}))_H \\ & - \frac{\beta}{k} (\nabla(w_{n-1} - w_{n-1}^{hk}), \nabla(u_{n-1} - u_{n-1}^{hk}))_H \\ & \leq C(\|\nabla(u_n - u_n^{hk})\|_H^2 + \|\nabla(\dot{u}_n - \delta u_n)\|_H^2 + \|u_n - u_n^{hk}\|_Y^2 \\ & + \|v_n - v_n^{hk}\|_Y^2 + \|v_n - z^h\|_Y^2 + \|\dot{v}_n - \delta v_n\|_Y^2 + \|w_n - w_n^{hk}\|_Y^2 \\ & + \|e_n - e_n^{hk}\|_Y^2 + \|\dot{u}_n - \delta u_n\|_Y^2 + \|\nabla(v_n - z^h)\|_H^2 \\ & + (\delta v_n - \delta v_n^{hk}, v_n - z^h)_Y + \|\nabla(e_n - r^h)\|_H^2 + \|\dot{w}_n - \delta w_n\|_Y^2 \\ & + \|\nabla(w_n - w_n^{hk})\|_H^2 + \|\nabla(\dot{w}_n - \delta w_n)\|_H^2 \\ & + (\delta e_n - \delta e_n^{hk}, e_n - r^h)_Y + \|\dot{e}_n - \delta e_n\|_Y^2 + \|e_n - r^h\|_Y^2 \\ & + \epsilon \|\nabla(v_n - v_n^{hk})\|_H^2 + \epsilon \|\nabla(w_n - w_n^{hk})\|_H^2) \\ & + \frac{\rho_1}{2k} \|v_{n-1} - v_{n-1}^{hk}\|_Y^2 + \frac{\xi}{2k} \|u_{n-1} - u_{n-1}^{hk}\|_Y^2 \\ & + \frac{\alpha}{2k} \|\nabla(u_{n-1} - u_{n-1}^{hk})\|_H^2 + \frac{\rho_2}{2k} \|e_{n-1} - e_{n-1}^{hk}\|_Y^2 \\ & + \frac{\xi}{2k} \|w_{n-1} - w_{n-1}^{hk}\|_Y^2 + \frac{\gamma}{2k} \|\nabla(w_{n-1} - w_{n-1}^{hk})\|_H^2. \quad (24) \end{aligned}$$

Using again that $\alpha\gamma - \beta^2 > 0$ we can choose $\zeta > 0$ such that $\frac{\beta}{\gamma} < \zeta < \frac{\alpha}{\beta}$. Thus, writing

$$\begin{aligned} & -2\beta(\nabla(w_n - w_n^{hk}), \nabla(u_n - u_n^{hk}))_H \\ & \leq \beta\zeta \|\nabla(u_n - u_n^{hk})\|_H^2 + \frac{\beta}{\zeta} \|\nabla(w_n - w_n^{hk})\|_H^2 \end{aligned}$$

we can conclude that

$$\begin{aligned} & \alpha \|\nabla(u_n - u_n^{hk})\|_H^2 + \gamma \|\nabla(w_n - w_n^{hk})\|_H^2 \\ & + 2\beta(\nabla(w_n - w_n^{hk}), \nabla(u_n - u_n^{hk}))_H \\ & \geq \left(\gamma - \frac{\beta}{\zeta}\right) \|\nabla(w_n - w_n^{hk})\|_H^2 + (\alpha - \beta\zeta) \|\nabla(u_n - u_n^{hk})\|_H^2. \end{aligned}$$

Therefore, multiplying estimates (24) by k and summing up to n , we obtain, for all $z^h = \{z_j^h\}_{j=0}^n \subset V^h$ and $r^h = \{r_j^h\}_{j=0}^n \subset V^h$,

$$\begin{aligned} & \|v_n - v_n^{hk}\|_Y^2 + k \sum_{j=1}^n \|\nabla(v_j - v_j^{hk})\|_H^2 + \|u_n - u_n^{hk}\|_Y^2 + \|\nabla(u_n - u_n^{hk})\|_H^2 \\ & + \|e_n - e_n^{hk}\|_Y^2 + \|w_n - w_n^{hk}\|_Y^2 + \|\nabla(w_n - w_n^{hk})\|_H^2 \\ & \leq Ck \sum_{j=1}^n (\|\nabla(u_j - u_j^{hk})\|_H^2 + \|\nabla(\dot{u}_j - \delta u_j)\|_H^2 + \|u_j - u_j^{hk}\|_Y^2 \\ & + \|v_j - v_j^{hk}\|_Y^2 + \|v_j - z_j^h\|_Y^2 + \|\dot{v}_j - \delta v_j\|_Y^2 + \|w_j - w_j^{hk}\|_Y^2 \\ & + \|e_j - e_j^{hk}\|_Y^2 + \|\dot{e}_j - \delta e_j\|_Y^2 + \|e_j - r_j^h\|_Y^2) \end{aligned}$$

$$\begin{aligned} & + \|v_j - v_j^{hk}\|_Y^2 + \|v_j - z_j^h\|_Y^2 + \|\dot{v}_j - \delta v_j\|_Y^2 \\ & + \|w_j - w_j^{hk}\|_Y^2 + \|e_j - e_j^{hk}\|_Y^2 + \|\nabla(v_j - z_j^h)\|_H^2 \\ & + (\delta v_j - \delta v_j^{hk}, v_j - z_j^h)_Y + \|\nabla(e_j - r_j^h)\|_H^2 \\ & + \|\nabla(w_j - w_j^{hk})\|_H^2 + \|\nabla(\dot{w}_j - \delta w_j)\|_H^2 + (\delta e_j - \delta e_j^{hk}, e_j - r_j^h)_Y \\ & + \|\dot{e}_j - \delta e_j\|_Y^2 + \|e_j - r_j^h\|_Y^2 + \|\dot{u}_j - \delta u_j\|_Y^2 + \|\dot{w}_j - \delta w_j\|_Y^2 \\ & + C(\|v_0 - v_0^h\|_Y^2 + \|u_0 - u_0^h\|_Y^2 + \|\nabla(u_0 - u_0^h)\|_H^2 + \|e_0 - e_0^h\|_Y^2 \\ & + \|w_0 - w_0^h\|_Y^2 + \|\nabla(w_0 - w_0^h)\|_H^2). \end{aligned}$$

Keeping in mind that

$$\begin{aligned} & k \sum_{j=1}^n (\delta v_j - \delta v_j^{hk}, v_j - z_j^h)_Y = \sum_{j=1}^n (v_j - v_j^{hk} - (v_{j-1} - v_{j-1}^{hk}), v_j - z_j^h)_Y \\ & = (v_n - v_n^{hk}, v_n - z_n^h)_Y + (v_0^h - v_0, v_1 - z_1^h)_Y \\ & + \sum_{j=1}^{n-1} (v_j - v_j^{hk}, v_j - z_j^h - (v_{j+1} - z_{j+1}^h))_Y, \\ & k \sum_{j=1}^n (\delta e_j - \delta e_j^{hk}, e_j - r_j^h)_Y = \sum_{j=1}^n (e_j - e_j^{hk} - (e_{j-1} - e_{j-1}^{hk}), e_j - r_j^h)_Y \\ & = (e_n - e_n^{hk}, e_n - r_n^h)_Y + (e_0^h - e_0, e_1 - r_1^h)_Y \\ & + \sum_{j=1}^{n-1} (e_j - e_j^{hk}, e_j - r_j^h - (e_{j+1} - r_{j+1}^h))_Y, \end{aligned}$$

applying a discrete version of Gronwall's inequality (see Campo et al., 2006), we obtain the following theorem which provides a priori error estimates.

