



Thermal-electric-elastic analyses of a thermoelectric material containing two circular holes

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ABSTRACT

We investigate the thermal-electric-elastic fields around two circular holes in a thermoelectric material. Solutions describing electric, thermal and elastic fields are obtained via complex variable and numerical methods. Our results show that the presence of the holes has the same effect on the effective electric and thermal conductivities but no effect whatsoever on the effective Seebeck coefficient and consequently on the thermoelectric figure of merit. The maximum values of electric potential and temperature around one hole are increased by the presence of the second hole but decrease sharply as the distance between the two holes continues to increase. For two holes of equal size, the interaction effects on the corresponding electric and thermal fields are negligible when the distance between the holes is twice that of the common radius. In comparison to the thermoelectric field, the interaction effect on the stress field decreases much slower with the increase of distance, especially when the difference in the radii between the two holes is large. Numerical results show that the interaction effect on hoop stress between two holes of equal size is five times that of the corresponding effect on the temperature field and that the interaction effect is essentially negligible if one hole is more than twice the size of the other. It is worth pointing out that the two adjacent holes generate an extreme stress distribution in thermoelectric materials when the distance between the two holes is far less than the smaller of the two radii which is in stark contrast to the analogous case encountered in classical thermoelastic problems.

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1. Introduction

Thermoelectric materials can directly interconvert heat and electricity via internal carriers whose advantages include simplicity in structure, stability in efficiency and the absence of any emissions (Liu et al., 2020; Shi and Chen, 2016). In recent years, the energy conversion efficiency of thermoelectric materials has been significantly improved in an effort to support their use in a number of applications including waste heat recovery, solar energy devices, and thermoelectric refrigeration equipment (Fernandes et al., 2020; Kim et al., 2006). The reliability of thermoelectric devices, however, has not received similar attention in the literature.

In thermoelectric materials, both electric and thermal loadings generate thermal stress due to the coupling transport of electric current and heat flux. Since thermoelectric materials are brittle, they fracture easily under thermal stress, especially when they incorporate inhomogeneities. Such inhomogeneities are generally

introduced into a thermoelectric material in order to improve energy conversion efficiency. For example, porous structures are commonly found in thermoelectric devices (Tang et al., 2010). When such a thermoelectric material is loaded with temperature difference and electric current, the embedded inhomogeneities result in an uneven temperature field around the inhomogeneities which often leads to a situation in which the thermal stress exceeds the yield stress of the material (Wang et al., 2019). To examine the failure mechanisms arising in thermoelectric composites, Song et al. (2015) derived general solutions for the corresponding thermal, electric and elastic fields in thermoelectric materials, and discussed the singularity of stress at a crack tip induced by remote electric and thermal loading. Shortly after, theoretical and numerical results describing the thermoelectric and thermoelastic fields of a thermoelectric material with an inclined elliptic hole were obtained (Wang and Wang, 2017). Noting the coupling interaction between heat and electricity in thermoelectric materials, additional electric current was introduced to counteract the stress intensity factors induced by thermal expansion at the crack tip (Song et al., 2019a). The aforementioned works focused on material properties around a single hole embedded in a thermo-

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electric material. The study of multiphysics problems involving porous thermoelectric materials, however, remains largely absent from the literature.

In porous structures, the interaction effects arising from physical fields between adjacent holes cannot be ignored when the holes are located relatively close together. As a result, the series-form solution for an infinite plate with two circular holes was obtained by introducing a conformal mapping function to map the physical region onto an annulus in the image plane (Haddon, 1967). Under the combined loadings of remote stresses and internal pressures on the boundary of the holes, the analytic solution for stresses in an infinite plate with two unequal circular holes was presented using the general expression for a biharmonic function in bipolar coordinates. In addition, the interaction force between the holes as well as the region of stress concentration have also been discussed in detail (Radi, 2011). Taking surface tension into consideration, Wang et al. (2020) studied the surface tension-induced stress field around two nanoscale holes in an unbounded material and concluded that the interaction between the two holes can be neglected when the distance between the two holes is more than three times the sum of their diameters. These works highlight the dominant role of interaction effects between two adjacent holes in mechanical stress analysis which motivates our present study of the interaction effects between holes in thermoelectric materials.

In this paper, we investigate the thermal-electric-elastic coupling problem of an infinite thermoelectric plate containing two circular holes. The partly analytical and partly numerical solutions describing electric, thermal and elastic fields around two circular holes are obtained via complex variable and numerical methods, and the interaction effects on the effective properties and thermal stress between the two holes are discussed using illustrative examples. We aim to provide guidance for the optimal design of thermoelectric devices and suggest a way forward for the study of multi-field coupling problems in porous structures.

2. Governing equations

2.1. Electric and thermal fields

In a Cartesian plane described by the coordinates (x, y) , the electric potential and temperature fields can be expressed, respectively, as $\phi(x, y)$ and $T(x, y)$, and the equations governing electric current density $\mathbf{J}$, heat flux $\mathbf{Q}$ and energy flux $\mathbf{U}$ are given by (Callen, 1960; Yang et al., 2015)

$$-\mathbf{J} = \sigma \nabla \phi + \sigma S \nabla T \quad (1)$$

$$\mathbf{Q} = TS\mathbf{J} - \kappa \nabla T \quad (2)$$

$$\mathbf{U} = \mathbf{Q} + \phi \mathbf{J} \quad (3)$$

where σ , S and κ are the electric conductivity, Seebeck coefficient and thermal conductivity, respectively. We assume that electric current and energy are conserved, and then

$$\nabla \cdot \mathbf{J} = 0 \quad (4)$$

$$\nabla \cdot \mathbf{U} = 0 \quad (5)$$

Note that since the material parameters in thermoelectric materials are generally assumed to be temperature-independent, we can rewrite Eqs. (4) and (5) using Eqs. (1)–(3) as (Zhang et al., 2017)

$$\nabla^2(\phi + ST) = 0 \quad (6)$$

$$\sigma[\nabla(\phi + ST)]^2 + \kappa \nabla^2 T = 0 \quad (7)$$

With the aid of complex variable methods, the expression $\phi + ST$ can be expressed in terms of a complex analytic function $f(z)$ as (Song et al., 2015)

$$\phi + ST = f(z) + \bar{f}(\bar{z}) \quad (8)$$

in which z represents the complex variable $z = x + iy$ ($i^2 = -1$) and the overbar denotes the complex conjugate. Substituting Eq. (8) into Eq. (7), the general solution describing the temperature field is given by

$$T = -\frac{\sigma}{\kappa} f(z) \bar{f}(\bar{z}) + g(z) + \bar{g}(\bar{z}) + M \quad (9)$$