Theorem 3.5. Under the assumptions of Theorem 2.1, if we denote by (v, e) and (v^{hk}, e^{hk}) the respective solutions to problems VP and VP^{hk}, we have the following a priori error estimates, for all $z^h = \{z_j^h\}_{j=0}^n \subset V^h$ and $r^h = \{r_j^h\}_{j=0}^n \subset V^h$,

$$\begin{aligned} & \max_{0 \leq n \leq N} \{ \|v_n - v_n^{hk}\|_Y^2 + \|u_n - u_n^{hk}\|_Y^2 + \|\nabla(u_n - u_n^{hk})\|_H^2 + \|e_n - e_n^{hk}\|_Y^2 \\ & + \|w_n - w_n^{hk}\|_Y^2 + \|\nabla(w_n - w_n^{hk})\|_H^2 \} + k \sum_{j=1}^N \|\nabla(v_j - v_j^{hk})\|_H^2 \\ & \leq Ck \sum_{j=1}^N (\|\dot{u}_j - \delta u_j\|_Y^2 + \|v_j - z_j^h\|_Y^2 + \|\dot{v}_j - \delta v_j\|_Y^2 \\ & + \|\nabla(v_j - z_j^h)\|_H^2 + \|\nabla(e_j - r_j^h)\|_H^2 + \|\dot{w}_j - \delta w_j\|_Y^2 \\ & + \|\dot{e}_j - \delta e_j\|_Y^2 + \|e_j - r_j^h\|_Y^2) \\ & + \frac{C}{k} \sum_{j=1}^{N-1} \|v_j - z_j^h - (v_{j+1} - z_{j+1}^h)\|_Y^2 + C \max_{0 \leq n \leq N} \|v_n - z_n^h\|_Y^2 \\ & + \frac{C}{k} \sum_{j=1}^{N-1} \|e_j - r_j^h - (e_{j+1} - r_{j+1}^h)\|_Y^2 + C \max_{0 \leq n \leq N} \|e_n - r_n^h\|_Y^2 \\ & + C(\|v_0 - v_0^h\|_Y^2 + \|u_0 - u_0^h\|_Y^2 + \|\nabla(u_0 - u_0^h)\|_H^2 + \|e_0 - e_0^h\|_Y^2 \\ & + \|w_0 - w_0^h\|_Y^2 + \|\nabla(w_0 - w_0^h)\|_H^2). \quad (25) \end{aligned}$$

Estimates (25) are the basis for the convergence analysis. As an example, note that the finite element space V^h is given in (14), and let us assume further regularity conditions on the solution to the continuous problem:

$$\begin{aligned} & u \in H^2(0, T; H^1(\Omega)) \cap H^3(0, T; Y) \cap C^1([0, T]; H^2(\Omega)), \\ & w \in H^2(0, T; H^1(\Omega)) \cap H^3(0, T; Y) \cap C^1([0, T]; H^2(\Omega)). \quad (26) \end{aligned}$$

Recalling that the discrete initial conditions u_0^h, v_0^h, w_0^h and e_0^h were defined in (15), we have the following.

Corollary 3.6. Let the assumptions of Theorem 3.5 still hold. Under the additional regularity conditions (26) the linear convergence of the algorithm is deduced; i.e. there exists a positive constant $C > 0$, independent of the discretization parameters h and k , such that

$$\max_{0 \leq n \leq N} \{ \|v_n - v_n^{hk}\|_Y + \|u_n - u_n^{hk}\|_Y + \|\nabla(u_n - u_n^{hk})\|_H + \|e_n - e_n^{hk}\|_Y + \|w_n - w_n^{hk}\|_Y + \|\nabla(w_n - w_n^{hk})\|_H \} \leq C(h+k).$$

The proof of Corollary 3.6 relies on the well-known estimates by finite elements (see (Ciarlet, 1991)):

$$\begin{aligned} k \sum_{j=1}^N & \left(\inf_{z_j^h \in V^h} \|v_j - z_j^h\|_Y^2 + \inf_{z_j^h \in V^h} \|\nabla(v_j - z_j^h)\|_H^2 + \inf_{r_j^h \in V^h} \|e_j - r_j^h\|_Y^2 \right. \\ & \left. + \inf_{r_j^h \in V^h} \|\nabla(e_j - r_j^h)\|_H^2 \right) \\ & + \max_{0 \leq n \leq N} \inf_{z_n^h \in V^h} \|v_n - z_n^h\|_Y^2 + \max_{0 \leq n \leq N} \inf_{r_n^h \in V^h} \|e_n - r_n^h\|_Y^2 \\ & \leq Ch^2 (\|v\|_{C([0,T];H^2(\Omega))}^2 + \|e\|_{C([0,T];H^2(\Omega))}^2). \end{aligned}$$

From the regularity conditions (26), it follows that

$$u_0, v_0, w_0, e_0 \in H^2(\Omega),$$

and therefore,

$$\begin{aligned} & \|v_0 - v_0^h\|_Y^2 + \|u_0 - u_0^h\|_Y^2 + \|\nabla(u_0 - u_0^h)\|_H^2 + \|e_0 - e_0^h\|_Y^2 \\ & + \|w_0 - w_0^h\|_Y^2 + \|\nabla(w_0 - w_0^h)\|_H^2 \\ & \leq Ch^2 (\|u_0\|_{H^2(\Omega)}^2 + \|v_0\|_{H^2(\Omega)}^2 + \|w_0\|_{H^2(\Omega)}^2 + \|e_0\|_{H^2(\Omega)}^2). \end{aligned}$$

Moreover, using again regularity conditions (26), we easily find that

$$\begin{aligned} k \sum_{j=1}^N & [\|\dot{u}_j - \delta u_j\|_V^2 + \|\dot{v}_j - \delta v_j\|_Y^2 + \|\dot{w}_j - \delta w_j\|_V^2 + \|\dot{e}_j - \delta e_j\|_Y^2] \\ & \leq ck^2 (\|u\|_{H^2(0,T;H^1(\Omega))}^2 + \|u\|_{H^3(0,T;Y)}^2 \\ & + \|w\|_{H^2(0,T;H^1(\Omega))}^2 + \|w\|_{H^3(0,T;Y)}^2). \end{aligned}$$

Finally, the last terms in estimates (25) are bounded using the ideas employed in Barboteu et al. (2005); Campo et al. (2006), obtaining

$$\begin{aligned} & \frac{1}{k} \sum_{j=1}^{N-1} \|v_j - z_j^h - (v_{j+1} - z_{j+1}^h)\|_Y^2 + \frac{1}{k} \sum_{j=1}^{N-1} \|e_j - r_j^h - (e_{j+1} - r_{j+1}^h)\|_Y^2 \\ & \leq Ch^2 (\|\dot{v}\|_{L^2(0,T;H^1(\Omega))}^2 + \|\dot{e}\|_{L^2(0,T;H^1(\Omega))}^2). \end{aligned}$$

Combining all these estimates, the linear convergence is deduced.