where $g(z)$ is an analytic function of the complex variable z and M is a real constant which represents a uniform temperature field. Substituting Eq. (9) into Eq. (8), the general solution describing the electric potential can be expressed as

$$\phi = \frac{\sigma S}{\kappa} f(z) \bar{f}(\bar{z}) + f(z) + \bar{f}(\bar{z}) - S(g(z) + \bar{g}(\bar{z})) - SM \quad (10)$$

in which the term $(-SM)$ can be considered as a uniform electric potential field. Substituting Eqs. (9)–(10) into Eqs. (1)–(3), we have

$$J_x - iJ_y = -2\sigma f'(z) \quad (11)$$

$$Q_x - iQ_y = 2\sigma S f'(z) \left(\frac{\sigma}{\kappa} f(z) \bar{f}(\bar{z}) - g(z) - \bar{g}(\bar{z}) - M \right) + 2\sigma f'(z) \times \bar{f}(\bar{z}) - 2\kappa g'(z), \quad (12)$$

$$U_x - iU_y = -2\sigma f(z) f'(\bar{z}) - 2\kappa g'(z) \quad (13)$$

2.2. Stress and displacement fields

The stress field in thermoelectric materials is induced mainly by an uneven temperature field. The equation governing the Airy stress function Φ and the temperature field T is given by (Parkus, 1968)

$$\nabla^4 \Phi + E\alpha \nabla^2 T = 0 \quad (14)$$

where E is Young's modulus, and α is the thermal expansion coefficient. Solving Eq. (14), we have (Wang and Wang, 2017)

$$\Phi = \frac{1}{2} [\bar{z} \varphi(z) + z \bar{\varphi}(\bar{z}) + \vartheta(z) + \bar{\vartheta}(\bar{z})] + \frac{E\alpha\sigma}{4\kappa} F(z) \bar{F}(\bar{z}) \quad (15)$$

where $\varphi(z)$ and $\vartheta(z)$ are complex analytic functions of z , while

$$F(z) = \int f(z) dz \quad (16)$$

From Eq. (15), the stress components can be expressed as (Muskhelishvili, 1975)

$$\sigma_x + \sigma_y = 4 \frac{\partial^2 \Phi}{\partial z \partial \bar{z}} = 2(\varphi'(z) + \bar{\varphi}'(\bar{z})) + \frac{E\alpha\sigma}{\kappa} f(z) \bar{f}(\bar{z}) \quad (17)$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 4 \frac{\partial^2 \Phi}{\partial z^2} = 2(\bar{z} \varphi''(z) + \psi'(z)) + \frac{E\alpha\sigma}{\kappa} f'(z) \bar{F}(\bar{z}) \quad (18)$$

where $\psi(z) = \vartheta'(z)$. Noting that the displacement field comprises free thermal expansion and the part corresponding to the stress field, we have

$$u + iv = \frac{1}{2G} [K\varphi(z) - z\bar{\varphi}'(\bar{z}) - \bar{\psi}(\bar{z})] + 2\alpha \int g(z) dz - \frac{E\alpha\sigma}{4\kappa G} F(z) \bar{f}(\bar{z}) \quad (19)$$

where $K = (3 - \mu)/(1 + \mu)$. Here, G and μ are the shear modulus and Poisson's ratio, respectively. For the plane strain problem, E , μ and α are replaced by $E/(1 - \mu^2)$, $\mu/(1 - \mu)$ and $(1 + \mu)\alpha$, respectively.

3. Solution for an infinite plate containing two circular holes

Consider a thermoelectric plane containing two circular holes and which is subjected to remote electric current and energy flux, as shown in Fig. 1. The radii of the holes are respectively R_1 and R_2 , while the corresponding boundaries are L_1 and L_2 . Note that the centers of the holes are respectively located at z_1 and z_2 so that

$$|z_1 - z_2| = R_1 + R_2 + d \quad (20)$$

where d is the distance between the two holes.

3.1. Boundary conditions

We assume that the holes are insulated, then the normal electric current and energy flux are zero, which yields

$$\int_{P_j}^{Q_j} J_r^j ds = 0, j = 1, 2 \quad (21)$$

$$\int_{P_j}^{Q_j} U_r^j ds = 0, j = 1, 2 \quad (22)$$

where P_j and Q_j are arbitrary points on L_j , while J_r and U_r are the normal electric current and energy flux. According to Eqs. (11) and (13), the integral forms of electric current and energy flux can be expressed as

$$\int J_r ds = i\sigma(f(z) - f(\bar{z})) \quad (23)$$

$$\int U_r ds = \frac{i\sigma}{2}(f(z)^2 - f(\bar{z})^2) + i\kappa(g(z) - g(\bar{z})) \quad (24)$$

On the other hand, the thermoelectric plane is subjected only to remote electric current and energy flux, and thus the boundaries of the holes are traction free,

$$i \int_{P_j}^{Q_j} (X^j + iY^j) ds = 0, j = 1, 2 \quad (25)$$

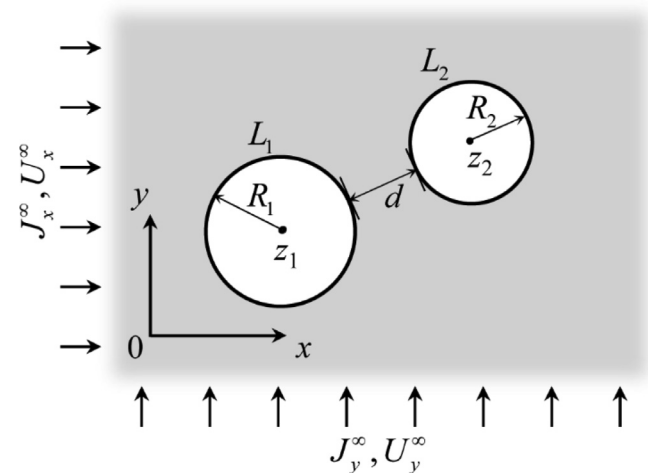


Fig. 1. Infinite thermoelectric plane containing two circular holes.

where X and Y are the components of surface force on the boundary. Noting Eq. (15), we have

$$i \int_P^Q (X + iY) ds = \varphi(z) + z \varphi'(z) + \psi(\bar{z}) + \frac{E\alpha\sigma}{2\kappa} f(\bar{z}) F(z) \quad (26)$$