4. Numerical results

In this section, we first describe the numerical scheme implemented in MATLAB in order to obtain the numerical approximations of Problem VP^{hk} and then, we present some numerical results to exhibit its accuracy in a simple one-dimensional example, with an exact solution known, and its behaviour in the simulation of other one-dimensional examples and also a two-dimensional one.

Considering the finite element space defined in (14), for $n = 1, 2, \dots, N$ and given $u_{n-1}^{hk}, v_{n-1}^{hk}, w_{n-1}^{hk}, e_{n-1}^{hk} \in V^h$, the discrete velocity fields at time $t = t_n$ of the viscoelastic and elastic materials, v_n^{hk} and e_n^{hk} respectively, are then obtained from Eqs. (16) and (17); namely, they solve the problem:

$$\begin{aligned} & \rho_1 (v_n^{hk}, z^h)_Y + k^2 \xi (v_n^{hk}, z^h)_Y - k^2 \xi (e_n^{hk}, z^h)_Y \\ & + k \xi^* (v_n^{hk}, z^h)_Y - k \xi^* (e_n^{hk}, z^h)_Y \\ & + k^2 \beta (\nabla e_n^{hk}, \nabla z^h)_H + k^2 \alpha (\nabla v_n^{hk}, \nabla z^h)_H + k \mu^* (\nabla v_n^{hk}, \nabla z^h)_H \\ & = k \rho_1 (F_{1n}, z^h) + \rho_1 (v_{n-1}^{hk}, z^h)_Y - k \xi (u_{n-1}^{hk}, z^h)_Y + k \xi (w_{n-1}^{hk}, z^h)_Y \end{aligned}$$

$$\begin{aligned} & - k \beta (\nabla w_{n-1}^{hk}, \nabla z^h)_H - k \alpha (\nabla u_{n-1}^{hk}, \nabla z^h)_H \quad \forall z^h \in V^h, \\ & \rho_2 (e_n^{hk}, r^h)_Y + k^2 \xi (e_n^{hk}, r^h)_Y - k^2 \xi (v_n^{hk}, r^h)_Y + k \xi^* (e_n^{hk}, r^h)_Y \\ & - k \xi^* (v_n^{hk}, r^h)_Y + k^2 \beta (\nabla v_n^{hk}, \nabla r^h)_H + k^2 \gamma (\nabla e_n^{hk}, \nabla r^h)_H \\ & = k \rho_2 (F_{2n}, r^h) + \rho_2 (e_{n-1}^{hk}, r^h)_Y - k \xi (w_{n-1}^{hk}, r^h)_Y + k \xi (u_{n-1}^{hk}, r^h)_Y \\ & - k \beta (\nabla u_{n-1}^{hk}, \nabla r^h)_H - k \gamma (\nabla w_{n-1}^{hk}, \nabla r^h)_Y, \quad \forall r^h \in V^h, \end{aligned}$$

where the discrete displacement fields are updated in the following form:

$$u_n^{hk} = k v_n^{hk} + u_{n-1}^{hk}, \quad w_n^{hk} = k e_n^{hk} + w_{n-1}^{hk}.$$

This problem leads to a linear system for a variable U in the product space which is solved by using classical Cholesky's method. This numerical scheme has been implemented on a 3.2 Ghz PC using MATLAB, and a typical 1D run ($h = k = 0.01$) takes about 0.51 s of CPU time.

4.1. First example: numerical convergence

As a first example, the following one-dimensional problem is considered.

Problem 1D. Find the displacement of the viscoelastic material u : $[0, 1] \times [0, 1] \rightarrow \mathbb{R}$ and the displacement of the elastic material w : $[0, 1] \times [0, 1] \rightarrow \mathbb{R}$ such that

$$\frac{\partial^2 u}{\partial t^2} - 2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 w}{\partial x^2} + (u - w) - \frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\partial(u - w)}{\partial t} = F_1$$

$$\text{in } (0, 1) \times (0, 1),$$

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - 2 \frac{\partial^2 w}{\partial x^2} - (u - w) - \frac{\partial(u - w)}{\partial t} = F_2 \quad \text{in } (0, 1) \times (0, 1),$$

$$u(0, t) = 0, \quad w(0, t) = 0 \quad \text{for } t \in (0, 1),$$

$$u(1, t) = 0, \quad w(1, t) = 0 \quad \text{for } t \in (0, 1),$$

$$u(x, 0) = x(x - 1), \quad w(x, 0) = x(x - 1) \quad \text{for } x \in (0, 1),$$

$$\frac{\partial u}{\partial t}(x, 0) = x(x - 1), \quad \frac{\partial w}{\partial t}(x, 0) = x(x - 1) \quad \text{for } x \in (0, 1),$$

where the volume forces F_1 and F_2 are given by

$$F_1(x, t) = e^t (x(x - 1) - 8),$$

$$F_2(x, t) = e^t (x(x - 1) - 6).$$

Problem 1D corresponds to Problem P with the following data:

$$\Omega = (0, 1), \quad T = 1, \quad \alpha = 2, \quad \mu^* = 1, \quad \xi = 1, \quad \xi^* = 1, \quad \beta = 1,$$

$$\gamma = 2, \quad \rho_1 = 1, \quad \rho_2 = 1, \quad u_0 = v_0 = w_0 = e_0 = x(x - 1).$$

We remark that the exact solution to Problem 1D can be easily calculated and it has the form:

$$u(x, t) = x(x - 1)e^t,$$

$$w(x, t) = x(x - 1)e^t \quad \text{for } (x, t) \in (0, 1) \times (0, 1).$$

Our aim here is to show the numerical convergence of the algorithm. Therefore, several uniform partitions of both the time interval and the domain, dividing $\Omega = (0, 1)$ into n segments, have been performed. We note that the spatial discretization parameter h equals to $\frac{1}{n}$.

The numerical errors, given by

$$\begin{aligned} \text{Error} = \max_{0 \leq n \leq N} & \{ \|u_n - u_n^{hk}\|_Y + \|\nabla(u_n - u_n^{hk})\|_H + \|\nabla(w_n - w_n^{hk})\|_H \\ & + \|w_n - w_n^{hk}\|_Y + \|e_n - e_n^{hk}\|_Y + \|v_n - v_n^{hk}\|_Y \}, \end{aligned}$$

and obtained for different discretization parameters h and k , are depicted (multiplied by 1000) in Table 1. Moreover, the evolution of the error depending on $h + k$ is plotted in Fig. 1. We notice that the convergence of the algorithm is clearly observed, and the linear convergence, stated in Corollary 3.6, is achieved.

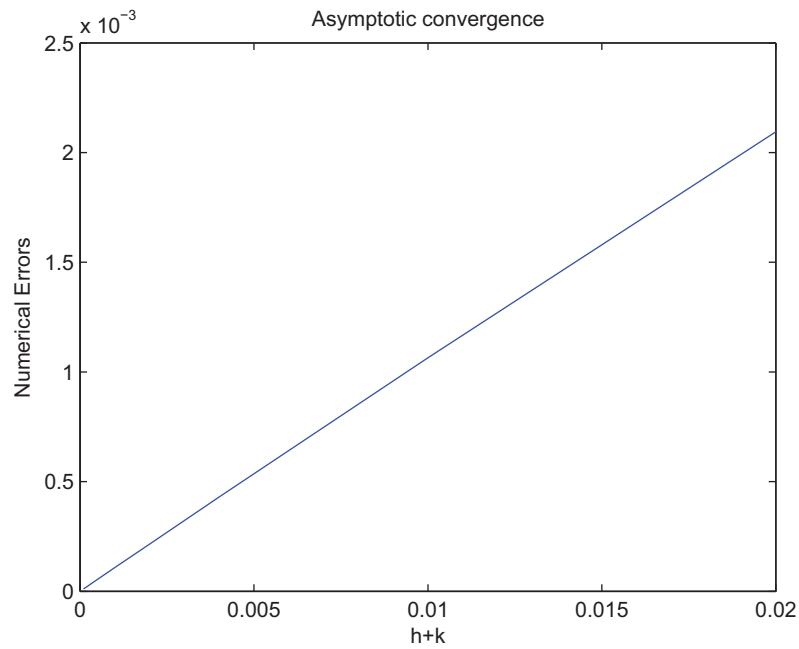


Fig. 1. Example 1: Asymptotic constant error.

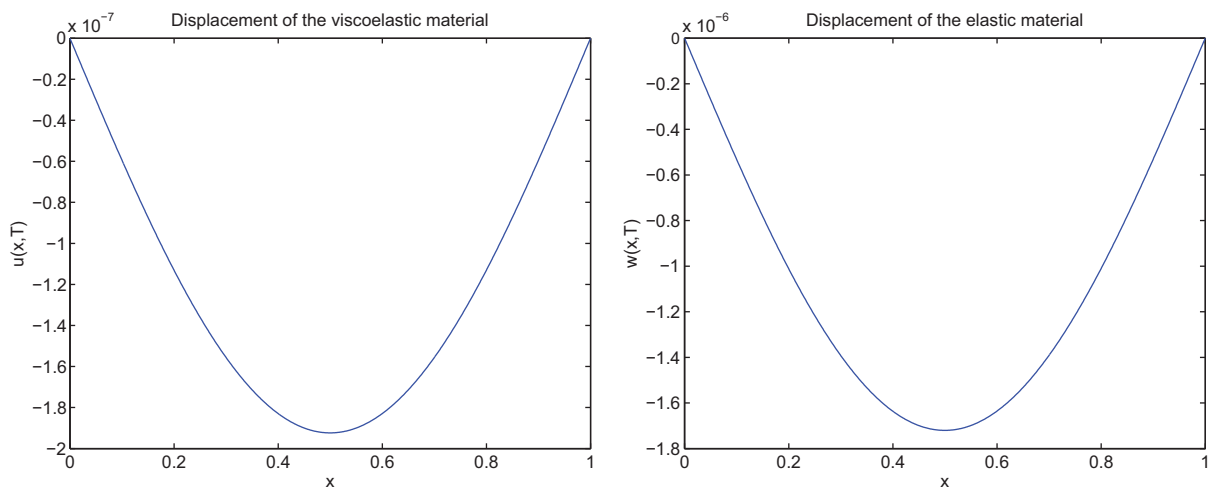


Fig. 2. Example 2: Displacements of the viscoelastic and elastic material at final time.