3.2. Solutions describing the electric and thermal fields

According to the electric current density and energy flux at infinity, the complex functions $f(z)$ and $g(z)$ in Eqs. (9) and (10) can be expressed as

$$f(z) = A_{01}z + f_1(z - z_1) + f_2(z - z_2) \quad (27)$$

$$g(z) = B_{02}z^2 + B_{01}z + g_1(z - z_1) + g_2(z - z_2) \quad (28)$$

where f_j and g_j are analytic functions outside L_j ($j = 1, 2$). In addition, A_{01} , B_{01} and B_{02} are introduced to represent the states of electric current density and energy flux at infinity, which can be determined by Eqs. (11) and (13) as

$$A_{01} = -\frac{J_x^\infty - iJ_y^\infty}{2\sigma}, \quad B_{01} = -\frac{U_x^\infty - iU_y^\infty}{2\kappa}, \quad B_{02} = -\frac{(J_x^\infty - iJ_y^\infty)^2}{8\sigma\kappa} \quad (29)$$

Since it is quite a formidable task to establish exact closed-form solutions describing electric and thermal fields around the two holes using the boundary conditions in Eqs. (21) and (22), we seek approximate solutions instead. To this end, we expand $f(z)$ and $g(z)$ into finite series as

$$f(z) = A_{01}z + \sum_{j=1}^n A_{1j}(z - z_1)^{-j} + \sum_{j=1}^n A_{2j}(z - z_2)^{-j} \quad (30)$$

$$g(z) = B_{02}z^2 + B_{01}z + \sum_{j=1}^n B_{1j}(z - z_1)^{-j} + \sum_{j=1}^n B_{2j}(z - z_2)^{-j} \quad (31)$$

where n is the greatest power of $1/(z - z_1)$ and $1/(z - z_2)$ in both $f(z)$ and $g(z)$. In the solution process, the accuracy of solutions describing electric and thermal fields depends mainly on n . Substituting Eq. (30) into Eq. (21), we have

$$A_{01}z + \sum_{j=1}^n A_{1j}(z - z_1)^{-j} + \sum_{j=1}^n A_{2j}(z - z_2)^{-j} - \left[\bar{A}_{01} \bar{z} + \sum_{j=1}^n \bar{A}_{1j} (\bar{z} - \bar{z}_1)^{-j} + \sum_{j=1}^n \bar{A}_{2j} (\bar{z} - \bar{z}_2)^{-j} \right] = 0, \quad z \in L_1, \quad (32)$$

$$A_{01}z + \sum_{j=1}^n A_{1j}(z - z_1)^{-j} + \sum_{j=1}^n A_{2j}(z - z_2)^{-j} - \left[\bar{A}_{01} \bar{z} + \sum_{j=1}^n \bar{A}_{1j} (\bar{z} - \bar{z}_1)^{-j} + \sum_{j=1}^n \bar{A}_{2j} (\bar{z} - \bar{z}_2)^{-j} \right] = 0, \quad z \in L_2. \quad (33)$$

Note that $z - z_1 = R_1 e^{i\theta} = R_1 w_1$ on L_1 and we can expand the $(z - z_2)^{-j}$ ($j = 1 \dots n$) into Fourier series as

$$(z - z_2)^{-j} = \sum_{k=-n'}^{n'} a_j^k w_1^k \quad (34)$$

where n' is carefully selected to cover the maximum power of w_1 in Eq. (32), while

$$a_j^k = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-ik\theta}}{(R_1 e^{i\theta} - z_2)^j} d\theta \quad (35)$$

Rewriting Eq. (32) using Eq. (34), we have

$$A_{01}(R_1 w_1 + z_1) + \sum_{j=1}^n A_{1j} R_1^{-j} w_1^{-j} + \sum_{j=1}^n A_{2j} \sum_{k=-n'}^{n'} a_j^k w_1^k - \left[\bar{A}_{01} (R_1 w_1^{-1} + \bar{z}_1) + \sum_{j=1}^n \bar{A}_{1j} R_1^{-j} w_1^j + \sum_{j=1}^n \bar{A}_{2j} \sum_{k=-n'}^{n'} \bar{a}_j^k w_1^{-k} \right] = 0. \quad (36)$$

Similarly, noting that $z - z_2 = R_2 e^{i\theta} = R_2 w_2$ on L_2 , we can rewrite Eq. (33) as

$$A_{01}(R_2 w_2 + z_2) + \sum_{j=1}^n A_{1j} \sum_{k=-n'}^{n'} b_j^k w_2^k + \sum_{j=1}^n A_{2j} R_2^{-j} w_2^{-j} - \left[\bar{A}_{01} (R_2 w_2^{-1} + \bar{z}_2) + \sum_{j=1}^n \bar{A}_{1j} \sum_{k=-n'}^{n'} \bar{b}_j^k w_2^{-k} + \sum_{j=1}^n \bar{A}_{2j} R_2^{-j} w_2^j \right] = 0, \quad (37)$$

where

$$(z - z_1)^{-j} = \sum_{k=-n'}^{n'} b_j^k w_2^k \quad (38)$$

and

$$b_j^k = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-ik\theta}}{(R_2 e^{i\theta} - z_1)^j} d\theta \quad (39)$$

Noting that the coefficients of w_j^k ($j = 1, 2$ and $k = 1 \cdots n$) are the complex conjugates of the coefficients of w_j^{-k} in Eqs. (36) and (37), we can obtain a system of equations with respect to A_{1j} and A_{2j} by equating the coefficients of w_j^k in Eqs. (36) and (37). Solving this system numerically, the complex function $f(z)$ can be determined entirely.

As the complex function $f(z)$ is identified, the key part of the solution describing the thermal field lies in how we determine the unknown function $g(z)$. Substituting Eqs. (30) and (31) into Eq. (22), and noting Eqs. (34) and (38), we have

$$\begin{aligned} & \frac{i\sigma}{2} \left[\left(A_{01}(R_1 w_1 + z_1) + \sum_{j=1}^n A_{1j} R_1^{-j} w_1^{-j} + \sum_{j=1}^n A_{2j} \sum_{k=-n'}^{n'} a_j^k w_1^k \right)^2 \right. \\ & \quad \left. - \left(\bar{A}_{01} (R_1 w_1^{-1} + \bar{z}_1) + \sum_{j=1}^n \bar{A}_{1j} R_1^{-j} w_1^j + \sum_{j=1}^n \bar{A}_{2j} \sum_{k=-n'}^{n'} \bar{a}_j^k w_1^{-k} \right)^2 \right] \\ & = -i\kappa \left[\left(B_{02}(R_1 w_1 + z_1)^2 + B_{01}(R_1 w_1 + z_1) + \sum_{j=1}^n B_{1j} R_1^{-j} w_1^{-j} \right. \right. \\ & \quad \left. \left. + \sum_{j=1}^n B_{2j} \sum_{k=-n'}^{n'} a_j^k w_1^k \right) - \left(\bar{B}_{02} (R_1 w_1^{-1} + \bar{z}_1)^2 + \bar{B}_{01} (R_1 w_1^{-1} + \bar{z}_1) \right. \right. \\ & \quad \left. \left. + \sum_{j=1}^n \bar{B}_{1j} R_1^{-j} w_1^j + \sum_{j=1}^n \bar{B}_{2j} \sum_{k=-n'}^{n'} \bar{a}_j^k w_1^{-k} \right) \right], \quad (40) \end{aligned}$$