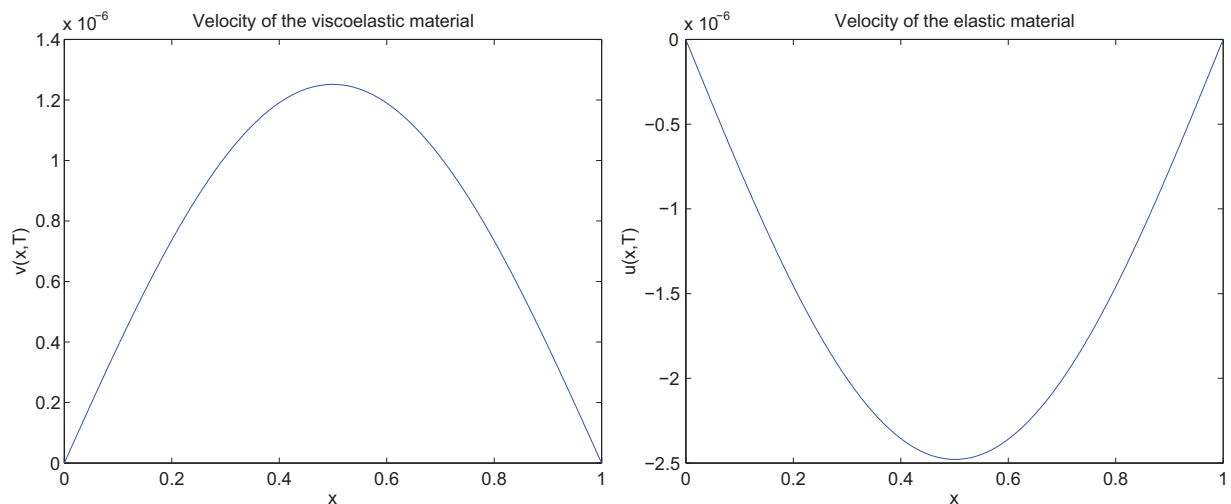
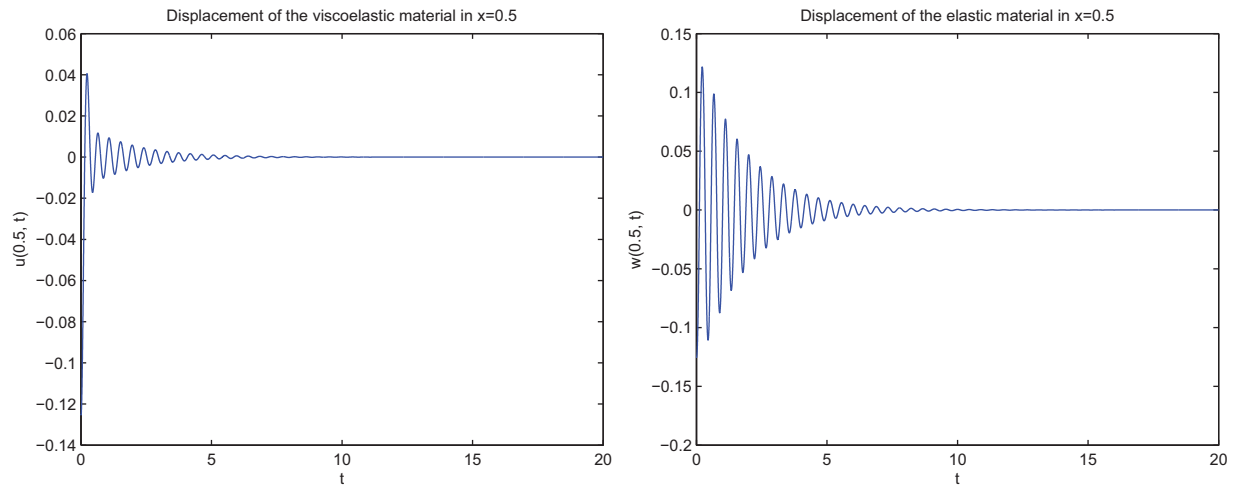
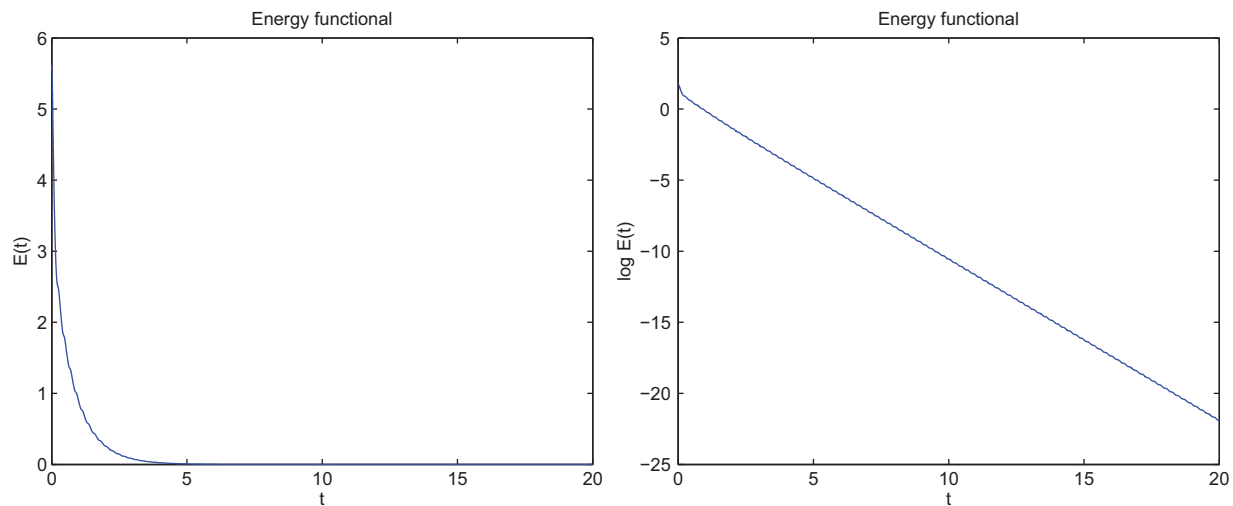


Fig. 3. Example 2: Velocities of the viscoelastic and elastic material at final time.

Table 1Example 1D-1: Numerical errors ($\times 1000$) for some h and k .

$h \downarrow k \rightarrow$	0.01	0.005	0.001	0.0005	0.0002	0.0001	0.00005
0.01	2.094946	1.007398	0.168197	0.090658	0.070414	0.080952	0.088682
0.005	2.153877	1.064716	0.197497	0.090833	0.033740	0.019638	0.017603
0.002	2.170650	1.081387	0.213352	0.105083	0.040208	0.018739	0.008527
0.001	2.173055	1.083787	0.215734	0.107444	0.042497	0.020857	0.010051
0.0005	2.173656	1.084388	0.216333	0.108042	0.043091	0.021445	0.010623
0.0002	2.173825	1.084556	0.216500	0.108209	0.043258	0.021612	0.010789
0.0001	2.173851	1.084581	0.216527	0.108236	0.043283	0.021637	0.010814
0.00005	2.173848	1.084579	0.216522	0.108232	0.043282	0.021636	0.010814

**Fig. 4.** Example 2: Evolution in time of the displacements of the viscoelastic and elastic material at the middle point.**Fig. 5.** Example 2: Evolution in time of the discrete energy in natural and semi-log scales.

4.2. Second example: energy decay property

As a second example, we solve another one-dimensional problem with the following data:

$$\begin{aligned} \Omega &= (0, 1), \quad \Gamma_D = \{0\}, \quad \Gamma_S = \{1\}, \quad T = 20, \\ \alpha &= 20, \quad \mu^* = 1, \quad \xi = 1, \\ \xi^* &= 1, \quad \beta = 1, \quad \gamma = 20, \quad \rho_1 = 1, \quad \rho_2 = 1, \quad F_1 = F_2 = 0, \\ u_0 &= v_0 = w_0 = e_0 = x^2(x-1). \end{aligned}$$

The discretization parameters used in this example are $h = k = 0.001$. Thus, in Fig. 2 the displacements of the viscoelastic material and the elastic material are plotted at final time. Moreover, the velocities of the viscoelastic and elastic materials are shown in Fig. 3.

The evolution in time of the displacement of the viscoelastic and elastic materials at the middle point $x = 0.5$ is plotted in Fig. 4. As can be seen, both displacements tend to zero due to the viscoelastic effect.

Finally, in Fig. 5 we show the evolution of the discrete energy given in (21) when time grows in both natural and semi-log scales. We can observe that it is a decreasing function and that it tends to zero. Moreover, the exponential decay seems to be achieved.

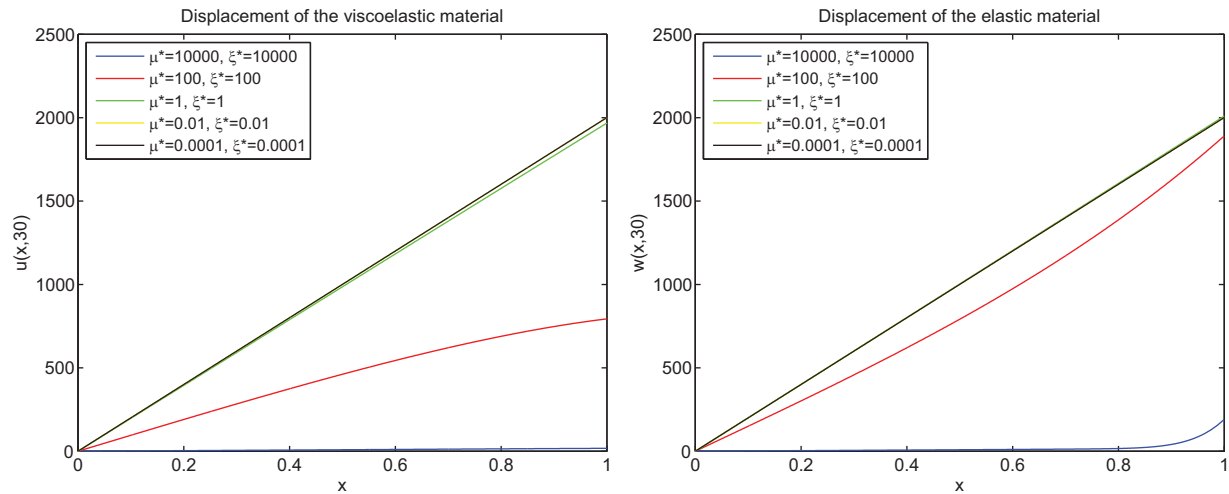


Fig. 6. Example 3: Displacements of the viscoelastic and elastic materials at final time for different values of the viscoelastic coefficients.

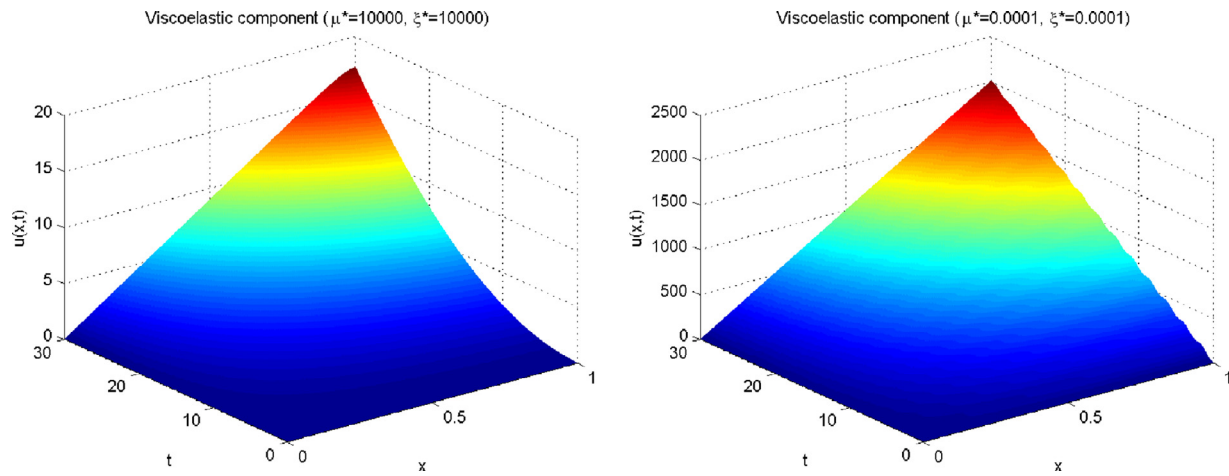


Fig. 7. Example 3: Displacements of the viscoelastic component for values $\mu^* = \xi^* = 10^4$ and $\mu^* = \xi^* = 10^{-4}$.

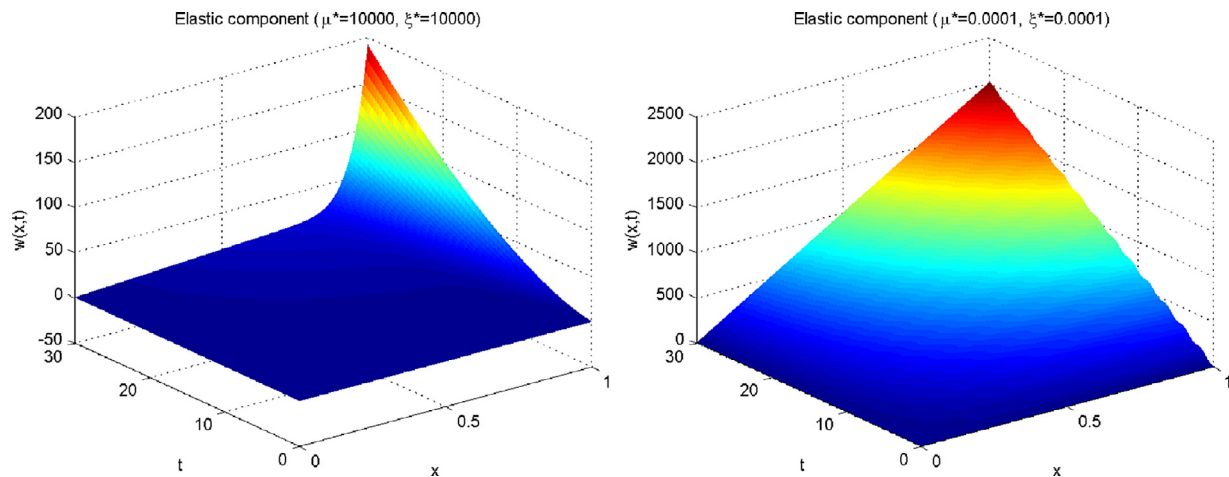


Fig. 8. Example 3: Displacements of the elastic component for values $\mu^* = \xi^* = 10^4$ and $\mu^* = \xi^* = 10^{-4}$.

4.3. Third example: dependence on the viscoelastic coefficients

As a final one-dimensional example, we consider the numerical resolution of a one-dimensional problem depending on the viscoelastic coefficients μ^* and ξ^* . The following remaining data have been

considered:

$$\Omega = (0, 1), \quad \Gamma_D = \{0\}, \quad \Gamma_S = \{1\}, \quad T = 30, \quad \alpha = 2, \quad \xi = 1, \\ \beta = 1, \quad \gamma = 2, \quad \rho_1 = 1, \quad \rho_2 = 1, \quad u_0 = v_0 = w_0 = e_0 = 0, \\ F_1 = F_2 = 0.$$