$$\begin{aligned} & \frac{i\sigma}{2} \left[\left(A_{01}(R_2 w_2 + z_2) + \sum_{j=1}^n A_{1j} \sum_{k=-n'}^{n'} b_j^k w_2^k + \sum_{j=1}^n A_{2j} R_2^{-j} w_2^{-j} \right)^2 \right. \\ & \quad \left. - \left(\bar{A}_{01} (R_2 w_2^{-1} + \bar{z}_2) + \sum_{j=1}^n \bar{A}_{1j} \sum_{k=-n'}^{n'} \bar{b}_j^k w_2^{-k} + \sum_{j=1}^n \bar{A}_{2j} R_2^{-j} w_2^j \right)^2 \right] \\ & = -i\kappa \left[\left(B_{02}(R_2 w_2 + z_2)^2 + B_{01}(R_2 w_2 + z_2) + \sum_{j=1}^n B_{1j} \sum_{k=-n'}^{n'} b_j^k w_2^k \right. \right. \\ & \quad \left. \left. + \sum_{j=1}^n B_{2j} R_2^{-j} w_2^{-j} \right) - \left(\bar{B}_{02} (R_2 w_2^{-1} + \bar{z}_2)^2 + \bar{B}_{01} (R_2 w_2^{-1} + \bar{z}_2) \right. \right. \\ & \quad \left. \left. + \sum_{j=1}^n \bar{B}_{1j} \sum_{k=-n'}^{n'} \bar{b}_j^k w_2^{-k} + \sum_{j=1}^n \bar{B}_{2j} R_2^{-j} w_2^j \right) \right]. \quad (41) \end{aligned}$$

Equating the coefficients of w_j^k in Eqs. (40) and (41), and solving this set of equations numerically, the unknown coefficients B_{1j} and B_{2j} in $g(z)$ are determined. Thus, the temperature and electric potential fields around two circular holes in a thermoelectric material are given by

$$\begin{aligned} T & = -\frac{\sigma}{\kappa} \left| A_{01}z + \sum_{j=1}^n A_{1j}(z - z_1)^{-j} + \sum_{j=1}^n A_{2j}(z - z_2)^{-j} \right|^2 \\ & \quad + 2\text{Re} \left[B_{02}z^2 + B_{01}z + \sum_{j=1}^n B_{1j}(z - z_1)^{-j} + \sum_{j=1}^n B_{2j}(z - z_2)^{-j} \right] + M, \quad (42) \end{aligned}$$

$$\begin{aligned} \phi & = \frac{\sigma S}{\kappa} \left| A_{01}z + \sum_{j=1}^n A_{1j}(z - z_1)^{-j} + \sum_{j=1}^n A_{2j}(z - z_2)^{-j} \right|^2 \\ & \quad + 2\text{Re} \left[A_{01}z + \sum_{j=1}^n A_{1j}(z - z_1)^{-j} + \sum_{j=1}^n A_{2j}(z - z_2)^{-j} \right. \\ & \quad \left. - S \left(B_{02}z^2 + B_{01}z + \sum_{j=1}^n B_{1j}(z - z_1)^{-j} + \sum_{j=1}^n B_{2j}(z - z_2)^{-j} \right) \right] - SM. \quad (43) \end{aligned}$$

3.3. Effective material parameters

It has been proved that the effective parameters are independent of the remote loading in thermoelectric composites (Song et al., 2019b). Thus, the effective electric conductivity, thermal conductivity and Seebeck coefficient can be derived from the rectangular area containing two circular holes, as shown in Fig. 2, in which the width and height of the selected area are respectively

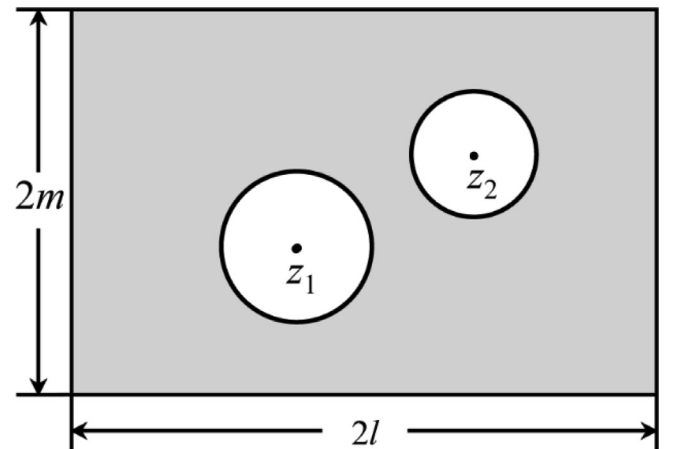


Fig. 2. Rectangular thermoelectric area contains two circular holes.

2l and 2m. Without loss of generality, the effective parameters are derived only in the y-direction (the effective parameters in other directions can be derived similarly). When we set the applied energy flux $U_y^\infty = 0$, the temperature difference between $y = -m$ and $y = m$ is zero, and then the effective electric conductivity σ^* can be derived from Eq. (1) as

$$\sigma^* = \frac{2m \int_{-l}^l J_y|_{y=-m} ds}{\int_{-l}^l \phi|_{y=-m} ds - \int_{-l}^l \phi|_{y=m} ds} U_y^\infty = 0 \quad (44)$$

In the same way, the effective thermal conductivity κ^* and Seebeck coefficient S^* can be deduced under the condition that $J_y^\infty = 0$. From Eqs. (1) and (2), we have

$$\kappa^* = \frac{2m \int_{-l}^l Q_y|_{y=-m} ds}{\int_{-l}^l T|_{y=-m} ds - \int_{-l}^l T|_{y=m} ds} \quad (45)$$

$$S^* = -\frac{\int_{-l}^l \phi|_{y=-m} ds - \int_{-l}^l \phi|_{y=m} ds}{\int_{-l}^l T|_{y=-m} ds - \int_{-l}^l T|_{y=m} ds} \quad (46)$$