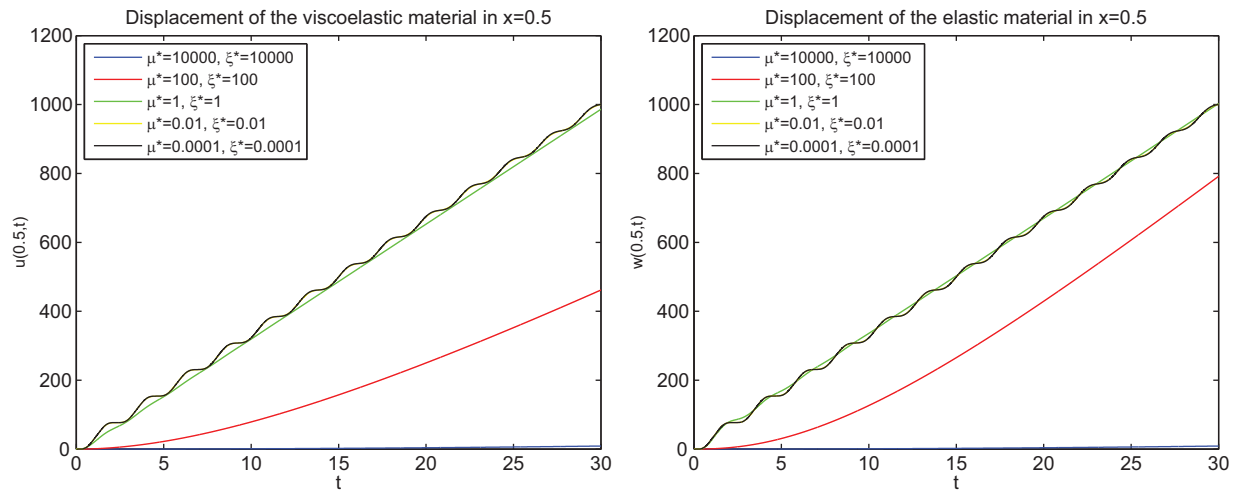


Fig. 9. Example 3: Evolution in time of the displacements of the viscoelastic and elastic material at the middle point for different values of the viscoelastic coefficients.

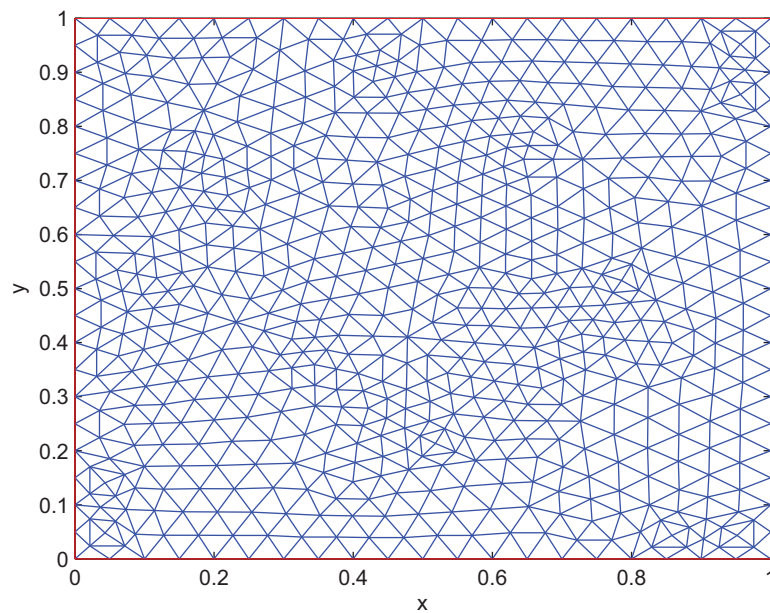


Fig. 10. Example 4: Finite element mesh.

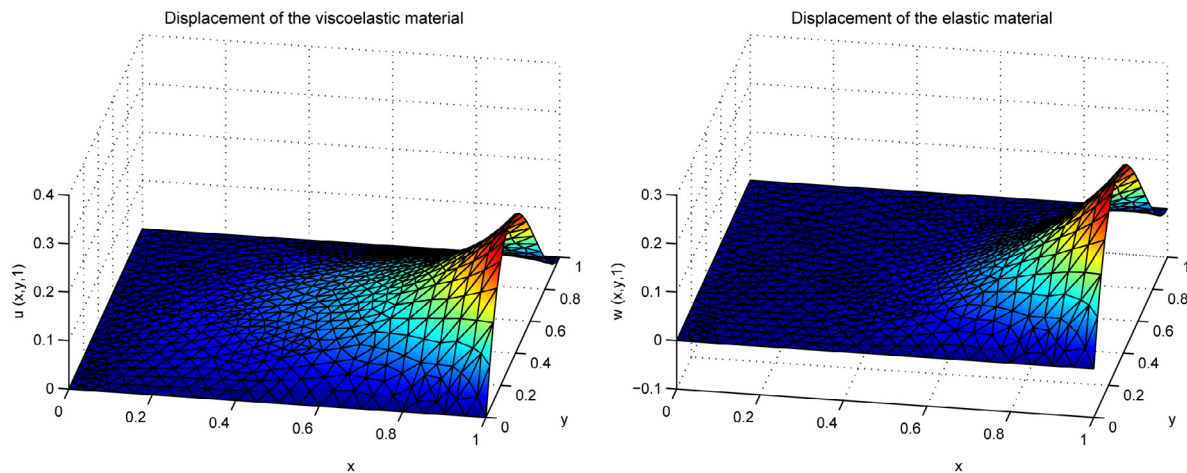


Fig. 11. Example 4: Displacements of the viscoelastic and elastic materials at final time.

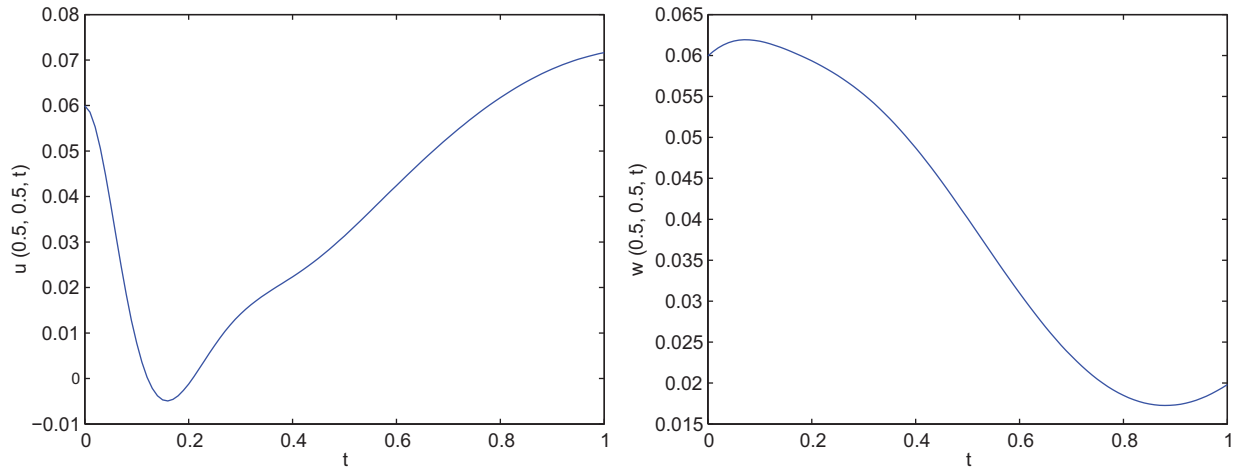


Fig. 12. Example 4: Evolution in time of the displacements of the viscoelastic and elastic material at the middle point $\mathbf{x} = (0.5, 0.5)$.