Ultimately, the effective thermoelectric figure of merit can be derived as (Tritt et al., 2008)

$$Z^* = \frac{\sigma^* S^{*2}}{\kappa^*} \quad (47)$$

3.4. Solutions for the stress field

According to Eqs. (17) and (18), the thermal stress generated by the temperature field firmly depends on the function $F(z)$. Noting Eq. (30), we have

$$F(z) = \frac{A_{01} z^2}{2} + A_{11} \ln(z - z_1) - \sum_{j=2}^n \frac{A_{1j} (z - z_1)^{-j+1}}{j-1} + A_{21} \ln(z - z_2) - \sum_{j=2}^n \frac{A_{2j} (z - z_2)^{-j+1}}{j-1}. \quad (48)$$

In order to neutralize the multi-valued terms in the stress and displacement fields, we express the unknown functions $\varphi(z)$ and $\psi(z)$ as

$$\varphi(z) = \chi_{11} \ln(z - z_1) + \chi_{21} \ln(z - z_2) + \varphi_1(z - z_1) + \varphi_2(z - z_2) \quad (49)$$

$$\psi(z) = \chi_{12} \ln(z - z_1) + \chi_{22} \ln(z - z_2) + \psi_1(z - z_1) + \psi_2(z - z_2) \quad (50)$$

where the coefficients χ_{jk} ($j = 1, 2, k = 1, 2$) can be determined using single-valued conditions of stress and displacement as

$$\begin{cases} \chi_{11} = -\frac{4G\alpha}{K+1} B_{11}, & \chi_{21} = -\frac{4G\alpha}{K+1} B_{21}, \\ \chi_{12} = \bar{\chi}_{11} + \frac{E\alpha\sigma}{2K} A_{11} f(z), & \chi_{22} = \bar{\chi}_{21} + \frac{E\alpha\sigma}{2K} A_{21} f(z). \end{cases} \quad (51)$$

We further expand $\varphi(z)$ and $\psi(z)$ into Fourier series as

$$\begin{aligned} \varphi(z) &= \chi_{11} \ln(z - z_1) + \chi_{21} \ln(z - z_2) + \sum_{j=1}^n C_{1j} (z - z_1)^{-j} \\ &\quad + \sum_{j=1}^n C_{2j} (z - z_2)^{-j} \end{aligned} \quad (52)$$

$$\begin{aligned} \psi(z) &= \chi_{12} \ln(z - z_1) + \chi_{22} \ln(z - z_2) + \sum_{j=1}^n D_{1j} (z - z_1)^{-j} \\ &\quad + \sum_{j=1}^n D_{2j} (z - z_2)^{-j} \end{aligned} \quad (53)$$

Substituting Eqs. (52) and (53) into Eq. (25), and noting Eqs. (34) and (38), we have

$$\begin{aligned} &2\chi_{11} \ln R_1 + 2\chi_{21} \ln |R_1 w_1 + z_1 - z_2| + \sum_{j=1}^n C_{1j} R_1^{-j} w_1^{-j} \\ &\quad + \sum_{j=1}^n C_{2j} \sum_{k=-n'}^{n'} a_j^k w_1^k + (R_1 w_1 + z_1) \left(\bar{\chi}_{11} R_1^{-1} w_1 + \bar{\chi}_{21} \sum_{k=-n'}^{n'} \bar{a}_1^k w_1^{-k} \right. \\ &\quad \left. - j \sum_{j=1}^n \bar{C}_{1j} R_1^{-j-1} w_1^{j+1} - j \sum_{j=1}^n \bar{C}_{2j} \sum_{k=-n'}^{n'} \bar{a}_{j+1}^k w_1^{-k} \right) + \sum_{j=1}^n \bar{D}_{1j} R_1^{-j} w_1^j \\ &\quad + \sum_{j=1}^n \bar{D}_{2j} \sum_{k=-n'}^{n'} \bar{a}_j^k w_1^{-k} + \frac{E\alpha\sigma}{2K} \left(\bar{A}_{01} (R_1 w_1^{-1} + \bar{z}_1) + \sum_{j=1}^n \bar{A}_{1j} R_1^{-j} w_1^j \right. \\ &\quad \left. + \sum_{j=1}^n \bar{A}_{2j} \sum_{k=-n'}^{n'} \bar{a}_j^k w_1^{-k} \right) \times \left(\frac{A_{01}}{2} (R_1 w_1 + z_1)^2 + 2A_{11} \ln R_1 \right. \\ &\quad \left. - \sum_{j=2}^n \frac{A_{1j} R_1^{-j+1} w_1^{-j+1}}{j-1} + 2A_{21} \ln |R_1 w_1 - z_2| - \sum_{j=2}^n \frac{A_{2j}}{j-1} \sum_{k=-n'}^{n'} a_{j-1}^k w_1^k \right) = 0, \end{aligned} \quad (54)$$

$$\begin{aligned} &2\chi_{11} \ln |R_2 w_2 + z_2 - z_1| + 2\chi_{21} \ln R_2 \\ &\quad + \sum_{j=1}^n C_{1j} \sum_{k=-n'}^{n'} b_j^k w_2^k + \sum_{j=1}^n C_{2j} R_2^{-j} w_2^{-j} + (R_2 w_2 + z_2) \\ &\quad \left(\bar{\chi}_{11} \sum_{k=-n'}^{n'} \bar{b}_1^k w_2^{-k} + \bar{\chi}_{21} R_2^{-1} w_2 - j \sum_{j=1}^n \bar{C}_{1j} \sum_{k=-n'}^{n'} \bar{b}_{j+1}^k w_2^{-k} - j \sum_{j=1}^n \bar{C}_{2j} R_2^{-j-1} w_2^{j+1} \right) \\ &\quad + \sum_{j=1}^n \bar{D}_{1j} \sum_{k=-n'}^{n'} \bar{b}_j^k w_2^{-k} + \sum_{j=1}^n \bar{D}_{2j} R_2^{-j} w_2^j + \frac{E\alpha\sigma}{2K} \\ &\quad \left(\bar{A}_{01} (R_2 w_2^{-1} + \bar{z}_2) + \sum_{j=1}^n \bar{A}_{1j} \sum_{k=-n'}^{n'} \bar{b}_j^k w_2^{-k} + \sum_{j=1}^n \bar{A}_{2j} R_2^{-j} w_2^j \right) \\ &\quad \times \left(\frac{A_{01}}{2} (R_2 w_2 + z_2)^2 + 2A_{11} \ln |R_2 w_2 - z_1| - \sum_{j=2}^n \frac{A_{1j}}{j-1} \sum_{k=-n'}^{n'} b_{j-1}^k w_2^k \right. \\ &\quad \left. + 2A_{21} \ln R_2 - \sum_{j=2}^n \frac{A_{2j}}{j-1} R_2^{-j+1} w_2^{-j+1} \right) = 0, \end{aligned} \quad (55)$$

where

$$\ln |R_1 w_1 + z_1 - z_2| = \sum_{j=-n'}^{n'} c_j w_1^j \quad (56)$$

$$\ln |R_2 w_2 + z_2 - z_1| = \sum_{j=-n'}^{n'} h_j w_2^j \quad (57)$$

in which

$$c_j = \frac{1}{2\pi} \int_0^{2\pi} \ln |R_1 e^{i\theta} + z_1 - z_2| e^{-ij\theta} d\theta \quad (58)$$

$$h_j = \frac{1}{2\pi} \int_0^{2\pi} \ln |R_2 e^{i\theta} + z_2 - z_1| e^{-ij\theta} d\theta \quad (59)$$

In this way Eq. (54) becomes the equation with respect to w_1 , while Eq. (55) is the equation for w_2 . Equating the coefficients of w_1^k ($k = -n \dots -1, 1 \dots n$) in Eq. (54) and w_2^k in Eq. (55), a system

containing $4n$ equations can be obtained. Solving this system numerically, the coefficients C_{1j} , C_{2j} , D_{1j} and D_{2j} ($j = 1 \cdots n$) can be determined completely.

4. Numerical analyses

In this section, we analyze the temperature and thermal stress around two circular holes in a thermoelectric material, in which the material parameters of the thermoelectric matrix surrounding the hole are listed in Table 1 (Wu et al., 2014). Without loss of generality, the remote electric current and energy flux are applied in the y -direction as $J_y^\infty = 3 \times 10^4$ A/m² and $U_y^\infty = 4 \times 10^3$ W/m².

We first investigate the temperature, electric potential and hoop stress on the boundary of hole 1 when $R_1 = R_2$ and $d = 0.4R_1$ for different values of n , while n' can be selected as $n + 1$ so that the highest powers of w_j ($j = 1, 2$) in Eqs. (34), (38), (56), and (57) are sufficient to be used in determining the unknown coefficients. Since the temperature and electric potential have the same rate of convergence with the increase of n , we illustrate in Fig. 3 only the temperature and hoop stress around hole 1. It can be seen that the distributions of the temperature field converge to a uniform field when $n > 5$, while the distributions of hoop stress tend to be uniform when $n > 15$. The results show that the thermal, electric and elastic fields around two circular holes in thermoelectric materials converge quickly with the increase of n . In order to avoid the error caused by numerical calculation, we set $n = 50$ in the following numerical examples.

In order to verify the correctness of the numerical results, we compare the temperature and hoop stress to previously published results in the literature by allowing our model to degenerate into the corresponding classical case. Firstly, the interaction effect between the two holes is negligible when we set $d \rightarrow \infty$, and then the thermal-electric-elastic problem around hole 1 can be considered as the corresponding problem around a single hole. Numerical results in Fig. 4 show that the solutions describing temperature

and hoop stress are consistent with the analytical solutions around a single hole (Wang and Wang, 2017) when $d \rightarrow \infty$. Secondly, the thermal-electric-elastic model will degenerate into the classical thermoelasticity model when $\sigma \rightarrow 0$. Noting this, we compare the temperature and hoop stress around hole 1 to the results from the classical thermoelasticity model (Song et al., 2021) for the case when $d = 0.4R_1$ and $\sigma \rightarrow 0$. The results in Fig. 4 indicate the correctness of the theoretical and numerical solutions obtained in this paper.

We analyze the effective electric conductivity, thermal conductivity and Seebeck coefficient under different conditions, such as remote loading and size of the selected region. The results show that the effective material parameters firmly depend on the proportion of the area between the circular holes and the rectangle, but not on the remote loading and absolute size of the selected region. For the case that $l/m = 2$, the effective electric conductivity, thermal conductivity and Seebeck coefficient versus the area proportion are illustrated in Fig. 5(a). As we can see, the effective electric and thermal conductivities decrease in the same way with the increase of v_c , and the rate of descent reaches a maximum when the two holes are equal in size. On the other hand, the two holes have no effect on the effective Seebeck coefficient, and thus have no effect on the effective thermoelectric figure of merit since their effect on the electric and thermal conductivities essentially cancel each other out in constituting the thermoelectric figure of merit. This conclusion agrees with the effective thermoelectric figure of merit around a single hole (Song et al., 2019b). We further illustrate the effective properties versus the distance between two holes in Fig. 5(b), where $l/m = 2$, $R_1 = R_2$, and the centers of two holes are located, respectively, on the x -axis, $y = x$ and on the y -axis. Results show that the adjacent two holes have a stronger influence on the effective electric and thermal conductivities when their centers are located on the x -axis yet, in all cases, they do not affect in any way, the effective thermoelectric figure of merit.

When the electric current and heat flux are blocked by insulated holes, they will bypass the holes which leads to uneven electric

Table 1
Material parameters of thermoelectric sample.

Sample	σ (Sm ⁻¹)	ε (VK ⁻¹)	κ (Wm ⁻¹ K ⁻¹)	E (GPa)	α (K ⁻¹)	μ
Bi ₂ Te ₃	100,000	0.00022	1.1	63.00	1.3×10^{-5}	0.23

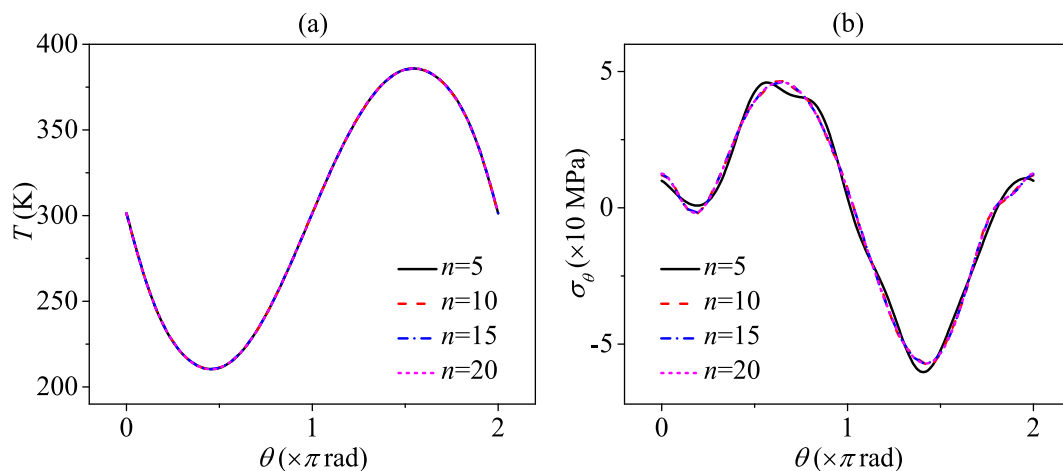


Fig. 3. (a) Temperature field and (b) hoop stress on the boundary of hole 1 for different values of n , where $n' = n + 1$, $R_1 = R_2$ and $d = 0.4R_1$.