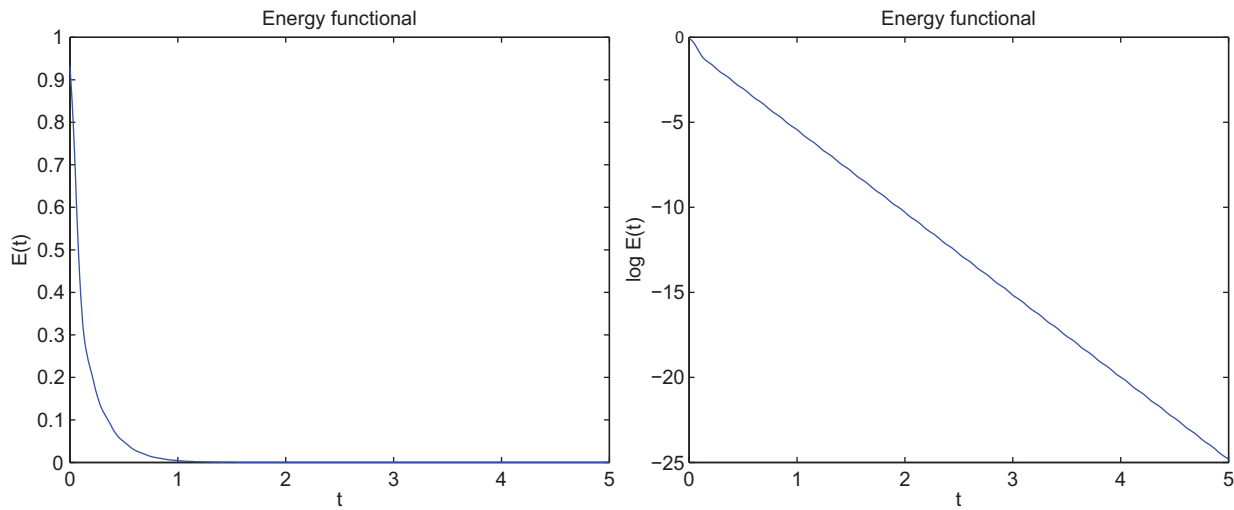


Fig. 13. Example 4: Evolution in time of the discrete energy in natural and semi-log scales.

In this example, we assume that a traction force is applied on the boundary Γ_S . That is, we impose the Neumann condition:

$$\begin{aligned} \alpha \frac{\partial u}{\partial x}(1, t) + \beta \frac{\partial w}{\partial x}(1, t) + \mu^* \frac{\partial^2 u}{\partial t \partial x}(1, t) &= g_1(t), \\ \beta \frac{\partial u}{\partial x}(1, t) + \gamma \frac{\partial w}{\partial x}(1, t) &= g_2(t). \end{aligned}$$

In particular, two forces of density $g_1(t) = g_2(t) = t$ are applied at the right point $x = 1$. Moreover, we consider that the viscoelastic coefficients μ^* and ξ^* are equal and that they vary between 10^{-4} and 10^4 (values $10^4, 10^2, 1, 10^{-2}$ and 10^{-4} have been used).

Using the discretization parameters $h = 0.0005$ and $k = 0.005$, in Fig. 6 the displacements of the viscoelastic and elastic materials are plotted at final time for different values of the viscoelastic coefficients. As expected, since the force is applied at the right point, the displacements are always increasing with respect to the spatial variable and, when the viscosity coefficients are higher, the displacements reduce drastically.

In Figs. 7 and 8 we show the displacements of the viscoelastic and elastic components, respectively, for values $\mu^* = \xi^* = 10^{-4}$ and $\mu^* = \xi^* = 10^4$. As expected, displacements are less when the elastic and viscoelastic coefficients are greater, and we also observe that the displacements of the elastic and viscoelastic components seem to have a different behaviour for large values of the constitutive coefficients.

Finally, in Fig. 9 the evolution in time of the displacements of the viscoelastic and elastic materials are plotted at the middle point $x = 0.5$ for different values of the viscoelastic coefficients. We can observe again that these are also increasing functions with respect to the time and that, when the viscous effects are higher, their values are very small.

4.4. Fourth example: two-dimensional simulations

As a first two-dimensional example, we solve Problem P with the following data:

$$\Omega = (0, 1) \times (0, 1), \quad \Gamma_D = \Gamma, \quad \Gamma_S = \emptyset,$$

$$T = 1, \quad \rho_1 = 1, \quad \mu^* = 1,$$

$$\alpha = 25, \quad \xi = 25, \quad \xi^* = 1, \quad \beta = 20, \quad \gamma = 25, \quad \rho_2 = 1,$$

with the initial conditions $u_0 = v_0 = w_0 = e_0 = xy(y-1)^2$.

Here, our aim is to show the behaviour of the constitutive equations in a two-dimensional example. Thus, we assume a non-homogenous boundary condition on Γ_D in the form:

$$\begin{cases} 0 & \text{if } x = 0, y \in [0, 1], \\ 0 & \text{if } y = 1, x \in [0, 1], \\ 0 & \text{if } y = 0, x \in [0, 1], \\ y(y-1)^2 & \text{if } x = 1, y \in [0, 1], \end{cases}$$

and that the volume forces are neglected, i.e. $F_1 = F_2 = 0$.

Using the time discretization parameter $k = 10^{-2}$ and the finite element mesh shown in Fig. 10, the displacements of the viscoelastic and elastic materials are plotted at final time in Fig. 11. As we can see, both increase with a quadratic form with respect to the Y -variable, although the elastic displacements seem to be higher. Moreover, in Fig. 12 we show the evolution in time of these displacements at the middle point $\mathbf{x} = (0.5, 0.5)$. An oscillation is observed, possibly due to the viscosity effects.

Finally, in order to show the energy decay in a two-dimensional case, we assume now a homogeneous Dirichlet boundary condition and we employ the initial conditions:

$$u_0(x, y) = w_0(x, y) = v_0(x, y) = e_0(x, y) = (x^2 - x)(y^2 - y).$$

Thus, using again the finite element mesh shown in Fig. 10 and the time discretization parameter $k = 10^{-2}$, in Fig. 13 the evolution of the discrete energy, given in (21), is plotted when time grows in both natural and semi-log scales. As in the one-dimensional case, we can observe that it is a decreasing function and that it tends to zero. The exponential decay also seems to be achieved.

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