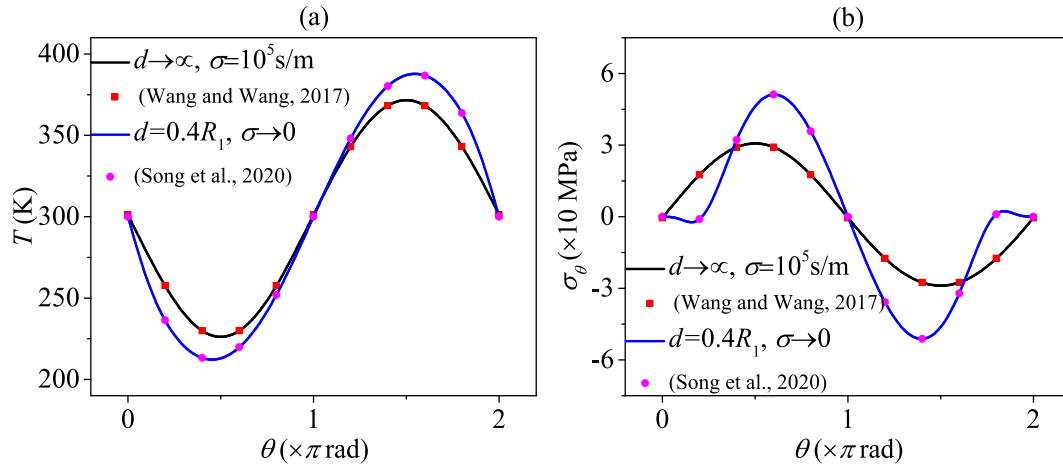


Fig. 4. (a) Temperature field and (b) hoop stress on the boundary of hole 1 for the particular conditions $d \rightarrow \infty$ and $\sigma \rightarrow 0$.

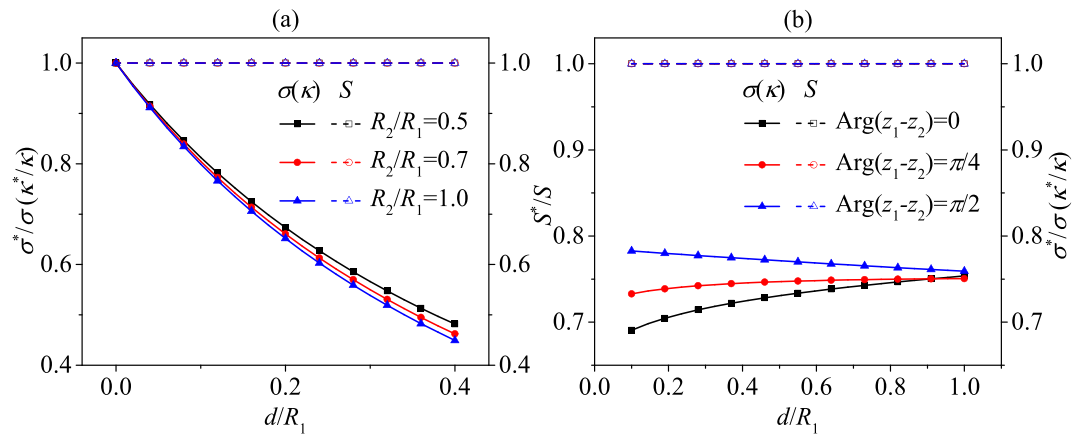


Fig. 5. Effective material parameters versus (a) area proportion and (b) distance between two holes, where v_c and v_s are respectively the total area of circular holes and area of selected region, while “Arg” is argument of the complex number from the real axis.

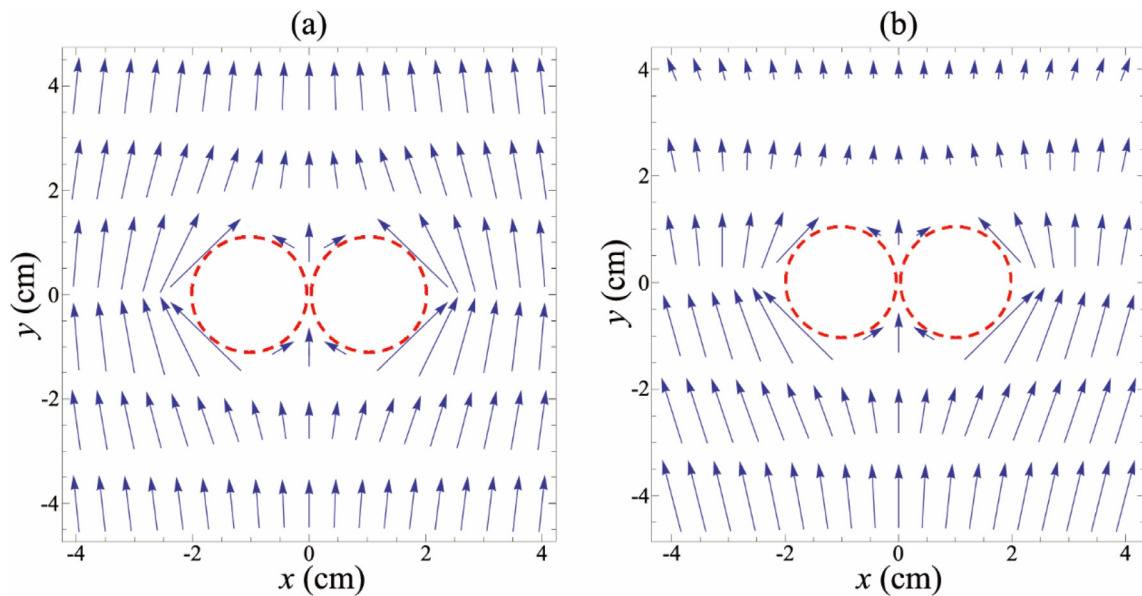


Fig. 6. Distributions of (a) electric current and (b) heat flux when $d = 0.1 R_1$.

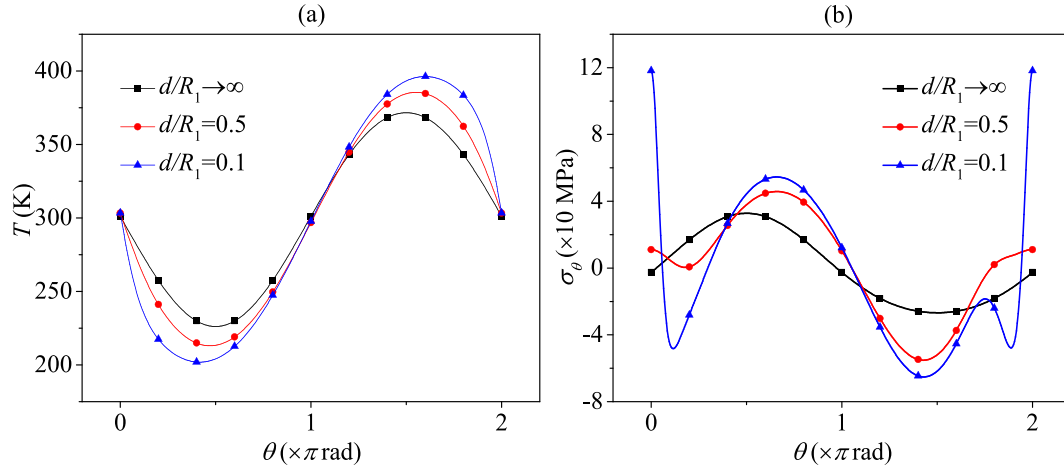


Fig. 7. (a) Temperature field and (b) hoop stress on the boundary of hole 1 when $R_1 = R_2$.

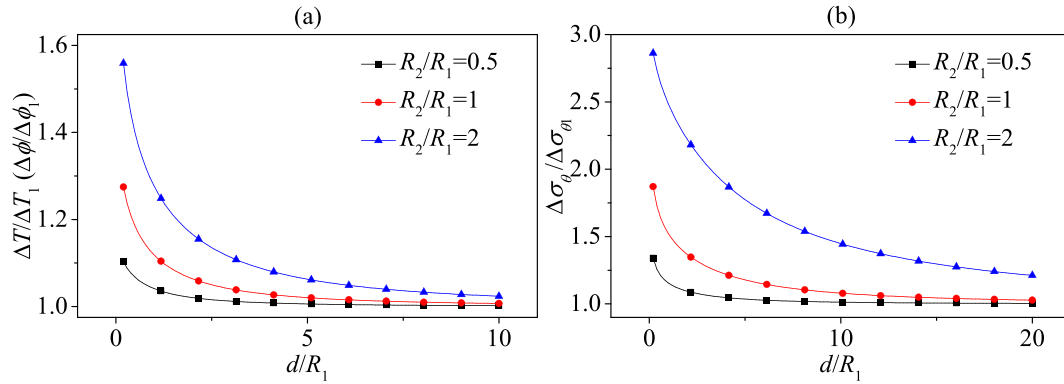


Fig. 8. Maximum difference of temperature (electric potential) and hoop stress on the boundary of hole 1 versus distance between two holes, where ΔT_1 and $\Delta \sigma_{\theta 1}$ are maximum temperature and hoop stress on the boundary of a single hole ($d \rightarrow \infty$).

potential and temperature fields. Since the electric charge is conserved in our system, the electric current gradually returns to the original direction and magnitude after going around the two holes, as shown in Fig. 6(a), where $d = 0.1R_1$. However, the heat flux continues to get smaller in the direction of conduction due to the interconversion between heat and electricity (see Fig. 6(b)), which is in stark contrast to classical heat conduction problems where the system is assumed to be heat conserved. We can see from Fig. 6(b) that the heat flux decreases by half when passing through the gap between the two holes, and thus the corresponding temperature difference in the upper half plane ($y \geq 0$) is quite different from the temperature difference in the lower half plane ($y < 0$). Such a temperature field generates incompatible thermal expansion, which stands a good chance to generate extreme thermal stress between the two holes, especially in the case that the distance between the two holes is much smaller than their size.

The temperature and hoop stress on the boundary of hole 1 for various distances between the two holes are illustrated in Fig. 7. Results show that both the maximum temperature and stress differences around hole 1 are enhanced by the presence of the adjacent hole, while the interaction effects increase with the decrease of distance between the two holes. From Fig. 7(a) we discover that the temperature gradient along the boundary of hole 1 increases with the decrease of d , especially near the area between the two holes. The higher temperature gradient generates higher hoop stress around hole 1, as shown in Fig. 7(b). In particular, the hoop stress at the point $\theta = 0$ increases rapidly with the decrease of d and surpasses triple the value of the maximum hoop stress around

a single hole (under the same conditions, the maximum hoop stress around a single hole is equal to that of the case when $d \rightarrow \infty$) when $d = 0.1R_1$. Detailed results show that the hoop stress at $\theta = 0$ will reach an extreme value when the distance between the two holes is sufficiently small which is again very different from the case in classical thermoelastic problems (Nakagawa et al., 1999). For example, the maximum hoop stress in the case $d = 0.01R_1$ is over 40 times that of the maximum hoop stress around a single hole in a thermoelectric material, but the maximum hoop stress around two adjacent holes is no more than four times of that around a single hole in classical thermoelasticity.

We have found that the interaction effects on electric, thermal and stress fields between two holes vanish when the distance d is much larger than the maximum size of the two holes. To further explore the relationship between distance and interaction effects, we illustrate in Fig. 8 the maximum differences of temperature (electric potential) and hoop stress around hole 1 versus the distance between the two holes. Our results show that the two holes have the same interaction effects on electric and thermal fields, and that a large hole shows a stronger effect on the temperature and stress fields around a relatively small hole. With the increase of d , the interaction effect on temperature field decreases sharply and is less than 7% when $d = 5R_1$ for all the cases shown in Fig. 8 (a). However, the interaction effect on the stress field is significantly greater than that on the temperature field. As can be seen from Fig. 8(b), the interaction effect between two holes of equal size is larger than 17% when $d = 5R_1$. When $R_2 = 2R_1$, the hoop stress around hole 1 continues to be enhanced by hole 2 by more

than 20% even for the case that $d = 20R_1$, which means that the interaction effect on thermal stress cannot be ignored when the difference in radii between the two holes is large.

5. Conclusions

We investigate the interaction effects on electric, thermal and elastic fields between two circular holes in a thermoelectric material. With the aid of complex variable and numerical methods, we obtain partly theoretical and partly numerical solutions, and conclude the following main results:

- The two circular holes have the same effect on the effective electric and thermal conductivities, and have no effect on the effective Seebeck coefficient and consequently no effect on the effective thermoelectric figure of merit.
- Due to the energy conversion between heat and electricity, the adjacent two holes lead to an uneven temperature field which induces an extreme stress distribution when the distance between the two holes is much less than their size; this is in stark contrast to classical thermoelasticity.
- With the increase of distance between the two holes, the interaction effects on the electric and thermal fields decrease sharply, while the interaction effects on the stress field decrease relatively slowly, especially for the cases when the difference in radii between the two holes is large.

We hope that our results can provide guidance for the optimal design of thermoelectric devices and lay the foundation for the study of multi-field coupling problems of porous structures.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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